Portfolio theory and risk management
Homework set 2

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General information

The homework set gives at most 3 points which are added to your result on the exam. You may work individually or in groups of at most two persons. To obtain the points you/the group must present the solution nicely in a report which clearly shows how the problems were solved. I will consider both correctness and the quality of the report important when I evaluate your work. The report, printed on paper, must be handed in on time in order to be accepted.

The solutions to the homework set must be handed in no later than Friday 1/10 10:15.

Exercises

Exercise 1. Consider the numerical evaluation of mean-variance approaches to optimal investments in Section 4.1.1 in the lecture notes. Change the mean vector $\boldsymbol{\mu}$ to values that you consider realistic (for yearly returns) so that the theoretical Sharpe ratio becomes bigger, approximately one. Do the corresponding computations and the plots corresponding to Figures 4.1, 4.2, and 4.3.

Exercise 2. Consider the mean-variance approach to optimal investments presented in Section 4.1 in the lecture notes. Suppose there is also a liability (subtract the random variable L from the value of the portfolio at the end of the time period).

(a) Determine and interpret the optimal solution to the problem corresponding to (4.4) in the lecture notes (now with a liability).

(b) Suppose that you are not allowed to take a short position in the bond. Determine and interpret the optimal solution to the problem corresponding to (4.4) in the lecture notes (now with a liability and no short sales of the bond).

1 Evaluating the methods on simulated data

Consider a vector $\mathbf{R}$ of percentage returns for two risky asset whose mean vector and covariance matrix are given by

$$\boldsymbol{\mu} = \begin{pmatrix} 1.025 \\ 1.075 \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2 0.5 \\ \sigma_1\sigma_2 0.5 & \sigma_2^2 \end{pmatrix},$$

where $\sigma_1 = 0.3$ and $\sigma_2 = 0.5$. Suppose that there also exists a risk-free asset with percentage return $R_{rf} = 1$.

Suppose that we want to invest according to the solution to one of the three versions of the investment problem that we have analyzed analytically. The first investment criterion is

$$\begin{array}{ll} \text{maximize} & cV_0(\boldsymbol{\mu}^{\mathrm{T}}\mathbf{w} + w_{R_{rf}}R_{rf}) - \frac{1}{2}\mathbf{w}^{\mathrm{T}}\Sigma\mathbf{w} \\ \text{subject to} & \mathbf{1}^{\mathrm{T}}\mathbf{w} + w_{R_{rf}} \leq V_0 \end{array} \qquad \text{(Trade-off)}$$

The second investment criterion is

$$\begin{array}{ll} \text{maximize} & \boldsymbol{\mu}^{\mathrm{T}}\mathbf{w} + w_{R_{rf}}R_{rf} \\ \text{subject to} & \mathbf{w}^{\mathrm{T}}\Sigma\mathbf{w} \leq \sigma_0^2 V_0^2 \\ \text{and} & \mathbf{1}^{\mathrm{T}}\mathbf{w} + w_{R_{rf}} \leq V_0 \end{array} \qquad \text{(MaxExp)}$$

with $\sigma_0 = 0.3$, i.e. we want the standard deviation of the percentage return for the portfolio to be smaller or equal to the smallest of those of the risky assets. The third investment criterion is

$$\begin{array}{ll} \text{minimize} & \mathbf{w}^{\mathrm{T}}\Sigma\mathbf{w} \\ \text{subject to} & \boldsymbol{\mu}^{\mathrm{T}}\mathbf{w} + w_{R_{rf}}R_{rf} \geq \mu_0 V_0 \\ \text{and} & \mathbf{1}^{\mathrm{T}}\mathbf{w} + w_{R_{rf}} \leq V_0. \end{array} \qquad \text{(MinVar)}$$

The parameters $c = 1.995897$ and $\mu_0 = 1.045092$ are chosen so that the optimal solutions to the three investment problems coincide. We may without loss of generality set $V_0 = 1$ which means that the solution $\mathbf{w}$ is the position in the risky assets per unit of initial capital. The common theoretical solution $\mathbf{w}$ to the investment problems (Trade-off), (MaxExp), and (MinVar) is given by

$$\begin{aligned} \mathbf{w} &= c\Sigma^{-1}(\boldsymbol{\mu} - R_{rf}\mathbf{1}) \\ &= \sigma_0 \frac{\Sigma^{-1}(\boldsymbol{\mu} - R_{rf}\mathbf{1})}{\sqrt{(\boldsymbol{\mu} - R_{rf}\mathbf{1})^{\mathrm{T}}\Sigma^{-1}(\boldsymbol{\mu} - R_{rf}\mathbf{1})}} \\ &= (\mu_0 - R_{rf}) \frac{\Sigma^{-1}(\boldsymbol{\mu} - R_{rf}\mathbf{1})}{(\boldsymbol{\mu} - R_{rf}\mathbf{1})^{\mathrm{T}}\Sigma^{-1}(\boldsymbol{\mu} - R_{rf}\mathbf{1})} \\ &\approx \begin{pmatrix} 0.074 \\ 0.577 \end{pmatrix} \end{aligned}$$

and $w_{R_{rf}} = 1 - \mathbf{1}^{\mathrm{T}}\mathbf{w} \approx 0.349$. However, we do not know $\boldsymbol{\mu}$ and Σ and therefore not the optimal solution $(\mathbf{w}, w_{R_{rf}})$.

Suppose that $\mathbf{R}$ is normally distributed with mean $\boldsymbol{\mu}$ and covariance matrix Σ and that we have observed 200 outcomes of independent copies of $\mathbf{R}$. From these observations we can compute estimates $\widehat{\boldsymbol{\mu}}$ and $\widehat{\Sigma}$ and obtain estimates $(\widehat{\mathbf{w}}, \widehat{w}_{R_{rf}})$ by replacing $\boldsymbol{\mu}$ and Σ by $\widehat{\boldsymbol{\mu}}$ and $\widehat{\Sigma}$ in the above expressions for the solutions to (Trade-off), (MaxExp), and (MinVar). In order to determine the accuracy of these estimates we repeat this scheme 3000 times and plot the estimated weights $\widehat{\mathbf{w}}$ for the solutions to the three investment problems.

The first three plots in Figure 1 are scatter plots of the 3000 portfolio weights in the risky assets for the three versions of the investment problems in the above order. In total, 3000 samples of 200 independent copies of $\mathbf{R}$ were generated. Each of the samples generated an estimate of $(\boldsymbol{\mu}, \Sigma)$ which in turn generated one point $\widehat{\mathbf{w}}$ for each of the three versions of the investment problem.

Notice that for (MaxExp) the risk constraint and the estimate $\widehat{\Sigma}$ force the solution $\widehat{\mathbf{w}}$ to be a point on the ellipse $\widehat{\mathbf{w}}^{\mathrm{T}}\widehat{\Sigma}\widehat{\mathbf{w}} = \sigma_0^2$. Since the estimates $\widehat{\Sigma}$ vary across the 3000 samples the points $\widehat{\mathbf{w}}$ of the scatter plot form a point cloud that is concentrated near the ellipse $\mathbf{w}^{\mathrm{T}}\Sigma\mathbf{w} = \sigma_0^2$. The points of the scatter plots for (Trade-off) and (MinVar) are more spread-out, especially for the problem (MinVar). Notice in particular that many of the the solutions $\widehat{\mathbf{w}}$ for (MinVar) based on the simulated samples are very far from the theoretical solution $\mathbf{w} = (0.074, 0.577)^{\mathrm{T}}$. The reason for this is that many of the estimated values $\widehat{\boldsymbol{\mu}} - R_{rf}\mathbf{1}$ are very close to $\mathbf{0}$ causing the weights to "explode" due to the values very close to 0 in the denominator that for (MinVar), unlike (MaxExp), are not canceled by the same small values in the nominator.

Each of the remaining three plots in Figure 1 are scatter plots of the 3000 points

$$(\sigma(\widehat{\mathbf{w}}), \mu(\widehat{\mathbf{w}})) = (\sqrt{\widehat{\mathbf{w}}^{\mathrm{T}}\Sigma\widehat{\mathbf{w}}}, \boldsymbol{\mu}^{\mathrm{T}}\widehat{\mathbf{w}} + (1 - \mathbf{1}^{\mathrm{T}}\widehat{\mathbf{w}})R_{rf}),$$

for the three versions of the investment problem, where each $\widehat{\mathbf{w}}$ is a point of the corresponding scatter plot in Figure 1. That is, the pairs of plots are (1,4), (2,5), and (3,6). For a given vector $\widehat{\mathbf{w}}$ of portfolio weights, $\sigma(\widehat{\mathbf{w}})$ and $\mu(\widehat{\mathbf{w}})$ are the standard deviation and expected value of the percentage return for that portfolio. Note that $(\sigma(\mathbf{w}), \mu(\mathbf{w})) = (\sigma_0, \mu_0)$. Since $\widehat{\mathbf{w}}$ is a function of the empirical mean vector $\widehat{\boldsymbol{\mu}}$ and covariance matrix $\widehat{\Sigma}$, $(\sigma(\widehat{\mathbf{w}}), \mu(\widehat{\mathbf{w}}))$ is a random vector.

The standard deviations and expected values $(\sigma(\widehat{\mathbf{w}}), \mu(\widehat{\mathbf{w}}))$ for the estimated solutions to (MaxExp) are much closer to the theoretical value (σ_0, μ_0) than for (MinVar), and also closer than for (Trade-off). We find that whereas the problem (MaxExp) is rather robust to noise perturbing

the parameter $\boldsymbol{\mu}$, this is not at all so for the problem (MinVar). However, it is interesting to note that the empirical Sharpe ratios

$$\frac{\mu(\widehat{\mathbf{w}}) - R_{rf}}{\sigma(\widehat{\mathbf{w}})} = \frac{(\boldsymbol{\mu} - R_{rf}\mathbf{1})^{\mathrm{T}}\widehat{\Sigma}^{-1}(\widehat{\boldsymbol{\mu}} - R_{rf}\mathbf{1})}{\sqrt{(\widehat{\boldsymbol{\mu}} - R_{rf}\mathbf{1})^{\mathrm{T}}\widehat{\Sigma}^{-1}\Sigma\widehat{\Sigma}^{-1}(\widehat{\boldsymbol{\mu}} - R_{rf}\mathbf{1})}}$$

coincide for the three versions of the investment problem.

Figure 2 shows the same thing as Figure 1 when the theoretical mean vector $\boldsymbol{\mu}$ was set to $\boldsymbol{\mu} = (1.1, 1.2)^{\mathrm{T}}$. Here, the difference in accuracy for the solutions to the three versions of the investment problem based on simulated data is much smaller. The reason is that here we do not find estimates $\widehat{\boldsymbol{\mu}} - R_{rf}\mathbf{1} \approx \mathbf{0}$ and therefore no exploding weights $\widehat{\mathbf{w}}$ due to "division by zero".

Let us look a bit closer at the accuracy of the estimation of means. For sake of clarity we consider the univariate case. Consider the simplest possible univariate case; given a sample $\{R_1, \ldots, R_m\}$ of independent random variables with common mean $\mathrm{E}[R_k] = \mu$ and variance $\mathrm{Var}(R_k) = \sigma^2$ we consider the problem of estimating μ. Set $\widehat{\mu} = (R_1 + \cdots + R_m)/m$, i.e. the standard estimator. Then $\mathrm{E}[\widehat{\mu}] = \mu$ and

$$\begin{aligned}
\mathrm{Var}(\widehat{\mu}) = \mathrm{E}[(\widehat{\mu} - \mu)^2] &= \mathrm{E}\left[\left(\frac{1}{m}\sum_{k=1}^{m}(R_k - \mu)\right)^2\right] \\
&= \frac{1}{m^2}\sum_{k=1}^{m}\mathrm{E}[(R_k - \mu)^2] + \frac{1}{m^2}\sum_{j\neq k}\underbrace{\mathrm{E}[(R_j - \mu)(R_k - \mu)]}_{=0} \\
&= \frac{\sigma^2}{m}.
\end{aligned}$$

Hence, the estimator $\widehat{\mu}$ has standard deviation $\sigma/\sqrt{m}$. In the simulation study above we have, for the first component, $\mu = 1.025$, $\sigma = 0.3$ and $m = 200$. In particular,

$$\mu - R_{rf} = 0.025 \approx 0.021 \approx \sigma/\sqrt{m} = \sqrt{\mathrm{Var}(\widehat{\mu} - R_{rf})}.$$

2 Investments in the presence of liabilities

We now consider optimal investments in the presence of liabilities. We consider the trade-off version of the optimal investment problem, high expected payoff is good and high variance of payoff is bad and a constant $c \geq 0$ determines the trade-off we aim for between the two. The optimization problem reads

$$\begin{array}{ll}
\text{maximize} & c\,\mathrm{E}[h_0 + \mathbf{h}^{\mathrm{T}}\mathbf{S}_1 - L] - \frac{1}{2}\,\mathrm{Var}(h_0 + \mathbf{h}^{\mathrm{T}}\mathbf{S}_1 - L) \\
\text{subject to} & h_0 B_0 + \mathbf{h}^{\mathrm{T}}\mathbf{S}_0 \leq V_0.
\end{array} \tag{1}$$

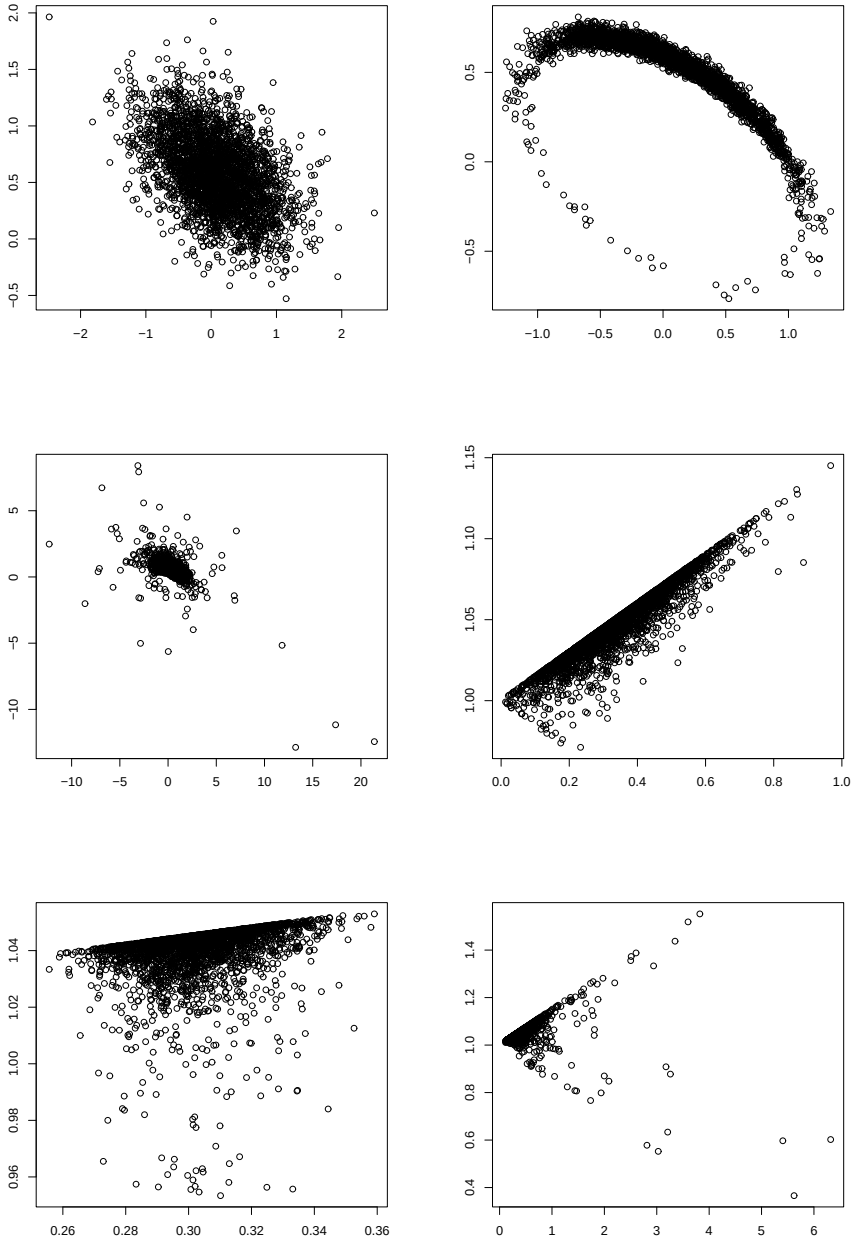

Figure 1: The first three plots show empirical optimal portfolio weights in risky assets based on 3000 samples of size 200 for the (Trade-off)-, (MaxExp)-, and (MinVar) version for $\boldsymbol{\mu} = (1.025, 1.075)^{\mathrm{T}}$. The remaining plots show the corresponding standard deviation-mean pairs.

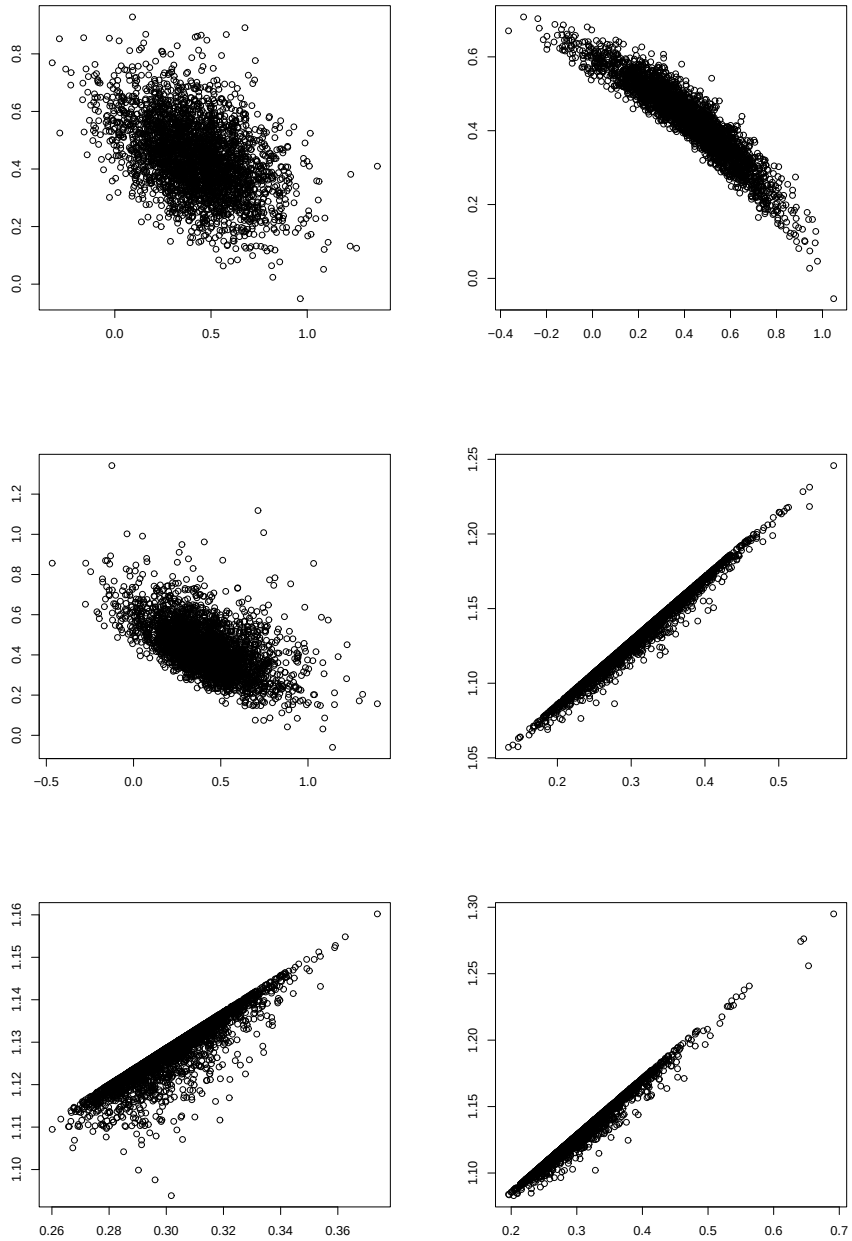

Figure 2: The first three plots show empirical optimal portfolio weights in risky assets based on 3000 samples of size 200 for the (Trade-off)-, (MaxExp)-, and (MinVar) version for $\boldsymbol{\mu} = (1.1, 1.2)^{\mathrm{T}}$. The remaining plots show the corresponding standard deviation-mean pairs.

We choose to formulate it as the convex optimization problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\big(\mathbf{h}^{\mathrm{T}}\Sigma_{\mathbf{S}_1}\mathbf{h} + \sigma_L^2 - 2\mathbf{h}^{\mathrm{T}}\Sigma_{L,\mathbf{S}_1}\big) - c\big(h_0 + \mathbf{h}^{\mathrm{T}}\boldsymbol{\mu}_{\mathbf{S}_1} - \mu_L\big) \\ \text{subject to} & h_0 B_0 + \mathbf{h}_0^{\mathrm{T}}\mathbf{S}_0 \leq V_0. \end{array}$$

The same arguments as in the case of no liability leads to the necessary and sufficient conditions

$$\begin{aligned} &\mathbf{h} = \Sigma_{\mathbf{S}_1}^{-1}\big(c\boldsymbol{\mu}_{\mathbf{S}_1} - \lambda_1\mathbf{S}_0 + \Sigma_{L,\mathbf{S}_1}\big), \\ &\lambda_1 B_0 - c = 0, \\ &\lambda_1(h_0 B_0 + \mathbf{h}^{\mathrm{T}}\mathbf{S}_0 - V_0) = 0, \\ &h_0 B_0 + \mathbf{h}^{\mathrm{T}}\mathbf{S}_0 \leq V_0, \end{aligned}$$

and therefore the optimal solution

$$\begin{aligned} \mathbf{h} &= c\Sigma_{\mathbf{S}_1}^{-1}\big(\boldsymbol{\mu}_{\mathbf{S}_1} - \frac{1}{B_0}\mathbf{S}_0\big) + \Sigma_{\mathbf{S}_1}^{-1}\Sigma_{L,\mathbf{S}_1}, \\ h_0 &= \frac{1}{B_0}\big(V_0 - \mathbf{h}^{\mathrm{T}}\mathbf{S}_0\big). \end{aligned}$$

We observe that the solution corresponds to the minimum variance hedge plus the optimal investment position without a liability. If the initial capital V_0 is insufficient to take this position, then this problem is solved by borrowing money (a short position in the risk-free bond).

A relevant question is: what is the right trade-off between hedging the liability and speculating in case borrowing money is not possible? This problem is the problem above with the inclusion of the constraint $h_0 \geq 0$ (or equivalently $-h_0 \leq 0$). The necessary and sufficient conditions for an optimal solution becomes

$$\begin{aligned} &\mathbf{h} = \Sigma_{\mathbf{S}_1}^{-1}\big(c\boldsymbol{\mu}_{\mathbf{S}_1} - \lambda_1\mathbf{S}_0 + \Sigma_{L,\mathbf{S}_1}\big), \\ &\lambda_1 B_0 - c - \lambda_2 = 0, \\ &\lambda_1(h_0 B_0 + \mathbf{h}^{\mathrm{T}}\mathbf{S}_0 - V_0) = 0, \\ &\lambda_2 h_0 = 0, \\ &\lambda_1, \lambda_2, h_0 \geq 0, \\ &h_0 B_0 + \mathbf{h}^{\mathrm{T}}\mathbf{S}_0 \leq V_0. \end{aligned}$$

If $\lambda_2 = 0$, then we have the solution above, with $h_0 \geq 0$. Therefore the interesting case is when $\lambda_2 > 0$ which implies $h_0 = 0$. In this case $\mathbf{h}^{\mathrm{T}}\mathbf{S}_0 = V_0$ since $\mathbf{h}^{\mathrm{T}}\mathbf{S}_0 < V_0$ would correspond to throwing away money rather than investing them in a risk-free bond, which is clearly sub-optimal. The optimal solution is

$$\begin{aligned} &\mathbf{h} = c\Sigma_{\mathbf{S}_1}^{-1}\big(\boldsymbol{\mu}_{\mathbf{S}_1} - \frac{1}{B_0}\mathbf{S}_0\big) - \frac{\lambda_2}{B_0}\Sigma_{\mathbf{S}_1}^{-1}\mathbf{S}_0 + \Sigma_{\mathbf{S}_1}^{-1}\Sigma_{L,\mathbf{S}_1}, \\ &\mathbf{h}^{\mathrm{T}}\mathbf{S}_0 = V_0. \end{aligned}$$

Combining the two equations give

$$c\mathbf{S}_0^{\mathrm{T}}\Sigma_{\mathbf{S}_1}^{-1}\boldsymbol{\mu}_{\mathbf{S}_1} - \frac{c}{B_0}\mathbf{S}_0^{\mathrm{T}}\Sigma_{\mathbf{S}_1}^{-1}\mathbf{S}_0 - \frac{\lambda_2}{B_0}\mathbf{S}_0^{\mathrm{T}}\Sigma_{\mathbf{S}_1}^{-1}\mathbf{S}_0 + \Sigma_{L,\mathbf{S}_1}^{\mathrm{T}}\Sigma_{\mathbf{S}_1}^{-1}\mathbf{S}_0 = V_0.$$

Since $\Sigma_{\mathbf{S}_1}$ is positive definite, so is $\Sigma_{\mathbf{S}_1}^{-1}$ and therefore $\mathbf{S}_0^{\mathrm{T}}\Sigma_{\mathbf{S}_1}^{-1}\mathbf{S}_0 > 0$. This means that if the position corresponding to the optimal solution without borrowing restriction is too expensive then taking $\lambda_2 > 0$ large enough gives a modified position that is affordable without borrowed money. We observe that the solution now is to take the position corresponding to the minimum variance hedge of the liability and the optimal speculative position in the risky assets but then to adjust the position if it turns out to be too expensive.

Summing-up we have found that the mean-variance approach to optimal investments in the presence of liabilities, and possibly borrowing restrictions, provides a rather simple solution and intuitive solution. The solution consists of computing the variance minimizing hedge and the optimal solution without a liability and taking the position which is the sum of the two positions. This may require borrowing money. If this option is not available, then $h_0 = 0$ and the position in the risky assets is modified by subtracting the position

$$\frac{\lambda_2}{B_0}\Sigma_{\mathbf{S}_1}^{-1}\mathbf{S}_0$$

for a number $\lambda_2 > 0$ such that the cost of the position in the risky assets is the initial capital V_0.

Example 1. Suppose that the risky assets can be divided into a set of hedging instruments (bonds say) and a set of pure investment assets (stocks say), where the values of the assets of the latter kind are uncorrelated with the liability. Write

$$\mathbf{S}_k = \begin{pmatrix} \mathbf{S}_k^i \\ \mathbf{S}_k^h \end{pmatrix} \text{ for } k = 1, 2 \text{ and } \mathbf{h} = \begin{pmatrix} \mathbf{h}^i \\ \mathbf{h}^h \end{pmatrix}.$$

This means that

$$\Sigma_{\mathbf{S}_1} = \begin{pmatrix} \Sigma_{\mathbf{S}_1^i} & 0 \\ 0 & \Sigma_{\mathbf{S}_1^h} \end{pmatrix} \text{ and } \Sigma_{\mathbf{S}_1}^{-1} = \begin{pmatrix} \Sigma_{\mathbf{S}_1^i}^{-1} & 0 \\ 0 & \Sigma_{\mathbf{S}_1^h}{-1} \end{pmatrix}$$

and therefore the solution to the optimal investment problem with a liability and no risk-free borrowing reads

$$\begin{aligned}
\mathbf{h}^i &= c\Sigma_{\mathbf{S}_1^i}^{-1}\big(\boldsymbol{\mu}_{\mathbf{S}_1^i} - \frac{1}{B_0}\mathbf{S}_0^i\big) - \frac{\lambda_2}{B_0}\Sigma_{\mathbf{S}_1^i}^{-1}\mathbf{S}_0^i, \\
\mathbf{h}^h &= c\Sigma_{\mathbf{S}_1^h}^{-1}\big(\boldsymbol{\mu}_{\mathbf{S}_1^h} - \frac{1}{B_0}\mathbf{S}_0^h\big) - \frac{\lambda_2}{B_0}\Sigma_{\mathbf{S}_1^h}^{-1}\mathbf{S}_0^h + \Sigma_{\mathbf{S}_1^h}^{-1}\Sigma_{L,\mathbf{S}_1^h}, \\
\mathbf{h}^{i\mathrm{T}}\mathbf{S}_0^i &+ \mathbf{h}^{h\mathrm{T}}\mathbf{S}_0^h = V_0.
\end{aligned}$$

This means that we can divide the original investment problem with a liability into two simpler problems, one with the liability and only the hedging instruments as risky assets, and one without a liability and only the pure investment assets as risky assets.