Portfolio theory and risk management
Homework set 3

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General information

The homework set gives at most 3 points which are added to your result on the exam. You may work individually or in groups of at most two persons. To obtain the points you/the group must present the solution nicely in a report which clearly shows how the problems were solved. I will consider both correctness and the quality of the report important when I evaluate your work. The report, printed on paper, must be handed in on time in order to be accepted.

Continuous compounding of interest rates is used.

The solutions to the homework set must be handed in no later than Friday 15/10 10:15.

Exercises

Exercise 1. Consider the option prices specified in Table 1. The option prices were listed 2010-08-27. The options are the actively traded European call and put options that day on the value of a stock market index on 2010-10-15.

Strike	980	990	1000	1020	1040
Call price	63	56	49.5	37.25	27
Strike	1060	1080	1100	1120	1140
Call price	18.25	11.75	6.75	3.65	1.65

Table 1: Prices 2010-08-27 of options maturing 2010-10-15.

Suppose that now is 2010-08-27. Let S be the random value of the index on 2010-10-15. You belive that $\mathrm{P}(S < 1000) \approx 0.30$ and $\mathrm{P}(S > 1045) \approx 0.30$. Suppose also that S can be considered lognormally distributed according to your view.

(a) You have $V_0 = 10000$ SEK that you want to invest by taking long positions (or no positions) in the call options listed above. You may also put some of your money on a cash account that pays no interest. You invest according to the investment criterion that says that you should maximize the expected value of the value V_1 of your position at the maturity time of the options subject to the risk constraint that $\mathrm{ES}_{0.05}(V_1 - V_0) \le c$. Choose c, motivate your choice, and solve the investment problem. Assume that you can buy fractions of options.

(b) Do the same thing as in (a) but now with a different investment criterion. Here you want to minimize $\mathrm{ES}_{0.05}(V_1 - V_0)$ subject to the constraint $\mathrm{E}[V_1] \ge \mu_0$. Choose μ_0, motivate your choice, and solve the investment problem.

(c) Suppose now that you may also invest (long positions) in the index. One share of the index costs 1020 SEK and the index pays no dividends until 2010-10-15. Invest optimally according to one of the criteria in (a) and (b).

Solution to Exercise 1

Consider the option prices specified in Table 1. The option prices correspond to actively traded European call options on the value of a stock market index 35 trading days from now. Let S be the random value of the index at

Strike	980	990	1000	1020	1040
Call price	63	56	49.5	37.25	27
Strike	1060	1080	1100	1120	1140
Call price	18.25	11.75	6.75	3.65	1.65

Table 2: Prices of call options maturing 35 days from now.

the maturity date of the options. You belive that $\mathrm{P}(S < 1000) \approx 0.30$ and $\mathrm{P}(S > 1045) \approx 0.45$. Suppose also that S can be considered lognormally distributed according to your view.

Suppose that we have $V_0 = 10000$ to invest by taking long positions (or no positions) in the call options listed above. We may also put some of our money on a cash account that pays no interest. We invest according to the investment criterion that says that we should maximize the expected value of the value V_1 of our position at the maturity time of the options subject to the risk constraint that $\mathrm{ES}_{0.05}(V_1 - V_0) \leq c$. For now, we let c be unspecified. We assume that we can buy fractions of options (not realistic but saves us from dealing with irrelevant details).

We have seen how to compute the expected value of $\max(S - K_j, 0)$ when S is $\mathrm{LN}(\mu, \sigma^2)$-distributed.

$$\begin{aligned} \mathrm{E}[\max(S - K_j, 0)] &= \mathrm{E}[I(S > K_j)S] - K_j \,\mathrm{P}(S > K_j) \\ &= e^{\mu + \sigma^2/2}\Phi(\sigma - \widetilde{K}_j) - K_j \Phi(-\widetilde{K}_j), \end{aligned}$$

where $\widetilde{K}_j = (\log K_j - \mu)/\sigma$ and Φ is the standard normal distribution function.

For a lognormally distributed $S = e^{\mu + \sigma Z}$ it holds that $\mathrm{P}(S \leq x) = \mathrm{P}(Z \leq (\log x - \mu)/\sigma) = \Phi((\log x - \mu)/\sigma)$. Here we get

$$\begin{pmatrix} 1 & \Phi^{-1}(0.3) \\ 1 & \Phi^{-1}(0.55) \end{pmatrix} \begin{pmatrix} \mu \\ \sigma \end{pmatrix} = \begin{pmatrix} \log 1000 \\ \log 1045 \end{pmatrix}$$

with the solution $(\mu, \sigma) \approx (6.943, 0.068)$. We find that $F_S^{-1}(0.05) \approx 927$ and therefore

$$\begin{aligned} \mathrm{ES}_{0.05}(V_1) &= \mathrm{ES}_{0.05}\Big(w_0 + \sum_{j \geq 1} \frac{w_j}{C_{j,0}}(S - K_j)_+\Big) \\ &= -w_0 - \sum_{j \geq 1} w_j \frac{1}{C_{j,0}} \frac{1}{0.05} \int_0^{0.05} (F_S^{-1}(p) - K_j)_+ dp \\ &= -w_0. \end{aligned}$$

Strike	980	990	1000	1020	1040
Expected return	1.047	1.040	1.030	1.021	1.012
Strike	1060	1080	1100	1120	1140
Expected return	1.035	1.066	1.180	1.331	1.722

Table 3: Expected returns for the 10 call options.

This gives $\mathrm{ES}_{0.05}(V_1 - V_0) = V_0 - w_0$. The investment problem therefore reads:

$$\begin{array}{ll} \text{maximize} & w_0 + \sum_{j\geq 1} w_j \psi_j \\ \text{subject to} & V_0 - w_0 \leq c \\ & w_0 + \sum_{j\geq 1} w_j \leq V_0 \\ & w_j \geq 0 \quad \text{for } j \geq 1. \end{array}$$

Since $w_0 + \sum_{j\geq 1} w_j < V_0$ would give a suboptimal solution (the objective function can be made larger without violating any of the constraints), $w_0 + \sum_{j\geq 1} w_j = V_0$ for an optimal solution. In particular, $V_0 - w_0 = \sum_{j\geq 1} w_j$ and we see that c is the maximum amount that we can accept to speculate with (long position in options). We also find that the objective function is maximized (subject to the constraints) by placing the amount c in the option that we consider to be the best deal (the jth option if $\psi_j = \max_k \psi_k$). Here this means buying options with strike price 1140 for the amount c and putting the remaining amount $10000 - c$ on the cash account.

Take for instance $c = 1000$ and suppose that we are also allowed to invest by taking a long position in the underlying index. Suppose further that one share of the index costs $S_0 = 1020$ now and that the index does not pay dividends until the maturity time of the options. Now

$$V_1 = w_0 + \sum_{j=1}^{10} w_j \frac{(S - K_j)_+}{C_{j,0}} + w_{11} \frac{S}{S_0},$$

where $C_{j,0}$ and S_0 are the current prices of the jth option and the underlying index, respectively. We find that

$$\psi_{11} := \frac{\mathrm{E}[S]}{S_0} = \frac{e^{\mu+\sigma^2/2}}{S_0} \approx 1.018 < \max_k \psi_k.$$

In particular, an index position is not the best deal in terms of expected percentage return. We also find that

$$\mathrm{ES}_{0.05}(V_1 - V_0) = V_0 - w_0 - w_{11} \frac{1}{S_0} \frac{1}{0.05} \int_0^{0.05} F_S^{-1}(p)dp,$$

where

$$\gamma := \frac{1}{S_0}\frac{1}{0.05}\int_0^{0.05} F_S^{-1}(p)dp \approx 0.884.$$

The investment problem takes the form

$$\begin{array}{ll} \text{maximize} & w_0 + \sum_{j=1}^{10} w_j\psi_j + w_{11}\psi_{11} \\ \text{subject to} & 10000 - w_0 - \gamma w_{11} \leq 1000 \\ & w_0 + \sum_{j\geq 1} w_j \leq 10000 \\ & w_j \geq 0 \quad \text{for } j \geq 0. \end{array}$$

Not using all initial capital will lead to a suboptimal solution. Therefore the investment problem can be formulated as

$$\begin{array}{ll} \text{maximize} & w_0 + \sum_{j=1}^{10} w_j\psi_j + w_{11}\psi_{11} \\ \text{subject to} & \sum_{j=1}^{10} w_j + (1-\gamma) w_{11} \leq 1000 \\ & w_0 + \sum_{j\geq 1} w_j = 10000 \\ & w_j \geq 0 \quad \text{for } j \geq 0. \end{array}$$

If we take as safer than necessary position, then the expected value will not be maximized. Moreover, each monetary unit invested in the options contribute equally to the risk as measured by $\mathrm{ES}_{0.05}$. Since the option with strike 1140 has the highest expected return, only this option needs to be considered. Therefore the investment problem can be formulated as

$$\begin{array}{ll} \text{maximize} & w_0 + w_{10}\psi_{10} + w_{11}\psi_{11} \\ \text{subject to} & w_{10} + (1-\gamma) w_{11} = 1000 \\ & w_0 + w_{10} + w_{11} = 10000 \\ & w_j \geq 0 \quad \text{for } j \geq 0. \end{array}$$

We find that $(1-\gamma)w_{11} = 1000$ gives $w_{11} = 1000/(1-\gamma) \approx 8597.52 < 10000$. This means that the solution will be either to invest as much as we can (without violating the risk constraint) in the index or in the option with strike 1140. The former alternative gives the expected value $w_0 + w_{11}\psi_{11} = 10000 - 1000/(1-\gamma) + 1000\psi_{11}/(1-\gamma) \approx 10156.14$. The latter alternative gives the expected value $w_0 + w_{10}\psi_{10} = 9000 + 1000\psi_{10} \approx 10721.98$. That is, the possibility to invest in the index is not used since it does not improve the optimal solution. Notice that here this could be spotted right away since investing everything in the index (which violates the risk contraint) would be suboptimal.

Another version of the investment problem would be to minimize $\mathrm{ES}_{0.05}(V_1 - V_0)$ subject to the constraint that $\mathrm{E}[V_1] \geq \mu_0$ for the optimal solution (and the budget constraint and the constraints ruling out short sales). The solution to this investment problem is to invest just as much as needed in the asset with the best expected return (and the remaining capital in the risk-free bond) so that $\mathrm{E}[V_1] = \mu_0$. Clearly, if μ_0 is greater than V_0 times the highest of the expected returns, then there is no solution at all.