KTH Mathematics

Tentamen i 5B1575 Finansiella Derivat.
Tisdag 22 maj 2007 kl. 14.00–19.00.

Answers and suggestions for solutions.

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1. (a) According to the First Fundamental Theorem the model is free of arbitrage if and only if there exists a martingale measure. We thus need to prove that there exist q_1, q_2, and q_3 all strictly between zero and one, and such that

$$\begin{aligned} s &= \frac{1}{1+r}\left[q_1 \cdot su + q_2 \cdot s + q_3 \cdot sd\right], \\ 1 &= q_1 + q_2 + q_3. \end{aligned}$$

Letting q_1 act as a parameter we obtain for q_2 and q_3

$$q_2 = 2(0.6 - q_1), \qquad q_3 = q_1 - 0.2.$$

From this we see that all values of q_1 such that $0.2 < q_1 < 0.6$ will result in a martingale measure, and therefore the model is free of arbitrage. (There are infinitely many martingale measures, but to show that the model is free of arbitrage you only need to find one, so if you have found one solution, say $q_1 = 0.3$, $q_2 = 0.6$ and $q_3 = 0.1$, you need not worry about the other solutions.)

(b) Since the interest rate is zero the option price is given by the following formula

$$C_{Bach}(0) = E^Q[\max\{S_T - K, 0\}] = E^Q[(S_T - K)I_{\{S_T - K \geq 0\}}].$$

Since

$$S_T - K = S_0 + \sigma S_0 V_T - K,$$

we see that $S_T - K \in N(S_0 - K, \sigma^2 S_0^2 T)$. Now apply the stated result with $\mu = S_0 - K$, "$\sigma^2 = \sigma^2 S_0^2 T$", $l = 0$, and $h = \infty$, and note that $1 - \Phi(x) = \Phi(-x)$ and $\phi(x) = \phi(-x)$.

(c) By definition we have that

$$\Delta_{call,Bach} = \frac{\partial C_{Bach}}{\partial s},$$

once you have substituted s for S_t in C_{Bach}. Differentiating you will find

$$\Delta_{call,Bach}(t) = \Phi\left(\frac{S_t - K}{\sigma S_0 \sqrt{T-t}}\right).$$

Here we have substituted back S_t to obtain todays value of Δ.

If you have sold the option Δ tells you how many you have to buy of the underlying in order for your portfolio to become delta neutral, i.e. insensitive to small changes in the stock price.

2. (a) We have the following equation

$$0 = \Pi(t; X) = e^{-r(T-t)} E^Q \left[S_T - f(t; T, S_T) \middle| \mathcal{F}_t \right].$$

Solving for the forward price we obtain (use that $f(t;T,S_T) \in \mathcal{F}_t$)

$$f(t; T, S_T) = E^Q[S_T | \mathcal{F}_t].$$

Since S/B is a Q-martingale and $B_t = e^{rt}$ we have that

$$f(t; T, S_T) = B_T E^Q \left[\frac{S_T}{B_T} \middle| \mathcal{F}_t \right] = B_T \frac{S_t}{B_t} = e^{r(T-t)} S_t.$$

(b) The payoff of the range forward can be written as

$$\begin{aligned} X &= \max\{\min\{S_T, K_2\}, K_1\} - f(0; T, S_T) \\ &= K_1 + \max\{S_T - K_1, 0\} - \max\{S_T - K_2, 0\} - f(0; T, S_T). \end{aligned}$$

The price is therefore given by

$$\begin{aligned} \Pi(t; X) &= e^{-r(T-t)} E^Q[K_1 + \max\{S_T - K_1, 0\} \\ &\quad - \max\{S_T - K_2, 0\} - f(0; T, S_T) | \mathcal{F}_t] \\ &= C(t, S_t, K_1, T, r, \sigma) - C(t, S_t, K_2, T, r, \sigma) \\ &\quad + e^{-r(T-t)}(K_1 - S_0 e^{rT}), \end{aligned}$$

where $C(t, s, K, T, r, \sigma)$ denotes the standard Black-Scholes price at time t of a European call option with exercise price K and expiry date T, when the current price of the underlying is s, the interest rate is r, and the volatility of the underlying is σ. For future use we let $P(t,s,K,T,r,\sigma)$ denote the price of the corresponding put option.

(c) Using put-call-parity at time T

$$\max\{K - S_T, 0\} = K + \max\{S_T - K, 0\} - S_T,$$

we can decompose the payoff of the range forward in the following way instead

$$\begin{aligned} X &= K_1 + \max\{S_T - K_1, 0\} - \max\{S_T - K_2, 0\} - f(0;T,S_T) \\ &= \max\{K_1 - S_T\} - \max\{S_T - K_2, 0\} + S_T - f(0;T,S_T). \end{aligned}$$

From this we see that the price can also be written in the following way

$$\Pi(t;X) = P(t,S_t,K_1,T,r,\sigma) - C(t,S_t,K_2,T,r,\sigma) + S_t - S_0 e^{rt}.$$

Since $S_t - S_0 e^{rt}$ is the price of a forward, we see that a range forward is equal to a portfolio composed of a long forward contract, a long put with strike K_1, and a short call with strike K_2.

3. (a) We have

$$\begin{aligned} p(t,T) &= E^Q\left[e^{-\int_t^T r(u)du} \middle| \mathcal{F}_t \right] \\ &= E^Q\left[e^{-\int_t^T [X_1(u)+X_2(u)]du} \middle| \mathcal{F}_t \right] \\ &= E^Q\Big[e^{-\int_t^T \left(e^{-\kappa_1(u-t)}X_1(t)+\kappa_1\theta_1\int_t^u e^{-\kappa_1(u-s)}ds+\sigma_1\int_t^u e^{-\kappa_1(u-s)}dW_1(s)\right)du} \times \\ &\qquad e^{-\int_t^T \left(e^{-\kappa_2(u-t)}X_2(t)+\kappa_2\theta_2\int_t^u e^{-\kappa_2(u-s)}ds+\sigma_2\int_t^u e^{-\kappa_2(u-s)}dW_2(s)\right)du} \Big| \mathcal{F}_t \Big] \\ &= e^{-\int_t^T e^{-\kappa_1(u-t)}du\cdot X_1(t)} e^{-\int_t^T e^{-\kappa_2(u-t)}du\cdot X_2(t)} \\ &\qquad E^Q\Big[e^{-\int_t^T \left(\kappa_1\theta_1\int_t^u e^{-\kappa_1(u-s)}ds+\sigma_1\int_t^u e^{-\kappa_1(u-s)}dW_1(s)\right)du} \times \\ &\qquad e^{-\int_t^T \left(\kappa_2\theta_2\int_t^u e^{-\kappa_2(u-s)}ds+\sigma_2\int_t^u e^{-\kappa_2(u-s)}dW_2(s)\right)du} \Big] \end{aligned}$$

which shows that the zero coupon bond prices have the desired form. (Actually the expectation can be computed directly, since the stochastic variable in the exponential is normally distributed.)

(b) The Itô formula applied to $p(t,T) = e^{A(t,T)-B(t,T)X_1(t)-C(t,T)X_2(t)}$ yields

$$
\begin{aligned}
dp^T &= (A_t - B_t X_1 - C_t X_2)p^T dt - Bp^T dX_1 - Cp^T dX_2 \\
&\quad + \frac{1}{2}B^2 p^T (dX_1)^2 + \frac{1}{2}C^2 p^T (dX_2)^2 \\
&= \Big[A_t - B_t X_1 - C_t X_2 - B\kappa_1(\theta_1 - X_1) - C\kappa_2(\theta_2 - X_2) \\
&\quad + \frac{1}{2}B^2\sigma_1^2 + \frac{1}{2}C^2\sigma_2^2 \Big] p^T dt + \ldots \\
&= \Big[A_t - B\kappa_1\theta_1 - C\kappa_2\theta_2 + \frac{1}{2}B^2\sigma_1^2 + \frac{1}{2}C^2\sigma_2^2 \\
&\quad - (B_t - B\kappa_1)X_1 - (C_t - C\kappa_2)X_2 \Big] p^T dt + \ldots
\end{aligned}
$$

Under Q we know that $p(t,T)/B(t)$ is a martingale, which means that $p(t,T)$ has to have local return equal to the short rate $r = X_1 + X_2$. Thus, for $t \geq 0$, and $x_i \in (-\infty, \infty)$ $i = 1,2$ the following equality has to hold

$$
\begin{aligned}
&\Big[A_t - B\kappa_1\theta_1 - C\kappa_2\theta_2 + \frac{1}{2}B^2\sigma_1^2 + \frac{1}{2}C^2\sigma_2^2 \Big] \\
&-[B_t - B\kappa_1 + 1]x_1 - [C_t - C\kappa_2 + 1]x_2 = 0
\end{aligned}
$$

The only way this is possible is if all three square brackets equal zero, this will give you the ordinary differential equations solved by A, B, and C. The boundary conditions are obtained from the condition that $p(T,T) = 1$. So to sum up we have

$$
\begin{cases}
B_t(t,T) - B(t,T)\kappa_1 = -1, \\
B(T,T) = 0,
\end{cases}
$$

$$
\begin{cases}
C_t(t,T) - C(t,T)\kappa_2 = -1, \\
C(T,T) = 0,
\end{cases}
$$

and

$$
\begin{cases}
A_t(t,T) = \kappa_1\theta_1 B(t,T) + \kappa_2\theta_2 C(t,T) - \frac{1}{2}\sigma_1^2 B^2(t,T) - \frac{1}{2}\sigma_2^2 C^2(t,T), \\
A(T,T) = 0.
\end{cases}
$$

4. (a) You can replicate the payoff $1/p_d(T,U)$ at time U by buying a domestic T-bond at time t and reinvesting the principal received at time T in $1/p_d(T,U)$ domestic U-bonds. The price of a roll bond is therefore $p_d(t,T)$.

(b) For $t \geq T$ we have

$$\begin{aligned}\Pi_{q-roll} &= E^Q\left[e^{-\int_t^U r_d(s)ds}\frac{1}{p_f(T,U)}\bigg|\mathcal{F}_t\right] \\ &= \frac{1}{p_f(T,U)}E^Q\left[e^{-\int_t^U r_d(s)ds}\bigg|\mathcal{F}_t\right] \\ &= \frac{p_d(t,U)}{p_f(T,U)},\end{aligned}$$

where we have used that $1/p_f(T,U) \in \mathcal{F}_t$ for $t \geq T$ to obtain the second equality. The price of a quanto roll bond for $t \geq T$ is thus $p_d(t,U)/p_f(T,U)$.

(c) We have that the value process of the portfolio is given by

$$\begin{aligned}V^h(t) &= h_1(t)p_d(t,U) + h_2(t)\tilde{p}_f(t,U) + h_3(t)\tilde{p}_f(t,T) \\ &= V(t;T,U).\end{aligned}$$

At time T the value process then equals

$$\begin{aligned}V^h(T) &= V(T;T,U) = \frac{p_d(T,U)p_f(T,T)G(T;T,U)}{p_f(T,U)} \\ &= \frac{p_d(T,U)}{p_f(T,U)},\end{aligned}$$

where we have have used that, as always, $p_f(T,T) = 1$, and that $G(T;T,U) = 1$. Thus the portfolio is equal to the desired claim at time T.

It remains to check that the portfolio is self-financing, i.e. that

$$dV^h(t) = h_1(t)dp_d(t,U) + h_2(t)d\tilde{p}_f(t,U) + h_3(t)d\tilde{p}_f(t,T)$$

or with the expressions for h_1, h_2 and h_3 inserted

$$dV^h(t) = V(t;T,U)\left(\frac{dp_d(t,U)}{p_d(t,U)} - \frac{d\tilde{p}_f(t,U)}{\tilde{p}_f(t,U)} + \frac{d\tilde{p}_f(t,T)}{\tilde{p}_f(t,T)}\right) \quad (1)$$

Using the Itô formula and that $\tilde{p}_f(t,S) = X(t)p_f(t,S)$ we obtain

$$\begin{aligned}d\tilde{p}_f^S &= Xd\tilde{p}_f^S + \tilde{p}_f^S dX + d\tilde{p}_f^S dX \\ &= r_d(t)\tilde{p}_f(t,S)dt + [\sigma(t) + \nu_f(t,S)]\tilde{p}_f(t,S)dW,\end{aligned}$$

and inserted in to (1) we get

$$dV^h(t) = V(t;T,U)\left(\frac{dp_d(t,U)}{p_d(t,U)} - \nu_f(t,U)dW + \nu_f(t,T)dW\right).$$

Now use that the dynamics of $p_f(t,S)$ to see that

$$\begin{aligned} dV^h(t) &= V(t;T,U)\left(\frac{dp_d(t,U)}{p_d(t,U)} - \frac{dp_f(t,U)}{p_f(t,U)} + \frac{dp_f(t,T)}{p_f(t,T)}\right. \\ &\qquad \left. + \sigma(t)[\nu_f(t,T) - \nu_f(t,U)]\right). \end{aligned}$$

Since $V^h(t) = V(t;T,U)$ and the dynamics in the formula above are the same as those of $V(t;T,U)$ given in the exercise, we conclude that the portfolio is self-financing and replicates the T-claim $p_d(T,U)/p_f(T,U)$.

5. If we use S as the new numeraire we know that Π/S is a martingale under Q^S. Writing down the martingale property we get

$$\frac{\Pi(t)}{S(t)} = E^S\left[\left.\frac{\Pi(T)}{S(T)}\right| \mathcal{F}_t\right],$$

where the super script S indicates that the expectation should be taken under Q^S. Thus,

$$\begin{aligned} \frac{\Pi(t)}{S(t)} &= E^S\left[\left.\frac{\max\left\{\frac{1}{T}\int_0^T S_u du - S(T), 0\right\}}{S(T)}\right| \mathcal{F}_t\right] \\ &= E^S\left[\max\left\{\left.\frac{\frac{1}{T}\int_0^T S_u du}{S(T)} - 1, 0\right\}\right| \mathcal{F}_t\right] \\ &= E^S\left[\max\left\{Z(T) - 1, 0\right\}\right| \mathcal{F}_t]. \end{aligned}$$

In order to apply Feynman-Kač we need the Q^S-dynamics of the process Z given by

$$Z(t) = \frac{\frac{1}{T}\int_0^t S(u)du}{S(t)}.$$

Recall that the Girsanov kernel which takes you from Q to Q^S is the volatility of S. Using this we find that the Q^S-dynamics of S are given by

$$dS_t = (r + \sigma^2)S_t dt + \sigma S_t dU_t,$$

where U is a Q^S-Wiener process. Let

$$Y(t) = \frac{1}{T}\int_0^t S_u du, \text{ i.e.} \qquad dY_t = \frac{1}{T} S_t dt.$$

The dynamics of Z can now be found by an application of the Itô formula

$$\begin{aligned} dZ_t &= \frac{1}{S_t} dY_t - \frac{Y_t}{S_t^2} dS_t + \frac{1}{2}\frac{2Y_t}{S_t^3}(dS_t)^2 \\ &= \left(\frac{1}{T} - rZ_t\right) dt - \sigma Z_t dU_t. \end{aligned}$$

Now Feynman-Kač's theorem tells us that

$$\frac{\Pi(t)}{S(t)} = F(t, Z(t)),$$

where F solves the PDE stated in the exercise.