KTH Mathematics

Tentamen i 5B1575 Finansiella Derivat.
Måndag 27 augusti 2007 kl. 14.00–19.00.

Examinator: Camilla Landén, tel 790 8466.

Tillåtna hjälpmedel: Inga.

Allmänna anvisningar: Lösningarna skall vara lättläsliga och **välmotiverade**. All införd notation skall vara förklarad. Problem rörande integrabilitet behöver ej redas ut.

OBS! Personnummer skall anges på försättsbladet. Endast en uppgift på varje blad. Numrera sidorna och skriv namn på varje blad!

25 poäng inklusive bonuspoäng ger säkert godkänt.

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1. (a) Compute the price of a European call option with strike price $K = 125$ kr and exercise time $T = 2$ years, using a binomial tree with two trading dates $t_1 = 0$ and $t_2 = 1$ (your portfolio at time $t_3 = 2$ is the same as your portfolio at time $t_2 = 1$) and parameters $s_0 = 100$, $u = 1.5$, $d = 0.5$, $r = 0$, and $p = 0.75$. (3p)

(b) Consider a standard Black-Scholes market described in detail in exercise 2. Describe how to replicate a European put option on the stock, with strike price K and exercise date T, using zero coupon bonds, the underlying stock, and call options. ..... (3p)

(c) Consider a standard Black-Scholes market described in detail in exercise 2.

i. Give the definition of a (risk neutral) martingale measure in this model. .... (2p)

ii. Define the logarithmic return at time t, R_t, by

$$R_t = \ln\left(\frac{S_t}{S_0}\right).$$

Determine the distribution of the logarithmic return under the martingale measure Q. ....(2p)

2. Consider a standard Black-Scholes market, i.e., a market consisting of a risk free asset, B, with P-dynamics given by

$$\begin{cases} dB_t &= rB_t dt, \\ B_0 &= 1, \end{cases}$$

and a stock, S, with P-dynamics given by

$$\begin{cases} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ S_0 &= s_0. \end{cases}$$

Here W denotes a P-Wiener process and r, α, and σ are assumed to be constants.

(a) Determine the arbitrage price at time t for $t \in [0, T]$ of a T-claim defined by

$$X = \phi(S_T) = S_T^{\beta},$$

where β is a constant. .... (6p)

(b) Explain how to hedge/replicate the claim (using the risk free asset B, and the stock S). .... (4p)

3. Consider a standard Black-Scholes model under the objective probability measure P:

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dB_t &= rB_t dt. \end{aligned}$$

Suppose that you want to price a European call option with exercise price K and exercise date T on the underlying stock. Furthermore, assume that the stock pays a continuous dividend yield δ, i.e. that the cumulative dividend process of the stock has the structure

$$dD_t = \delta S_t dt,$$

where $\delta > 0$ is a constant.

(a) Discuss verbally what relation you would expect to hold between the option price in the case when there is a dividend yield, and the option price in the case when the dividend yield is zero. (Which would be greater and why?) .... (1p)

(b) Derive a pricing formula (as explicit as possible) for the call option in the case of a nonzero dividend yield. You are allowed to use the standard Black-Scholes formula without proof. .... (7p)

4. Assume that we are in a world with *stochastic* interest rates, and that the short rate follows a Ho-Lee model

$$\begin{cases} dr_t &= \theta(t)dt + \rho dU_t, \\ r_0 &= r^*, \end{cases}$$

under a martingale measure Q. Here θ is a deterministic function, ρ is a constant, and U is a Q-Wiener process.

Apart from this, however, we assume a Black-Scholes setting. More precisely, we assume that we have a stock, S, and a risk free asset, B, with Q-dynamics

$$\begin{aligned} dS_t &= r(t)S_t dt + \sigma S_t dW_t, \\ dB_t &= r(t)B_t dt, \end{aligned}$$

with the short rate defined as above, and where W is a Q-Wiener process independent of U.

(a) Derive a closed form formula for the bond prices in this economy (dependent only on r and the parameters of the model). You do **not** have to fit the model to the initial term structure. .... (5p)
Hint: Affine term structure.

(b) Determine the arbitrage price of a European call option on S. As usual let K denote the strike price, and T the maturity date of the option. .... (7p)
Hint: Change measures and use the extension of the standard Black-Scholes formula found at the end of the exam.

5. Consider an arbitrage free interest rate model under a martingale measure Q, and denote the (instantaneous) forward rates by $f(t,T)$. The interest rate model is said to exhibit *parallel shifts* if the forward rate process has the following structure

$$f(t,T) = Z_t + H(T-t),$$

where H is a deterministic function of time to maturity $(T-t)$ and Z is a random process. We normalize so that $H(0) = 0$.

(a) Show that the process Z is in fact the short rate process, i.e. show that $Z_t = r_t$. .... (2p)

(b) Now, assume that the model exhibits parallel shifts and consider the result in (a) to be proved, i.e. assume that

$$f(t,T) = r_t + H(T-t).$$

Also assume that r satisfies the SDE

$$dr_t = \mu(t,r_t)dt + \sigma(t,r_t)dW_t,$$

where W is a Q-Wiener process. Your task is to find out what can be said about μ and σ under these assumptions. In other words: What must a short rate model look like in order to generate a parallel shift term structure? .... (8p)
Hint: Use the HJM drift condition.

Lycka till!

Hints:
You are free to use the following in any of the above exercises.

- The standard Black-Scholes formula for the price $\Pi(t)$ of a European call option with strike price K and time of maturity T is $\Pi(t) = F(t, S(t))$, where

  $$F(t,s) = s\Phi[d_1(t,s)] - e^{-r(T-t)}K\Phi[d_2(t,s)].$$

  Here Φ is the cumulative distribution function for the $N(0,1)$ distribution and

  $$\begin{aligned} d_1(t,s) &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\}, \\ d_2(t,s) &= d_1(t,s) - \sigma\sqrt{T-t}. \end{aligned}$$

- If the constant σ in the standard Black-Scholes model is replaced by a (nice) deterministic function $\sigma(t)$ the Black-Scholes formula given above is still valid if you replace $\sigma^2(T-t)$ with

  $$\int_t^T \sigma^2(u)du.$$