KTH Mathematics

Exam in SF2975 Financial Derivatives.
Monday August 25 2008 14.00-19.00.

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Answers and suggestions for solutions.

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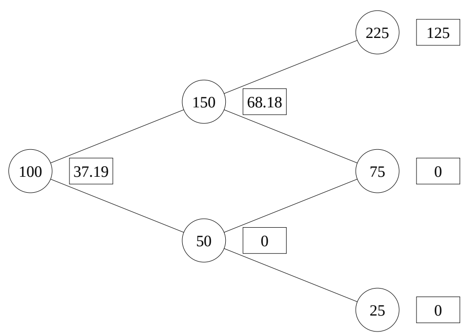
**1.** (a) For the martingale probabilities we have

$$q = \frac{1+r-d}{u-d} = 0.6.$$

Using them we obtain the following binomial tree where the value of the stock is written in the nodes, and the value of the option is written in the adjacent boxes. The value 68.18 adjacent to the node with stock price 150 is obtained as $\frac{1}{1.1}(0.6 \cdot 125 + 0.4 \cdot 0)$, and the other values are obtained analogously.




The price of the call option is thus 37.19 kr.

(b) The market is incomplete. Let $Z$ denote a random variable which equals $u$ with probability $p$, 1 with probability $q$, and $d$ with probability $1-p-q$. If the market was complete it would be possible to solve the following set of equations for every function $\phi(z)$ (then $\mathbf{h} = (x, y)$ would be a replicating portfolio).

$$\begin{cases} x(1+r) + ysu &= \phi(u), \\ x(1+r) + ys &= \phi(1), \\ x(1+r) + ysd &= \phi(d). \end{cases}$$

Let $\phi$ be given by

$$\phi(z) = \begin{cases} K, & \text{if } z = u \text{ or } z = 1, \\ 2K, & \text{if } z = d. \end{cases}$$

Solving for $x$ and $y$ using the first two equations gives $x = K/(1+r)$ and $y = 0$, but this does not satisfy the third equation. This means that there are claims which cannot be replicated in this model and therefore the model is not complete.

(c) i. False. Implied volatility is the volatility implied by an option price observed in the market.
   ii. True.
   iii. False. In order for the dynamics of the value process of a certain portfolio to have the stated form, the portfolio has to be self-financing.
   iv. True.

**2.** (a) The payoff of the collar option can be written as (draw a picture of the payoff function)

$$\min\{\max\{S_T, K_1\}, K_2\} = K_1 + \max\{S_T - K_1, 0\} - \max\{S_T - K_2, 0\}.$$

Since the LHS is equal to the RHS the prices must agree and thus if $\Pi(t)$ denotes the price of the collar option at time $t$, and $c(t, s, K, T, r, \sigma)$ denotes the standard Black-Scholes price at time $t$ of a European call option with exercise price $K$ and expiry date $T$, when the current price of the underlying is $s$, the interest rate is $r$, and the volatility of the underlying is $\sigma$ then

$$\Pi(t) = e^{-r(T-t)}K_1 + c(t, S_t, K_1, T, r, \sigma) - c(t, S_t, K_2, T, r, \sigma).$$

(b) The price of the claim at time $t \in [0, T]$ is given by

$$\begin{aligned}\Pi_t[X] &= e^{-r(T-t)}E^Q\left[S(T) - \frac{1}{T}\int_0^T S_u du \middle| \mathcal{F}_t\right] \\ &= S(t) - \frac{e^{-r(T-t)}}{T}\int_0^t S_u du - \frac{e^{-r(T-t)}}{T}E^Q\left[\int_t^T S_u du \middle| \mathcal{F}_t\right]\end{aligned}$$

Change the order of integration in the last term to obtain

$$\Pi_t[X] = S(t) - \frac{e^{-r(T-t)}}{T}\int_0^t S_u du - \frac{e^{-r(T-t)}}{T}\int_t^T E^Q\left[S_u \middle| \mathcal{F}_t\right] du$$

Now you can either use that the dynamics of $S$ under $Q$ are given by

$$dS_t = rS_t dt + \sigma S_t dV_t,$$

where $V$ is a $Q$-Wiener process to compute $E^Q[S_u|\mathcal{F}_t]$, or you can use that $S/B$

is a $Q$-martingale and that $B_t = e^{rt}$ in the following way

$$\begin{aligned}
\Pi_t[X] &= S(t) - \frac{e^{-r(T-t)}}{T}\int_0^t S_u du - \frac{e^{-r(T-t)}}{T}\int_t^T E^Q\left[S_u \middle| \mathcal{F}_t\right] du \\
&= S(t) - \frac{e^{-r(T-t)}}{T}\int_0^t S_u du - \frac{e^{-r(T-t)}}{T}\int_t^T e^{ru} E^Q\left[\frac{S_u}{e^{ru}} \middle| \mathcal{F}_t\right] du \\
&= S(t) - \frac{e^{-r(T-t)}}{T}\int_0^t S_u du - \frac{e^{-r(T-t)}}{T}\int_t^T e^{ru}\frac{S_t}{e^{rt}} du \\
&= S(t) - \frac{e^{-r(T-t)}}{T}\int_0^t S_u du - \frac{e^{-rT}S_t}{T}\int_t^T e^{ru} du \\
&= S(t) - \frac{e^{-r(T-t)}}{T}\int_0^t S_u du - \frac{S_t(1-e^{-r(T-t)})}{rT}
\end{aligned}$$

The price of the claim at time $t$ is thus given by

$$\Pi_t[X] = S(t) - \frac{e^{-r(T-t)}}{T}\int_0^t S_u du - \frac{S_t(1-e^{-r(T-t)})}{rT}$$

**3.** (a) According to the First and Second Fundamental Theorem of arbitrage pricing we need to show that there exists (abscence of arbitrage) a unique (completeness) martingale measure.

The Itô formula gives the following $P$-dynamics for $Z^1(t) = X(t)/B(t)$ and $Z^2(t) = Y(t)/B(t)$

$$\begin{aligned}
dZ_t^1 &= \frac{1}{B_t}dX_t - \frac{X_t}{B_t^2}dB_t \\
&= (\alpha - r)Z_t^1 dt + \rho Z_t^1 dV_t + \sigma Z_t^1 dW_t, \\
dZ_t^2 &= \frac{1}{B_t}dY_t - \frac{Y_t}{B_t^2}dB_t \\
&= (\beta - r)Z_t^2 dt + \gamma Z_t^2 dV_t.
\end{aligned}$$

Here $V$ and $W$ denote two independent $Q$-Wiener processes. Define a Girsanov transformation by

$$dQ = L(t)dP, \qquad \text{on } \mathcal{F}_t,$$

where

$$\begin{cases} dL_t &= g_t L_t dV_t + h_t L_t dW_t, \\ L_0 &= 1. \end{cases}$$

From Girsanov's theorem we have that

$$\begin{bmatrix} dV_t \\ dW_t \end{bmatrix} = \begin{bmatrix} g_t \\ h_t \end{bmatrix} dt + \begin{bmatrix} dV_t^Q \\ dW_t^Q \end{bmatrix},$$

where $V^Q$ and $W^Q$ are two independent $Q$-Wiener processes. Thus, the $Q$-dynamics of $Z^1$ and $Z^2$ are given by

$$\begin{aligned}
dZ_t^0 &= (\alpha - r + \rho g_t + \sigma h_t)Z_t^1 dt + \rho Z_t^1 dV_t^Q + \sigma Z_t^1 dW_t^Q, \\
dZ_t^1 &= (\beta - r + \gamma g_t)Z_t^1 dt + \gamma Z_t^2 dV_t^Q.
\end{aligned}$$

In order for $Z^1$ and $Z^2$ to be $Q$-martingales the drift terms of their stochastic differentials must equal zero. This gives the following expressions for $g$ and $h$

$$\begin{aligned} g &= \frac{r-\beta}{\gamma}, \\ h &= \frac{r-\alpha}{\sigma} - \rho\frac{r-\beta}{\sigma\gamma}. \end{aligned} \tag{1}$$

Since the filtration is the natural filtration the converse of the Girsanov Theorem guarantess that the martingale measure is unique.

(b) We have the following identity

$$\begin{aligned} Z_{call,max} - Z_{put,max} &= \max\{\max(X_T, Y_T) - K, 0\} - \max\{K - \max(X_T, Y_T), 0\} \\ &= \max(X_T, Y_T) - K. \end{aligned}$$

Let $C_{max}(t; K)$ and $P_{max}(t; K)$ denote the prices at time $t$ of the call and the put with strike $K$, respectively, then we have that

$$\begin{aligned} C_{max}(t; K) - P_{max}(t; K) &= e^{-r(T-t)} E^Q[\max(X_T, Y_T) - K|\mathcal{F}_t] \\ &= e^{-r(T-t)} E^Q[\max(X_T, Y_T)|\mathcal{F}_t] - e^{-r(T-t)} K. \end{aligned}$$

The trick now is to realise that since $X > 0$ and $Y > 0$ we have

$$\max(X_T, Y_T) = \max\{\max(X_T, Y_T) - 0, 0\}$$

which menas that

$$\begin{aligned} C_{max}(t; K) - P_{max}(t; K) &= e^{-r(T-t)} E^Q[\max(X_T, Y_T)|\mathcal{F}_t] - e^{-r(T-t)} K \\ &= C_{max}(t; 0) - e^{-r(T-t)} K. \end{aligned}$$

**4.** (a) A replication portfolio for the contingent $T$-claim $X$ is a self-financing portfolio $h$, such that

$$V^h(T) = X \quad P - a.s.$$

for the corresponding value process $V^h$.

(b) We have that

$$V^h(t) = h^S(t)p(t, S) + h^T p(t, T),$$

where

$$h^S(t) = \Phi(d), \qquad \text{and} \qquad h^T(t) = -K\Phi(d - \sigma_p),$$

and

$$\begin{aligned} d &= \frac{1}{\sigma_p} \ln\left(\frac{p(t, S)}{p(t, T)K}\right) + \frac{1}{2}\sigma_p \\ \sigma_p &= \frac{1}{\kappa}\left\{1 - e^{-\kappa(S-T)}\right\} \sqrt{\frac{\sigma_r^2}{2\kappa}\{1 - e^{-2\kappa(T-t)}\}}. \end{aligned}$$

What we need to show is that

i. $V^h(T) = \max\{p(T, S) - K, 0\} \quad P - a.s.$, and
ii. that $dV^h(t) = h^S(t)dp(t, S) + h^T dp(t, T)$, i.e. that the portfolio is self-financing.

i. If $p(T,S) \geq K$ then
$$V^h(T) = \Phi(\infty)p(t,S) - K\Phi(\infty) = p(t,S) - K,$$
and if $p(T,S) < K$ then
$$V^h(T) = \Phi(-\infty)p(t,S) + K\Phi(-\infty) = 0.$$
Thus, $V^h(T) = \max\{p(T,S) - K, 0\} \quad P-a.s.$

ii. Now use that $V^h(t) = \Pi_{call}(t) = f(t, p(t,S), p(t,T))$ for a function $f(t,x,y)$ to obtain (using the Itô formula)
$$dV^h = f_t dt + f_x dp^S + f_y dp^T + \frac{1}{2} f_{xx}(dp^S)^2 + f_{xy} dp^S dp^T + \frac{1}{2} f_{yy}(dp^T)^2.$$
Here sub indicies denote partial derivatives w.r.t the variables $t$, $x$, and $y$, respectively, and super indices indicate the maturity of the bond we are working with. Now, insert the dynamics
$$dp^T(t) = r(t)p^T(t)dt + \sigma_p^T(t)p^T(t)dV_t,$$
to obtain
$$\begin{aligned} dV^h &= \Bigg( f_t + \frac{1}{2} f_{xx}(\sigma_p^S p^S)^2 + f_{xy}\sigma_p^S \sigma_p^T p^S p^T + \frac{1}{2} f_{yy}(\sigma_p^T p^T)^2 \\ &\quad + r \underbrace{[f_x p^S + f_y p^T]}_{=f \text{ by hint}} \Bigg) dt + f_x \sigma_p^S p^S dV + f_y \sigma_p^T p^T dV \end{aligned}$$
Since we know that the local rate of return of $\Pi_{call}$ is $r$ we see that all second degree terms must vanish, and we are left with
$$dV^h = f_x dp^S + f_y dp^T.$$
All that remains is to check that $f_x = \Phi(d)$ and that $f_y = -K\Phi(d - \sigma_p)$. This turns out to be true (it is easy but cumbersome to do the calculations), and so we are done.

**5.** (a) The arbitrage free price at time $t$ of the contract is given by
$$\Pi(t) = p(t,T)E^T[F(T,T_1)|\mathcal{F}_t],$$
where the superindex $T$ indicates that the expectation should be taken under the $T$-forward measure $Q^T$ (the formula follows from the fact that $\Pi(t)/p(t,T)$ is a $Q^T$-martingale).

To price the contract we need the dynamics of $F(t,T_1)$ under $Q^T$. The Girsanov transformation between $Q$ and $Q^T$ is given by
$$dQ^T = L_t^T dQ \qquad \text{on } \mathcal{F}_t,$$
where
$$\begin{cases} dL_t^T &= \sigma_p(t,T)L_t^T dV_t, \\ L_0^T &= 1. \end{cases}$$
Here $\sigma_p(t,T)$ denotes the volatility at time $t$ of a bond with maturity $T$. To obtain this volatility note that the Vasiček model has an affine term structure, which means that
$$p(t,T) = e^{A(t,T) - B(t,T)r_t}.$$

Itô's formula gives us that

$$dp(t,T) = \dots dt - \sigma_r B(t,T)p(t,T)dV_t$$

(we actually know that the drift has to equal the short rate $r$, but that is not what we are interested in right now). Note that $\sigma_p$ is deterministic. In one of the hints the equation for $B$ is given. With the parameters for the Vasiček model inserted it has the following appearance

$$\begin{aligned} B_t(t,T) - \kappa B(t,T) &= -1 \\ B(T,T) &= 0. \end{aligned}$$

The solution of this equation is

$$B(t,T) = \frac{1}{\kappa}[1 - e^{-\kappa(T-t)}].$$

Girsanov's theorem states that

$$dV_t = \sigma_p(t,T)dt + dV_t^T,$$

where $V^T$ is a $Q^T$-Wiener process. The dynamics under $Q^T$ of the futures price of a futures contract for delivery at time $T_1$ are therefore

$$dF(t,T_1) = \sigma_F \sigma_p(t,T)F(t,T_1)dt + \sigma_F F(t,T_1)dV_t^T.$$

Integrating this we obtain

$$F(u,T_1) = \int_t^u \sigma_F \sigma_p(s,T)F(s,T_1)ds + \int_t^u \sigma_F F(s,T_1)dV_s^T.$$

Take the expectation of this conditional on $\mathcal{F}_t$ to obtain

$$E^T[F(u,T_1)|\mathcal{F}_t] = \int_t^u \sigma_F \sigma_p(s,T)E^T[F(s,T_1)|\mathcal{F}_t]ds.$$

Let $m(u) = E^T[F(u,T_1)|\mathcal{F}_t]$ and take the derivative with respect to $u$ and you will have the following ODE for $m$

$$\begin{cases} m'(u) = \sigma_F \sigma_p(u,T)m(u), \\ m(t) = F(t,T_1). \end{cases}$$

The solution to this equation is

$$m(u) = F(t,T_1)\exp\left\{\int_t^u \sigma_F \sigma_p(s,T)ds\right\}$$

The price of the contract is therefore

$$\begin{aligned} \Pi(t) &= p(t,T)F(t,T_1)\exp\left\{\int_t^T \sigma_F \sigma_p(s,T)ds\right\} \\ &= p(t,T)F(t,T_1)\exp\left\{-\sigma_F \frac{\sigma_r}{\kappa}[T - t - (1 - e^{\kappa(T-t)})/\kappa]\right\}. \end{aligned}$$

(b) The contracts are not the same. Since the price of a futures contract is zero at all times receiving one is not worth anything and the price of the second contract is therefore zero at all times.