KTH Mathematics

Exam in SF2975 Financial Derivatives.
Monday August 25 2008 14.00-19.00.

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Aids: None.

General instructions: The solutions should be legible, easy to follow and not lacking in explanations. In particular, all notation used should be explained. Problems concerning integrability need not be treated.

N.B. Number the pages and write your name on each sheet. Write only one exercise per sheet and state the number of the exercise on the sheet.

25 points including the bonus guarantees a passing grade.

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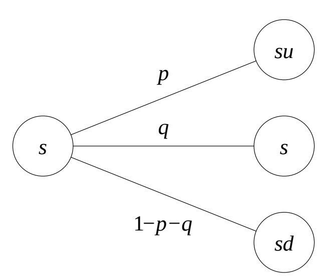
1. (a) Compute the price of a *European call option* with strike price $K = 100$ kr and exercise date $T = 2$ years, using a binomial tree with two trading dates $t_1 = 0$ and $t_2 = 1$ (your portfolio at time $t_3 = 2$ is the same as your portfolio at time $t_2 = 1$) and parameters $s_0 = 100$, $u = 1.5$, $d = 0.5$, $r = 10\%$, and $p = 0.75$. .... (3p)

(b) Consider a one period model very similar to the one period binomial model, the only difference being that the stock price S can also stay the same with a certain probability q, as depicted in the figure below.



Given that $d < 1 + r < u$ is the model complete or not? A proof or counterexample is required. .... (3p)

(c) For each of the following four statements indicate whether it is true or false *in a standard Black-Scholes setting.* It is also possible not to answer at all. A correct answer will be rewarded with one point, an incorrect answer results in minus one point, and no answer gives zero points. If the total sum of points is negative you will get zero points.

i. Implied volatility is estimated from historical data on prices of the underlying asset.

ii. The delta of a contingent T-claim X is the rate of change of its price with respect to the price of the underlying asset.

iii. Assume that $h = (h_1, \ldots, h_n)$ and $S = (S_1, \ldots, S_n)$ are adapted processes with stochastic differentials representing a portfolio process and a price process. If $V = \sum_{i=1}^{n} h_i S_i$, then

$$dV = hdS.$$

iv. The local rate of return of an ideally traded asset is always equal to the short rate under the risk-neutral martingale measure Q.

.... (4p)

2. Consider a standard Black-Scholes market, i.e. a market consisting of a risk free asset, B, with P-dynamics given by

$$\begin{cases} dB_t &= rB_t dt, \\ B_0 &= 1, \end{cases}$$

and a stock, S, with P-dynamics given by

$$\begin{cases} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ S_0 &= s_0. \end{cases}$$

Here W denotes a P-Wiener process and r, α and σ are assumed to be constants.

(a) Let $K_2 > K_1 > 0$ be fixed real numbers. The payoff at maturity time T of a *collar option* is given by

$$X = \min\{\max\{S_T, K_1\}, K_2\}.$$

Compute the arbitrage price of the collar option for $t \leq T$. .... (4p)

(b) The term Asian option is a generic name for a class of options whose terminal payoffs are based on average asset values during some period of the option's lifetime. Due to their averaging feature Asian options are suitable for assets which are not traded liquidly. Pricing Asian options is difficult. Your task is to determine the arbitrage price at time t for $t \leq T$ of the simpler T-claim defined by

$$X = S(T) - \frac{1}{T}\int_0^T S(u)du.$$

(The claim is thus averaging, but there is no option feature.) .... (6p)

3. Consider a two-dimensional Black-Scholes market i.e. a market consisting of a risk-free asset, B, with P-dynamics given by

$$\begin{cases} dB_t &= rB_t dt, \\ B_0 &= 1, \end{cases}$$

(r is constant) and two stocks, X and Y, with P-dynamics given by

$$\begin{cases} dX_t &= \alpha X_t dt + \rho X_t dV_t + \sigma X_t dW_t, \\ dY_t &= \beta Y_t dt + \gamma Y_t dV_t. \end{cases}$$

Here W and V are two independent P-Wiener processes and α, ρ, σ, β and γ are assumed to be constants.

(a) Assume that the filtration is the natural filtration generated by the Wiener processes W and V. Show that this model is free of arbitrage and complete given that $\sigma\gamma \neq 0$. .... (6p)

(b) In general the term *rainbow option* refers to a derivative exposed to two or more sources of uncertainty. Usually a rainbow option is a call or put on the best or worst of n underlying assets, or an option which pays the best or worst of n assets. The payoff at maturity T of a call with strike K on the maximum of the two assets in the model from (a) is

$$Z_{call,max} = \max\{\max(X_T, Y_T) - K, 0\}.$$

Now suppose that you can price a European version of the above call option *for all* $K \geq 0$. In order to price the put corresponding to the call, i.e. the T-claim with payoff

$$Z_{put,max} = \max\{K - \max(X_T, Y_T), 0\},$$

without doing more computations, you need a put–call parity. Derive such a put–call parity. (You may use the prices of B, X, Y, the prices of the calls, and the price of the put of course). .... (4p)

4. Consider a market where the short rate, r, is given by the Vasiček model under the equivalent martingale measure Q, that is

$$dr = \kappa(\alpha - r)dt + \sigma_r dV_t,$$

where κ, α and σ_r are constants and V is a one-dimensional standard Q-Wiener process. It is then known that the price of a European call option with time of maturity T and strike price K on an S-bond where $T < S$ is given by

$$\Pi_{call}(t) = p(t,S)\Phi(d) - p(t,T)K\Phi(d - \sigma_p)$$

where

$$\begin{aligned} d &= \frac{1}{\sigma_p}\ln\left(\frac{p(t,S)}{p(t,T)K}\right) + \frac{1}{2}\sigma_p \\ \sigma_p &= \frac{1}{\kappa}\left\{1 - e^{-\kappa(S-T)}\right\}\sqrt{\frac{\sigma_r^2}{2\kappa}\{1 - e^{-2\kappa(T-t)}\}}. \end{aligned}$$

Here $p(t, T)$ denotes the price at time t of a zero coupon bond with maturity T, and Φ denotes the cumulative distribution function for the $\mathrm{N}(0, 1)$ distribution.

(a) Define what is meant by a **replicating** or **hedging** portfolio for a contingent T-claim X. .... (2p)

(b) Show that a replicating portfolio for the call option is one that holds

- $\Phi(d)$ units of the S-bond, and
- $-K\Phi(d - \sigma_p)$ units of the T-bond.

.... (8p)

Hint: It might help to use that

$$\Pi_{call}(t) = f(t, p(t, S), p(t, T)),$$

and that if

$$f(t, \lambda x, \lambda y) = \lambda f(t, x, y)$$

then

$$f(t, x, y) = f_x(t, x, y)x + f_y(t, x, y)y.$$

It may also help to recall that you know what the drift of any (ideally traded) price process is under the risk neutral martingale measure Q.

5. Consider a market where the short rate, r, is given by the Vasiček model under the equivalent martingale measure Q, that is

$$dr = \kappa(\alpha - r)dt + \sigma_r dV_t,$$

where κ, α and σ_r are constants and V is a one-dimensional standard Q-Wiener process. Furthermore, let $F(t, T)$ denote the futures price at time t, of a futures contract written on some fixed asset also present on the market, for delivery at time T. Suppose that the dynamics of these futures prices are given by

$$dF(t, T) = \sigma_F F(t, T)dV_t,$$

where σ_F is a constant.

(a) Fix two points in time T and T_1, with $T < T_1$. Your task is to find the arbitrage free price at time t, $t \in [0, T]$ of a contract which pays $F(T, T_1)$ dollars to the holder of the contract at time T. The price of the contract at time t is allowed to be expressed in terms of bond prices and futures prices observable at time t. .... (8p)

Hint: A change of measure might simplify the computations.

(b) Suppose that instead of $F(T, T_1)$ dollars, the holder receives a futures contract with time of delivery T_1 at time T. Is this the same contract as the one described above? If not, what is the arbitrage free price at time t, $t \in [0, T]$ of this contract? .... (2p)

Hints:
You are free to use the following in any of the above exercises.

- The density function of a normally distributed random variable with expectation m and variance σ^2 is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/(2\sigma^2)}.$$

- Let Φ denote the cumulative distribution function for the N(0, 1) distribution. Then

$$\Phi(-x) = 1 - \Phi(x).$$

- The standard Black-Scholes formula for the price $\Pi(t)$ of a European call option with strike price K and time of maturity T is $\Pi(t) = F(t, S(t))$, where

$$F(t, s) = s\Phi[d_1(t, s)] - e^{-r(T-t)} K\Phi[d_2(t, s)].$$

  Here Φ is the cumulative distribution function for the $N(0, 1)$ distribution and

$$\begin{aligned} d_1(t, s) &= \frac{1}{\sigma\sqrt{T-t}} \left\{ \ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t) \right\}, \\ d_2(t, s) &= d_1(t, s) - \sigma\sqrt{T-t}. \end{aligned}$$

- Consider a short rate model where the Q-dynamics are given by

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dV(t).$$

  Proposition 0.1 (Affine term structure) *Assume that μ and σ are of the form*

$$\begin{cases} \mu(t, r) = a(t)r + b(t), \\ \sigma^2(t, r) = c(t)r + d(t). \end{cases}$$

  Then the model admits an affine term structure, i.e. $p(t, T) = e^{A(t,T) - B(t,T)r(t)}$, *where A and B satisfy the system*

$$\begin{cases} A_t(t, T) = b(t)B(t, T) - \frac{1}{2}d(t)B^2(t, T), \\ A(T, T) = 0, \end{cases} \tag{1}$$

$$\begin{cases} B_t(t, T) + a(t)B(t, T) - \dfrac{1}{2}c(t)B^2(t, T) = -1, \\ B(T, T) = 0. \end{cases} \tag{2}$$

Good luck!