KTH Mathematics

Exam in SF2975 Financial Derivatives.
Monday August 24 2009 14.00-19.00.

Answers and suggestions for solutions.

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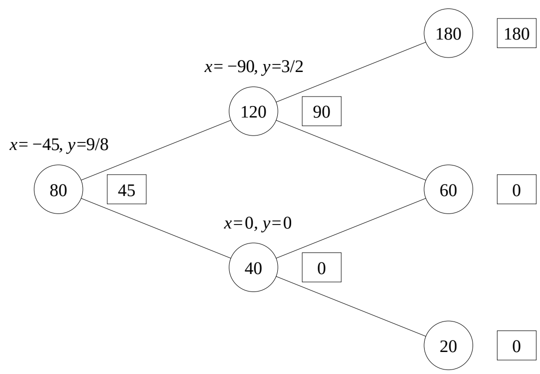
1. (a) To obtain the replicating portfolio at $t = 0$ we have to solve the following set of equations

$$\begin{cases} x + y \cdot 120 &= 90, \\ x + y \cdot 40 &= 0, \end{cases}$$

since regardless of whether the stock price goes up or down the value of the portfolio should equal the value of the option. This yields

$$x = -45, \qquad y = \frac{9}{8}.$$

Using the same method we find the rest of the replicating portfolio strategy and it is shown in the figure below.



That the portfolio strategy is self-financing is seen from the following equations

$$\begin{cases} -45 + \frac{9}{8} \cdot 120 &= -90 + \frac{3}{2} \cdot 120, \\ -45 + \frac{9}{8} \cdot 40 &= 0 + 0 \cdot 40. \end{cases}$$

(b) Denoting $E\left[r(t)\right]$ by $x(t)$, and using standard technique we obtain the ODE

$$\frac{dx_t}{dt} = \alpha - \beta x_t.$$

Thus

$$\begin{aligned} E\left[r(t)\right] &= x(t) = e^{-\beta t} r_0 + \int_0^t e^{-\beta(t-s)} \alpha ds \\ &= e^{-\beta t} r_0 + \frac{\alpha}{\beta}\left[1 - e^{-\beta t}\right] \end{aligned}$$

(c) If we denote by $C(t, K)$ the price at time t of a European call option with strike price K and exercise date T written on the stock, and by $P(t, K)$ the price at time t of a European put option with the same strike price and exercise date as the call, and also having the stock as underlying, then the price of the box spread strategy is given by

$$\Pi_{box}(t) = C(t, K_1) - C(t, K_2) - P(t, K_1) + P(t, K_2).$$

According to put-call parity we have

$$P(t, K) = Ke^{-r(T-t)} + C(t, K) - S(t).$$

Using this we obtain that the price of the box spread strategy is

$$\Pi_{box}(t) = e^{-r(T-t)}(K_2 - K_1).$$

2. (a) The only difference between the gap option and a European call option is that in the payoff function the strike price K is replaced by the cash amount A at one point. The price of the gap option is therefore $\Pi(t) = F(t, S(t))$, where

$$F(t, s) = s\Phi[d_1(t, s)] - e^{-r(T-t)} A\Phi[d_2(t, s)].$$

Here Φ is the cumulative distribution function for the $N(0, 1)$ distribution and

$$\begin{aligned} d_1(t, s) &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\}, \\ d_2(t, s) &= d_1(t, s) - \sigma\sqrt{T-t}. \end{aligned}$$

(b) Suppose that

$$C(K_1) - c(K_1) < C(K_2) - c(K_2).$$

Then selling short the RHS (write the American call and buy the European call, both with strike K_2) and buying the LHS (buy the American call and write the European call, both with strike K_1) will result in a strictly positive profit, which we invest in the bank account. If the American option with exercise price K_2 is not exercised before maturity the position can be held to maturity and then the cash flows will net out at zero. If the American option with exercise price K_2 is exercised before maturity (recall that this decision is made by the holder of the option) you can exercise the American option with strike K_1 and thus receive $K_2 - K_1 > 0$ at say time $\tau < T$, where T is the time of maturity of the options. This profit can be invested in the bank account and at maturity you will be left with at least

$$(K_2 - K_1)e^{r(T-\tau)} - (K_2 - K_1) > 0.$$

Thus this strategy leads to a sure profit at no cost, i.e. an arbitrage. If there is to be no arbitrage we therefore have to have

$$C(K_1) - c(K_1) \geq C(K_2) - c(K_2).$$

3. (a) A short rate model is said to have an affine term structure if zero coupon bond prices can be written on the following form

$$p(t,T) = e^{A(t,T)-B(t,T)r_t},$$

where A and B are deterministic functions.

Sufficient conditions on μ and σ which guarantee the existence of an affine term structure are that μ and σ^2 are affine in r (and that there exists solutions to two certain ordinary differential equation, see below), i.e.

$$\begin{cases} \mu(t,r) &= a(t)r + b(t), \\ \sigma^2(t,r) &= c(t)r + d(t). \end{cases}$$

To see this insert these expression into the term structure equation

$$\begin{cases} F_t^T + \mu F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T - rF^T = 0, \\ F(T,r) = 1. \end{cases}$$

(Here $F^T(t,r_t) = p(t,T)$, and we have used the notation $F_t^T = \partial F^T/\partial t$, etc.) This will after some rewriting give you

$$\begin{cases} A_t(t,T) - b(t)B(t,T) + \frac{1}{2}d(t)B^2(t,T) \\ + \left(1 + B_t(t,T) + a(t)B(t,T) - \frac{1}{2}c(t)B^2(t,T)\right) r = 0, \\ e^{A(T,T)-B(T,T)r} = 0. \end{cases}$$

This equation should hold for all t and r, so there will be an affine term structure if A and B solve the following ordinary differential equations

$$\begin{cases} A_t(t,T) &= b(t)B(t,T) - \frac{1}{2}d(t)B^2(t,T), \\ A(T,T) &= 0, \end{cases} \tag{1}$$

and

$$\begin{cases} B_t(t,T) + a(t)B(t,T) - \dfrac{1}{2}c(t)B^2(t,T) &= -1, \\ B(T,T) &= 0. \end{cases} \tag{2}$$

(b) The price $\Pi(t;X)$ of the caplet is given by

$$\begin{aligned}
\Pi(t;X) &= E^Q\left[e^{-\int_t^T r_s ds} X \middle| \mathcal{F}_t \right] \\
&= E^Q\left[e^{-\int_t^T r_s ds} \max\left\{ \frac{1-p(S,T)}{p(S,T)} - R(T-S), 0 \right\} \middle| \mathcal{F}_t \right] \\
&= E^Q\left[E^Q\left[e^{-\int_t^S r_s ds} e^{-\int_S^T r_s ds} \max\left\{ \frac{1-p(S,T)}{p(S,T)} - R(T-S), 0 \right\} \middle| \mathcal{F}_S \right] \middle| \mathcal{F}_t \right] \\
&= E^Q\left[e^{-\int_t^S r_s ds} \max\left\{ \frac{1-p(S,T)}{p(S,T)} - R(T-S), 0 \right\} E^Q\left[e^{-\int_S^T r_s ds} \middle| \mathcal{F}_S \right] \middle| \mathcal{F}_t \right] \\
&= E^Q\left[e^{-\int_t^S r_s ds} \max\left\{ 1 - p(S,T) - p(S,T)R(T-S), 0 \right\} \middle| \mathcal{F}_t \right] \\
&= [1 + R(T-S)] E^Q\left[e^{-\int_t^S r_s ds} \max\left\{ \frac{1}{1+R(T-S)} - p(S,T), 0 \right\} \middle| \mathcal{F}_t \right]
\end{aligned}$$

For the third equality we have used iterated expectation, note that $\mathcal{F}_t \subseteq \mathcal{F}_S$. The fourth equality is obtained using that $e^{-\int_t^S r_s ds} \in \mathcal{F}_S$ and $X \in \mathcal{F}_S$ (although it is a T-claim). To obtain the fifth equality note that

$$p(S,T) = E^Q\left[e^{-\int_S^T r_s ds} \middle| \mathcal{F}_S \right].$$

The price of the caplet is hence equal to the price of $\alpha = 1+R(T-S)$ European put options written on the zero coupon bond with maturity at T. The strike price of the put option is $K = 1/[1+R(T-S)]$ and the exercise date is S. Thus holding a portfolio of α of said put options is equivalent to holding the caplet. Since the portfolio is constant it is self-financing and therefore a replicating portfolio for the caplet.

4. (a) Under Q^d the following processes should be martingales

$$\frac{B^d}{B^d}, \quad \frac{\widetilde{B}^f}{B^d} = \frac{XB^f}{B^d}, \quad \frac{\widetilde{S}^f}{B^d} = \frac{XS^f}{B^d}.$$

This means that the processes B^d, $\widetilde{B}^f$, and $\widetilde{S}^f$ should have a local rate of return equal to r_d. Perform a Girsanov transformation according to

$$dQ^d = L^d(t)dP \qquad \text{on } \mathcal{F}_t,$$

where

$$\begin{cases} dL_t^d &= L_t^d \varphi^* d\overline{W}_t, \\ L_0^d &= 1. \end{cases}$$

Here $*$ denotes transpose. Using Itô's formula and Girsanov's theorem we find the Q^d-dynamics of the above processes to be

$$\begin{aligned} d\widetilde{B}^f &= \widetilde{B}^f(r_f + \alpha_X + \sigma_X\varphi)dt + \widetilde{B}^f\sigma_X d\overline{V}, \\ d\widetilde{S}^f &= \widetilde{S}^f(\alpha_f + \alpha_X + \sigma_f\sigma_X^* + [\sigma_f + \sigma_X]\varphi)dt + \widetilde{S}^f(\sigma_f + \sigma_X)d\overline{V}. \end{aligned}$$

Here $\overline{V} = (V_1, V_2)^*$ is the two dimensional Q^d-Wiener process, defined by $d\overline{V} = d\overline{W} - \varphi dt$. Setting the local rates of return equal to r_d gives the following system of equations

$$\begin{aligned} \sigma_X\varphi &= r_d - r_f - \alpha_X, \\ (\sigma_f + \sigma_X)\varphi &= r_d - \alpha_f - \alpha_X - \sigma_f\sigma_X^*. \end{aligned}$$

Thus, under Q^d we have

$$\begin{aligned} dS^f &= (\alpha_f + \sigma_f\varphi)S^f dt + S^f\sigma_f d\overline{V} \\ &= S^f(r_f - \sigma_f\sigma_X^*)dt + S^f\sigma_f d\overline{V}. \end{aligned}$$

(b) The price of the foreign equity call struck in domestic currency

$$\Pi(t;Y) = e^{-r_d(T-t)}E^d\left[\max\{X_T S_T^f - K^d, 0\} \middle| \mathcal{F}_t \right].$$

From 4a we know that the dynamics of $\widetilde{S}^f = XS^f$ under Q^d are given by

$$d\widetilde{S}^f = \widetilde{S}^f r_d dt + \widetilde{S}^f(\sigma_f + \sigma_X)d\overline{V}.$$

or

$$d\widetilde{S}^f = \widetilde{S}^f r_d dt + \widetilde{S}^f \|\sigma_f + \sigma_X\| dU.$$

where U is a one-dimensional Q^d-Wiener process and $\|\cdot\|$ denotes the Euclidean norm. The price is therefore given by $c(t, \widetilde{S}_t^f, K^d, T, r_d, \|\sigma_f + \sigma_X\|)$, where $c(t, s, K, T, r, \sigma)$ denotes the standard Black-Scholes price at time t of a European call option with exercise price K and expiry date T, when the current price of the underlying is s, the interest rate is r, and the volatility of the underlying is σ.

5. The Z-gain process has the following appearance in this case

$$\begin{aligned} G^Z(t) &= \frac{\Pi(t)}{B(t)} + \int_0^t \frac{1}{B(s)} ds \\ &= g(r_t)\exp\{-\int_0^t r_s ds\} + \int_0^t \exp\{-\int_0^s r_u du\} ds. \end{aligned}$$

Let

$$\begin{aligned} X_t &= \int_0^t r_s ds, \\ Y_t &= \int_0^t \exp\{-\int_0^s r_u du\} ds. \end{aligned}$$

The Z-gain process can then be written as

$$G^Z(t) = g(r_t)e^{-X_t} + Y_t = \Psi(r_t, X_t, Y_t).$$

Using Itô's formula we obtain

$$\begin{aligned} dG^Z &= \Psi_r dr + \Psi_X dX + \Psi_Y dY + \frac{1}{2}\Psi_{rr}(dr)^2 \\ &= g'(r)e^{-X} q r^{3/2} dV - g(r)e^{-X} r dt + 1 \cdot e^{-X} dt + \frac{1}{2} g''(r) e^{-X} q^2 r^3 dt. \end{aligned}$$

Now, since the Z-gain process is a Q-martingale the dt-term must equal zero, that is

$$-g(r)e^{-X} r + e^{-X} + \frac{1}{2} g''(r) e^{-X} q^2 r^3 = 0.$$

We thus have the following ODE for g

$$-rg(r) + 1 + \frac{1}{2} q^2 r^3 g''(r) = 0.$$

Inserting $g(r) = A/r$ we obtain the following equation for A

$$-A + 1 + q^2 A = 0.$$

The solution to this equation is

$$A = \frac{1}{1 - q^2}.$$