KTH Mathematics

Exam in SF2975 Financial Derivatives.
Monday May 18 2009 08.00-13.00.

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Aids: None.

General instructions: The solutions should be legible, easy to follow and not lacking in explanations. In particular, all notation used should be explained. Problems concerning integrability need not be treated.

N.B. Number the pages and write your name on each sheet. Write only one exercise per sheet and state the number of the exercise on the sheet.

25 points including the bonus guarantees a passing grade.

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1. (a) Compute the price of an *American put option* with strike price $K = 90$ kr and exercise date $T = 2$ years, using a binomial tree with two trading dates $t_1 = 0$ and $t_2 = 1$ (your portfolio at time $t_3 = 2$ is the same as your portfolio at time $t_2 = 1$) and parameters $s_0 = 100$, $u = 1.5$, $d = 0.5$, $r = 0$, and $p = 0.75$. . .(3p)

(b) Consider the Vasiček model

$$\begin{aligned} dr(t) &= [\alpha - \beta r(t)]\,dt + \sigma dW(t), \\ r(0) &= r_0, \end{aligned}$$

where α, β, σ, and r_0 are constants. Determine the distribution of $r(t)$. . (3p)

(c) Consider the standard Black-Scholes setting described in detail in Exercise 2. A portfolio is a vector process $\mathbf{h} = (h^B, h^S)$ which is adapted and sufficiently integrable. Here h^B is the number of risk free assets in the portfolio and h^S is the number of stocks. The corresponding value process is given by

$$V(t; \mathbf{h}) = h^B(t)B(t) + h^S(t)S(t).$$

i. Define what is meant by an **arbitrage strategy** and a **replicating strategy** in this setting. . . . . (2p)

ii. Define what is meant by that the model is **free of arbitrage** and that the model is **complete**. . . . . (2p)

2. Consider a standard Black-Scholes market, i.e. a market consisting of a risk free asset, B, with P-dynamics given by

$$\begin{cases} dB_t &= rB_t dt, \\ B_0 &= 1, \end{cases}$$

and a stock, S, with P-dynamics given by

$$\begin{cases} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ S_0 &= s_0. \end{cases}$$

Here W denotes a P-Wiener process and r, α and σ are assumed to be constants.

(a) Determine the arbitrage price of a so called *straddle*, which is a contingent T-claim defined by

$$X = \phi(S_T) = |S_T - K|.$$

A straddle is thus a contract you would buy if you believed that there would be large movements in the stock price, but you were not sure in which direction. .... (4p)

(b) Determine the arbitrage price of a *powered call*, which is a T-claim $X = \phi(S_T)$ with contract function ϕ given by

$$\phi(s) = \begin{cases} (s-K)^2, & \text{if } s > K \\ 0, & \text{otherwise.} \end{cases}$$

Hint: The payoff function can be written as $\phi(s) = (s-K)^2 I\{s > K\}$, where I denotes the indicator function. If you develop the square and compare this with the payoff of a standard call option $\phi_{call}(s) = (s-K)I\{s > K\}$ you will find that only one term will require work when pricing. .... (6p)

3. Consider a standard HJM model, under the risk neutral martingale measure Q, of the form

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t$$

where W for simplicity is assumed to be a scalar Q-Wiener process. We assume that $\sigma(t,T)$ is a **deterministic** function of the two variables t and T. Let $p(t,T)$ denote the bond price at t of a zero coupon bond maturing at T.

(a) The bond price dynamic induced by the forward rate dynamics above can, after we have used the HJM drift condition, be written as

$$dp(t,T) = p(t,T)r(t)dt + p(t,T)v(t,T)dW_t$$

Compute the bond volatility $v(t,T)$ in terms of the forward rate volatility σ. .... (5p)

(b) Compute as explicitly as possible, in terms of the forward rate volatility σ, the expectation

$$E^Q\left[\left.\frac{1}{p(S,T)}\right|\mathcal{F}_t\right],$$

where $t \leq S \leq T$. This expression turns up in connection with the pricing of Eurodollar Futures. .... (5p)
Hint: It may be profitable to use the relation $1 = p(S,S)$ in the expectation above.

4. Consider a two-dimensional Black-Scholes market i.e. a market consisting of a risk-free asset, B, with P-dynamics given by

$$\begin{cases} dB_t &= rB_t dt, \\ B_0 &= 1, \end{cases}$$

(r is constant) and two stocks, S^1 and S^2, with P-dynamics given by

$$\begin{cases} dS_t^1 &= \alpha_1 S_t^1 dt + \sigma_1 S_t^1 dW_t^1, \\ dS_t^2 &= \alpha_2 S_t^2 dt + \sigma_2 \rho S_t^2 dW_t^1 + \sigma_2\sqrt{1-\rho^2} S_t^2 dW_t^2. \end{cases}$$

Here W^1 and W^2 are two independent P-Wiener processes and α_i, σ_i, $i = 1,2$ and ρ are assumed to be constants. The asset with price process S^1 is assumed to be a non-dividend paying stock, whereas the asset with price process S^2 is assumed to be an index which pays a continuous dividend yield δ, i.e. the cumulative dividend process of the index has the structure

$$dD_t = \delta S_t dt,$$

where $\delta > 0$ is a constant.

(a) Write down the dynamics of S^1 and S^2 under the risk neutral martingale measure Q. .... (2p)

(b) Compute the arbitrage price of a European put option on S^1 with an uncertain exercise price depending on S^2. More precisely, compute the arbitrage price of the T-claim X with payoff

$$X = \max\{KS_T^2 - S_T^1, 0\}.$$

where $K > 0$ is a constant. .... (8p)
Hint: The payoff the claim factorizes as

$$X = KS_T^1 \max\left\{\frac{S_T^2}{S_T^1} - \frac{1}{K}, 0\right\}.$$

5. (a) Forward rate models have to be specified with some care to avoid introducing arbitrage possibilities. It is well known that if the forward rates are modeled as

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t, \quad T \geq 0,$$

where W is a (one-dimensional) Wiener process under the risk neutral martingale measure Q the Heath-Jarrow-Morton drift condition has to be satisfied in order for there to be no arbitrage. State the Heath-Jarrow-Morton drift condition (under Q). .... (2p)

(b) It is also well known that an arbitrary specification of the forward rate volatilities will lead to a non-Markovian short rate. There are some special cases though. Define the forward rate volatilities by

$$\sigma(t,T) = \eta(r_t)e^{-\int_t^T \kappa(x)dx},$$

where $\eta(x)$ is a deterministic function, r_t denotes the short rate at time t, and $\kappa(x)$ is a deterministic (integrable) function. Now define the process Z according to

$$Z_t = \int_0^t \sigma^2(s,t)ds.$$

Show that the dynamics of the short rate r and the process Z satisfies the following two-dimensional SDE

$$d\begin{pmatrix} r_t \\ Z_t \end{pmatrix} = \begin{pmatrix} \mu(t, r_t, Z_t) \\ \eta^2(r_t) - 2\kappa(t)Z_t \end{pmatrix} dt + \begin{pmatrix} \eta(r_t) \\ 0 \end{pmatrix} dW_t$$

where

$$\mu(t, r_t, Z_t) = \kappa(t)[f(0,t) - r_t] + Z_t + \frac{\partial}{\partial t}f(0,t).$$

The short rate itself is thus not Markovian, but there is a two-dimensional vector including the short rate which is. .... (8p)

Hint: In essence this is deriving dr from df using the specific volatility structure. The following "trick" is useful

$$g(s,t) = g(s,s) + \int_s^t \frac{\partial}{\partial u}g(s,u)du.$$

Hints:
You are free to use the following in any of the above exercises.

- The density function of a normally distributed random variable with expectation m and variance σ^2 is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-m)^2/(2\sigma^2)}.$$

- Let Φ denote the cumulative distribution function for the N(0,1) distribution. Then

$$\Phi(-x) = 1 - \Phi(x).$$

- The standard Black-Scholes formula for the price $\Pi(t)$ of a European call option with strike price K and time of maturity T is $\Pi(t) = F(t, S(t))$, where

$$F(t,s) = s\Phi[d_1(t,s)] - e^{-r(T-t)}K\Phi[d_2(t,s)].$$

Here Φ is the cumulative distribution function for the $N(0,1)$ distribution and

$$\begin{aligned} d_1(t,s) &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\}, \\ d_2(t,s) &= d_1(t,s) - \sigma\sqrt{T-t}. \end{aligned}$$

- Suppose that there exist processes $X(\cdot, T)$ for every $T \geq 0$ and suppose that Y is a process defined by

$$Y(t) = \int_t^{T_0} X(t,s)ds$$

Then we have the following version of Itô's formula

$$dY_t = -X(t,t)dt + \int_t^{T_0} dX(t,s)ds.$$

- Consider a short rate model where the Q-dynamics are given by

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dV(t).$$

Proposition 0.1 (Affine term structure) *Assume that μ and σ are of the form*

$$\begin{cases} \mu(t,r) = a(t)r + b(t), \\ \sigma^2(t,r) = c(t)r + d(t). \end{cases}$$

Then the model admits an affine term structure, i.e. $p(t,T) = e^{A(t,T)-B(t,T)r(t)}$, where A and B satisfy the system

$$\begin{cases} A_t(t,T) = b(t)B(t,T) - \frac{1}{2}d(t)B^2(t,T), \\ A(T,T) = 0, \end{cases} \tag{1}$$

$$\begin{cases} B_t(t,T) + a(t)B(t,T) - \frac{1}{2}c(t)B^2(t,T) = -1, \\ B(T,T) = 0. \end{cases} \tag{2}$$

Good luck!