KTH Matematik

SOLUTION TO EXAMINATION IN SF2975 FINANCIAL DERIVATIVES 2011-05-27.

Problem 1

(a) For the martingale probabilities we have

$$q = \frac{1+r-d}{u-d} = 0.5.$$

Using them we obtain the following binomial tree for the share price and for the option price The price of the option is 9.09.

100	140	196		9.09	0	0
	80	112			20	0
		64				36

Table 1: Binomial tree for the spot price (left) and for the options price (right).

(b) Since $\max(S_T - K, 0) = S_T - K + \max(K - S_T, 0)$ it follows by linearity of the price Π that

$$\Pi(\max(S_T - K, 0)) = \Pi(S_T) - \Pi(K) + \Pi(K - S_T, 0).$$

In the Black-Scholes model this becomes

$$c(t, T, S_0, K, r, \sigma) = S_0 - e^{-r(T-t)}K + p(t, T, S_0, K, r, \sigma),$$

where c is the price of a call and p the price of a put.
(c) Since the market is free of arbitrage the spot price must drop by 5% at time t_0. Let $S(t_0)$ be the price just after the dividend is paid and $S(t_0-)$ the price just before. Then $S(t_0) = 0.95S(t_0-)$. Up to time t_0- we have a standard Black-Scholes model, so

$$dS(t) = rS(t)dt + \sigma S(t)dW^Q(t),\ S(0) = s,\ t < t_0,$$

and after t_0 we also have a standard Black-Scholes model so,

$$dS(t) = rS(t)dt + \sigma S(t)dW^Q(t),\ S(t_0) = 0.95S(t_0-), t \geq t_0.$$

Problem 2

Under the equivalent martingale measure Q we have

$$dS(t) = (r-\delta)S(t)dt + \sigma S(t)dW^Q(t),$$

and the price of the option is given by

$$100e^{-rT}E^Q[I\{S(T) \le K\}] = 100e^{-rT}Q(S(T) \le K).$$

Since $S(T)$ has the same distribution as

$$S(0)\exp\Big\{(r-\delta-\sigma^2/2)T + \sigma\sqrt{T}Z\Big\},$$

where $Z \sim N(0,1)$ it follows that the price is

$$100e^{-rT}N(d), \text{ with } d = \frac{\log(K/S(0)) - (r-\delta-\sigma^2/2)T}{\sigma\sqrt{T}}.$$

Problem 3

The model has affine term structure if

$$p(t,T) = e^{A(t,T)-B(t,T)r(t)},$$

for some deterministic functions $A(t,T)$ and $B(t,T)$. The term structure equation leads to

$$\begin{aligned} &B_t(t,T) = -1, \quad B(T,T) = 0 \\ &A_t(t,T) = \Theta(t)B(t,T) - \frac{\sigma^2}{2}B^2(t,T), \quad A(T,T) = 0. \end{aligned}$$

The solution to these equations are

$$\begin{aligned} B(t,T) &= T - t, \\ A(t,T) &= \int_t^T \Theta(s)(s-T)ds + \frac{\sigma^2}{2}\frac{(T-t)^3}{3}. \end{aligned}$$

This gives the price $p(t,T)$ of a zero-coupon bond.

Problem 4

(a) The LIBOR rate $L_i(t)$ is such that if you put in 1 at time T_{i-1} you receive $1 + L_i(t)(T_i - T_{i-1})$ at T_i. That is, $L_i(t)$ is the solution to

$$1 + L_i(t)(T_i - T_{i-1}) = \frac{p(t,T_{i-1})}{p(t,T_i)}.$$

Then we see

$$L_i(t) = \frac{p(t,T_{i-1}) - p(t,T_i)}{(T_i - T_{i-1})p(t,T_i)}.$$

(b) One can first note that under the forward measure $Q^{(T_i)}$ the LIBOR rate must be a martingale, because it is a price process divided by the numeraire. For the zero-coupon bonds we have, from Ito's formula,

$$dp(t,T) = \{\dots\}dt + p(t,T)(-B(t,T)\sigma)dW^Q,$$

and by Ito's formula we have in particular

$$d\frac{p(t,T_{i-1})}{p(t,T_i)} = \{\dots\}dt + \frac{p(t,T_{i-1})}{p(t,T_i)}(-B(t,T_{i-1}) + B(t,T_i))\sigma dW^Q.$$

Since the diffusion term does not change when changing measure from Q to $Q^{(T_i)}$ we also have

$$d\frac{p(t,T_{i-1})}{p(t,T_i)} = \{\dots\}dt + \frac{p(t,T_{i-1})}{p(t,T_i)}(-B(t,T_{i-1}) + B(t,T_i))\sigma dW^{Q^{(T_i)}}.$$

We know the drift must be 0 since it is a martingale, and putting in the expression for $B(t,T)$ we get

$$\begin{aligned} dL_i(t) &= \frac{1}{T_i - T_{i-1}} d\Big(\frac{p(t,T_{i-1})}{p(t,T_i)} - 1\Big) \\ &= \frac{p(t,T_{i-1})}{(T_i - T_{i-1})p(t,T_i)}(-B(t,T_{i-1}) + B(t,T_i))\sigma dW^{Q^{(T_i)}} \\ &= (L_i(t)(T_i - T_{i-1}) + 1)\sigma dW^{Q^{(T_i)}}. \end{aligned}$$

Problem 5

Since $S(t) = s\exp\{(r - \sigma^2/2)t + \sigma W^Q(t)\}$ it follows that

$$\int_0^T \log S(t)dt = T\log s + (r - \sigma^2/2)\frac{T^2}{2} + \sigma \int_0^T W^Q(t)dt.$$

The last term is the complicated one. Since

$$d(tW^Q(t)) = W^Q(t)dt + tdW^Q(t),$$

it follows that

$$\int_0^T W^Q(t)dt = TW^Q(T) - \int_0^T tdW^Q(t) = \int_0^T (T - t)dW^Q(t),$$

which has a normal distribution with variance

$$\int_0^T (T - t)^2 dt = \frac{T^3}{3}.$$

That is, $\int_0^T \log S(t)dt$ has the same distribution as

$$T\log s + (r - \sigma^2/2)\frac{T^2}{2} + \sigma\frac{T^{3/2}}{\sqrt{3}}Z,$$

where $Z \sim N(0,1)$.
As in the derivation of the Black-Scholes formula we have

$$E^Q[\max(e^{a+bZ} - K, 0)] = e^{a+b^2/2}N(d_1) - KN(d_2),$$

with $d_1 = (-\log K + a)/b$ and $d_2 = d_1 - b$. It follows that the price of the Asian option is given by

$$\begin{aligned}
&e^{-rT} E^Q[\max(\exp\{T \log s + (r - \sigma^2/2)\frac{T^2}{2} + \sigma \frac{T^{3/2}}{\sqrt{3}} Z\} - K, 0)] \\
&\quad = e^{-rT}(e^{a+b^2/2} N(d_1) - K N(d_2)),
\end{aligned}$$

where

$$\begin{aligned}
a &= T \log s + (r - \sigma^2/2)\frac{T^2}{2}, \\
b &= \sigma \frac{T^{3/2}}{\sqrt{3}}.
\end{aligned}$$