KTH Mathematics

Exam in SF2975 Financial Derivatives.
Thursday May 30 2013 14.00-19.00.

Examiner: Camilla Landén, tel 790 6197.

Aids: None.

General instructions: The solutions should be legible, easy to follow and not lacking in explanation or motivation. In particular, all notation used should be explained. Problems concerning integrability need not be treated.

N.B. Number the pages and write your name on each sheet. Write only one exercise per sheet and state the number of the exercise on the sheet.

To pass the exam you need 25 points. A chance to take a supplementary examination is given if you have 23 or 24 points (to take the supplementary examination you must contact Camilla Landén within two weeks from the posting of the results of the exam).

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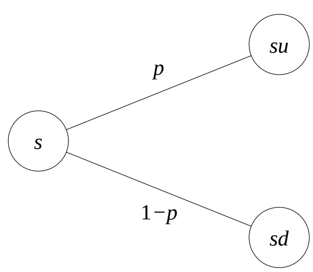
1. (a) Show that the one period binomial model is arbitrage free when $d < 1 + r < u$ (you may use the First Fundamental Theorem without proving it).



.... (3p)

(b) Consider the standard Black-Scholes model (described in detail in exercise 2 (a)). Suppose that you are given the following information: An American call option with the stock as underlying, with strike price 100 kr, and with exercise date in one year, currently trades at 16.13 kr. Today's stock price is 100 kr, and the price of a zero-coupon bond with maturity date in one year is 0.95 kr.

Now try to answer the following question: Given the above information, would you be able to price a European call option written on the stock S, with strike price 110 kr, and exercise date 6 months from now? If so, describe how you would go about the task, and if not, specify what additional information you would need. .... (3p)

(c) In the standard Black-Scholes setting (described in detail in exercise 2), compute the delta of a portfolio consisting of a long position in a European call option on the stock, with strike price K and exercise date T, and a short position in a European put option with the same strike price, exercise date, and underlying as the call option. .... (4p)

2. Consider a standard Black-Scholes market, i.e., a market consisting of a risk free asset, B, with P-dynamics given by

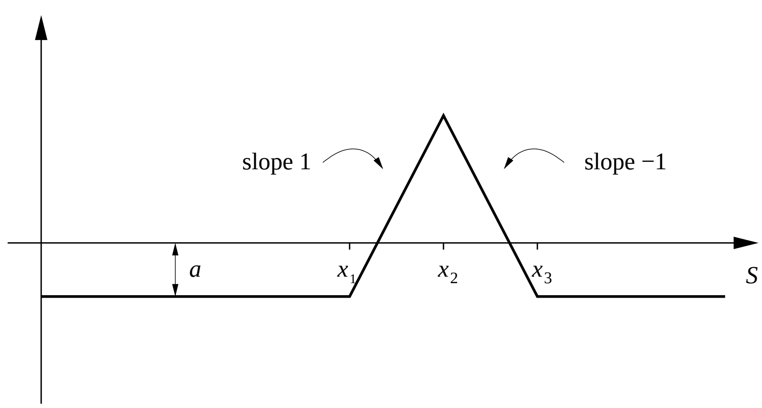
$$\begin{cases} dB_t &= rB_t dt, \\ B_0 &= 1, \end{cases}$$

and a stock, S, with P-dynamics given by

$$\begin{cases} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ S_0 &= s_0. \end{cases}$$

Here W denotes a P-Wiener process and r, α and σ are assumed to be constants.

(a) Suppose that you for some reason are fairly certain that the stock price will not move much until time T. In fact you are so certain of this you are willing to bet on it. Thus you would like to enter a *butterfly spread*, which is a T-contract with the payoff structure depicted in the figure below.



A natural choice for x_2 is today's stock price S_t and given that you have no information about whether a rise in the stock price is more likely than a fall you would set $x_2 = (x_1 + x_3)/2$. How narrow you want the interval $[x_1, x_3]$ depends on how sure you are that the stock price will not move. For this exercise set $x_1 = 0.95S_t$.

Now suppose that you do not have much money at the moment, and therefore would not want to pay anything entering the contract. Determine how much you must be willing to loose at most, i.e. the constant a, if you do not want to pay anything for the contract today (at time t). The expression derived for

a should be given only in terms of the parameters of the problem, but it may contain the density or distribution function for the $N(0,1)$-distribution. . (4p)

(b) Determine the arbitrage price of the contingent T-claim $X = \phi(S_T)$ with contract function ϕ given by

$$\phi(s) = \begin{cases} s^2, & \text{if } s > K \\ 0, & \text{otherwise.} \end{cases}$$

. . . . (6p)

3. (a) Consider a generalization of the standard Black-Scholes market, in which there exists a risk free asset, B, with interest rate 0 (for simplicity), and where the stock, S, has the following dynamics under risk neutral martingale measure Q

$$\begin{cases} dS_t &= \sigma_t S_t dW_t^Q, \\ S_0 &= s_0. \end{cases}$$

Here W^Q denotes a Q-Wiener process and σ is an arbitrary (up to regularity conditions, that you need not worry about) stochastic process.

Consider the simple contingent T-claim $X = g(S_T)$. Let $\sigma_{BS} \in \mathbb{R}$ be given and define the function $F : [0,T] \times \mathbb{R} \mapsto \mathbb{R}$ as the solution to the (familiar) partial differential equation

$$\begin{cases} \dfrac{\partial F}{\partial t}(t,s) + \dfrac{1}{2}\sigma_{BS}^2 s^2 \dfrac{\partial^2 F}{\partial s^2}(t,s) = 0, \\ \\ F(T,s) = g(s) \end{cases}$$

Finally, consider a trading strategy h that holds $h_S(t) = F_s(t,S(t))$ (subscript denotes differentiation of F) units of the stock and $h_B(t) = F(t,S(t)) - S(t)F_s(t,S(t))$ units of the bank account at time t.

Show that this trading strategy replicates the pay-off of the claim, i.e. that its value process, V^h, satisfies $V^h(T) = g(S(T))$.

Show that the self-financing condition for this strategy boils down to the equation

$$\frac{1}{2}\left[\sigma^2(t) - \sigma_{BS}^2\right] S^2(t) F_{ss}(t,S(t)) = 0$$

holding (almost everywhere, in an appropriate sense, that you need not worry about).

Argue that "usual results" are obtained when $\sigma(t) = \sigma_{BS}$, and that in general the h-strategy can be interpreted as "trying to replicate as if it were the Black-Scholes model". . . . . (6p)

(b) Again consider the standard Black-Scholes model, but now suppose that interest rates are really $s$tochastic and that the short rate under the risk neutral martingale measure Q follows a Ho-Lee model

$$\begin{cases} dr_t &= \theta(t)dt + \rho dU_t, \\ r_0 &= r^*, \end{cases}$$

where θ is a deterministic function, ρ is a constant, and U is a Q-Wiener process.

A risk manager who **did not** take this course, is still pricing simple contingent claims, with the stock as the underlying, as if the short rate were constant! That is, if he were to price say a European call option at time t, he would, after observing the short rate r_t, just use $r = r_t$ in the Black-Scholes formula.

Assume that you can trade directly with this risk manager. Would you then have an arbitrage opportunity? Answer yes or no, and give a precise argument to support your view. .... (4p)

4. Consider an interest rate model described by

$$dr = \mu(t,r)dt + \sigma(t,r)dV, \tag{1}$$

under a risk-neutral martingale measure Q. Here V denotes a Q-Wiener process.

(a) Define what is meant by an *Affine Term Structure* (ATS), and derive conditions which are sufficient to guarantee the existence of an ATS for the model above. .... (5p)

(b) Given that the short rate model possesses an affine term structure what is the zero coupon bond price volatility? (The expression for the volatility may contain μ and σ from equation (1) and functions which can be computed given specific μ and σ.) .... (2p)

(c) Compute zero coupon bond prices for the Vasiček model

$$\begin{cases} dr_t &= (b - ar_t)dt + \sigma dV_t, \\ r_0 &= r^0, \end{cases}$$

where a, b and σ are constants. The answer may contain integrals of constants and of exponential functions. .... (3p)

5. This exercise concerns forward measures. Given is a financial market with a risk-free asset with price process B and zero coupon bonds of all maturities T. Let $p(t,T)$ denote the price at time t of a zero coupon bond with maturity T. The dynamics of the risk-free asset are assumed to be given by

$$\begin{cases} dB_t &= r_t B_t dt, \\ B_0 &= 1. \end{cases}$$

Note that r is now a stochastic process. Furthermore it is assumed that under the risk-neutral martingale measure Q the price of the zero coupon bond maturing at time T satisfies

$$\begin{cases} dp(t,T) &= r(t)p(t,T)dt + \nu(t,T)p(t,T)dV_t, \\ p(0,T) &= p^*(0,T). \end{cases}$$

(a) Which process is used as numeraire process under the T-forward measure Q^T? .... (1p)

(b) Determine explicitly the Girsanov transformation between the risk-neutral martingale measure Q and the T-forward measure Q^T. .... (4p)

(c) Derive a pricing formula containing an expectation taken under Q^T rather than the usual expectation under Q. .... (2p)

(d) Compute the price of the claim $X = r_T$, using the Vasiček model for the short rate, i.e. when the Q-dynamics of the short rate are given by

$$\begin{cases} dr_t &= (b - ar_t)dt + \sigma dV_t, \\ r_0 &= r^0, \end{cases}$$

where a, b and σ are constants. To avoid making the same calculations all over again you may want to look back at exercises 4 (b) and (c). Your answer may contain integrals of constants and of exponential functions. .... (3p)

Good luck!

Hints:
You are free to use the following in any of the above exercises.

- The density function of a normally distributed random variable with expectation m and variance σ^2 is given by

$$\varphi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/(2\sigma^2)}.$$

- Let Φ denote the cumulative distribution function for the N$(0,1)$ distribution. Then

$$\Phi(-x) = 1 - \Phi(x).$$

- The standard Black-Scholes formula for the price $\Pi(t)$ of a European call option with strike price K and time of maturity T is $\Pi(t) = F(t, S(t))$, where

$$F(t,s) = s\Phi[d_1(t,s)] - e^{-r(T-t)} K\Phi[d_2(t,s)].$$

Here Φ is the cumulative distribution function for the $N(0,1)$ distribution and

$$\begin{aligned} d_1(t,s) &= \frac{1}{\sigma\sqrt{T-t}} \left\{ \ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t) \right\}, \\ d_2(t,s) &= d_1(t,s) - \sigma\sqrt{T-t}. \end{aligned}$$

- Suppose that there exist processes $X(\cdot, T)$ for every $T \geq 0$ and suppose that Y is a process defined by

$$Y(t) = \int_t^{T_0} X(t,s)ds$$

Then we have the following version of Itô's formula

$$dY_t = -X(t,t)dt + \int_t^{T_0} dX(t,s)ds.$$

- A linear SDE of the form

$$\begin{aligned} dX_t &= (aX_t + b_t)dt + \sigma_t dW_t, \\ X_0 &= x_0, \end{aligned}$$

where a is a constant and b and σ are deterministic functions, has the solution

$$X_t = e^{at}x_0 + \int_0^t e^{a(t-s)} b_s ds + \int_0^t e^{a(t-s)} \sigma_s dW_s.$$