KTH Mathematics

Examination in SF2943 Time Series Analysis, June 1, 2015, 08:00–13:00.

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Allowed aids: Pocket calculator, "Formulas and survey, Time series analysis" by Jan Grandell, without notes.

Any notation introduced must be explained and defined. Arguments and computations must be detailed so that they are easy to follow.

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Problem 1

Consider noisy observations $\{Y_t\}$ of a causal AR(1) process $\{X_t\}$:

$$X_t = \phi X_{t-1} + Z_t, \quad \{Z_t\} \sim \mathrm{WN}(0, \sigma_z^2),$$
$$Y_t = X_t + W_t, \quad \{W_t\} \sim \mathrm{WN}(0, \sigma_w^2),$$

where the noise sequences $\{Z_t\}$ and $\{W_t\}$ are independent.

(a) Show that $\{Y_t\}$ is stationary and compute its autocovariance function. (5 p)

(b) Show that $\{Y_t\}$ can be expressed as an ARMA(p, q) process and determine p, q, $\phi_1, \dots, \phi_p, \theta_1, \dots, \theta_q$ and the white noise variance. (5 p)

Problem 2

Consider two samples of size 400, the first one from an AR(2) process with $(\phi_1, \phi_2) = (0.5, 0.4)$, and the second one from an AR(2) process with $(\phi_1, \phi_2) = (-0.4, 0.5)$. What do the two periodograms based on the two samples look like (roughly), and how do you identify which periodogram belongs to which AR(2) process? (10 p)

Problem 3

Consider the AR(1) process $X_t = 0.5X_{t-1} + Z_t$, $\{Z_t\} \sim \mathrm{WN}(0, 1)$. To handle a missing observation X_t, the following alternatives are suggested: (1) $\widehat{X}_t^{(1)} := 0$, (2) $\widehat{X}_t^{(2)} := (X_{t+1} + X_{t-1})/2$, (3) $\widehat{X}_t^{(3)}$ is the best linear predictor of X_t based on X_{t+1} and X_{t-1}. Compare the three predictors $\widehat{X}_t^{(k)}$, $k = 1, 2, 3$, of the missing value X_t in terms of their mean squared prediction errors. (10 p)

Problem 4

Consider the time series $\{X_t\}$ given by $X_t - \theta X_{t-1} - 2\theta^2 X_{t-2} = Z_t + \theta Z_{t-1}$, where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$ and Z_t is independent of X_s for $s < t$. For which values of θ is $\{X_t\}$ a causal AR(p) process? Determine p and the coefficients $\phi_1, \dots, \phi_p$. (10 p)

Problem 5

Consider a time series $\{Y_t\}$ expressed as $Y_t = ct + X_t$, where $\{X_t\}$ is a causal and invertible ARMA(p, q) process. Consider the differenced series $\{\nabla Y_t\}$.

Show that $\{\nabla Y_t\}$ is a causal but not invertible ARMA(p', q') process and determine p' and q'. (10 p)

Problem 1

$\mathrm{E}[Y_t] = 0$ for all t and

$$\begin{aligned}\mathrm{Cov}(Y_{t+h}, Y_t) &= \mathrm{Cov}(X_{t+h} + W_{t+h}, X_t + W_t) = \{\{X_t\} \text{ and } \{W_t\} \text{ independent}\}\\ &= \mathrm{Cov}(X_{t+h}, X_t) + \mathrm{Cov}(W_{t+h}, W_t) = \gamma_X(h) + I\{h = 0\}\sigma_w^2.\end{aligned}$$

So $\{Y_t\}$ is stationary, $\gamma_Y(0) = \sigma_z^2/(1-\phi^2) + \sigma_w^2$ and $\gamma_Y(h) = \gamma_X(h) = \sigma_z^2\phi^{|h|}/(1-\phi^2)$ for $h \neq 0$.
Set $U_t := Y_t - \phi Y_{t-1} = Z_t + W_t - \phi W_{t-1}$. Similar computations to the above show that $\{U_t\}$ is stationary, $\gamma_U(0) = \sigma_z^2 + (1+\phi^2)\sigma_w^2$, $\gamma_U(h) = -\phi\sigma_w^2$ for $|h| = 1$, and $\gamma_U(h) = 0$ for $|h| \geq 2$. This is the autocovariance function of an MA(1) process: $U_t = V_t + \theta V_{t-1}$, $\{V_t\} \sim \mathrm{WN}(0, \sigma_v^2)$. It remains to solve

$$(1+\theta^2)\sigma_v^2 = \sigma_z^2 + (1+\phi^2)\sigma_w^2, \tag{1}$$
$$\theta\sigma_v^2 = -\phi\sigma_w^2 \tag{2}$$

for (θ, σ_v^2). Combining (1) and (2) yields

$$\theta^2 - c(\phi)\theta + 1 = 0, \quad c(\phi) := -\frac{\sigma_z^2 + (1+\phi^2)\sigma_w^2}{\phi\sigma_w^2} = -\frac{\sigma_z^2}{\phi\sigma_w^2} - \Big(\frac{1}{\phi} + \phi\Big)$$

with real-valued roots $\theta = c(\phi)/2 \pm \sqrt{(c(\phi)/2)^2 - 1}$ (since $|c(\phi)| > 1$). Selecting the root with $|\theta| < 1$ we have shown that $\{Y_t\}$ is a causal and invertible ARMA$(1,1)$ process

$$Y_t - \phi Y_{t-1} = V_t + \theta V_{t-1}, \quad \{V_t\} \sim \mathrm{WN}(0, \sigma_v^2),$$

where $\sigma_v^2 = -\sigma_w^2\phi/\theta > 0$.

Problem 2

Set $x_0 = 1$, $x_1 = 0.5x_0$ and $x_n = 0.5x_{n-1} + 0.4x_{n-2}$ for $n \geq 2$. Then $x_n \geq 0$ and does not exhibit any oscillating behavior. Alternatively, an oscillating behavior with infinite cycle length $c = \infty$ corresponding to the frequency $2\pi/c = 0$. The spectral density $f(\lambda)$ has a maximum at $\lambda = 0$. Similarly for the periodogram of the sample from the AR(2) process with $(\phi_1, \phi_2) = (0.5, 0.4)$.
Set $x_0 = 1$, $x_1 = -0.4x_0$ and $x_n = -0.4x_{n-1} + 0.5x_{n-2}$ for $n \geq 2$. Then x_n oscillates around the value 0 with cycles of length $c = 2$ corresponding to the frequency $2\pi/c = \pi$. The spectral density $f(\lambda)$ has a maximum at $\lambda = \pi$. Similarly for the periodogram of the sample from the AR(2) process with $(\phi_1, \phi_2) = (-0.4, 0.5)$.

Problem 3

Set $(\phi, \sigma^2) = (0.5, 1)$. The mean squared prediction errors are of the form

$$\begin{aligned}\mathrm{E}[(X_t - aX_{t-1} + bX_{t+1})^2] &= \mathrm{Var}(X_t - aX_{t-1} + bX_{t+1})\\ &= (1 + a^2 + b^2)\gamma(0) - 2(a+b)\gamma(1) + 2ab\gamma(2)\\ &=: f(a,b),\end{aligned}$$

where $\gamma(h) = \sigma^2\phi^{|h|}/(1-\phi^2) = 2^{-|h|}4/3$. Notice that the predictor $\widehat{X}_t^{(1)}$ corresponds to $a = b = 0$ which gives $f(a,b) = 4/3 \approx 1.33$. Notice that the predictor $\widehat{X}_t^{(2)}$

corresponds to $a = b = 1/2$ which gives $f(a, b) = 5/6 \approx 0.83$. Notice that the predictor $\widehat{X}_t^{(3)}$ corresponds to the (a, b) minimizing $f(a, b)$. Computing

$$\frac{\partial}{\partial a} f(a,b) = 2a\gamma(0) + 2b\gamma(2) - 2\gamma(1),$$
$$\frac{\partial}{\partial b} f(a,b) = 2b\gamma(0) + 2a\gamma(2) - 2\gamma(1),$$

setting these expressions to zero and solving for (a, b), gives $a = b = \gamma(1)/(\gamma(0) + \gamma(2)) = 2/5$ which gives $f(a, b) = 4/5 = 0.8$.

Problem 4

We need to determine the zeros of $\phi(z) = 1 - \theta z - 2\theta^2 z^2$.

$$\phi(z) = 0 \Leftrightarrow z^2 + \frac{1}{2\theta} - \frac{1}{2\theta^2} = 0 \Leftrightarrow z = -\frac{1}{4\theta} \pm \sqrt{(1/4\theta)^2 + 1/2\theta^2} = \begin{cases} 1/2\theta \\ -1/\theta \end{cases}$$

In particular, $|\theta| < 1/2$ is necessary for having both zeros outside the unit circle. However, $\theta(z) = 1 + \theta z$ has a zero at $z = -1/\theta$. In particular,

$$\frac{\phi(z)}{\theta(z)} = 1 - 2\theta z$$

so $X_t - 2\theta X_{t-1} = Z_t$. Therefore, $\{X_t\}$ is a causal AR(1) process with parameter 2θ if $|\theta| < 1/2$.

Problem 5

Notice that $\nabla ct = ct - c(t-1) = c$ and

$$\nabla X_t = (1 - B)X_t = (1 - B)\frac{\theta(B)}{\phi(B)} Z_t = \frac{\theta'(B)}{\phi(B)} Z_t,$$

where

$$\begin{aligned}\theta'(B) &= (1 - B)(1 + \theta_1 B + \cdots + \theta_q B^q) \\ &= 1 + (\theta_1 - 1)B + (\theta_2 - \theta_1)B^2 + \cdots + (\theta_q - \theta_{q-1})B^q - \theta_q B^{q+1}.\end{aligned}$$

Hence, $\{\nabla Y_t\}$ is an ARMA$(p, q + 1)$ process. Since $\phi'(z) = \phi(z)$, the process is causal. Since $\theta'(z)$ is a polyomial with a zero at $z = 1$ (not outside the unit circle), the ARMA$(p, q + 1)$ process is not invertible.