EXAMINATION IN SF2943 TIME SERIES ANALYSIS

Date: 2016-05-30, 08:00-13:00

Suggested solutions.

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Problem 1

We use Yule-Walker estimation to find $\hat{\varphi}_1$ and $\widehat{\sigma^2}$. Using the formulas on page 19 in FS we get

$$\begin{aligned}\hat{\gamma}(1) &= \hat{\varphi}_1\hat{\gamma}(0)\\ \widehat{\sigma^2} &= \hat{\gamma}(0) - \hat{\varphi}_1\hat{\gamma}(1)\end{aligned}$$

To get $\hat{\gamma}(h)$ for $h = 0, 1$ we use

$$\hat{\gamma}(h) = \frac{1}{n}\sum_{t=1}^{n-h}(x_t - \bar{x}_n)(x_{t+h} - \bar{x}_n)$$

(page 16 in FS) with $n = 5$. Using the given sample we get

$$\bar{x}_5 = -0.615$$

and

$$\begin{aligned}\hat{\gamma}(0) &= 0.481\\ \hat{\gamma}(1) &= 0.00329.\end{aligned}$$

Using the Yule-Walker estimators we arrive at

$$\begin{aligned}\hat{\varphi}_1 &= 0.00685\\ \widehat{\sigma^2} &= 0.481.\end{aligned}$$

Problem 2

We have

$$\varphi(z) = 1 - (0.2 + \alpha)z + 0.2\alpha z^2 = \left(1 - \frac{z}{5}\right)(1 - \alpha z)$$

and

$$\theta(z) = 1 - 0.8z.$$

The zeros of the polynomial $\varphi(z)$ are equal to

$$z_1 = 5 \text{ and } z_2 = \frac{1}{\alpha}.$$

(a) When $\alpha = 0$, the model becomes an ARMA$(1,1)$, so we must have $\alpha \neq 0$. The two polynomials $\varphi(z)$ and $\theta(z)$ should not have any common factors, so we must also have $\alpha \neq 0.8$. Finally, (X_t) should be stationary, which is equivalent to the fact that $\varphi(z)$ should not have any zeros on the unit circle, so we must have $\alpha \neq \pm 1$. To summarize: $\{X_t\}$ is a well defined ARMA(2,1) process for every $\alpha \in \mathbb{R} \backslash \{-1, 0, 0.8, 1\}$.

(b) In order for the given process to be a causal ARMA$(2,1)$ process the zeros of $\varphi(z)$ should lie outside the unit circle. This is satisfied if we have

$$\left| \frac{1}{\alpha} \right| > 1 \quad \Leftrightarrow \quad |\alpha| < 1.$$

We know from (a) that for any $\alpha \neq 0.8$ and $\alpha \neq 0$ satisfying $|\alpha| < 1$ the time series $\{X_t\}$ is a well defined ARMA$(2,1)$ process. Hence, $\{X_t\}$ is a causal ARMA$(2,1)$ process if and only if $\alpha \in (-1,1) \backslash \{0, 0.8\}$.

Problem 3

Series 1 has no apparent drop in neither the sample ACF nor in the sample PACF up to and including lag 5, so it does not look like the underlying time series is an AR(p) or MA(q) for some $p, q = 1, \ldots, 5$. Both the ACF and the PACF decays smoothly towards zero, so we rule out WN and conclude that an ARMA(1,1) model best fits the observed ACF and PACF.

For Series 2 there is a geometrical decay in the ACF and the PACF has a jump after lag 1 down to values close to zero for lags $h = 2, \ldots, 11$. Hence, in this case an AR(1) process seems to be the best model.

Problem 4

We let $E_t[\cdots]$ denote the mean where the information up to and including time t is used. We get

$$\begin{aligned}
E[X_t] &= E[E_{t-1}[X_t]] \\
&= E[E_{t-1}[\sigma_t Z_t]] \\
&= E\left[E_{t-1}\left[\sqrt{1.24 + 0.75 X_{t-1}^2}\, Z_t\right]\right] \\
&= E\left[\sqrt{1.24 + 0.75 X_{t-1}^2}\, E_{t-1}[Z_t]\right] \\
&= \{Z_t \text{ is independent of previous values}\} \\
&= E\left[\sqrt{1.24 + 0.75 X_{t-1}^2}\, E[Z_t]\right] \\
&= \{E[Z_t] = 0\} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Var}(X_t) &= \{E[X_t] = 0\} \\
&= E\left[X_t^2\right] \\
&= E\left[\sigma_t^2 Z_t^2\right] \\
&= E\left[E_{t-1}\left[(1.24 + 0.75X_{t-1}^2)Z_t^2\right]\right] \\
&= E\left[(1.24 + 0.75X_{t-1}^2)E_{t-1}\left[Z_t^2\right]\right] \\
&= \{Z_t^2 \text{ is independent of previous values}\} \\
&= E\left[(1.24 + 0.75X_{t-1}^2)E\left[Z_t^2\right]\right] \\
&= \{E\left[Z_t^2\right] = 1\} \\
&= E\left[1.24 + 0.75X_{t-1}^2\right] \\
&= 1.24 + 0.75E\left[X_{t-1}^2\right] \\
&= 1.24 + 0.75\mathrm{Var}(X_{t-1}).
\end{aligned}$$

Since $\alpha_1 = 0.75 < 1$ we know that $\{X_t\}$ is a stationary time series. It follows that

$$\mathrm{Var}(X_t) = 1.24 + 0.75\mathrm{Var}(X_t) \quad \Rightarrow \quad \mathrm{Var}(X_t) = \frac{1.24}{1 - 0.75} = 4.96.$$

Problem 5

(a) We know that $\gamma(h) = 0$ when $|h| > 2$. We also know that $\gamma(-h) = \gamma(h)$. Hence, we need to calculate $\gamma(0)$, $\gamma(1)$ and $\gamma(2)$. Using the fact that $\{Z_t\} \sim \mathrm{WN}(0, 0.16)$ we get

$$\begin{aligned}
\gamma(0) &= E\left[X_t^2\right] \\
&= E\left[(Z_t - 1.5Z_{t-1} + 0.56Z_{t-2})^2\right] \\
&= E\left[Z_t^2\right] + 1.5^2E\left[Z_{t-1}^2\right] + 0.56^2E\left[Z_{t-2}\right] \\
&= (1 + 2.25 + 0.3136) \cdot 0.16 \\
&= 0.570,
\end{aligned}$$

$$\begin{aligned}
\gamma(1) &= E\left[X_{t+1}X_t\right] \\
&= E\left[(Z_{t+1} - 1.5Z_t + 0.56Z_{t-1})(Z_t - 1.5Z_{t-1} + 0.56Z_{t-2})\right] \\
&= -1.5E\left[Z_t^2\right] - 0.56 \cdot 1.5E\left[Z_{t-1}^2\right] \\
&= -(1.5 + 0.84) \cdot 0.16 \\
&= -0.374
\end{aligned}$$

and

$$\begin{aligned}
\gamma(2) &= E\left[X_{t+2}X_t\right] \\
&= E\left[(Z_{t+2} - 1.5Z_{t+1} + 0.56Z_t)(Z_t - 1.5Z_{t-1} + 0.56Z_{t-2})\right] \\
&= 0.56E\left[Z_t^2\right] \\
&= 0.56 \cdot 0.16 \\
&= 0.0896.
\end{aligned}$$

To conclude:

$$\gamma(h) = \begin{cases} 0.570 & \text{when } h = 0 \\ -0.374 & \text{when } h = \pm 1 \\ 0.0896 & \text{when } h = \pm 2 \\ 0 & \text{when } |h| > 2. \end{cases}$$

(b) We get

$$\begin{aligned} f(\lambda) &= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma(h) \\ &= \frac{1}{2\pi} \sum_{h=-2}^{2} e^{-ih\lambda} \gamma(h) \\ &= \frac{1}{2\pi} \sum_{h=-2}^{2} \cos(h\lambda) \gamma(h) \\ &= \frac{\gamma(0)}{2\pi} + \frac{2}{2\pi} \Big(\gamma(1) \cos \lambda + \gamma(2) \cos(2\lambda) \Big) \\ &= \{\text{Use the values of } \gamma(0), \gamma(1) \text{ and } \gamma(2) \text{ from (a)}\} \\ &= 0.0908 - 0.119 \cos \lambda + 0.0285 \cos(2\lambda). \end{aligned}$$

Alternatively, we can calculate $f(\lambda)$ in the following way. The MA polynomial is

$$\theta(z) = 1 - 1.5z + 0.56z = (1 - 0.7z)(1 - 0.8z);$$

hence the roots to $\theta(z) = 0$ are

$$z_1 = \frac{1}{0.7} = 1.43 \text{ and } z_2 = \frac{1}{0.8} = 1.25.$$

This means that $\theta(z) \neq 0$ when $|z| \leq 1$, so $\{X_t\}$ is invertible and we can use the formula for the spectral density on page 10 in FS. Since $\varphi(z) \equiv 1$ we get (we keep the parameters and insert numerical values at the end)

$$\begin{aligned} f(\lambda) &= \frac{\sigma^2}{2\pi} \left|1 + \theta_1 e^{-i\lambda} + \theta_2 e^{-2i\lambda}\right|^2 \\ &= \frac{\sigma^2}{2\pi} \left(1 + \theta_1 e^{-i\lambda} + \theta_2 e^{-2i\lambda}\right) \left(1 + \theta_1 e^{i\lambda} + \theta_2 e^{2i\lambda}\right) \\ &= \frac{\sigma^2}{2\pi} \left(1 + \theta_1 e^{i\lambda} + \theta_2 e^{2i\lambda} + \theta_1 e^{-i\lambda} + \theta_1^2 + \theta_1\theta_2 e^{i\lambda} + \theta_2 e^{-2i\lambda} + \theta_1\theta_2 e^{-i\lambda} + \theta_2^2\right) \\ &= \frac{\sigma^2}{2\pi} \left(1 + \theta_1^2 + \theta_2^2 + 2\theta_1 \cos\lambda + 2\theta_2 \cos(2\lambda) + 2\theta_1\theta_2 \cos\lambda\right) \\ &= \frac{\sigma^2}{2\pi} \left(1 + \theta_1^2 + \theta_2^2 + 2\theta_1(1 + \theta_2) \cos\lambda + 2\theta_2 \cos(2\lambda)\right). \end{aligned}$$

Inserting the numerical values we get

$$f(\lambda) = 0.0908 - 0.119 \cos \lambda + 0.0285 \cos(2\lambda).$$