EXAMINATION IN SF2943 TIME SERIES ANALYSIS

Suggested solutions

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Problem 1

(a) The PACF drops down close to zero after the second lag – which is what we expect from an AR(2)-process.

(b) To estimate the parameters in an AR(2)-process using Yule-Walker, we need to solve for $\hat{\varphi} = [\hat{\varphi}_1, \hat{\varphi}_2]^T$ and $\hat{\sigma}^2$ in

$$\hat{\Gamma}_2 \hat{\varphi} = \gamma_2 \text{ and } \hat{\sigma}^2 = \gamma(0) - \hat{\varphi}^T \gamma_2,$$

or

$$\begin{bmatrix} \hat{\gamma}(0) & \hat{\gamma}(1) \\ \hat{\gamma}(1) & \hat{\gamma}(0) \end{bmatrix} \begin{bmatrix} \hat{\varphi}_1 \\ \hat{\varphi}_2 \end{bmatrix} = \begin{bmatrix} \hat{\gamma}_1 \\ \hat{\gamma}_2 \end{bmatrix} \text{ and } \hat{\sigma^2} = \gamma(0) - \hat{\varphi}^T \gamma_2.$$

To find the $\hat{\gamma}(h)$'s we use

$$\hat{\gamma}(h) = \frac{1}{n} \sum_{t=1}^{n-h} (x_t - \bar{x}_n)(x_{t+h} - \bar{x}_n), \; h = 0, 1, 2.$$

We get

$$\hat{\gamma}(0) = 0.2996, \; \hat{\gamma}(1) = 0.1748 \text{ and } \hat{\gamma}(2) = 0.0250.$$

Solving the system of equations yields

$$\begin{bmatrix} \hat{\varphi}_1 \\ \hat{\varphi}_2 \end{bmatrix} = \begin{bmatrix} 0.8265 \\ -0.4028 \end{bmatrix}.$$

Finally we get

$$\hat{\sigma}^2 = \hat{\gamma}(0) - \hat{\varphi}^T \hat{\gamma}_2 = 0.1622.$$

Problem 2

(a) Since $\{X_t\}$ is a causal process we must have

$$\psi_j = 0, \; j = 0, -1, -2, \ldots.$$

We also know that causality for the given AR(1) process is equivalent to $|\varphi| < 1$. Iterating the time series equation yields

$$\begin{aligned} X_t &= \varphi X_{t-1} + Z_t \\ &= \varphi^2 X_{t-2} + \varphi Z_{t-1} + Z_t \\ &= \cdots \\ &= \varphi^n X_{t-n} + \sum_{j=0}^{n-1} \varphi^j Z_{t-j}. \end{aligned}$$

Since $|\varphi| < 1$ we have $\varphi^n \to 0$ as $n \to \infty$. Hence, a candidate for the ψ_j's are

$$\psi_j = \varphi^j,\ j = 0, 1, \ldots.$$

We now show that

$$X_t = \sum_{j=0}^{\infty} \psi^j Z_{t-j}$$

indeed satisfies the time series equation. We get

$$\begin{aligned}
X_t &= \sum_{j=0}^{\infty} \varphi^j Z_{t-j} \\
&= \sum_{j=1}^{\infty} \varphi^j Z_{t-j} + Z_t \\
&= \varphi \sum_{j=1}^{\infty} \varphi^{j-1} Z_{t-j} + Z_t \\
&= \varphi \sum_{j=0}^{\infty} \varphi^j Z_{t-1-j} + Z_t \\
&= \varphi X_{t-1} + Z_t.
\end{aligned}$$

To summarize:

$$\psi_j = \begin{cases} \varphi^j & \text{if } j = 0, 1, \ldots \\ 0 & \text{if } j = -1, -2, \ldots. \end{cases}$$

(b)

$\boxed{h = 0}$ We get

$$\begin{aligned}
\gamma(0) &= \mathrm{Var}(X_t) \\
&= \mathrm{Cov}(X_t, X_t) \\
&= \mathrm{Cov}(\varphi X_{t-1} + Z_t, \varphi X_{t-1} + Z_t) \\
&= \varphi^2 \mathrm{Var}(X_t) + \sigma^2 \\
&= \varphi^2 \gamma(0) + \sigma^2.
\end{aligned}$$

It follows that

$$\gamma(0) = \frac{\sigma^2}{1 - \varphi^2}.$$

$\boxed{h > 0}$ Now

$$\begin{aligned}
\gamma(h) &= \mathrm{Cov}(X_t, X_{t-h}) \\
&= \mathrm{Cov}(\varphi X_{t-1} + Z_t, X_{t-h}) \\
&= \varphi \mathrm{Cov}(X_{t-1}, X_{t-h}) + 0 \\
&= \cdots \\
&= \varphi^h \mathrm{Cov}(X_{t-h}, X_{t-h}) \\
&= \varphi^h \mathrm{Var}(X_{t-h}) \\
&= \varphi^h \mathrm{Var}(X_t) \\
&= \varphi^h \gamma(0) \\
&= \sigma^2 \frac{\varphi^h}{1-\varphi^2}.
\end{aligned}$$

$\boxed{h < 0}$ Since

$$\gamma(-h) = \gamma(h)$$

we get

$$\gamma(h) = \sigma^2 \frac{\varphi^{-h}}{1-\varphi^2}$$

when $h < 0$.

Problem 3

We can write the time series equation as

$$\varphi(B)X_t = \theta(B)Z_t,$$

where B is the lag operator,

$$\varphi(z) = 1 + 0.60z - 0.16z^2 = (1 + 0.80z)(1 - 0.20z)$$

and

$$\theta(z) = 1 + 0.15z$$

respectively.

(a) The zeros of $\varphi(z)$ are

$$z_1 = -\frac{1}{0.8} = -1.25 \text{ and } z_2 = \frac{1}{0.2} = 5,$$

and the zero of $\theta(z)$ is

$$z_3 = -\frac{1}{0.15} = 6.67.$$

Since the two polynomials $\varphi(z)$ and $\theta(z)$ does not have any common zeros and since $\varphi(z) \neq 0$ when $|z| = 1$, the time series equation represents a well defined ARMA$(2, 1)$ process.

(b) The zeros of $\varphi(z)$ are outside the unit circle; hence $\{X_t\}$ is a causal process.

(c) The zero of $\theta(z)$ is outside the unit circle; hence $\{X_t\}$ is an invertible process.

Problem 4

To find $\hat{X}_4$ we use the fact that

$$\hat{X}_4 = \sum_{i=1}^{3} \varphi_{3,i} X_{4-i},$$

where

$$\varphi_3 = \begin{bmatrix} \varphi_{3,1} \\ \varphi_{3,2} \\ \varphi_{3,3} \end{bmatrix}$$

solves

$$\Gamma_3 \varphi_3 = \gamma_3$$

with

$$\Gamma_3 = \begin{bmatrix} \gamma(0) & \gamma(1) & \gamma(2) \\ \gamma(1) & \gamma(0) & \gamma(1) \\ \gamma(2) & \gamma(1) & \gamma(0) \end{bmatrix}$$

and

$$\gamma_3 = \begin{bmatrix} \gamma(1) \\ \gamma(2) \\ \gamma(3) \end{bmatrix}.$$

Since $\{X_t\}$ is an MA(1) process we have

$$\gamma(h) = \begin{cases} (1+\theta^2)\sigma^2 & \text{if } h = 0 \\ \theta\sigma^2 & \text{if } |h| = 1 \\ 0 & \text{if } |h| > 1 \end{cases} = \begin{cases} 0.1040 & \text{if } h = 0 \\ 0.0200 & \text{if } |h| = 1 \\ 0 & \text{if } |h| > 1. \end{cases}$$

Solving the system of equations yields

$$\varphi_3 = \begin{bmatrix} 0.1040 & 0.0200 & 0 \\ 0.0200 & 0.1040 & 0.0200 \\ 0 & 0.0200 & 0.1040 \end{bmatrix}^{-1} \begin{bmatrix} 0.0200 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.2000 \\ -0.0399 \\ 0.0077 \end{bmatrix}.$$

Using the data

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.6301 \\ 0.2652 \\ 0.7592 \end{bmatrix}$$

we get

$$\hat{X}_4 = 0.2000 \cdot 0.7592 - 0.0399 \cdot 0.2652 + 0.0077 \cdot 0.6301 = 0.1461.$$

Problem 5

We write the time series equation as

$$\Phi(B)\mathbf{X}_t = \mathbf{Z}_t,$$

where B is the lag operator and

$$\Phi(z) = I - \Phi_1 z = \begin{bmatrix} 1-\alpha z & 0 \\ \beta & 1-\beta z \end{bmatrix}.$$

The time series $\{\mathbf{X}_t\}$ is a causal 2-variate AR(1) process if and only if

$$\det \Phi(z) \neq 0 \text{ for every } z \text{ such that } |z| \leq 1.$$

Here

$$\det \Phi(z) = (1-\alpha z)(1-\beta z),$$

and the zeros of $\det \Phi(z)$ are

$$z_1 = \frac{1}{\alpha} \text{ and } z_2 = \frac{1}{\beta}.$$

We see that if $|\alpha| < 1$ and $|\beta| < 1$ then $\det \Phi(z) \neq 0$ for every $|z| \leq 1$. Since $\alpha, \beta \neq 0$ we find that $\{\mathbf{X}_t\}$ is a causal 2-variate AR(1) process if and only if

$$\alpha \in (-1,0) \cup (0,1) \text{ and } \beta \in (-1,0) \cup (0,1).$$