Structured products: optimal allocation in different market climates

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Abstract

The purpose of this thesis is to investigate if there is any difference in optimal allocation in structured products in different market climates. There are four different market climates considered; low/high interest rate and low/high implied volatility, where three assets are available; a risk less asset (bond), a risky asset (equity index) and a structured product (bond and derivative). This is accomplished by extracting the risk neutral density from the option market for the two different implied volatility levels. The risk neutral density is then transformed to a real world density corresponding to different expected risk premiums. Utility relations are explored for investments in structured products and a representative investor is defined. Portfolio optimization is performed on each scenario where the objective is to maximize terminal expected utility.
Keywords: Structured products, implied risk neutral density, utility theory, portfolio optimization

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Chapter 1

Introduction

The turmoil in the financial markets in 2008 has shaken the core of the financial system. Major banks around the world have suffered massive losses. The lack of credit have been devastating with a considerable number of banks going bankrupt. In this market climate investors got extremely risk averse. For example T-bills in U.S.A. went to negative yields. This means that the investor directly looses money by holding these instruments. The year 2008 went to the history as one of the worst year for global financial markets.

Financial instruments such as structured products can be attractive in this market climate . With these the investor can have a capital guarantee but can also participate if the market changes direction. The characteristic of structured products is the repackaging of strategies that involve positions in derivatives and other types of assets into an investment vehicle that is easily accessible by investors. Researches have shown that structured products can enhance an investor's portfolio in terms of reducing risk or increasing return in comparison to a traditional portfolio with only linear exposure to underlying markets.

The purpose of this thesis is not to motivate the existence of structured products in an effective portfolio but to investigate if there is any difference in optimal allocation in structured products in different market climates. There are four different market climates considered; low/high interest rate and low/high implied volatility and three assets; a risk less asset (bond), a risky asset (equity index) and a structured product (bond and derivative).

The participation rate of a structured product decides how much the investor takes part in the underlying assets performance. This is decreasing in volatility and increasing in interest rate, where interest rate dominates volatility. For example in a high volatility and low interest rate market the participation rate is low. However, the high volatility implies larger probability for large returns and the low interest rate causes a low opportunity cost. So, should the investor consider the participation rate when making investment decisions? The emphasis in this thesis is to outline the relation between optimal allocation in structured products with respect to the market climates, thus different participation rates, which creates interesting scenarios.

To obtain a probability density function for the asset movement the implied risk neutral density from the option market is extracted at certain points in time. Two time periods are selected which reflects high implied volatility and low implied volatility. This creates two scenarios that reflect the market's expectations not only in volatility but also the shape of the risk neutral distribution. However, the risk neutral distribution cannot be used in the portfolio optimization so some transformation must be made.

The transformation between the risk neutral world and the real world involves dependencies with the investor's utility. This transformation is made with different parameters that imply different expected returns. Consideration is also taken to historical risk premiums to evaluate the result. The portfolio optimization is established where the objective is to maximize terminal expected utility under short selling restriction. The utility function is chosen for an investor with a sharp bound on negative returns.

Chapter 2 explains the financial assets used in this thesis, especially the structured product. Chapter 3 outlines the approach of how the assets are modeled and in Chapter 4 this is described in more detail. In Chapter 5 this model is applied and calibrated to actual market option prices. Chapter 6 concerns terms of how risk/reward is measured. Finally, in Chapter 7 the portfolio optimization is performed and the result is presented.

Chapter 2

Financial assets

In this thesis three assets are available; a risk less asset (bond), a risky asset (equity index) and a structured product (bond and derivative). The bond is a theoretical asset where the default probability is zero. In reality no such asset exists. The closest one get is government bonds issued by high graded countries[1], like USA, Germany or Japan. The equity index represents the universe of risky assets. In this index all available equities are included. In practice this could be a broad market equity index where companies across different sectors are included. The structured product is a combination of the bond and a derivative on the equity index. In the following part the structured product is reviewed in more detail.

2.1 Structured products

The characteristic of structured products is the repackaging of strategies that involve long and short positions in derivatives and the underlying asset into an investment vehicle easily accessible by investors, [24]. The first occurrence of structured products was the option based portfolio insurance (OBPI). This consists basically of combining a risky asset with a put written on it. Today the most common package of a structured product in the retail segment is the combination of a zero coupon bond and a call option linked to equity, commodity or currency. This will in theory create the exact same strategy as the OBPI in the case of at the money (ATM) strikes; recall the put-call parity.

The purpose of a structured product is to create an investment vehicle that is tailored to the investors risk profile. This could be features like capital guarantee, which means that the notional amount is at least repaid at maturity. We also have highly geared products which give the investor leveraged exposure to underlying market. A product with capital guarantee is generally called a note otherwise a certificate.

[1] Based on five year sovereign credit default swap (CDS) rates gathered from Bloomberg.

In this thesis the simplest form of a structured product is considered. This is by combining a risk less bond, in the form of a zero coupon bond, with a plain vanilla ATM call option. Thus creating a 100 % capital guarantee of the invested amount. The bond represents the safety net and the long position in the call option the opportunity for future return on capital. Figure 2.1 illustrates this relation.

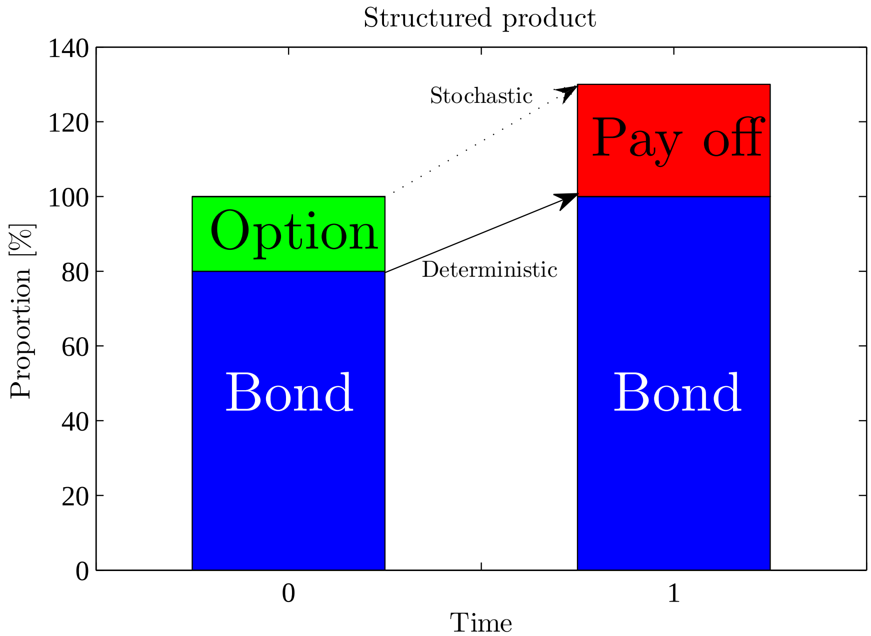

Figure 2.1: The figure illustrates the composition of the structured product.

Depending on the price of the zero coupon bond different amounts of the option can be bought within the structure. This causes different exposures depending on the market climate. The part of which the investor participates with the underlying performance is called participation rate. Let B be the price of a zero coupon bond and C[2] be the price of a call option. The participation rate, k, is then

$$k = \frac{1 - B}{C}.$$

For example if the participation rate is 0.6 then the investor takes part in 60 % of the underlying assets positive performance at maturity. The amount paid to the investor at maturity, in per cent of invested amount, is

$$1 + k \times \max(0, \text{asset performance}).$$

[2]When dealing with this type of products the option price is usually quoted in percent. This simply means that the option price is divided by the spot price of the underlying asset.

Figure 2.2 shows how the participation rate is varying with different volatilities (option prices), with interest rate at 4 %. The participation rate is more sensitive to changes with low volatility and less with high.

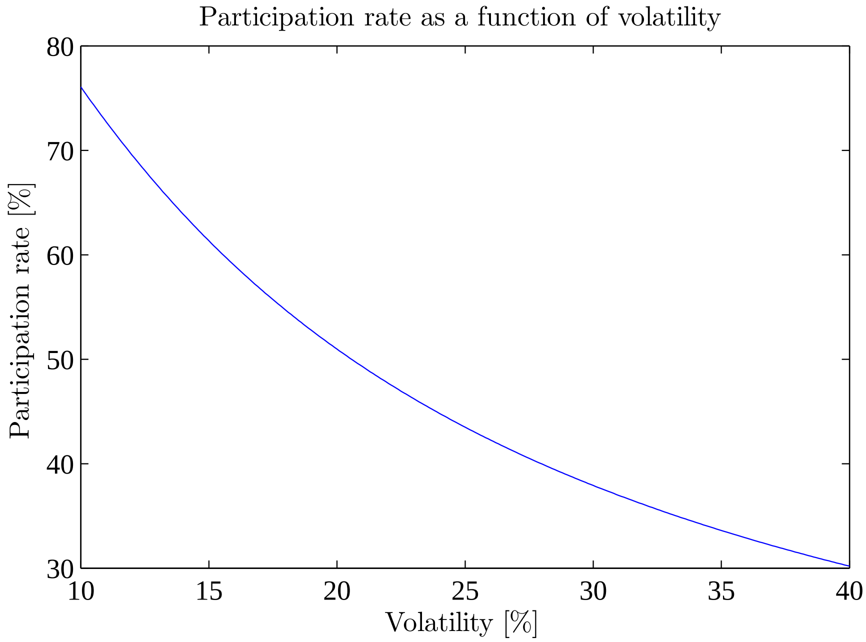

Figure 2.2: The graph shows the relation between participation rate and volatility.

Figure 2.3 shows how the participation rate is varying with different interest rates (bond prices), with volatility at 25 %. It is clear from this figure, compared to Figure 2.2, that the participation rate is more sensitive to changes in interest rate than from volatility. Figure 2.4 shows these relations in a surface diagram. The marked spots in each corner illustrate the different scenarios that are considered in this thesis. The question in this thesis is to answer if it matters for the investor in which market climate (level of interest rate and volatility) that prevails for optimal allocation in structured products.

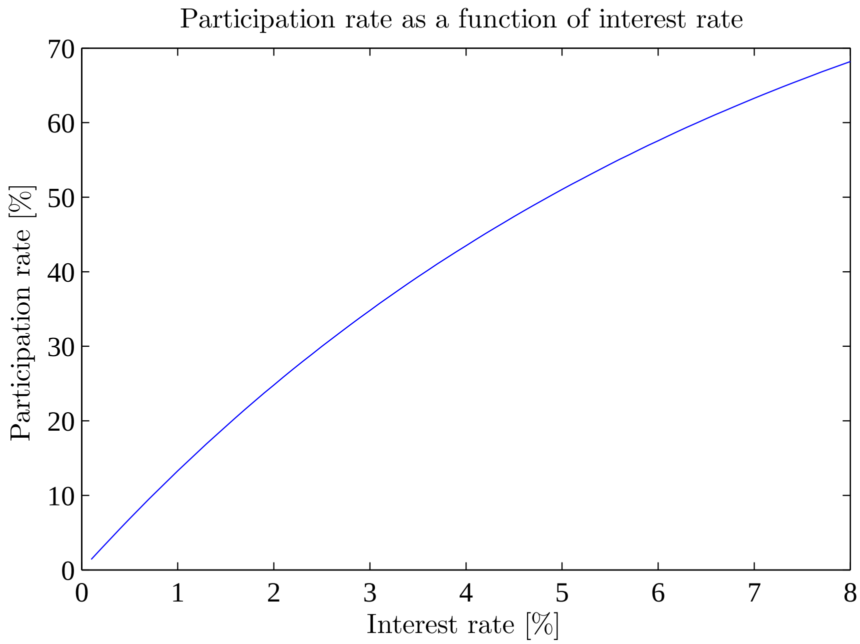

Figure 2.3: The graph shows the relation between participation rate and interest rate.

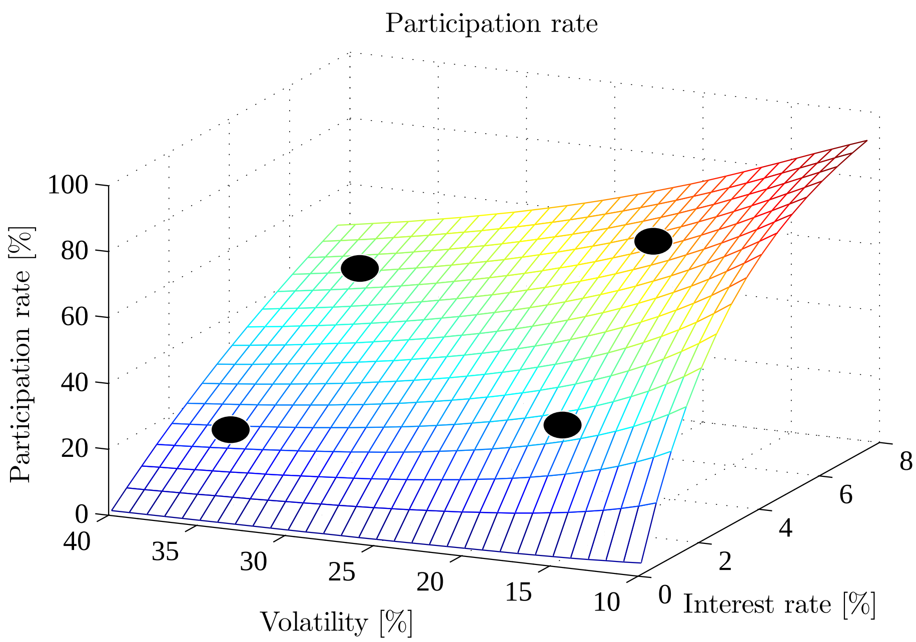

Figure 2.4: The surface diagram shows the relation between participation rate, interest rate and volatility. The four scenarios considered in this thesis are marked in each corner.

Chapter 3

Modeling financial assets

In this chapter it is described how the risky asset is modeled. Usually in portfolio optimization a parametric assumption is made about the distribution and parameters are estimated from historical data. The standard literature on the subject uses exclusively historical data to fit distributions to assets returns imposing distributional assumptions, [31]. For example the assumption about returns are normally distributed with parameters estimated from historical data. To make an assumption about the distribution from historical data ignores the market's expectations about the volatility and the shape of the distribution.

Financial time series show evidence to differ significantly from the normal distribution, [21]. Distribution of returns also shows excess kurtosis and skew. This behavior of financial data has been incorporated by option traders for a long time. However, prior to the stock market crash in October 1987 the volatility smile was nearly flat. After the crash traders got concerned about the possibility of another crash and priced options accordingly, [16]. This led to the suggestion that one reason for the volatility smile is fear for a similar crash.

For equity index options the typical shape is a negative skew, this means that the implied volatility is decreasing in strike. In [17] the authors outline four reasons for this:

- Leverage effect – when equity prices fall the company's debt-to-equity ratio in market value tends to rise.

- Correlation effect – individual equity returns become more highly correlated when equity prices fall.

- Wealth effect – investors become less wealthy when equity prices fall and of this more risk averse and respond more strongly to news.

- Risk effect – because investors become more risk averse they demand a higher expected return and this lead to reduction in equity prices.

In this thesis financial assets returns are modeled by estimating the implied distribution from the option market and transform this to a real world distribution using a utility function. The option market has the expected future distribution rather than estimation from historical data. This creates a distribution that will change over time to catch the dynamics of changing expectations. However, it is important to point out that many researches have shown that the implied distribution does not predict certain movements but rather reflects the market's sentiment, [12]. On the other hand, others have shown that the implied distribution has higher likelihood than historical densities, [20].

In the following part several methods are described for estimating the implied risk neutral density (RND) from the option market and how to transform the RND to a real world density and further outline methods to extract market data from option prices.

3.1 Techniques for estimating the implied RND

In this section various techniques are presented for estimating the implied RND from option prices. There are basically three main approaches:

- The parametric approach – assumption is made about the underlying distribution.
- The semi-parametric approach – weak assumption is made about the underlying distribution.
- The non-parametric approach – no assumption is made about the underlying distribution.

All of these techniques use the result, in some way, from Breeden and Litzenberger [6]. They show that if the underlying price has a continuous probability distribution then the state price is proportional to the second derivative of a related option. When applied across all states the second derivative equals the discounted RND. For additional details on this result see Appendix A. However, options are only traded at discrete strikes and for a limited range. Hence, there are many RNDs that can fit the market prices and it really all comes down to the completion of the option pricing function.

Currently there is no consensus of a method for extracting the implied RND but many have been described in the literature. The following methods are described briefly in this section:

- Interpolate the observed option prices.
- Interpolate the implied volatility curve.
- Assumption about the underlying stochastic process.

- Assumption about the terminal distribution.
- Implied trees.

3.1.1 Interpolate the observed option prices

A straightforward approach would be by direct application of the Breeden and Litzenberger result, [6], to the call pricing function. However, the call pricing function is not directly observable and requires to be consistent with the monotonicity and convexity conditions and that can be differentiated twice. In [3] the authors constructed a non-parametric estimator for the RND. It means that they have no parametric restriction and do not need to make any assumptions about the underlying stochastic process or terminal RND. The method they use is based on kernel regression and takes a rather complicated form and is highly data-intensive. In [5] the author takes a more primitive approach; interpolate the observed option prices using cubic splines. Because of the complex form of the call pricing function this approach needs a large number of degrees of freedom and can therefore cause practical problems.

3.1.2 Interpolate the implied volatility curve

This method is similar to the previous one but instead of interpolating the option prices interpolation is applied to the implied volatility curve. As [7] points out, the translation of option prices into implied volatilities eliminates a lot of the non-linearity in the option price relation. In [32] the author, the first originator of this approach, means that implied volatility tends to be smoother than option prices. One shortfall of the author's, [32], method is that the model does not prevent negative implied probabilities which would mean arbitrage opportunities. However, there exist methods that prevent this. For example [7] came up with an approach that is, under weak constraints, consistent with the absence of arbitrage. One advantage with this approach is that generally no assumption has to be made about the underlying distribution; this approach would then fall into the non-parametric approach.

3.1.3 Assumption about the underlying stochastic process

The procedure with this technique is to make an assumption about the underlying stochastic process, a parametric approach. This model will then imply a specific RND. The RND can be obtained in closed form under strong assumptions about how the underlying price evolves. An example of this is the Black and Scholes assumption about a geometric Brownian motion (GBM) with constant drift and volatility, this would imply a lognormal RND. If one would use the GBM and add a jump-diffusion model and model the probability of jumps, with a non-stochastic size, as a Bernoulli distribution and only allow one jump per run, this would result in a mixture of two lognormal distributions. One large disadvantage with this procedure is that an assumption must be made about the stochastic process and the result will only be as good as the model assumed and calibration routines. Generally the terminal RND cannot be computed in closed form when assuming a more sophisticated model like a stochastic-volatility jump-diffusion model. It can also be hard to calibrate the model to observed option prices, [9]. The main advantage with this method is that any option can be priced when the whole stochastic price process is revealed.

3.1.4 Assumption about the terminal distribution

To start with an assumption about the terminal distribution is a more general approach rather than the previous described procedure. This is because a stochastic process implies a unique terminal distribution, but the opposite is not true. This means that any terminal RND is consistent with many stochastic processes. The use of a two lognormal mixture model is widely used in the literature, for example [4] and [33]. As pointed out in [4] the lognormal has good statistical properties and given that the distribution of observed prices are in the neighborhood of the lognormal distribution. However, the lognormal distribution has limited possibilities to model fat tailed behaved data even though a mixture model can be used. A mixture model may produce a good fit but the shape of the distribution can get inconsistent where the two distributions mix. There are other models used in the literature such as skewed Student-t, [10], generalized beta distribution, [1] and Weibull, [30]. More recent works with generalized gamma distribution, [34] and generalized extreme value distribution, [22]. These distributions have several parameters which imply that more shapes can be obtained in comparison to the lognormal distribution.

3.1.5 Implied trees

The implied binomial trees approach was introduced by [29] and uses a non-parametric technique to construct a binomial tree for the evolution of the underlying price process. This is based on finding a RND that leads to a best fit to observed option prices due to some smoothness criterion. The method demands a prior guess of the RND and these are set according to a lognormal distribution. However, it is shown that the priors have clearly less influence on the implied binomial tree as the number of options used. The authors in [27] extended this method and use a trinomial tree model which is a semi-recombining version of the technique presented in [29]. The main advantage with this approach is that path dependent options can be priced.

3.2 Model selection

The common path among researches is to limit the assumptions made but in most cases the limitation in assumptions makes the procedure more complicated. As stated before there is no consensus about a model and is therefore more in a question about how sophisticated model one wants and in what circumstances. However, it is very easy to compare different approaches and simply select one based on the result.

In this thesis an assumption is made about the terminal distribution, a parametric approach, and with this evaluate some work of previous mentioned authors by comparing several of the distributions mentioned. The reason for this selection is that there exist many suitable distributions for this purpose and a better understanding for the shape of the distribution can be obtained in contrast to interpolation methods for example. The goal is to find a model that is as simple as possible that provides a reasonable good fit.

Sometimes an analytic expression can be found for various distributions which make the calibration faster. This approach is consistent with the absence of arbitrage when this constraint is implied by the model. The degrees of freedom is limited to five which would mean two parameters per distribution if a mixture model is used or five if only one distribution is used. The reason is for the calibration to work in a smooth way.

3.3 Extracting the implied RND

As described in [13] a European option can be valued on the basis that the world is risk neutral under no arbitrage assumption. This means that the equilibrium price of a European option in an arbitrage free economy will be the discounted value of the expected value of the pay off under risk neutrality.

Thus, the price of a European call option and a put option for a stochastic variable, S, can be valued according to

$$C(K,\tau) = e^{-r\tau}E^Q\left[(S-K)^+\right] = e^{-r\tau}\int_K^\infty f_S(s)(s-K)\mathrm{d}s, \tag{3.1}$$

$$P(K,\tau) = e^{-r\tau}E^Q\left[(K-S)^+\right] = e^{-r\tau}\int_0^K f_S(s)(K-s)\mathrm{d}s, \tag{3.2}$$

where r is risk-free rate, τ is time to maturity, $f_S(s)$ is the risk neutral probability density function for S and K is the strike. The following martingale condition must be satisfied in an arbitrage free economy

$$F = E^Q[S]. \tag{3.3}$$

where F is the forward price of the underlying asset.

To calibrate the model to observed option prices some optimization method must be defined. The following optimization methods are outlined in [17]

$$\min \sum_i [C(K_i,\tau) - \hat{C}_i]^2 + \sum_i [P(K_i,\tau) - \hat{P}_i]^2 \qquad \textit{quadratic}, \tag{3.4}$$
$$\min \sum_i \left[\frac{C(K_i,\tau) - \hat{C}_i}{\hat{C}_i}\right]^2 + \sum_i \left[\frac{P(K_i,\tau) - \hat{P}_i}{\hat{P}_i}\right]^2 \ \textit{goodness of fit},$$
$$\min \sum_i |C(K_i,\tau) - \hat{C}_i| + \sum_i |P(K_i,\tau) - \hat{P}_i| \qquad \textit{absolute difference.}$$

Quadratic will give higher precision in terms of percentage deviation for deep in the money (ITM) calls and deep ITM puts. This is because these have higher prices which results in large penalty in absolute terms for deviation. Goodness of fit is basically the opposite of quadratic and will penalize deviations in terms of percentage deviation instead of absolute terms. This will result in a more accurate fit to for deep out of the money (OTM) calls and deep OTM puts in terms of absolute difference. The third method, absolute difference, is more like the quadratic method but will not penalize large deviations in such way. This will create an overall good fit but tolerate large deviations.

In [4] the author proposes the quadratic method with an extra penalty term for the martingale-condition (3.3) in the case of a two lognormal mixture model.

$$\begin{aligned}\min \sum_i [C(K_i,\tau) - \hat{C}_i]^2 &+ \sum_i [P(K_i,\tau) - \hat{P}_i]^2 \\ &+ \left[\theta e^{\alpha_1 + \frac{1}{2}\beta_1^2} + (1-\theta)e^{\alpha_2 + \frac{1}{2}\beta_2^2} - e^{r\tau}\right]^2,\end{aligned}$$

where θ is the weight and $\alpha_{1,2}$ and $\beta_{1,2}$ are parameters of the lognormal distributions.

3.4 Transforming the RND into a real world density

It is important to point out that the implied probability density extracted from the option market is risk neutral. This means that there is no risk premium to worry about. In reality investors are not risk neutral but risk aversive which mean that the RND does not equal the true market distribution. To obtain a real world density one could specify the aggregate market utility function and determine (estimate) the corresponding coefficient of risk aversion. Here one must separate between the transformation of aggregate market utility and subjective preferences of single investors. One can think of the market to be homogeneous or heterogeneous where investors have the same beliefs or different.

This can also be much more complicated than just specifying a general utility function because the investor's beliefs can be hard to transform explicit. It can be done by just to define a real world shape and then assume this from some unspecified utility function. For example by just changing the drift (mean) of the distribution.

In this thesis the transformation is made with respect to the aggregate market utility and the investor's specific preferences are defined later in the utility based portfolio optimization. The transformation between the risk neutral world and the real world involves an adjustment to the discount factor, [20]. Let $\zeta(s)$ be the stochastic discount factor for a payoff as

$$\zeta(s) = k\frac{dU(s)}{ds}, \tag{3.5}$$

where k is a constant and $U(s)$ is a utility function. In the real world the price of a European call option is

$$C(K,\tau) = E^P\left[\zeta(S)(S-K)^+\right] = \int_K^\infty \zeta(s) f_{S,real}(s)(s-K)\mathrm{d}s, \tag{3.6}$$

where $f_{S,real}$ is the real world density function. Using (3.6) and (3.1), $\zeta(s)$ yields

$$\zeta(s) = e^{-r\tau}\frac{f(s)}{f_{S,real}(s)}. \tag{3.7}$$

Using (3.5) and (3.7) with $\int_0^\infty f_{S,real}(s)\mathrm{d}s = 1$, the relation between the real world and risk neutral distribution is

$$f_{S,real}(s) = \frac{f(s)}{U'(s)}\frac{1}{\int_0^\infty f(z)/U'(z)\mathrm{d}z}. \tag{3.8}$$

To determine the coefficient of risk aversion is hard. It is well known from the equity risk premium puzzle that in order to achieve the high risk premium over government bonds, individuals must have implausibly high risk aversion according to benchmark economic models, [26]. In [31] the author proposed that the coefficient of risk aversion is instead calibrated from an assumed expected return in implicit form according to

$$r = \frac{E^P[S]}{S_0} - 1 = \frac{1}{S_0}\int_0^\infty s f_{S,real}(s)\mathrm{d}s - 1, \tag{3.9}$$

where S_0 is the initial price of the asset.

In this thesis no assumption is made about the expected return, rather derived from different coefficients. This is because the return has varied a lot historically and makes a huge impact on the result. Thus, the portfolio optimization is made with respect to different risk aversions corresponding to different implied expected returns. However, consideration is taken to historical risk premiums for a realistic comparison.

For doing this transformation some utility function must be specified. The power utility is widely used in the literature. This falls in the category constant relative risk aversion utility functions. One type of power utility is given by

$$U(x) = \begin{cases} \frac{x^{1-\lambda}}{1-\lambda} & \lambda \geq 0, \lambda \neq 1, \\ \ln(x) & \lambda = 1. \end{cases} \tag{3.10}$$

3.5 Extracting market data from options

When pricing derivatives some variables must be determined besides the distribution of the asset. This is the dividend and the risk-free interest rate. The dividend is in practice usually estimated by traders and financial analysts for input in the model. The risk-free interest rate is in reality not the corresponding government benchmark-rates but for example inter-bank swap rates for longer maturities. Instead of doing subjective estimates of these the implied data from the option market is used.

There exist some arbitrage conditions which can be exploited to determine the market's anticipated dividend and interest rate. From the put-call parity relation both the dividend and the risk-free interest rate can be estimated. The Box-spread relation is used to determine how accurate the risk-free interest rate estimation is from the put-call parity relation. For calculations of the implied volatility curve these variables are necessary to satisfy certain conditions. This must be in a reasonable range or the volatility curve will be unrealistic. One easy way to check this is to compare the curves calculated for calls and puts; they should be identical.

3.5.1 Put-call parity

There is a theoretical relation between European call and put options. This relation holds under no-arbitrage arguments. Put-call parity for an asset providing a dividend yield, q, equals

$$C + Ke^{-r} = P + Se^{-q}, \tag{3.11}$$

where C is the call price, P the put price, K the strike, S the spot price and r the risk-free interest rate.

Under this relation the market's anticipated interest rate and dividend yield can be obtained. This relation does not hold exactly in reality due to spreads and different estimates. To perform this practically a mean square optimization can be performed.

3.5.2 Box-spread

A box-spread is an arbitrage strategy. This means that the trade only takes place if an arbitrage opportunity exists. The payoff in a long position in a box-spread should equal lending at the risk free rate and a short position should equal borrowing at the risk-free rate. This could sometimes for example make that hedge funds have cheaper funding from this position then borrowing other ways.

A box-spread is a combination of a call spread with strikes K_1 and K_2 and a put spread with the same two strikes. The payoff will always equal $K_2 - K_1$. This implies that the price of a box-spread is the present value of this position;

$$e^{-r\tau}(K_2 - K_1). \tag{3.12}$$

Chapter 4

Probability distributions

There have been many probability distributions described in the literature for fitting the terminal RND to observed option prices. In [10] the authors use a skewed Student-t distribution, [34] the generalized gamma distribution and [22] the generalized extreme value distribution. For calibration purposes analytical formulas are preferred but this is not always possible due to complicated expressions for the density function. The following distributions are reviewed:

- Mixture of two lognormal distributions.
- Generalized extreme value (GEV) distribution.
- Generalized gamma distribution.
- Normal inverse Gaussian (NIG) distribution.
- Skew Student-t distribution.

The lognormal distribution is outlined separately even though this is a special case in some of the generalized distribution. This is because the lognormal distribution is a benchmark distribution in financial literature and is therefore easy to relate to in comparison with the other distributions.

4.1 Lognormal distribution

In the case of Black and Scholes assumption about the evolvement of the price as a GBM the terminal distribution is lognormal and log returns are normally distributed. The lognormal density function is

$$f_X(x|\alpha,\beta) = \frac{1}{x\beta\sqrt{2\pi}} e^{-\frac{(\log(x)-\alpha)^2}{2\beta^2}},$$

where α is the mean and β the volatility of $\log(x)$. Figure 4.1 shows the probability density for the price and logarithmic returns. The price of a

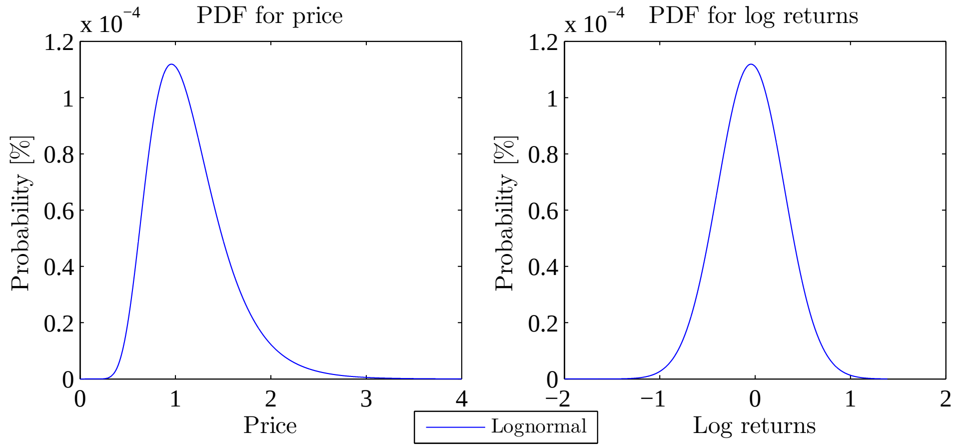

Figure 4.1: The figure shows the shape for price and log return for the lognormal distribution.

European call option, Black and Scholes option formula, is given by, in the usual form,

$$C(K,\tau) = \Phi\left[\frac{(\mu+\sigma^2/2)\tau - \log[K]}{\sigma\sqrt{\tau}}\right] - \mathrm{K}e^{-\mu\tau}\Phi\left[\frac{(\mu-\sigma^2/2)\tau - \log[K]}{\sigma\sqrt{\tau}}\right], \tag{4.1}$$

where μ is continous compounding interest rate and σ is the volatility.

4.2 Mixture of two lognormal distributions

The mixture of two lognormal distributions is a linear combination of two lognormal distributions. This makes that different shapes can be obtained. The probability density function is

$$f_X(x|\alpha_1,\beta_1,\alpha_2,\beta_2) = \theta\frac{1}{x\beta_1\sqrt{2\pi}}e^{-\frac{(\log(x)-\alpha_1)^2}{2\beta_1^2}} + c(1-\theta)\frac{1}{x\beta_2\sqrt{2\pi}}e^{-\frac{(\log(x)-\alpha_2)^2}{2\beta_2^2}},$$

where θ is the weight. The price of a European call option is thus a linear combination of (4.1) according to

$$C(K,\tau) = \theta C_1(K,\tau|\alpha_1,\beta_1)) + (1-\theta)C_2(K,\tau|\alpha_2,\beta_2)).$$

4.3 Generalized extreme value distribution

The GEV is a family of distributions that combines distributions within extreme value theory. This is a robust framework to model the tail behavior of distributions. For this reason the loss distribution is modeled. The GEV distribution takes three parameters. The probability density function is

$$f_X(x|\xi,\mu,\sigma) = \frac{1}{\sigma}\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-1-1/\xi}\exp\left(-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-1/\xi}\right), \quad \xi \neq 0 \tag{4.2}$$

for $1+\xi\frac{x-\mu}{\sigma} > 0$ and when $\xi \to 0$ then $(1+\xi x)^{-1/\xi} \to e^x$ we have

$$f_X(x|0,\mu,\sigma) = \frac{1}{\sigma}\exp(-e^{-(x-\mu)/\sigma} - (x-\mu)/\sigma), \qquad \xi = 0. \tag{4.3}$$

When $\xi > 0$ the associated distribution is called Fréchet. In the case when $\xi \to 0$ the distribution is the Gumbel class. If $\xi < 0$ we have the distribution class Weibull. Figure 4.2 shows the PDF for the price and for log returns for the special case when $\xi > 0$. A closed form solution can be obtained for the

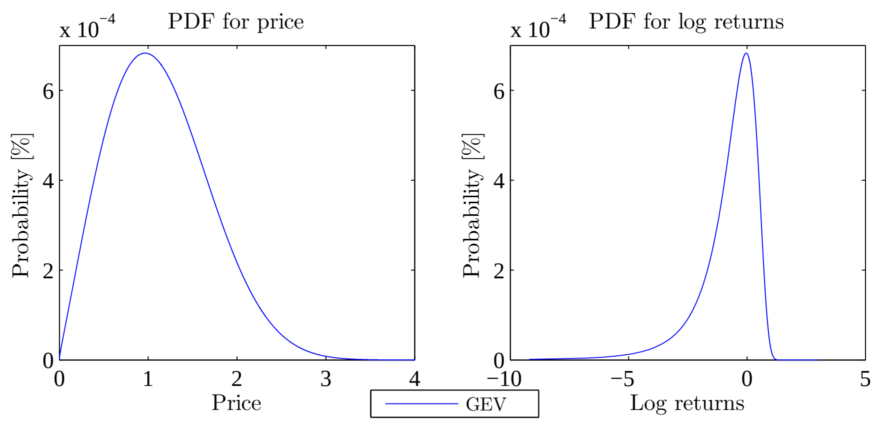

Figure 4.2: The figure shows the shape for price and log return for the GEV distribution.

price of a European option. This is derived in [22] and is for a call option

$$C(K,\tau) = e^{-r\tau}\left[\left((1-\mu+\sigma/\xi)e^{-H^{-1/\xi}} - \frac{\sigma}{\xi}\Gamma\left(1-\xi, H^{-1/\xi}\right)\right) - Ke^{-H^{-1/\xi}}\right],$$

where $\Gamma(\cdot,\cdot)$ is the incomplete gamma function and H is $1+\frac{\xi}{\sigma}(1-K-\mu)$. And for a put option

$$\begin{aligned} P(K,\tau) = e^{-r\tau}\Big[&K\Big(e^{-h^{-1/\xi}} - e^{-H^{-1/\xi}}\Big) + \frac{\sigma}{\xi}\Gamma\Big(1-\xi, h^{-1/\xi}H^{-1/\xi}\Big) \\ &- (1-\mu+\sigma/\xi)\Big(e^{-H^{-1/\xi}} - e^{-h^{-1/\xi}}\Big)\Big], \end{aligned} \tag{4.4}$$

where h is $1+\xi/\sigma(1-\mu)$.

4.4 Generalized gamma distribution

The generalized gamma distribution is a three parameter distribution. This distribution is not very common in the statistics literature but it incorporates a wide range of distributions, which are well known. The probability density function is

$$f_X(x|\alpha,\beta,k) = \frac{1}{\Gamma(k)}\frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{k\beta-1} e^{-\left(\frac{x}{\alpha}\right)^{\beta}},$$

where $\Gamma(\cdot)$ is the gamma function. Special cases are the gamma distribution when $\beta = 1$, the Weibull distribution when $k = 1$, the exponential distribution when $k = 1$ and $\beta = 1$, the Rayleigh distribution when $k = 2$ and $\beta = 2$ and the lognormal distribution when $k = 0$. Thus, a wide range of well known distributions. Figure 4.3 illustrates the shape for the special case gamma distribution. A closed form solution can be obtained for the price

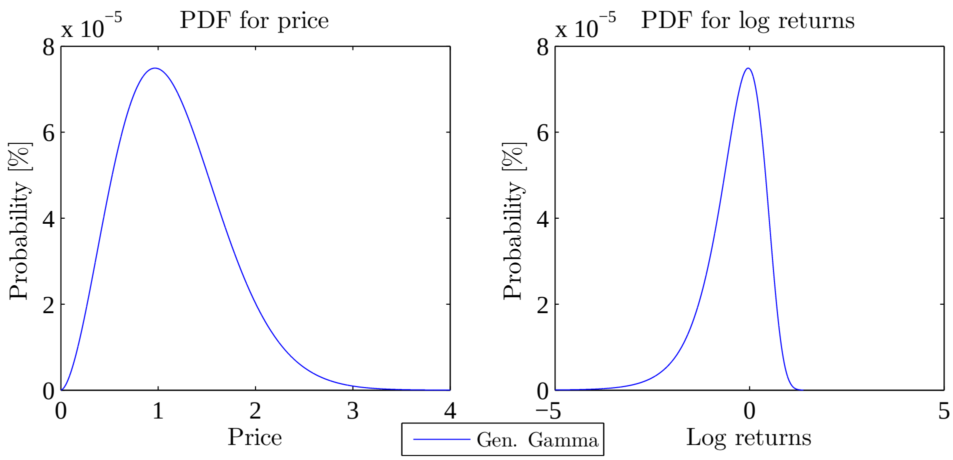

Figure 4.3: The figure shows the shape for price and log return for the generalized gamma distribution.

of a European option. This is for a call option

$$C(K,\tau) = e^{-r\tau}\frac{1}{\Gamma(k)}\left[\alpha\Gamma\left(\frac{1+k\beta}{\beta},\left(\frac{K}{\alpha}\right)^{\beta}\right) - K\Gamma\left(k,\left(\frac{K}{\alpha}\right)^{\beta}\right)\right]$$

and for a put option

$$\begin{aligned} P(K,\tau) = e^{-r\tau}\frac{1}{\Gamma(k)}\Bigg[&K\Gamma(k) - \alpha\Gamma\left(k+\frac{1}{\beta}\right) - K\Gamma\left(k,\left(\frac{K}{\alpha}\right)^{\beta}\right) \\ &+ \alpha\Gamma\left(k+\frac{1}{\beta},\left(\frac{K}{\alpha}\right)^{\beta}\right)\Bigg]. \end{aligned}$$

4.5 Normal inverse Gaussian distribution

The NIG distribution is a special case of the generalized hyperbolic (GH) family. It is a normal variance-mean mixture where the mixing density is the inverse Gaussian distribution. This is a four parameter distribution which can describe a wide range of shapes of the distribution. The probability density function is

$$f_X(x|\alpha,\beta,\mu,\delta) = \frac{\delta\alpha e^{\delta\gamma+\beta(x-\mu)}}{\pi\sqrt{\delta^2+(x-\mu)^2}} K_1\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right),$$

where $\gamma = \sqrt{\alpha^2-\beta^2}$ and K_1 is the modified Bessel function of the third kind. Figure 4.4 illustrates the shape for the NIG distribution. A closed form solution for the price of an European option can not be obtained so pricing must be integrated numerically according to (3.1) and (3.2).

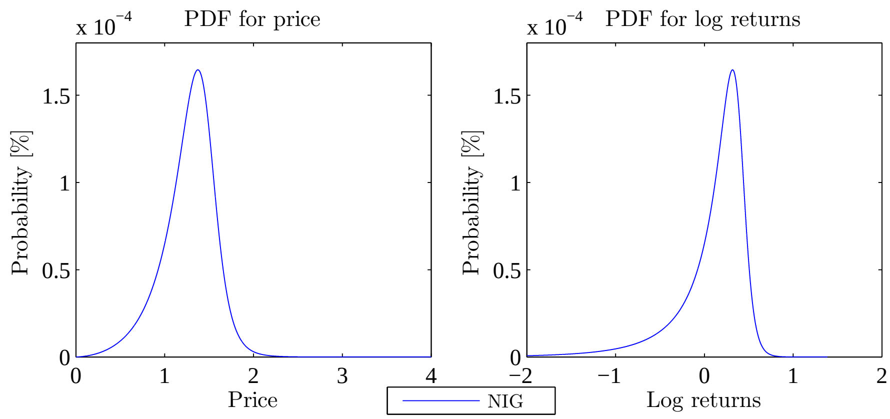

Figure 4.4: The figure shows the shape for price and log return for the NIG distribution.

4.6 Skew Student-t distribution

The skew Student-t distribution is also a special case of the GH family. This distribution has been proposed in the literature but its statistics is not well known, specifically its special tail behavior that one tail has polynomial and the other exponential behavior. The skew Student-t distribution is a normal variance-mean mixture where the mixing density is the inverse chi-square distribution. This is a four parameter distribution which can describe a wide range of shapes. The probability density function is

$$f_X(x|\nu,\beta,h,\mu) = \begin{cases} f_X(x-\mu|\nu,\beta,h) & \beta \neq 0, \\ \frac{1}{h} t_\nu\left(\frac{x-\mu}{h}\right) & \beta = 0. \end{cases}$$

where

$$f_X(x|\nu,\beta,h) = e^{\beta x}\frac{h^\nu \nu^{\nu/2}|\beta|^{(\nu+1)/2}K_{(\nu+1)/2}\left(|\beta|\sqrt{x^2+h^2\nu}\right)}{\sqrt{\pi}\Gamma(\nu/2)2^{(\nu+1)/2-1}\left(\sqrt{x^2+h^2\nu}\right)^{(\nu+1)/2}}$$

and where t_ν is the Student-t distribution with ν degrees of freedom and $K_\nu(\cdot)$ is the modified Bessel function of the second type of order ν. Figure 4.5 illustrates the shape for the skew Student-t distribution. A closed form

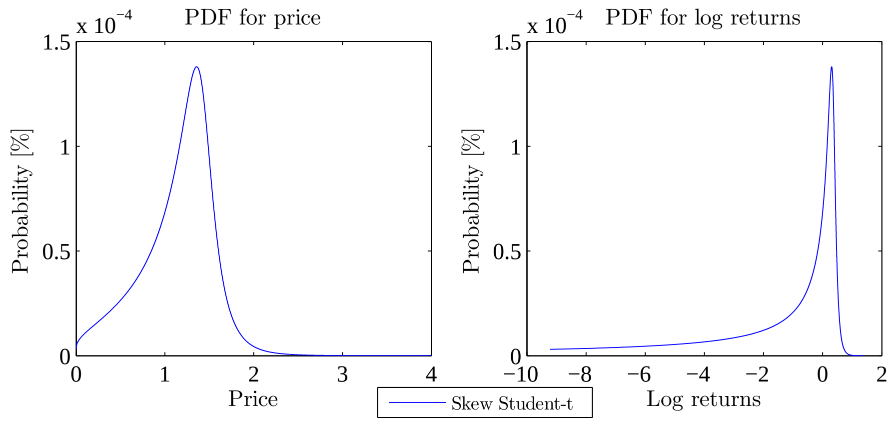

Figure 4.5: The figure shows the shape for price and log return for the skew Student-t distribution.

solution for the price of an European option can not be obtained so pricing must be integrated numerically according to (3.1) and (3.2).

Chapter 5

Model calibration

In this chapter the market option data is described which the RND is extracting from. The result from the fitting procedure is presented and described in detail. Finally, the transformation from the RND to a real world density is depicted and described. It is this probability density function that is further used in the portfolio optimization.

5.1 Market option data

The Dow Jones Euro Stoxx 50 index futures and options traded on Eurex[1] are considered as the most liquid derivative assets in the world. Options are available at a wide range of strikes with high liquidity which secures low spreads and reliable prices. This is the reason why in this thesis have chosen to use option prices from this market.

Two states are composed which defines a low volatility and a high volatility state. The background of this choice is described later in Chapter 7.

The options have a time to maturity of two years. The low volatility set has 41 calls and 41 puts and the high volatility set has 39 calls and 39 puts. The sets are chosen so there is a similar distance between strikes so the fit is not over weighted to close ATM strikes where usually more options are traded. This would cause the optimization to prioritize this area. The high volatility set has wider range of strikes than the low volatility. This is because of the large moves in the underlying index between the sets and may also be explained by that the probability for large moves are higher with high volatility that increases the demand for more and wider strikes.

Before the fitting procedure is started the market's anticipated interest rate and dividend yield is estimated using a least square optimization with the arbitrage conditions, (3.11) and (3.12), mentioned in Chapter 3. Equation (3.12) is only used to check the result. This confirms deviations from

[1] For further information see www.eurexchange.com

the theory which indicates that a perfect fit is not obtainable but this error is very small so it should not cause any problems.

5.2 Fitting

In this part the actual fitting is described and the result is presented. All of the distributions described in Chapter 4 are evaluated. The fitting is divided into two parts with respect to the volatility sets. This is because one could expect different properties of these and then also the result of fitting for different distributions. The optimization method used is (3.4), described in Chapter 3. Because of the wide range of strikes that includes deep ITM calls and puts the martingale condition, (3.3), will be satisfied enough for not adding an extra penalty term. All numeric procedures are performed in Matlab where Optimization Toolbox and Statistics Toolbox are required.

To evaluate the result of each individual fit the mean square error (MSE) is presented and also the implied volatility from the model compared to the market volatility. The comparison of the volatility is more easy to relate to and gives a good overview of the fit from comparing the actual option prices.

The implied volatility as a function of strike, also known as volatility smile, is illustrated in Figure 5.1.

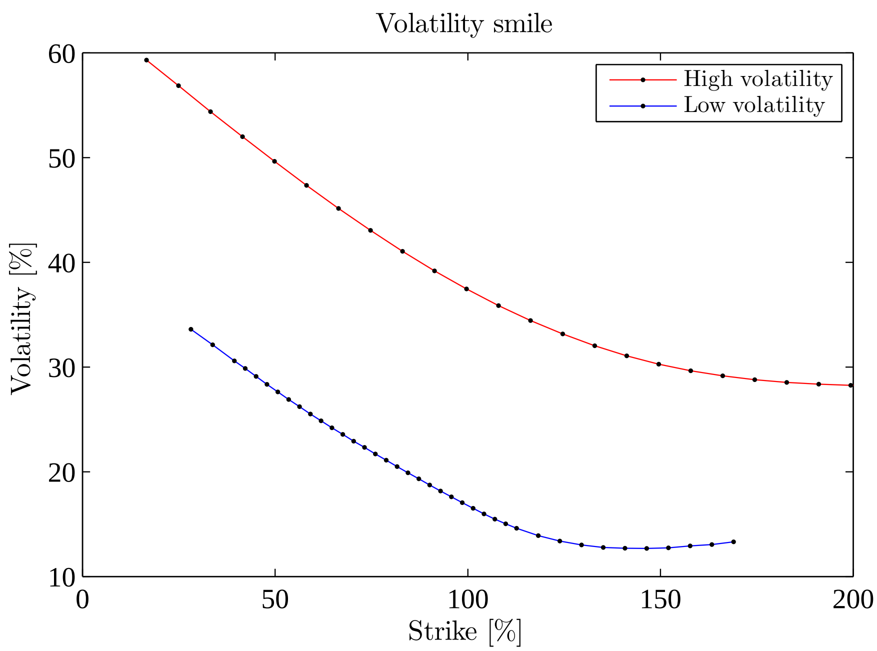

Figure 5.1: The figure shows the implied volatility curve for the two volatility sets.

5.2.1 Low volatility set

The mixture of two lognormal distributions turns out to have a somewhat good fit, the MSE is 0.305. However, the shape where the two distributions mix, even though it is not so apparent, does not show a reasonable behavior and shape. The RND is shown in Figure B.1 and in Figure B.2 the fitted volatility is shown. The inconsistency in the fitted curve below around 90 % strike is caused where the two lognormal mixes.

The GEV distribution fit is rather good with a MSE at 0.243. The best fit is when K is larger than zero which suggests a Frechet distribution. In Figure B.3 the RND is illustrated and in Figure B.4 the fitted volatility is shown. The fit is pretty good but with some errors in both low and high strikes compared to the market. The shape outside the observed volatility shows to be reasonable.

The fit with the generalized gamma distribution has a MSE at 0.368. The best fit is none of the special cases. In Figure B.5 the RND is illustrated and it shows that the left tail is more fat than the best fit GEV distribution while the right tail is less fat. The implied volatility curve, Figure B.6, shows a large underestimation for higher strikes compared to the market.

The NIG proofs to be an excellent fit with a MSE of 0.0836. Compared to the other distributions this fitting process takes a longer time to converge to the best fit. The estimation error is very small and can partly be described by inconsistencies in the market option data. The RND is shown in Figure 5.2 and is rather symmetric but with a less fat right tail. In Figure 5.3 the implied volatility curve is illustrated. The fit is overall very good except a short range for low strikes. However, this difference is very small.

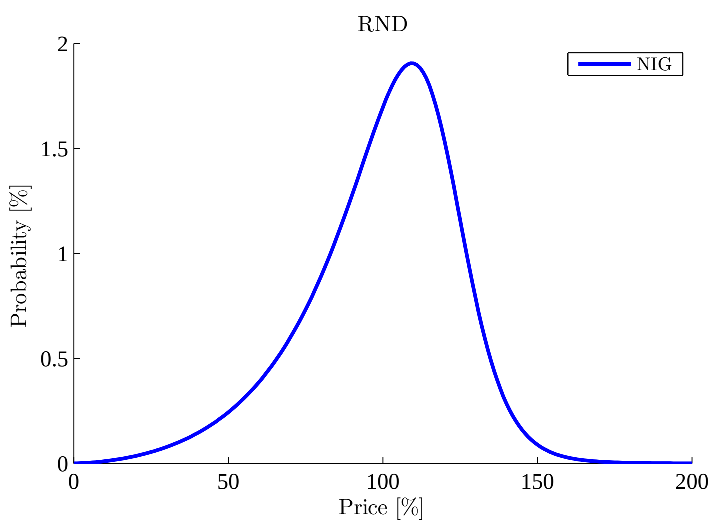

Figure 5.2: The figure shows the RND best fit with the NIG.

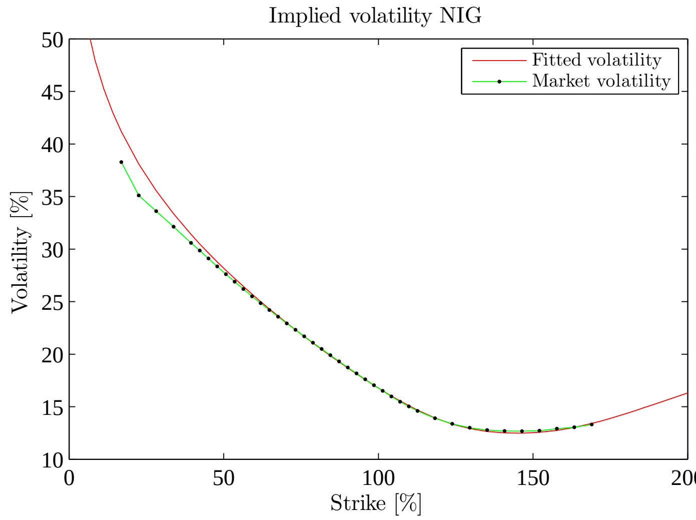

Figure 5.3: The figure illustrates comparison between best fit implied volatility.

Skew Student-t has a MSE of 0.161 and shows to be a pretty good fit. Similar to the NIG distribution this fitting process takes longer time to converge to the best fit, even longer than the NIG. The shape of the RND is depicted in Figure B.7 and shows similar behavior as the NIG RND. In Figure B.8 the implied volatility curve is illustrated. As shown is it a relative poor fit for low strikes which mean that the best fit with the skew Student-t distribution has a to fat left tail for this data set.

5.2.2 High volatility set

The MSE with the mixture of two lognormal distributions is 0.206 and it is a very good fit. Again with a problem with the shape where the two distributions mix, this time very clear. This does not show a reasonable behavior and shape, the RND is shown in Figure B.9. Figure B.10 shows the implied volatility, the lognormal mixture has an excellent fit except where the two distributions mix and the extrapolation for low strikes.

The GEV distribution has a MSE of 0.550 which is a very poor fit. K is again larger than zero which suggests a Frechet distribution. In Figure B.11 the RND is illustrated. In Figure B.12 this poor fit is explained, where the implied volatility is showed. The GEV distribution can not capture the flattening in volatility for high strikes.

The fit with the generalized gamma distribution is very bad, MSE showes to be 0.621. The best fit is none of the special cases. In Figure B.13 the RND is illustrated and in Figure B.14 the volatility.

The fit with the NIG is not very good but not as worse as GEV and

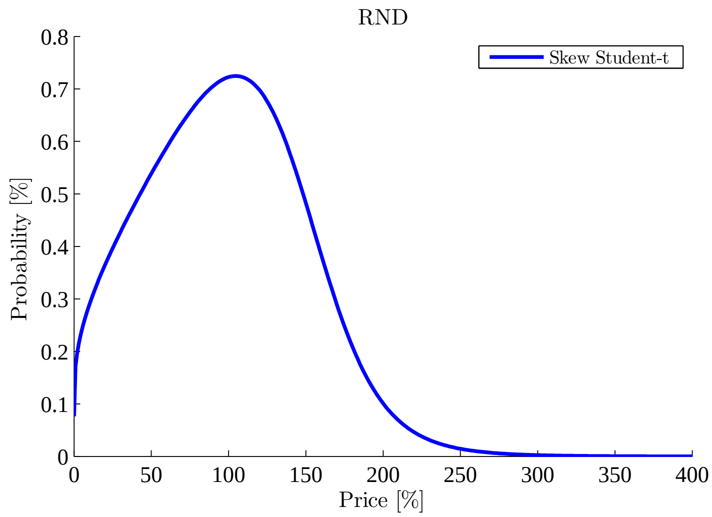

Figure 5.4: The figure shows the RND best fit with the skew Student-t.

the generalized gamma, the MSE is 0.477. Figure B.15 illustrates the RND. NIG can not capture the overall shape and has errors across all strikes. The fit seems to show similar errors as GEV and the generalized gamma except they are not as large, see Figure B.16

Skew Student-t has a MSE at 0.272. It takes very long time for the solution to converge to the best fit. The fit is very good but not as good as the mixture of two lognormal distribution. The RND is illustrated in Figure

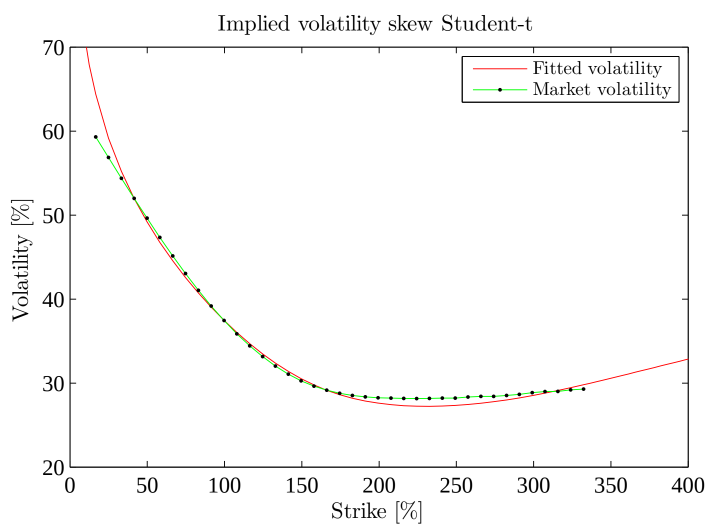

Figure 5.5: The figure illustrates comparison between best fit implied volatility.

5.4. In Figure 5.5, where one can see the implied volatility, the fit is proven to be very good. There are a few larger errors in the low strike area but rather small in the high strike area.

5.2.3 Summary

The best fit to the low volatility set is the NIG distribution. It showed an outstanding fit, superior to the other distributions evaluated. The completion of the RND shows a good and reasonable shape.

The best fit to the high volatility set is the two lognormal mixture. It showed a very good fit somewhat better than the Student-t distribution. However, the shape of the two lognormal mixture distribution is questionable with inconsistencies in the curve. Because of this the two lognormal distribution is not used, instead the skew Student-t is. The fit with the skew Student-t is slightly worse than the two lognormal mixture but much better than the other distributions. This distribution explains the shape very well outside the observed prices.

On the high volatility data set it is several options with strikes that have almost zero probability to be reached, according to the RND. This is all the calls with a strike of higher than 300 %. This causes that these option prices are not so important to get a good fit to for explaining the shape of the RND. This is more in terms of having the correct inputs of interest rate and dividend. The NIG and skew Student-t have similar properties and have been showed to provide a much better fit than the other distributions on these data sets. The NIG distribution can in these terms been seen as a low volatility distribution and the skew Student-t as a high volatility distribution. These distributions have been shown to be well suited to model this kind of data presented. Table 5.1 shows a summary of the fitting MSE.

Table 5.1: Summary of the MSE.

Distribution	**MSE high vol. set**	**MSE low vol. set**
Two lognormal mixture	0.206	0.304
GEV	0.550	0.243
Generalized gamma	0.621	0.368
NIG	0.480	0.084
Skew Student-t	0.272	0.161

5.3 Calibrating the real world distribution

In the scenarios two different states of interest are used. On both option data sets the interest rate does not match and the dividend yield is larger than zero. To be able to model these distributions the expected value with the RND should equal the corresponding interest rate. This is accomplished by refitting the best fit distributions on recalculated option prices corresponding to each interest rate and the dividend set to zero. With this the expected value with the RND will equal the corresponding interest rate.

When going from the RND to a real world density a utility function must be defined according to (3.8). This transformation is done with the power utility defined to (3.10) and the expected return is calculated using (3.9). Figure 5.6 shows real world distributions for different λ for the low volatility set. As can be seen, the expected return increases with λ while the downside risk decreases. There are two distributions for this set but only one is shown because they a very similar. Figure 5.7 shows the implied annual expected

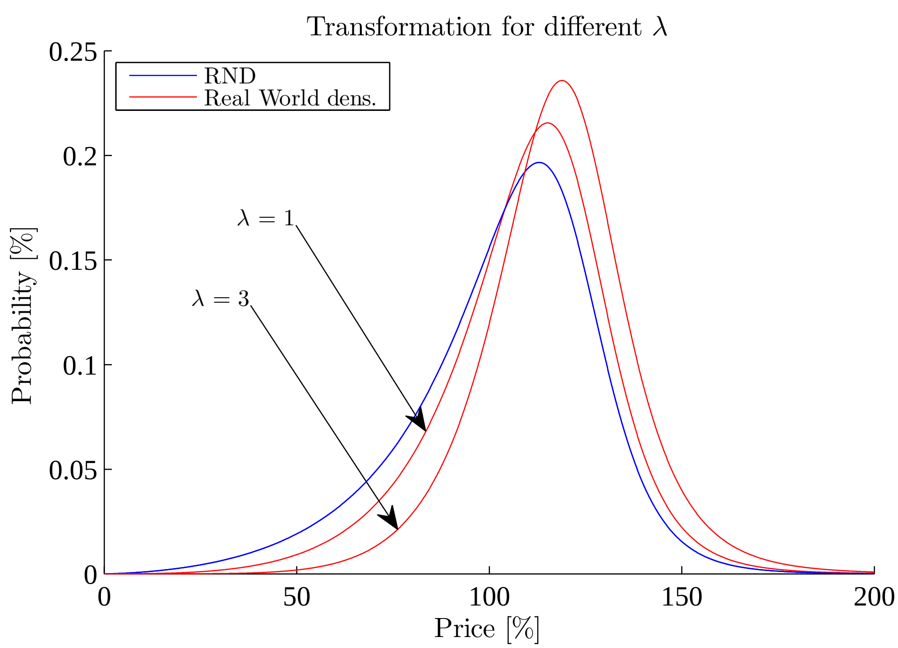

Figure 5.6: The graph shows the transformation for different λ for the low volatility set.

return corresponding to different λ. When $\lambda = 0$ then the implied expected return equals the expected return of the RND, that is the risk-free interest rate[2]. The increase in return with λ shows the same behavior between the

[2]The interest rate is recalculated to annual compounding which causes a larger value from the stated that is continuous compounding.

low interest rate but with a bump upwards.

Figure 5.8 shows real world distributions for different λ for the high

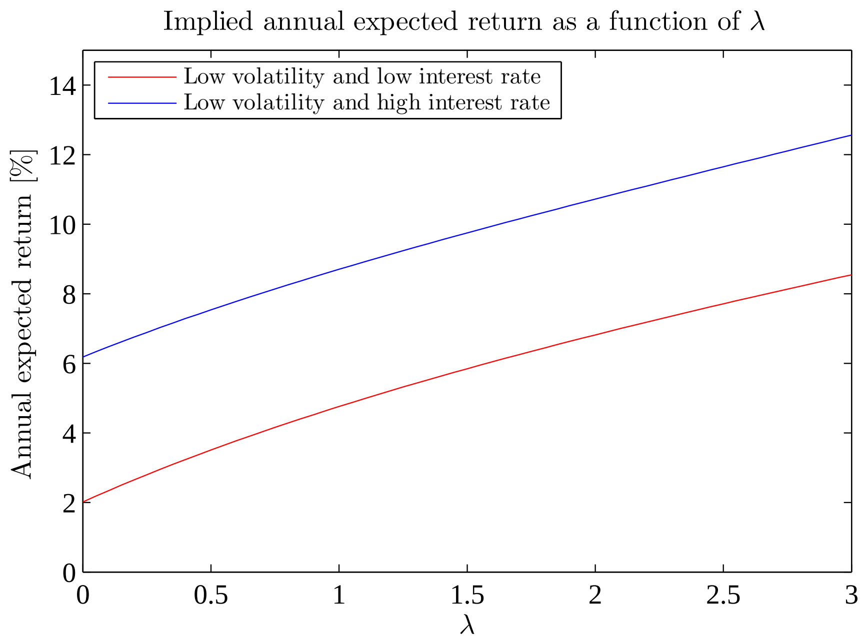

Figure 5.7: The graph shows implied expected return for different λ for the low volatility set.

volatility set. Because of the same reason as before, only one is shown. The increase in return is very sensitive to λ. This is why a smaller range of λ is used to illustrate the relation. This could be expected when the high volatility implies greater probabilities for larger returns. Figure 5.9 shows the implied annual return corresponding to different λ. The increase in return with λ shows the same behavior between the low interest rate but with a bump upwards. Compared to the low volatility set this show much higher returns for smaller λ.

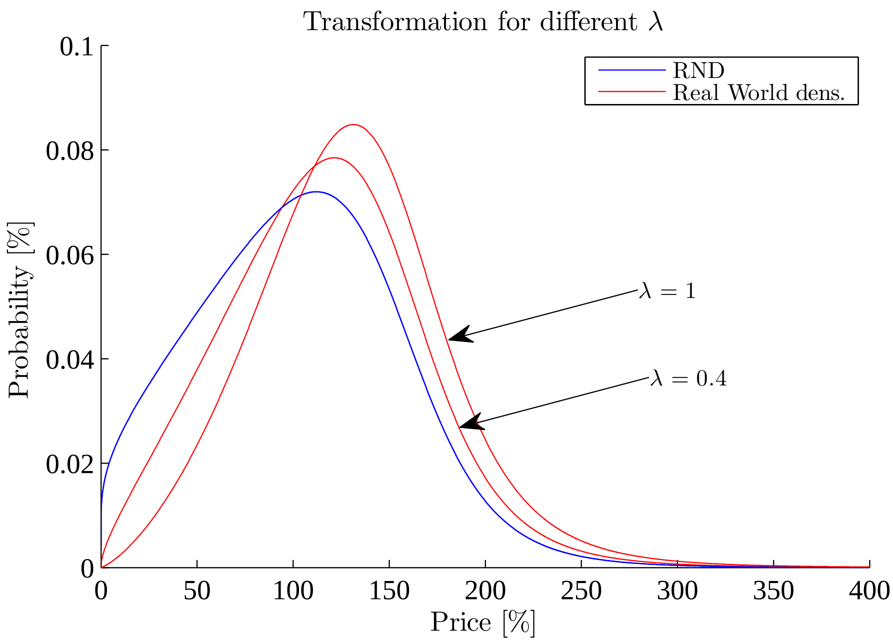

Figure 5.8: The graph shows the transformation for different λ for the high volatility set.

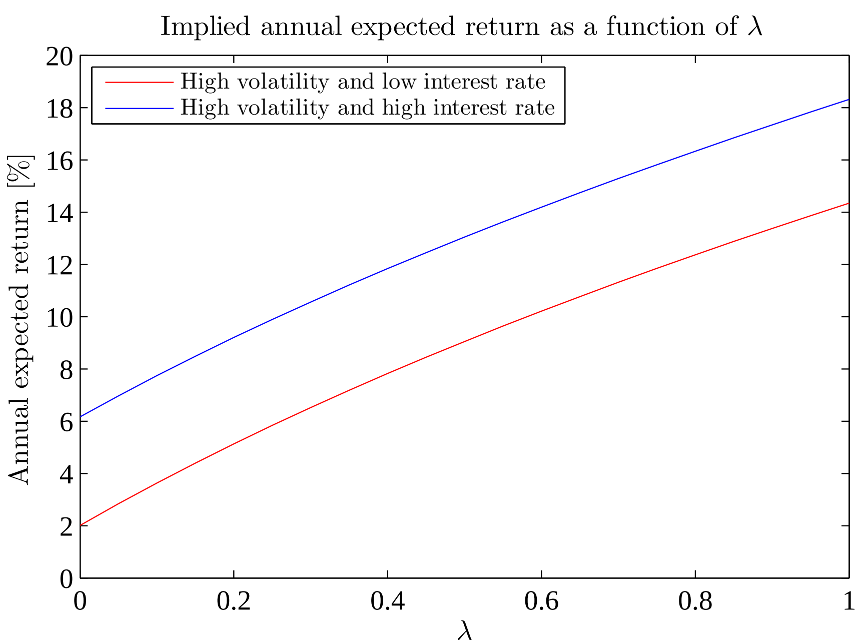

Figure 5.9: The graph shows implied expected return for different λ for the high volatility set.

Chapter 6

Utility theory

There is no unique definition of risk but risk is a very important part in the financial world. To be able to clarify the relation between risk and reward is essential to make good investment decisions. It is very common in the financial community to define risk as standard deviation, or volatility, when talking about financial assets. The relation between risk and reward then becomes the relation between the volatility and future expected returns. Another way to do this is to define a function that measures or ranks future returns and losses. This is achieved by using a utility function.

In this chapter the basics of utility theory is outlined and the representative investor is defined that is later used in the portfolio optimization.

6.1 Basic utility theory

Utility is a measure of happiness for the investor. A utility function can describe the relation between happiness and future wealth. The investor can systematically rank different investment opportunities corresponding to different wealth levels, thus a way to apply the principle of risk aversion. Risk aversion means that the investor prefers less risk against more wealth. For a utility function to illustrate risk aversion it is required to be concave.

There are some measures that describe how a utility function can explain the investor's behavior in the financial market. These are:

- Risk aversion – explains how the investor behaves in the equity market.
- Prudence – explains the investor's behavior when he makes precautionary savings.
- Cautiousness – explains the investor's tendency to trade derivatives, according to [15].

The authors in [28] and [2] developed the measure of risk aversion. The higher risk aversion an investor has the larger risk premium he demands for taking risk. This is for absolute risk aversion

$$ARA(W) = -\frac{u''(W)}{u'(W)}$$

and for relative risk aversion

$$RRA(W) = -W\frac{u''(W)}{u'(W)}.$$

The author in [18] developed the measure of prudence. The higher prudence an investor has the more precautionary savings he makes in relation to risk in his wealth. This is

$$P(W) = -\frac{u'''(W)}{u''(W)}.$$

The measure of cautiousness was introduced by [35]. This is the first derivative of risk tolerance, where risk tolerance is the inverse of absolute risk aversion. This is

$$P(W) = -\left(\frac{1}{ARA(W)}\right)'.$$

It is now widely accepted that investors should have decreasing ARA behaved utility function, [15]. It has also been shown in [14] that increasing cautiousness may be more likely if increasing cautiousness implies decreasing relative risk aversion given that marginal utility of zero wealth is infinity.

6.1.1 Utility and derivatives

How to motivate the existence of derivatives in the economy is not obvious from a theoretical point of view. The optimal allocation in derivatives in this sense has only been addressed in a few papers. This is partly because of the complexity of the problem and partly because of the major success of arbitrage based models for pricing derivatives. The optimal position in derivatives is usually either indeterminate or infinite based on these models. This is because if the investor agrees with the price of the derivative then he could either buy the derivative or dynamically trade the underlying asset to obtain the same pay off. If the investor disagrees with the price then he buys or sells infinite of the contract.

The authors in [8] set up a framework with a single period economy with the existence of three asset classes, a risk less asset, a risky asset and European options of all strikes. The inability to trade continuously in this framework makes it possible to invest in all investment classes. The objective is to maximize the expected utility of terminal wealth. They showed that under reasonable market conditions derivatives comprise an important, interesting and separate asset class imperfectly correlated with other asset classes.

6.2 The representative investor

The representative investor in this thesis will have a utility function on the form

$$U(W) = \begin{cases} \frac{1}{1-\lambda}(\tau\lambda + W)^{1-\lambda} & W \geq -\lambda\tau, \\ -\infty & W < -\lambda\tau, \end{cases}$$

for some $0 < \lambda < 1$. When τ is zero than we the have power utility, see (3.10). If τ is less than zero will the investor invest as to create a floor on terminal wealth, the floor becomes $-\lambda\tau$.

Figure 6.1 illustrates the shape for different floors with λ equal to 0.6. When creating a floor on terminal wealth the investor will first choose between the bond and the structured product for securing the floor. Then will the investor also consider the asset with downside risk, the equity index.

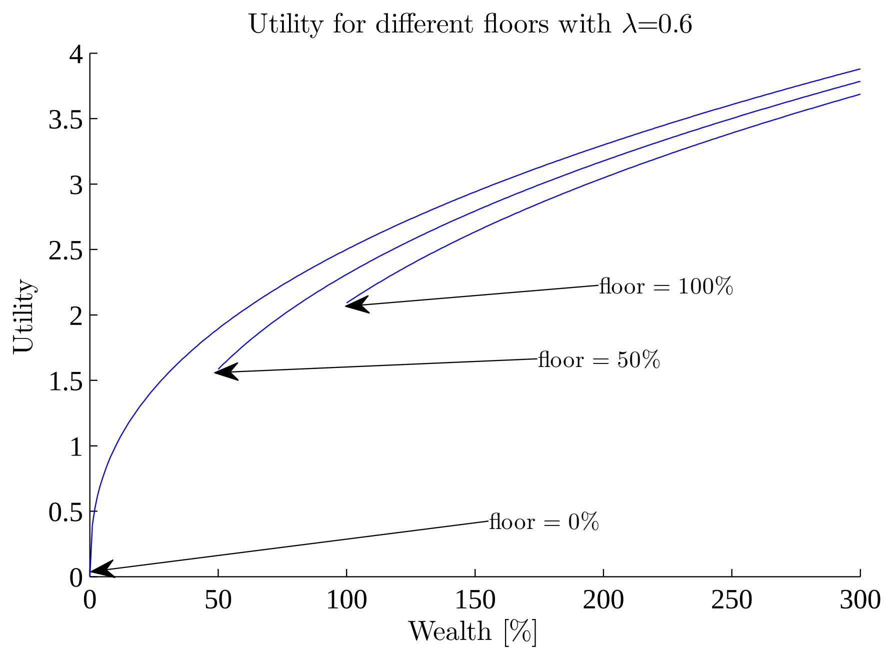

Figure 6.1: The figure illustrates different shapes of the utility function for different floors.

To be able to determine what kind of investor who would prefer to buy the structured product two portfolios are created. This with starting point that the investor wants to secure all amount invested. The first portfolio is the package of the bond and option with 100 % capital guarantee, which is the structured product. The second portfolio is created with the same idea about 100 % capital guarantee but to a traditional portfolio consisting of the bond and equity index. This creates a position where 100 % of the capital is secured. However, large losses in the risky part have very small probabilities. This also makes that large profits are very limited. Profit and loss diagram is showed in Figure 6.2.

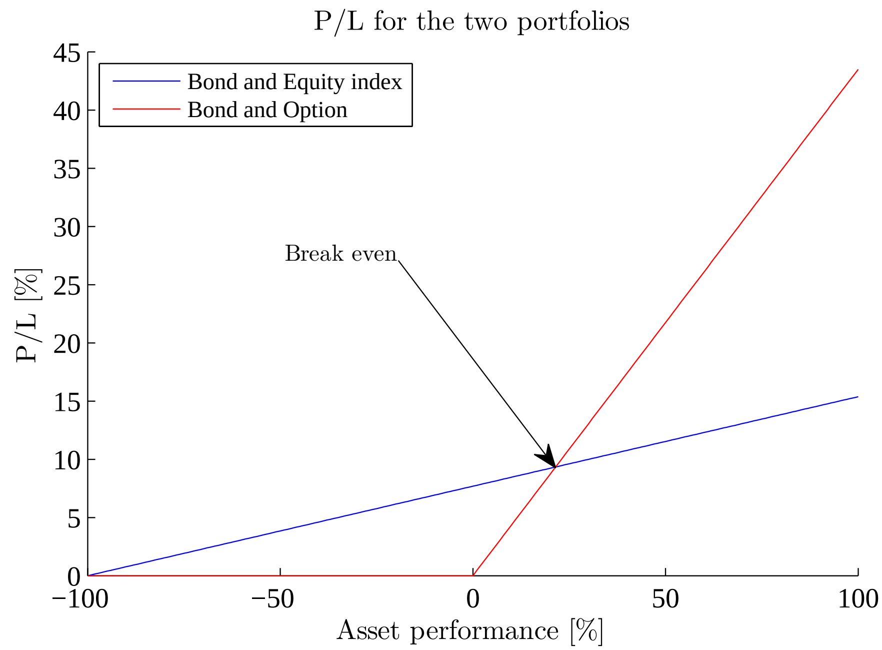

Figure 6.2: Profit and loss diagram for the two portfolios.

In this example the bond and the option mature in two years. The present value of the bond and the option price[1] is approximately 92 % respectively 18 %. These figures correspond to an interest rate of 4 % and volatility at 25%. This means that the traditional portfolio would consist of 92 % bond and 8 % equity index. A structured product, the package of a bond and an option, would have a participation rate of 43 %. Break even is given by

$$\frac{1-B}{k-(1-B)},$$

where B is the price of a zero coupon bond and k the participation rate. Break even is 21 % according to the numbers presented in the example. Figure 6.3 illustrates break even for different values for volatility and interest rate.

It is obvious from the relation shown in Figure 6.2 that an investor who wants to preserve capital and does not value high returns wants to invest in a portfolio consisting of a bond, which grows to the floor, and the equity index that can give a higher pay off. An investor who is satisfied with just preserving all of his initial wealth but wants higher return chooses the combination of a bond and an option.

[1] Valued with Black and Scholes formula.

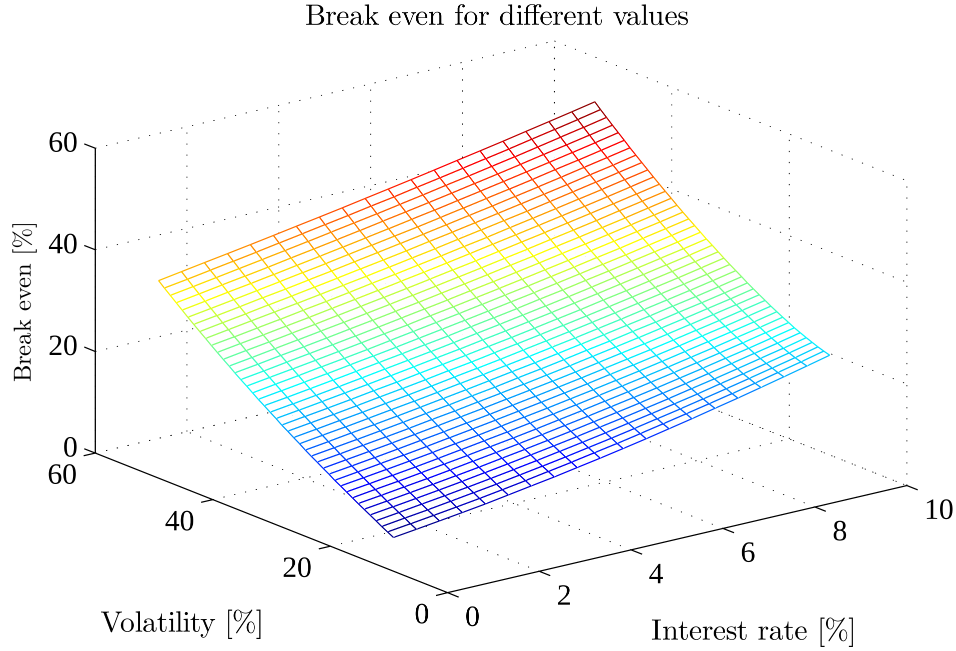

Figure 6.3: The surface diagram shows break even for different volatility and interest rate.

These types of structured products have grown massively during the last years so there is real world evidence that investors like or prefer these kind of products. It could be the combination of regret and greed which these products capture very well. The investor does not loose any money but he can still earn a lot compared to the same downside exposure in traditional investments, bond and equity index. We also have the aspect of an investor that have a short period of time until retirement where he cannot afford to loose any money but he still wants to participate in the performance of the equity market.

So what investor would prefer this pay off function? The following assumptions can be made:

- Preserve capital – the investor does not want to loose any money.
- High returns – the investor believes that the equity market will perform well (better than the market thinks) or fear bad (worse than the market thinks). This might be translated into a high future volatility view.
- Not low returns – the investor does not value small returns or think they have small probability to occur. (Practically the same preference as previous).

The utility function is calibrated to the investor's preferences who would prefer the structured product instead of the the combination of the bond and equity index. Because of both portfolios are 100 % capital guaranteed so makes it no sense to calibrate the floor, only λ.

This relation is of course dependent on the probability of each event so some calibration distribution must be selected. A lognormal RND is selected, with an interest rate of 4 %, volatility at 20 % and a time period of two years. The transformation to a real world density is made with (3.8), where the transformation parameter is calibrated for different excess returns.

It turns out that the level of λ so the investor would prefer the structured product is highly dependent of the expected excess return. The higher expected excess return the more the investor wants the structured product. The difference in expected utility as a function of λ is illustrated in Figure 6.4 for different excess returns. The values on the y-axis represent the difference in expected utility between the two portfolios. When the difference is positive then the investor would prefer the structured product to the other portfolio.

For expected excess returns of slightly higher than 4% and smaller the investor would prefer the bond and equity index portfolio for all λ . For expected excess returns from around 5% and up the investor prefers the structured product for all λ. Given historical risk premiums a reasonable range would be between 4% and 8%, see next section.

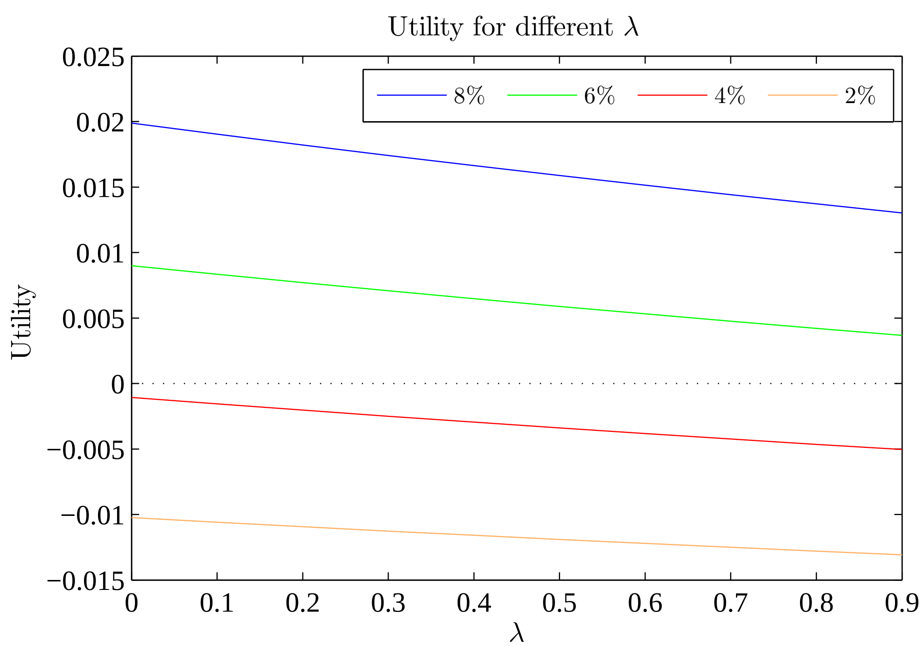

Figure 6.4: The figure illustrates the difference in expected utility for the two portfolios for different parameters.

λ is chosen to be 0.6 as a good reference. This procedure works just as a guideline for having some understanding of what parameter of λ that is reasonable to use. Because it is possible to create a third portfolio consisting of the bond and the structured product which would also create a 100 % capital guaranteed portfolio but with infinite number of weights between the two assets.

Further on is the floor set to 70 % which makes it possible to invest in the equity index as well. If the floor would be set to 100 % then the equity index has an upper bound equal one minus the price of a zero coupon bond.

6.3 Risk Premium

Risk premium is the excess return earned by a risky asset over a risk-free asset. During the post World War II period the U.S. stock market had a mean real return[2] of 8.4 % while a relatively risk less T-bill earned 0.6 %. This means an equity risk premium of 7.8 %. Other developed countries show the same pattern of high risk premiums. Table 6.1 shows numbers for different countries.

Table 6.1: Historical returns for markets worldwide, data are from [25].

Country	**Period**	**Equity Index**	**T-bill**	**Risk Premium**
United Kingdom	1947-99	5.7%	1.1%	4.6%
Japan	1970-99	4.7%	1.4%	3.3%
Germany	1978-97	9.8%	3.2%	6.6%
France	1973-98	9.0%	2.7%	6.3%
U.S.A.	1947-00	8.4%	0.6%	7.8%

The equity risk premium puzzle originates from [26] work. They concluded that the historical U.S. equity premium is an order of magnitude greater than can be explained in the context of the standard neoclassical economics. The relation to this in this thesis is the transformation from the RND to a real world density with the link to the investor's utility. To obtain the high risk premium observed historically would equal implausible high risk aversion.

The risk premium has varied a lot during time, in some periods negative and in some very large. The authors in [26] addressed the ex post (realized) risk premium but what is more interesting from an investor's point of view is the ex ante (expected) risk premium given the current state of the economy. In relation, after a bull market the ex post risk premium is high but the ex ante equity premium is likely to be low. This should be clear when returns

[2]Real return is inflation adjusted return.

to stock have been documented to be mean reverting, [25]. In this thesis no consideration will be taken to the current state of the economy but accept the ex post risk premium to be a good reference for future risk premium. The purpose is not to show relations of the current state of the economy in terms of expected return but to build a generalized model based on prior stated criteria.

In this thesis a transformation with rather high risk aversion will be accepted to be able to obtain a reasonable high risk premium. As [25] states, over the long term, the equity premium is likely to be similar to what it has been in the past and returns to investment in equity will continue to substantially dominate returns in T-bills for investors with a long planning horizon.

Chapter 7

Portfolio optimization

Modern portfolio theory was originated by Markowitz work who presented the mean-variance approach to asset allocation in 1952, see [23]. This is still a cornerstone in teaching and practice of classical financial theory. The mean-variance approach rests on firm theoretical grounds if either investors exhibit quadratic utility[1] or returns are multivariate normal. If investors exhibit quadratic utility then they ignore non-normality in the data. However, quadratic utility is not always plausible because utility is not strictly increasing in wealth from some scenarios. It means that investors will not always prefer more wealth to less. In practice there are many known shortfalls with this approach discussed in [31].

Variance as a measure of risk also has a big disadvantage in the sense that it treats both good and bad outcomes in the same way. Even Markowitz realized this and mentioned in the footnotes that other measures can be used like semi-variance. There have been many attempts to replace variance as a measure of risk such as Value-at-Risk (VaR), minimum regret and conditional Value-at-Risk (CVaR). All of these risk measures have in common that focus is on the left tail of the distribution, which means extreme losses.

When dealing with investment products that have non-linear payoffs this must be considered in terms of how risk is measured. Risk measures such as CVaR and VaR is suitable for this. One shortfall by just focusing on extreme events is that the whole distribution is not considered. When using a utility function every possible outcome is considered, not only the really bad ones. This is the reason in this thesis to consider an utility function to represent the investor's preferences.

Portfolio optimization involving structured products has been treated for example in [19] and [24]. Both of their focus is to create a simulation model where CVaR is minimized in the portfolio optimization.

[1]Quadratic utility is on the form $u(x) = x - \frac{b}{2}x^2$.

7.1 Optimization algorithm/problem

Consider an investment universe where three assets are available: a risk less asset, a risky asset and a structured product on the risky asset. Assume a single period investment, where investments are made at time 0 with all payoffs received at time 1. The objective is to maximize terminal expected utility and the problem becomes

$$\begin{aligned} \max_{z_1,z_2,z_3} \quad & \int_0^\infty U\left[W(s)\right] f_S(s)\mathrm{d}s \\ s.t. \quad & B_0 \int_0^\infty W(s)\overline{f}_S(s)\mathrm{d}s &= W_0 \\ & z_i &\geq 0 \\ & \textstyle\sum_i z_i &= 1 \end{aligned}$$

where z_1, z_2 and z_3 are the weights for each asset, $W(s)$ is the pay off function, W_0 initial wealth, $\overline{f}_S(s)$ the RND and $f_S(s)$ the real world density. The first constraint ensures that all assets are priced according market conditions so that real world probabilities are consistent with risk neutral probabilities. It also ensures not to overspend on assets. The other two constraints make that all initial wealth is spended and that short selling is restricted.

7.2 Scenario definition

The four scenarios are defined in this part. This is two volatility regimes and two states of interest rates.

The volatility regimes are defined with respect to historical implied volatilities derived from the VSTOXX Volatility Index[2]. This is an index which reflects the Euro Stoxx 50 option market implied volatility. Historical volatilities from VSTOXX are shown with the short term index, which has more history, and the 24 month index, Figure 7.1. The defined periods are a low volatility period in December of 2006 and a high volatility period during the credit crisis in December of 2008. These volatility regimes define points in time where the implied RND reflects the market sentiment from one normal and one distressed market.

In opposite from the volatility the interest rate is assumed with respect to historical interest rates in the Swedish fixed income market. Figure 7.2 shows historical 2 year Swedish swap rates and the level of the interest rates that are selected for the scenarios.

[2]For further information see www.stoxx.com

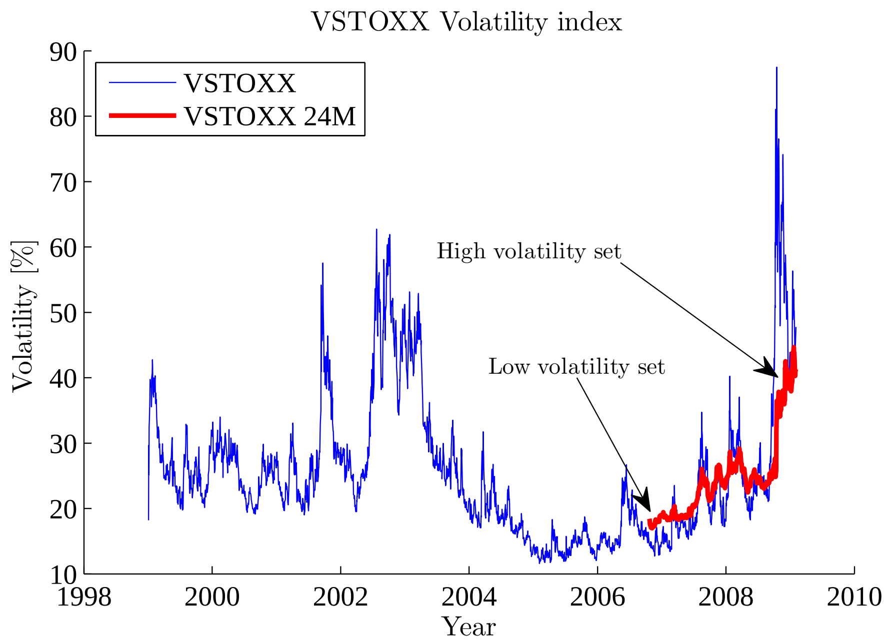

Figure 7.1: The figure illustrates historical implied volatilities.

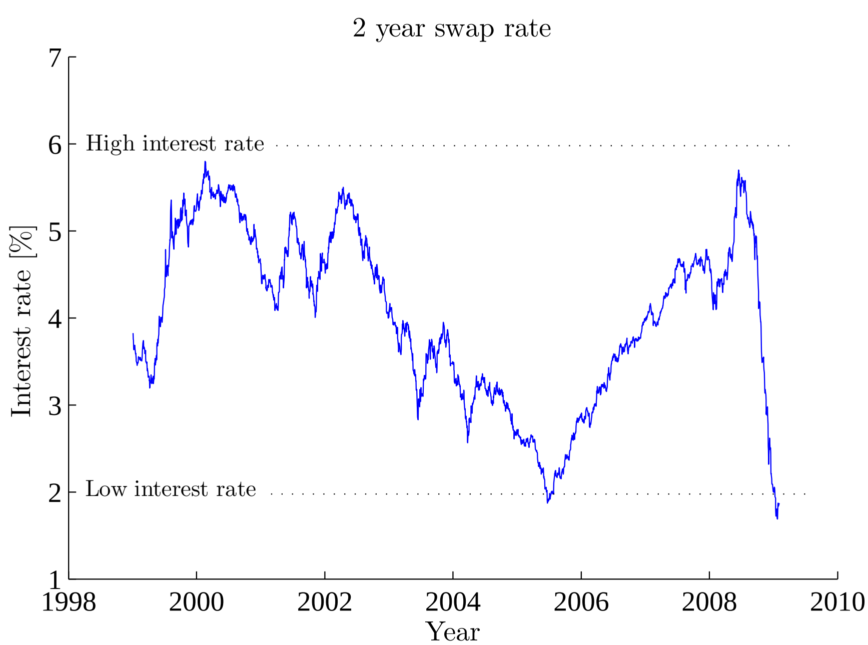

Figure 7.2: The figure illustrates historical swap rates.

7.3 Case study

Four scenarios have been created, summarized in Figure 7.3. These include two volatility regimes and two states of interest-rates. The corresponding interest rate, ATM implied volatility and the participation rate are shown. It is important to point out that the ATM implied volatility is risk neutral and also does not say anything about the shape of the distribution but only to outline the relation. The representative investor is defined to have the

	Low interest rate 2%	High interest rate 6%
Low volatility 17%	**Normal market** Participation rate 35%	**Normal market** Participation rate 72%
High volatility 37%	**Distressed market** Participation rate 17%	**Distressed market** Participation rate 44%

Figure 7.3: The figure summarize the four different scenarios.

utility function as discussed in Chapter 6, illustrated in Figure 7.4. This is with λ equal to 0.6 and the floor set to 70 %.

The optimal allocation for the scenarios is shown in Figure 7.5. The implied expected excess return on the x-axis is with respect to the equity index, not the optimal portfolio. The implied expected excess return is calculated with a range of λ between zero and one, (this λ used in the transformation must not be confused with the λ for the representative investor). When the implied expected excess return is calculated with the same variables means that the range on the x-axis is assumed to be equivalent across the different scenarios.

The optimal allocations show the same pattern. For lower excess return no allocation is made to the structured product but after a threshold the allocation goes from zero weight quick up to 70 %, where the floor is. This level is reached for the highest transformation value in the low volatility and high interest rate scenario.

One way to see if there is any difference is to determine the points where the allocation in the structured product first differ from zero. The threshold when allocation is made to the structured product is dependent on the relation discussed in Chapter 6. This means for lower excess returns the portfolio consisting of only bond and equity index is preferred against the

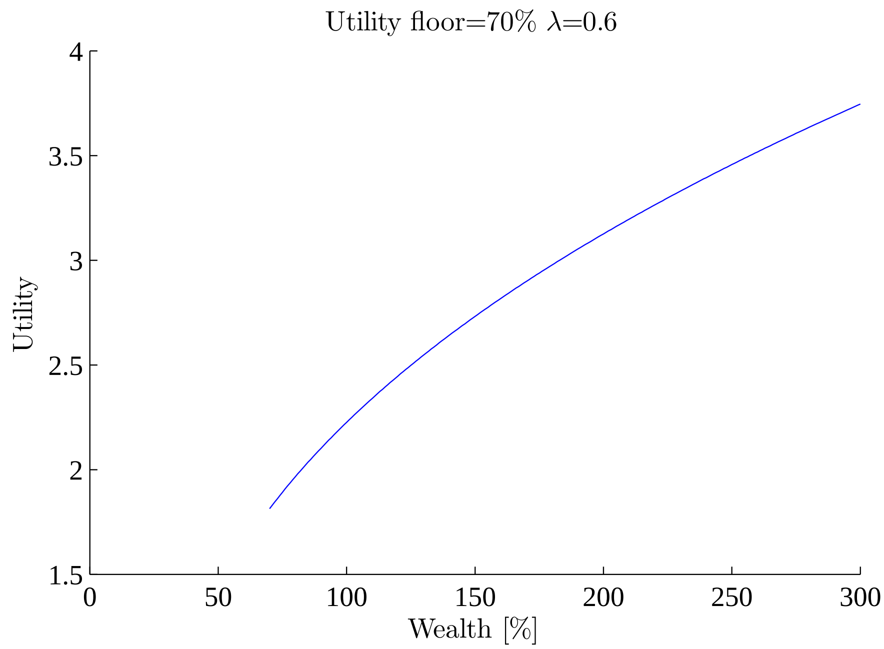

Figure 7.4: The figure illustrates the shape of the representative investor's utility function used in the scenarios.

structured product. For higher expected excess returns the structured product is prefered. From the figure there is no significant difference at all. The range where the allocation in bonds are flat for a short range and then lowers are almost identical across the scenarios. The difference in allocation is independent on the market climates but only if one would assume different expected excess return in different market climates. That is, the parameter which transforms the RND to a real world density is the same.

Another way to to see if there is any difference is to determine the range where the allocation in the structured product first equal the floor. The difference is most significant between the two interest rate states. In the high interest state scenarios the increase from zero allocation is more flat. The low interest rate scenarios show a more steep rise. This means that the investor demands a higher expected excess return for investing a large proportion in structured products in a high interest rate market.

The optimal weights are highly dependent of the choice of parameters, floor and λ. However, the weights show the same pattern and have the impact one can expect. For example, if one lower the floor then the weight in the structured product lowers and vice versa. The weight in the structured product is decreasing in λ. That is if λ decreases the weight in the structured product increases.

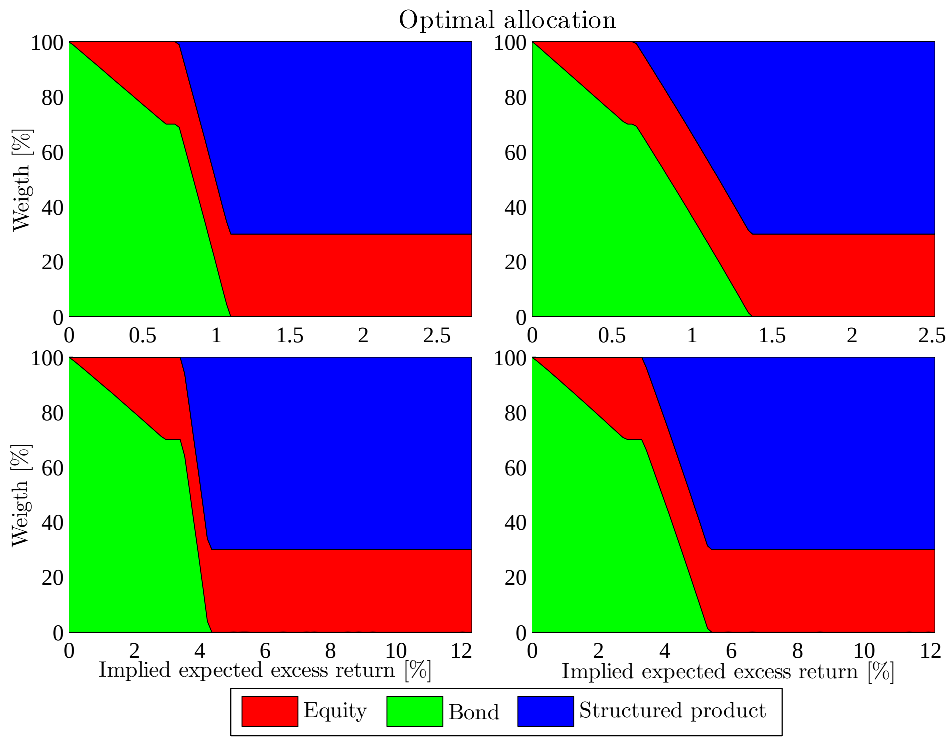

Figure 7.5: The figure illustrates the optimal allocation across the scenarios.

7.4 Conclusion

Modeling the risky asset by the implied risk neutral density from the option market permits to capture the market's sentiment. However, the transformation from the risk neutral world to the real world can be hard to interpret. It is possible to get a deeper understanding when translating this to an expected return instead of referring to different coefficients of risk aversion.

Several probability distributions were fitted to the two different implied volatility sets. The fit with the Normal inverse Gaussian distribution was selected to represent the low volatility set and the skew Student t distribution the high volatility set. These distributions have good properties and showed very good fits to the option prices but only to respective volatility set.

In the portfolio optimizaton a representative investor was considered who would prefer the structured product. It is obvious that the optimal allocation becomes biased towards the structured product but the aim is to show the difference with respect to the market climates not to motivate the existence of structured products in an effective portfolio.

The difference in optimal allocation between the four scenarios is shown to be negligible. That is, if one compare the points where allocation in the structured product first differ from zero. If one compare the points where the allocation in the structured product first equal the floor is the result somewhat different. Between the two interest rate states the difference is most significant. In the high interest state the increase from zero allocation is more flat. The low interest rate scenarios show a more steep rise. This means that the investor demands a higher expected excess return for investing a large proportion in structured products in a high interest rate market.

It was difficult to determine how the scenarios should be compared. The question about equivalence between expected excess return or the same calibration parameter arose. The latter was preferred because it seems to be a more reasonable assumption about different expected excess returns in different market climates than the same.

The result is thus based on the assumption that different market climates have different expected excess return but over the same transformation parameter, coefficient of risk aversion. Thus, the aggregate market utility risk aversion is the same independent of market climate.

The argument for high volatility implies higher probability for higher returns must hold. This is perhaps not true when equity prices tend to fall when volatility is high. Volatility is usually not regarded as a good thing in the financial world. With this assumption it does not matter for the investor what the volatility is. The investor can just rebalance his portfolio to match his risk preferences when the expected return increases with volatility.

The impact of using the risk neutral density implied by the option market in the portfolio optimization was not reviewed in detail. These distributions were accepted and assumed to be the best predictions of future movements. One alternative would be to use historical densities but this cannot reflect the market's expectation in the same way.

The level on allocation in structured products is shown to not be significantly dependent on the market climate. The participation rate is then not of crucial importance for investment decisions. It is shown that the most important factor is the expected excess return. If the expected excess return is low, relative to the transformation parameter, then no allocation is made to the structured product. If it is high then the structured product is preferred. However, this pattern is clear across the scenarios which imply independence between optimal allocation in structured products and the market climate.

In conclusion, the optimal allocation is not significantly dependent on the market climate, thus the participation rate. If an investor has invested in a structured product when the participation rate was high then there is no reason for him not to do it when the participation rate is low. The large excess return we have seen historically should justify and argue for investments in structured products regardless of what market climate that prevails.

Bibliography

[1] S. D. Aparicio and S. Hodges. Implied risk-neutral distribution: A comparison of estimation methods. *FORC Preprint*, 1998.

[2] K. J. Arrow. The role of securities in the optimal allocation of risk-bearing. *Review of Economic Studies*, 31(2):91–96, 1964.

[3] Y. Aït-Sahalia and A. Lo. Nonparametric estimation of state-price densities implicit in financial asset prices. *Journal of Finance*, 53(2):499–547, 1998.

[4] B. Bahra. Implied risk-neutral probability density functions from option prices: theory and application. *Bank of England*, 1997.

[5] D. Bates. The crash of 87: Was it expected? the evidence from option markets. *Journal of Finance*, 46(3):1009–1044, 1991.

[6] D. T. Breeden and R. H. Litzenberger. Prices of state-contingent claims implicit in option prices. *Journal of Business*, 51(4):621–51, 1978.

[7] B. Brunner and R Hafner. Arbitrage-free estimation of the risk-neutral density from the implied volatility smile. *Journal of Computational Finance*, 7(1):75–106, 2003.

[8] P. Carr and F. Madan. Optimal positioning in derivative securities. *Quantitative Finance*, 1(1):19–37, 2000.

[9] R. Cont and P. Tankov. Calibration of jump-diffusion option pricing models: a robust non-parametric approach. *Journal of Computational Finance*, 7(3):1–49, 2004.

[10] C. De Jong and R. Huisman. From skews to a skewed-t: Modelling option-implied returns by a skewed student-t. *ERIM Report Series*, 2000.

[11] G. Debreu. *Theory of Value*. Wiley, 1959.

[12] G. Gemmil and A. Saflekos. How useful are implied distributions? evidence from stock-index options. *Journal of Derivatives*, 7:83–98, 2000.

[13] J. M. Harrison and S. R Pliska. Martingales and stochastic integrals in the theory of continuous trading,. *Stochastic Processes Application*, 11:215–260, 1981.

[14] J. Huang. Relationship between risk aversion, prudence and cautiousness. *Working Paper*, 2000.

[15] J. Huang. Cautiousness and tendency to buy options. *Working Paper*, 2004.

[16] C. J. Hull. *Options Futures and Other derivatives.* Pearson/Prentice Hall, 6th edition, 2006.

[17] J. Jackwerth and M. Rubinstein. Recovering probabilities and risk aversion from option prices and realized returns. *Review of Financial Studies*, 13(2):433–451, 2003.

[18] M. Kimball. Precautionary saving in the small and in the large. *Econometrica*, 58:53–73, 1990.

[19] M. Larsson. Portfolio optimization with structured products using extreme value theory. Master's thesis, The Royal Institute of Technology, 2008.

[20] X. Liu, M. Shackleton, S. Taylor, and X. Xu. Closed form transformation from risk-neutral to real-world distributions. University of Essex, revised 2005.

[21] F. Longin. The asymptotic distribution of extreme stock market returns. *Journal of Business*, 69(3):383–408, 1996.

[22] S. Markose and A. Alentorn. The generalized extreme value (gev) distribution implied tail index and option pricing. Master's thesis, University of Essex, 2005.

[23] H. Markowitz. Portfolio selection. *Journal of Finance*, 7(1):77–91, 1952.

[24] L. Martellini, K. Simsek, and F. Goltz. Structured forms of investment strategies in institutional investors portfolios. Technical report, EDHEC Risk and Asset Management Research Centre, 2005.

[25] R. Mehra. The equity premium: Why is it a puzzle? *Financial Analysts Journal*, 59(1):54–69, 2003.

[26] R. Mehra and E. Prescott. The equity premium: A puzzle. *Journal of Monetary Economics*, 15:145–161, 1985.

[27] I. Nagot and R. Trommsdorff. The tree of knowledge. *RISK*, 12(8):99–102, 1999.

[28] J. W. Pratt. Risk aversion in the small and in the large. *Econometrica*, 32:122–136, 1964.

[29] M. Rubinstein. Implied binomial trees. *Journal of Finance*, 49(3):771–818, 1994.

[30] R. Savickas. Evidence on delta hedging and implied volatilities for the black-scholes, gamma and weibull option-pricing models. *Journal of Financial Research forthcoming*, 2004.

[31] B. Scherer. *Portfolio Construction and Risk Budgeting*. RiskBooks, 3rd edition, 2007.

[32] D. Shimko. Bounds of probability. *Risk*, 6:33–37, 1993.

[33] R. Stein and Y Hecht. Distribution of the exchange rate implicit in option prices: Application to tase. *Working paper, Bank of Isreal*, 2003.

[34] R. Tunaru and G. Albota. Estimating risk neutral density with a generalized gamma distribution. Master's thesis, Cass Business School, 2005.

[35] R. Wilson. The theory of syndicates. *Econometrica*, 36:119–132, 1968.

Appendices

Appendix A

The related result by Breeden and Litzenberger

In [6] the authors show that if the underlying price at time T has a continuous probability distribution, then the state price at state is determined by the second derivative of the call pricing function for the underlying asset with respect to the exercise price, X. This means

$$q(S) = e^{r\tau} \frac{[C(S + \Delta S, \tau) - C(S, \tau)] - [C(S, \tau) - C(S - \Delta S, \tau)]}{(\Delta S)^2}$$

where r is risk-free rate, τ is time to maturity, ΔS is the 'spread' and $C(S, \tau)$ is the price of a European call option with strike S.

The state price was introduced by [2] and [11] and it is the price of an elementary claim, also known as 'Arrow-Debreu' security. An elementary claim is an asset that pays one unit at a future time if the underlying asset takes a certain value or state. The state price is directly proportional to the risk-neutral probabilities of each possible state.

The elementary claim can be replicated using a butterfly spread. The price of a butterfly spread, centered on state $S = X$ is

$$P(S, \tau|\Delta S) = \frac{[C(S + \Delta S, \tau) - C(S, \tau)] - [C(S, \tau) - C(S - \Delta S, \tau)]}{\Delta S}$$

and when

$$\lim_{\Delta S \to 0} \frac{P(S, \tau|\Delta S)}{\Delta S} = \left. \frac{\partial C^2 (X, \tau)}{\partial X^2} \right|_{X=S}$$

follows the risk neutral probability for every state

$$\lim_{\Delta S \to 0} q(S) = e^{-r\tau} \left. \frac{[C(S + \Delta S, \tau) - C(S, \tau)] - [C(S, \tau) - C(S - \Delta S, \tau)]}{(\Delta S)^2} \right|_{X=S}$$

where ΔS is the 'spread'.

Appendix B

Fitting result

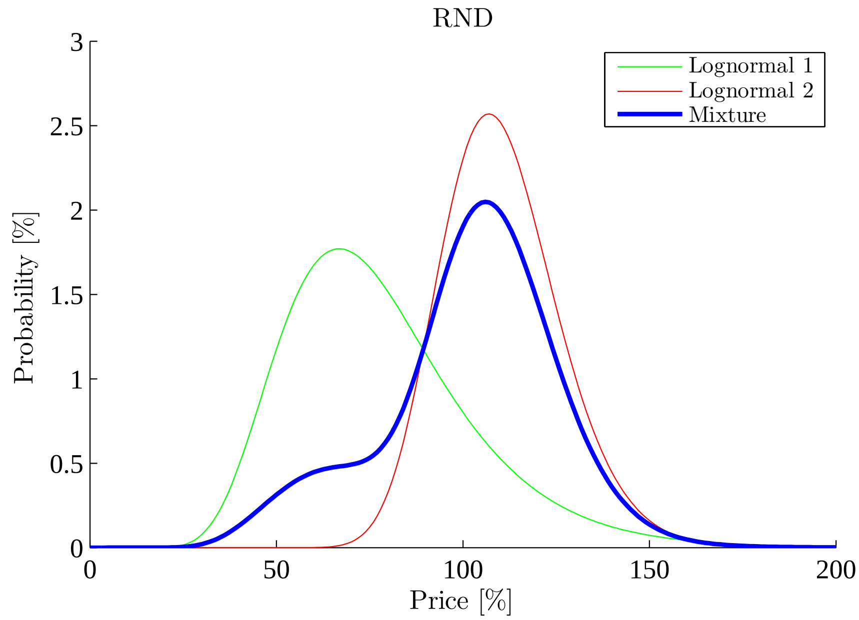

Figure B.1: The figure shows the RND best fit with the two lognormal mixture.

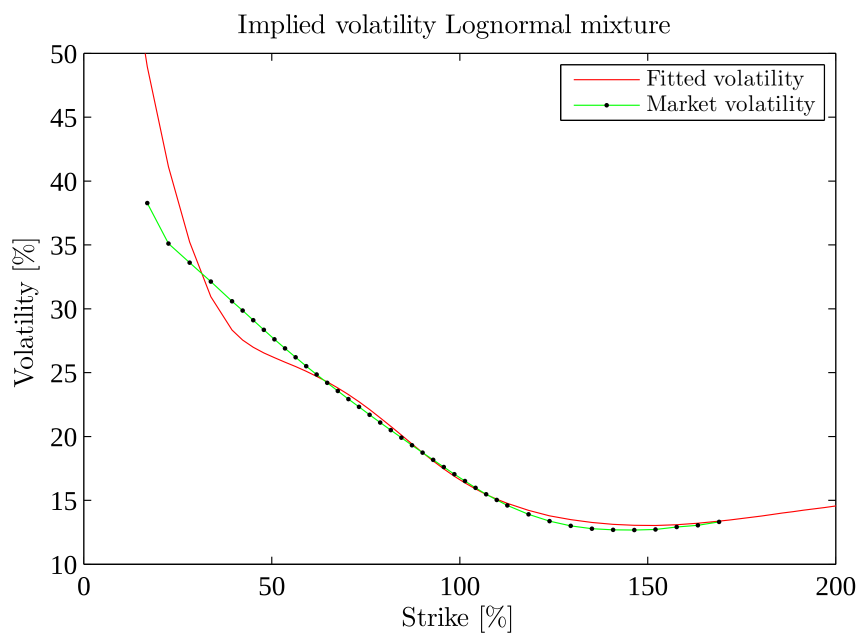

Figure B.2: The figure illustrates comparison between best fit implied volatility.

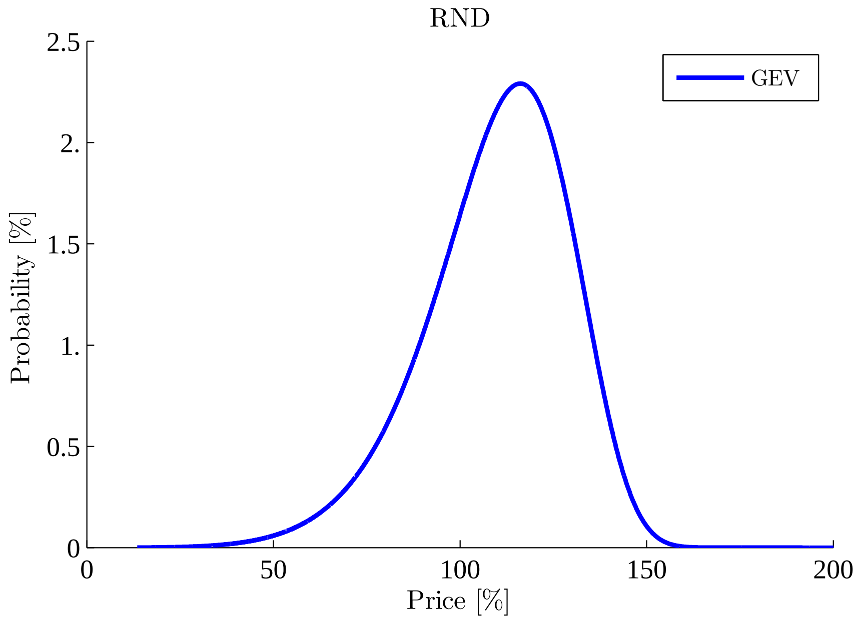

Figure B.3: The figure shows the RND best fit with the GEV.

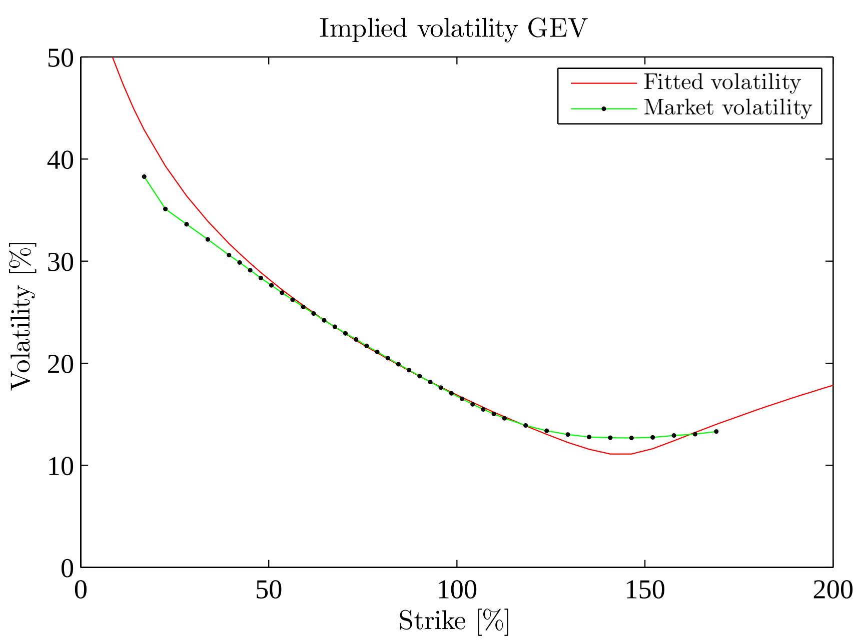

Figure B.4: The figure illustrates comparison between best fit implied volatility.

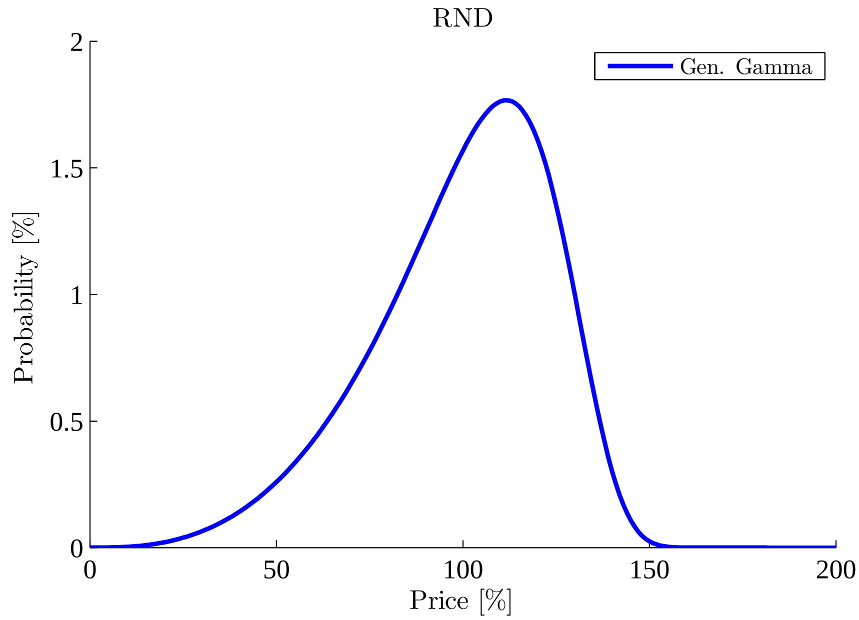

Figure B.5: The figure shows the RND best fit with the generalized gamma.

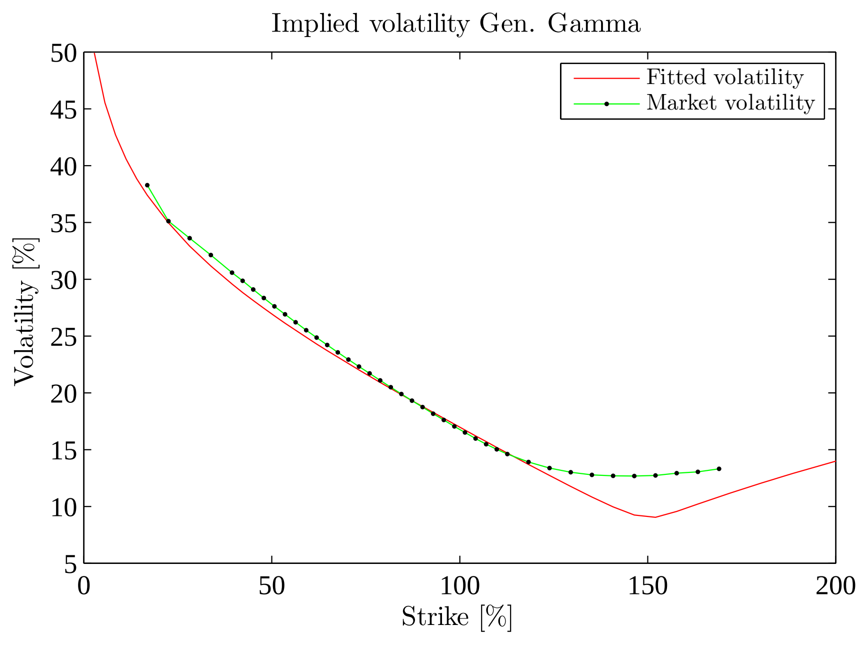

Figure B.6: The figure illustrates comparison between best fit implied volatility.

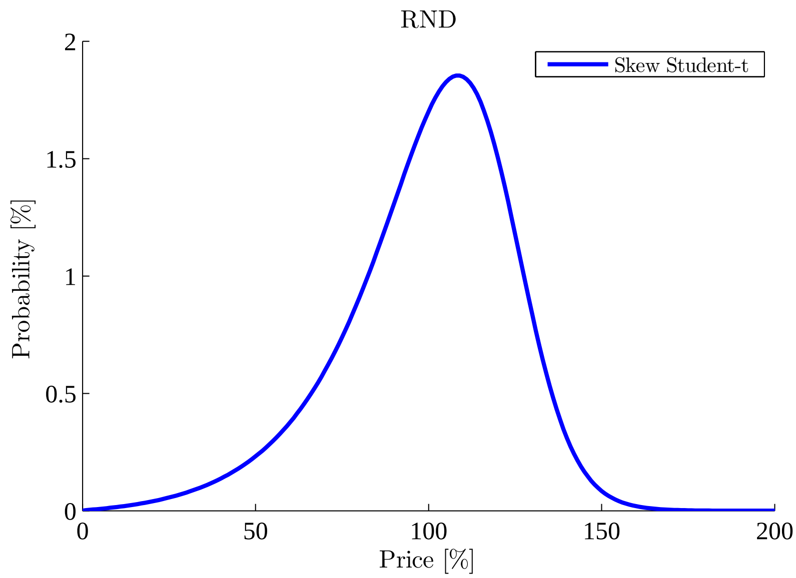

Figure B.7: The figure shows the RND best fit with the skew Student-t.

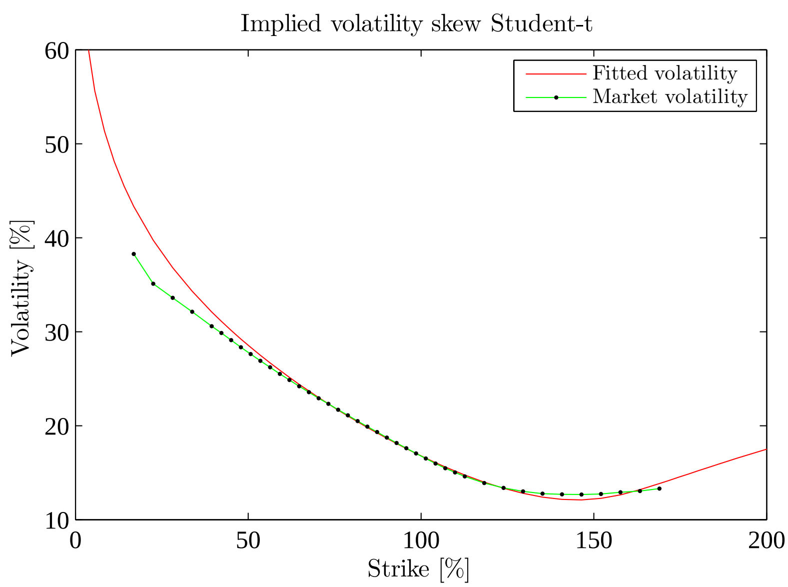

Figure B.8: The figure illustrates comparison between best fit implied volatility.

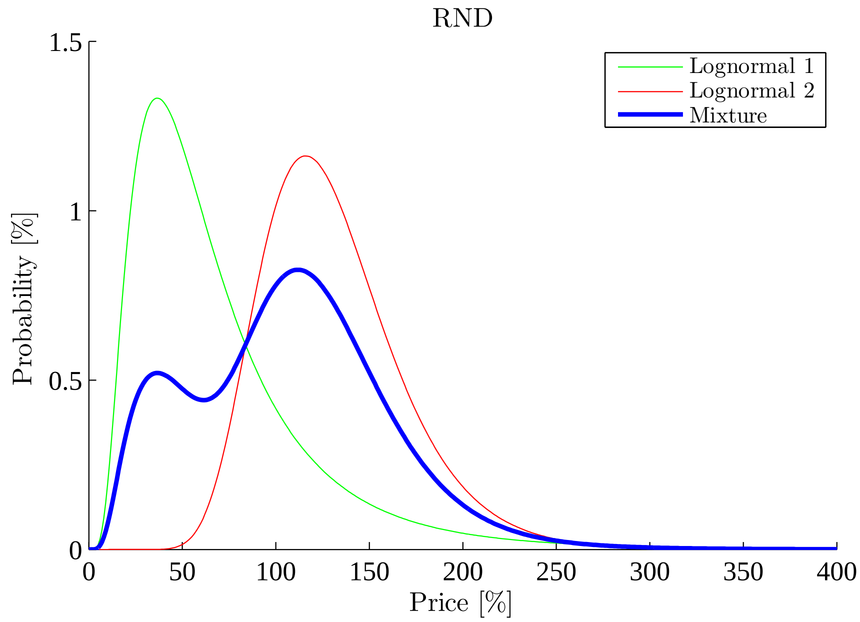

Figure B.9: The figure shows the RND best fit with the two lognormal mixture.

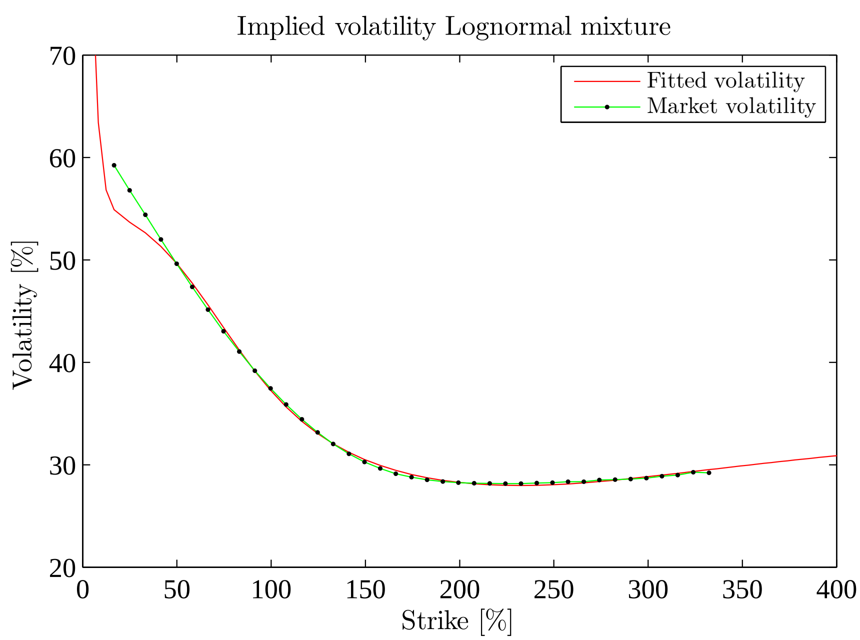

Figure B.10: The figure illustrates comparison between best fit implied volatility.

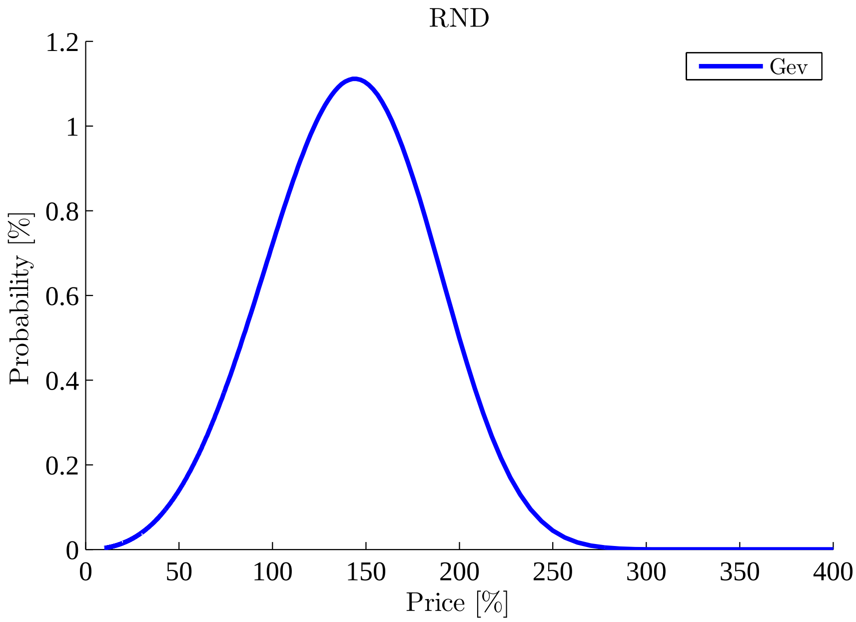

Figure B.11: The figure shows the RND best fit with the GEV.

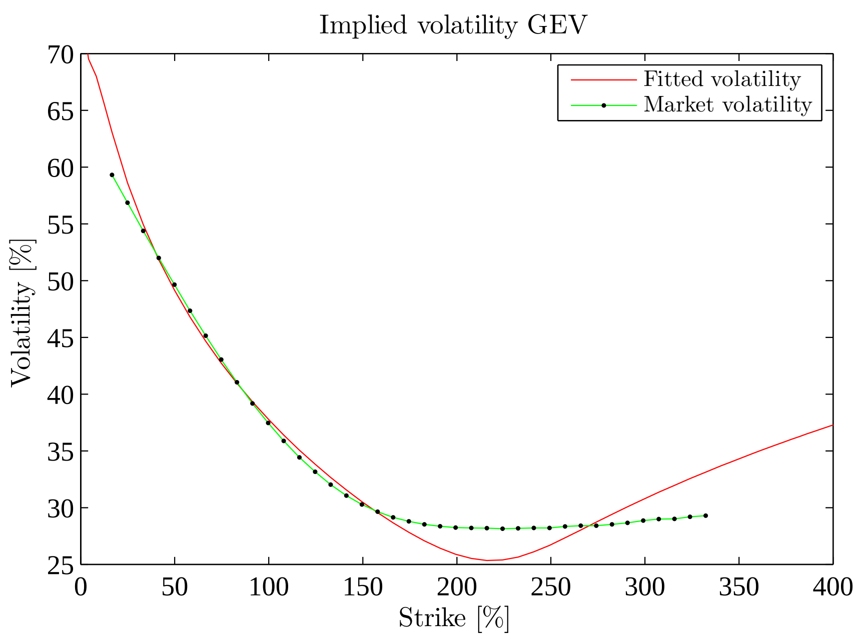

Figure B.12: The figure illustrates comparison between best fit implied volatility.

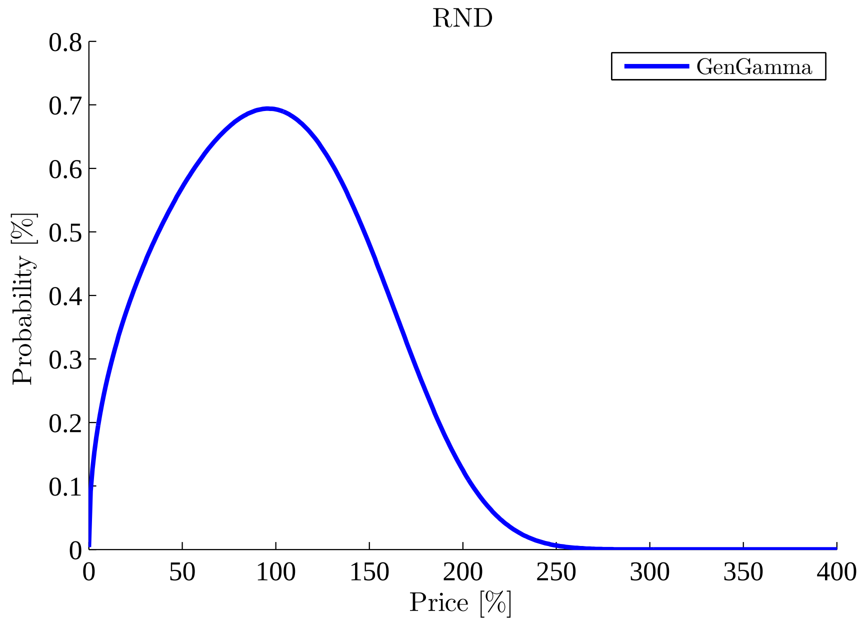

Figure B.13: The figure shows the RND best fit with the generalized gamma.

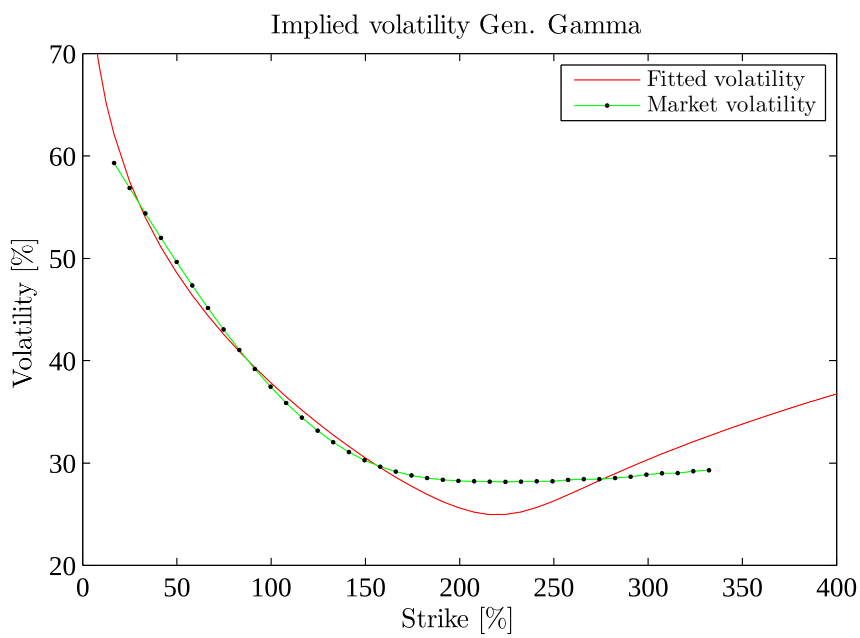

Figure B.14: The figure illustrates comparison between best fit implied volatility.

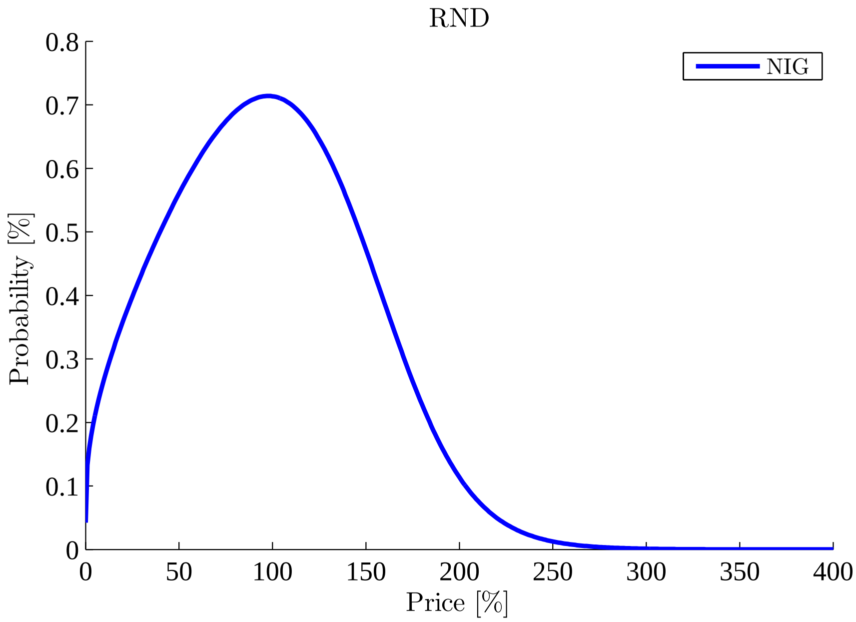

Figure B.15: The figure shows the RND best fit with the NIG.

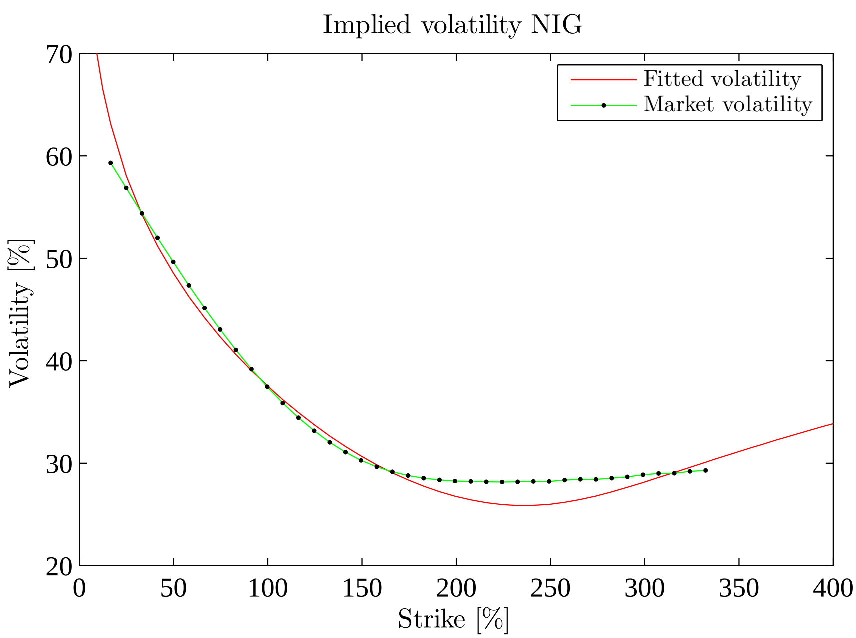

Figure B.16: The figure illustrates comparison between best fit implied volatility.