Valuation of Life Insurance Contracts with Simulated Guaranteed Interest Rate

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Abstract

We use Black and Scholes options theory to obtain market valuation of typical life insurance contracts. The contracts are specified as in Grosen & Lochte Jorgensen (2001), but with a slight extension where the guaranteed rate is simulated from market models of short interest rate such as Vasicek, Cox-Ingesoll-Ross and Ho-Lee models. Another extension is that we assume the guaranteed rate is equal to the risk free interest rate. Except the above extensions, we keep other factors of model similar as Grosen & Lochte Jorgensen (2001). First, the liability holders have prior claim for company asset than equityholders. Second, a regulatory mechanism is added into the model in order to reduce insolvency risk. Finally, we derive valuation formulas and give numerical examples for initial fair contracts and market values of contracts at different time points.

Contents

1 Introduction

The fair valuation of life insurance liabilities has been discussed in the past decades. One reason is how to give a suitable structure of liabilities contracts, which is also the main purpose of this paper.

For example, most insurance companies will promise to give policy holders an explicit interest rate added to their account, named as guaranteed interest rate. An unsuitable guaranteed rate will certainly bring insolvency risk to insurance company. The purpose of this paper is to discuss the effects of the variability of the guaranteed rate on the insolvency risk, and we will make some extension in such point. In our model, we assume that the guaranteed rate is a deterministic function, which means we can use Black and Scholes option theory to derive price formulas. Then, we use short rate models, such as Vasicek, Cox-Ingesoll-Ross, and Ho-Lee models, to simulated the above guaranteed rate and obtain discrete time point valuations. Finally, we inserted them into the formulas and obtain corresponding valuation results for this simulated guaranteed rate. Furthermore, we will present a reasonable connection between it and risk free rate.

Another mechanism we will introduce, which was also introduced in the literatures, is regulatory restriction. However, the difference here is that the regulatory line is a random process, rather than deterministic exponential function as in Grosen and Jorgensen (2001).

Finally, we provide a conclusion about the valuations of policyholders' claims under the following interest rate models: Vasicek model, Cox-Ingesoll-Ross model and Ho-Lee model.

2 Basic Model

In this section, we will present a model to deal with the problem discussed in the introduction. The basic framework is based on Briys and de Varenne(1997), but we test the short rate and guarantee rate by using stochastic process.

At t=0, policyholders and equityholders form a mutual company, the life insurance company. The total Asset is A_0, a fraction $L_0 = \alpha A_0 (0 \leq \alpha \leq 1)$ is invested by policyholders and $E_0 = (1-\alpha)A_0$ is invested by equityholders. The parameter α is called the wealth distribution coefficient.

Table 1

Assets	Liabilities
A_0	$L_0 \equiv \alpha A_0$
	$E_0 \equiv (1-\alpha)A_0$

Then the agents acquire a claim for a payoff on or before the maturity date T. Therefore, these claims are very similar to financial derivatives with the company's assets as the well-defined underlying asset. Hence, first, we will give a precise description of claims and then use the powerful apparatus of contingent claims valuation to price them. Second, we will choose suitable parameters in the valuation formula in order to form a fair contract at t=0, which means that initial investments equal to the time zero market value of the claims.

3 Contract Specifications

In this section we describe the details of the stakeholders' claims on the company's assets, as suggested in Briys & Varenne(1997) and Lochte & Jorgensen(2001). We begin with a specification of the liabilityholders' claim.

3.1 The liabilityholders' Claim

As mentioned in the introduction, the life insurance company promises the liabilityholders a continuously compounded return on the initial market value of the liabilities of r_G during the life of the contract: $L_t^G = L_0 e^{\int_0^t r_G(s)ds}$ at time t.

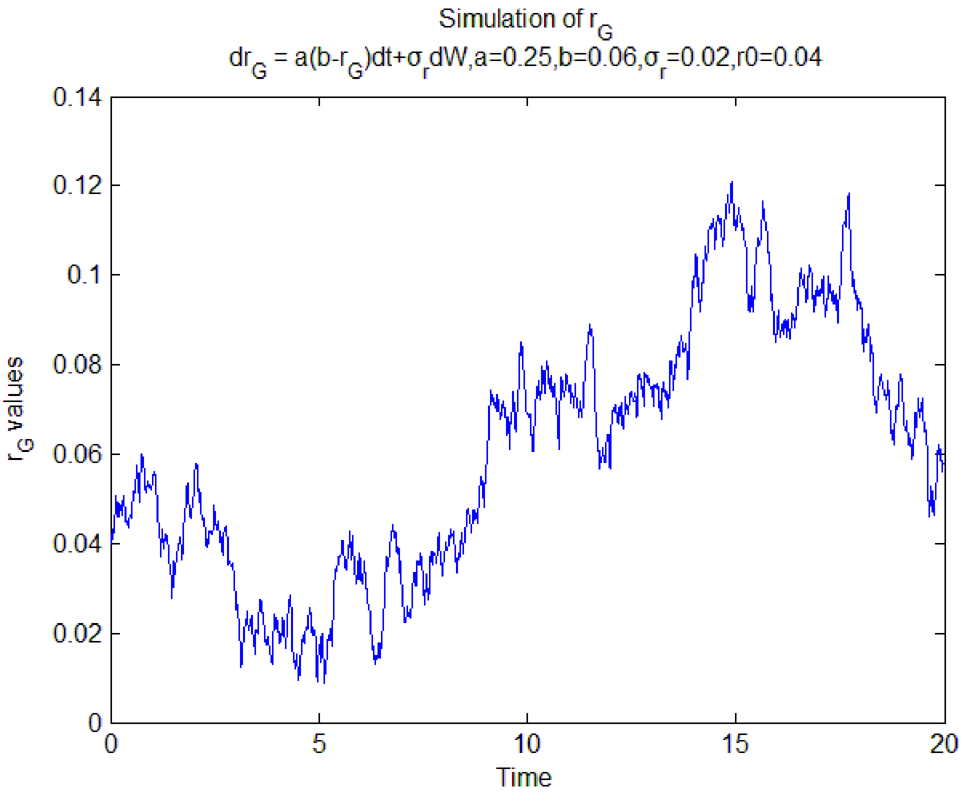
Furthermore, another important assumption in our model is that the risk free rate equals to interest rate guarantee: $r \equiv r_G$.

In Figure1, r_G is simulated from the Vasicek short rate models:

$$dr_G = a\,(b - r_G)\,dt + \sigma_r dW, \qquad (1)$$

where we fix the parameters at the following values a=0.25, b=0.06, $\sigma_r = 0.02$, r0=0.04. These values imply a positively sloped curve rising from 4% towards approximately 6% as time to maturity tends to infinity.

Figure 1



This translates into a guaranteed final payment of $L_T^G = L_0 e^{\int_0^T r_G(s)ds}$. When there are not any external guarantors, the company's promise can be realized only if it turns out that $A_T > L_T^G$ at time T. In the opposite case, $A_T \leq L_T^G$, and in the event that the company has not been declared insolvey by regulators, the policyholders receive A_T and the equityholders receive nothing.

Besides, in addition to the promised maturity payment L_T^G, liabilityholders are also generally given bonus if the market value of the assets evolves sufficiently favorable.

$$\delta[\alpha A_T - L_T^G] \tag{2}$$

From (2) it is clear that in final states where the policyholders' 'share' of total value A_T exceeds the promised payment of L_T^G they will receive a fraction, δ, of this surplus. The parameter δ is called as the participation coefficient and should be in the interval [0 1].

To sum up, the total maturity payoff to policyholders, $\Psi_L(A_T)$, can be described as

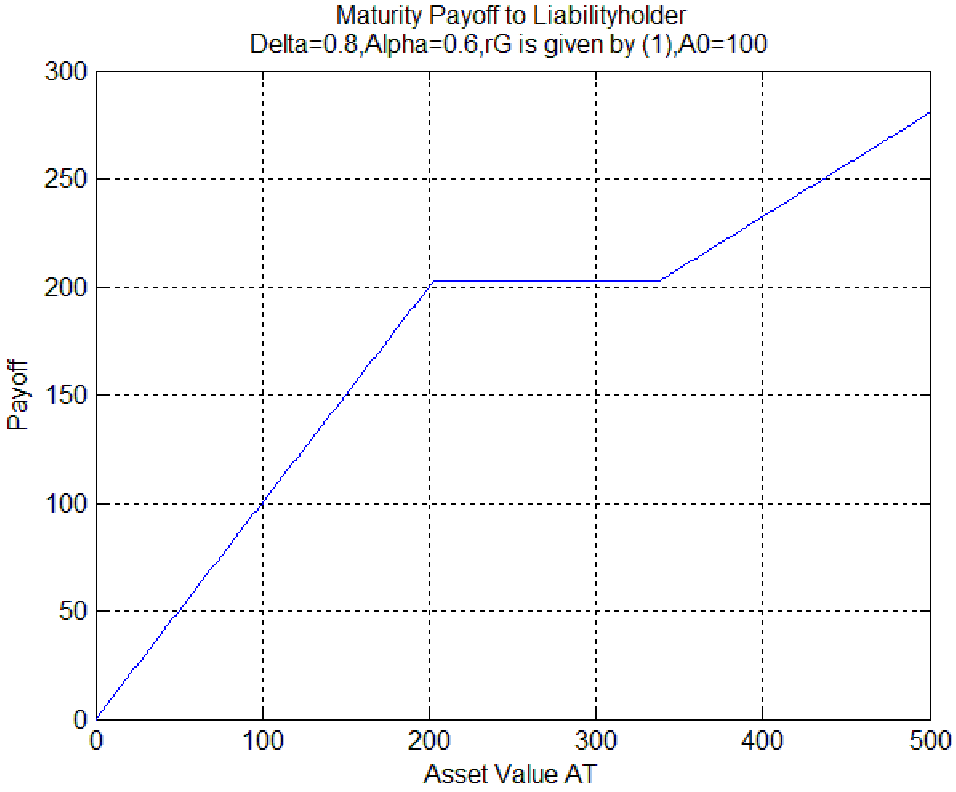
$$\Psi_L(A_T) = \begin{cases} A_T, & A_T < L_T^G, \\ L_T^G, & L_T^G < A_T < \frac{L_T^G}{\alpha}, \\ L_T^G + \delta[\alpha A_T - L_T^G], & A_T > \frac{L_T^G}{\alpha}, \end{cases} \tag{3}$$

or

$$\Psi_L(A_T) = \delta[\alpha A_T - L_T^G]^+ + L_T^G + [L_T^G - A_T]^+. \tag{4}$$

The first part of the right-hand side has already been named as the bonus option. The two remaining terms are fixed maturity payment and shorted put option respectively. Figure 2 is a graphical illustration of (4), where r_G is given by the Vasicek model (1).

Figure 2



In the next step, we will impose a regulatory restriction. Technically, suppose that in the framework above, the asset must evolve above a certain level:

$$A_t > \lambda L_0 e^{\int_0^t r_G(s)ds} \equiv B_t \qquad (5)$$

The curve $\{B(t)\}_{0 \leq t \leq T}$, will be called the regulatory boundary.

The purpose of adding the regulatory boundary is to determine an absolute lower bound of the assets, under which the company is declared insolvency: $L_0 e^{\int_0^t r_G(s)ds}$ is the policyholders' initial deposit compounded with the guaranteed rate of return up to time t. Therefore, only if that the total assets A_t at all times have been sufficient to cover $L_0 e^{\int_0^t r_G(s)ds}$ multiplied by some prerequisite constant, λ, will the stakeholders' options keep evolving.

In the opposite event, assets will at some point in time, τ, equal to B_τ, $A_\tau = B_\tau$. In this situation regulatory authorities close down the company immediately and distribute the wealth to stakeholders.

At this point we note that there are two interesting cases. For $\lambda \geq 1$ and in the event of a boundary hit, liabilityholders receive $L_0 e^{\int_0^\tau r_G(s)ds}$. At the same time equityholders receive $(\lambda - 1) L_0 e^{\int_0^\tau r_G(s)ds}$. Conversely, for $\lambda < 1$ and in the event of a boundary hit, liabilityholders receive $\lambda L_0 e^{\int_0^\tau r_G(s)ds}$. At the same time equityholders receive nothing. The various situations are illustrated in the figures below.

Figure 3a
For $\lambda < 1$

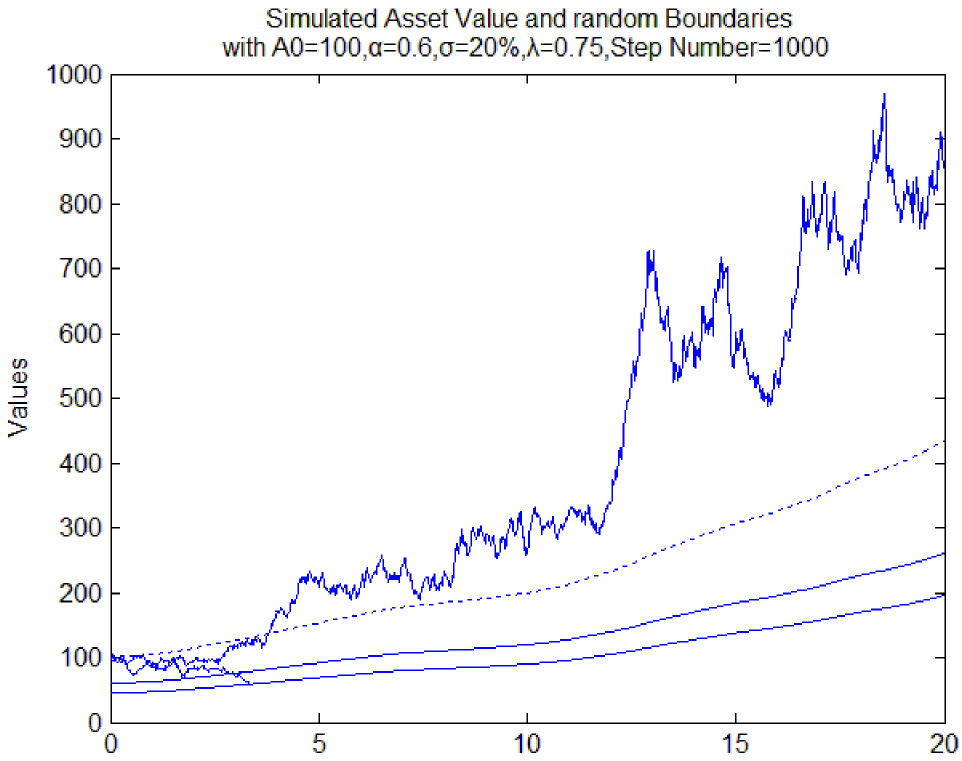
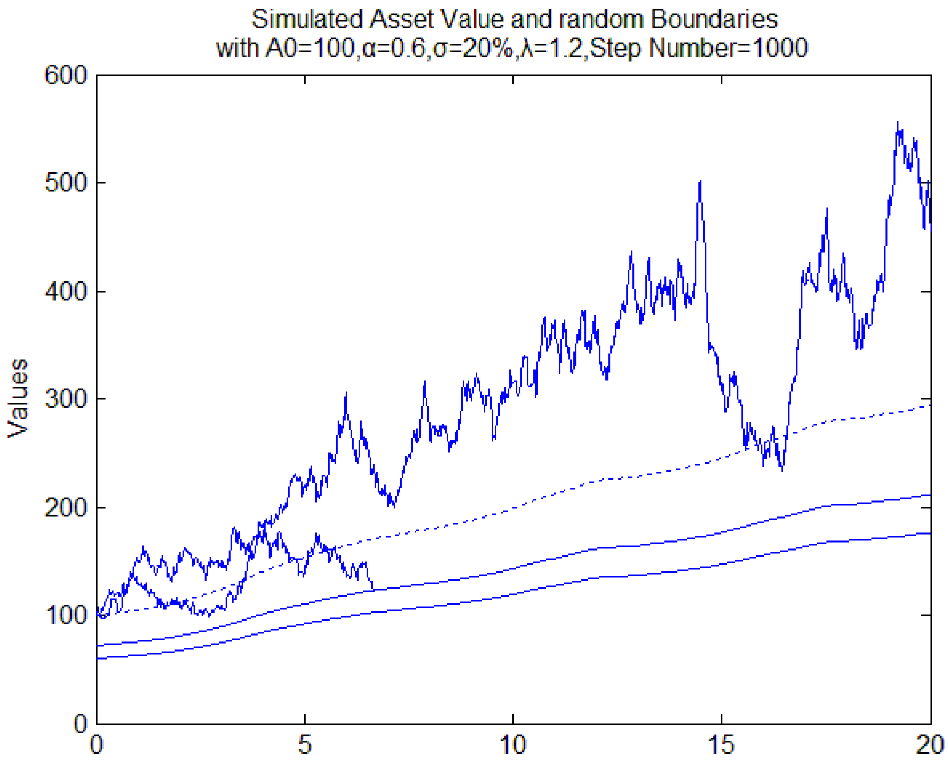


Figure 3b
For $\lambda \geq 1$



As described above there will be a rebate to liabilityholders in the event of premature closure at time τ. This rebate, $\Theta_L(\tau)$, is given as

$$\Theta_L(\tau) = \begin{cases} L_0 e^{\int_0^\tau r_G(s)ds}, & \lambda \geq 1, \\ \lambda L_0 e^{\int_0^\tau r_G(s)ds}, & \lambda < 1, \end{cases} \tag{6}$$

or

$$\Theta_L(\tau) = (\lambda \wedge 1)\, L_0 e^{\int_0^\tau r_G(s)ds}. \tag{7}$$

Before considering the valuation of claim, $(\Psi_L(A_T), \Theta_L(\tau))$, we describe the details regarding the equityholders' claim.

3.2 The Equityholder's claim

As residual claimants, equityholders will receive a payoff at the maturity date conditional on no premature closure as follows:

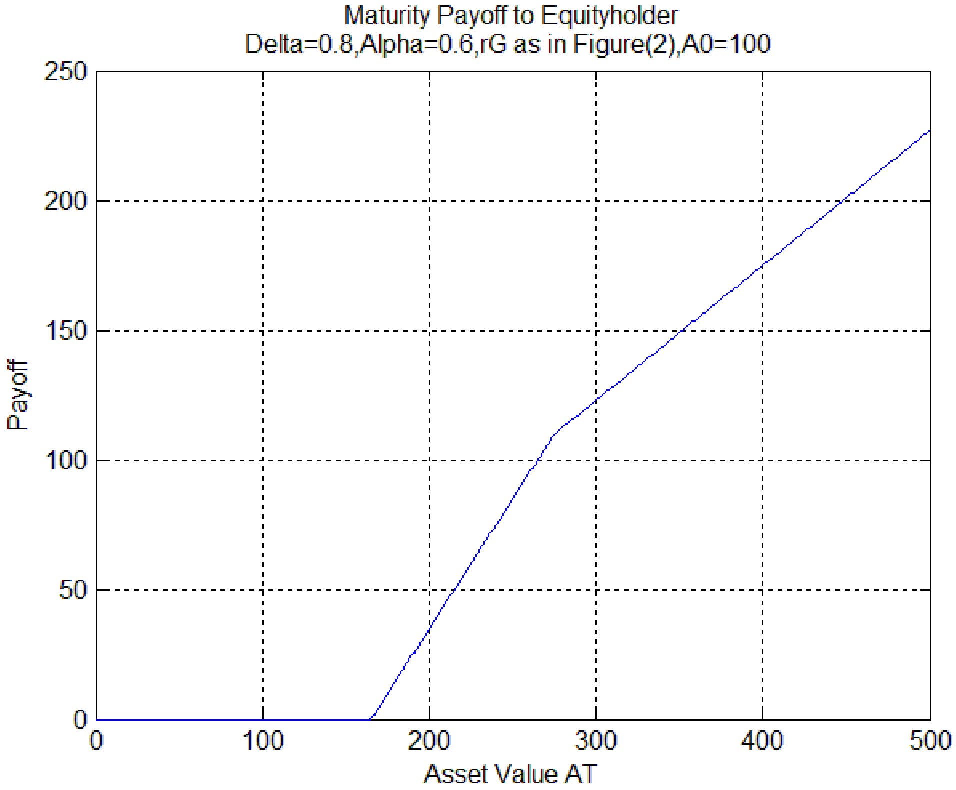
$$\Psi_E(A_T) = \begin{cases} 0, & A_T < L_T^G, \\ A_T - L_T^G, & L_T^G < A_T < \frac{L_T^G}{\alpha}, \\ A_T - L_T^G - \delta[\alpha A_T - L_T^G], & A_T > \frac{L_T^G}{\alpha}. \end{cases} \tag{8}$$

or

$$\Psi_E(A_T) = [A_T - L_T^G]^+ - \delta[\alpha A_T - L_T^G]^+. \tag{9}$$

The above payoff function is depicted in Figure 4 below.

Figure 4



As described above there will be a rebate to equityholders in the event of technical insolvency at time τ. This rebate, $\Theta_E(\tau)$, is given as

$$\Theta_E(\tau) = \begin{cases} (\lambda - 1)L_0 e^{\int_0^\tau r_G(s)ds}, & \lambda \geq 1, \\ 0, & \lambda < 1. \end{cases} \tag{10}$$

or

$$\Theta_E(\tau) = [(\lambda - 1) \vee 0]\, L_0 e^{\int_0^\tau r_G(s)ds}. \tag{11}$$

4 Valuations

To these contracts we apply the basic framework of Black and Scholes (1973) where all activity occurs on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. Under such assumption, the dynamic evolution of the assets is described by the stochastic differential equation:

$$dA_t = \mu A_t dt + \sigma A_t dW_t^p \tag{12}$$

where μ and σ are constants and $\{W_t^p\}$ is a standard Brownian motion under P. Given the riskless interest rate, r_t, we have

$$dA_t = r_t A_t dt + \sigma A_t dW_t^Q \tag{13}$$

where $\{W_t^Q\}$ is a standard Brownian motion under the risk-neutral probability measure Q.

Here, $V_i(A_t, t), i = L, E$, denote the time t value of the liability- and equityholders' claims respectively. We can write (see e.g. Ingersoll (1987), p. 369–370)

$$V_i(A_t, t) = E^Q[e^{-\int_t^T r(s)ds} \Psi_i(A_T) 1_{\tau \geq T} + e^{-\int_t^\tau r(s)ds} \Theta_i(\tau) 1_{\tau < T} | \mathcal{F}_t] =$$

$$e^{-\int_t^T r(s)ds} \int_0^\infty \Psi_i(A_T) \cdot f(A_T, T; A_t, t) dA_t + \int_t^T e^{-\int_t^\tau r(s)ds} \Theta_i(\tau) \cdot g(\tau; A_t, t) d\tau \tag{14}$$

where we assume r_t in the above formula a deterministic function and the following theorems are also based on the assumption.

Moreover, $f(A_T, T; A_t, t)$ denotes the risk-neutral density for the value of the assets at time T with an absorbing barrier $\{B_t\}_{0 \leq t \leq T}$. Similarly, $g(\tau; A_t, t)$ denotes the risk-neutral first hitting time density of asset through the absorbing barrier conditional on the position A_t at time t.

The explanation of these two densities is deferred to the Appendix.

Theorem 1.

We have the following value of the Liabilityholders' Claim $\Psi_L(A_T)$: A bonus (call) option element, a fixed payment, and a shorted put option element. If a premature barrier is hit there will be a rebate payment specified by $\Theta_L(\tau)$ in (6). $V_L(A_t,t)$ is the sum of the time t value of these four elements as given below in parts I-IV where we used the notation:

$$d_\gamma^{\mp}(x,t)=\frac{\ln(x)+\int_t^T(r(s)-\gamma\mp\frac{1}{2}\sigma^2)ds}{\sigma\sqrt{(T-t)}},$$

I: The time t value of the bonus (call) option element is

$$\delta\alpha\{A_t N\left(d_0^+\left(\frac{A_t}{X},t\right)\right)-\frac{L_T^G}{\alpha}e^{-\int_t^T r(s)ds}N(d_0^-(\frac{A_t}{X},t))$$

$$-(\frac{A_t}{B_t})\cdot[\frac{B_t^2}{A_t}N(d_0^+\left(\frac{\frac{B_t^2}{A_t}}{X},t\right))-\frac{L_T^G}{\alpha}e^{-\int_t^T r(s)ds}N(d_0^+\left(\frac{\frac{B_t^2}{A_t}}{X},t\right))]\}.$$

where, $X\equiv L_T^G(\lambda\vee\frac{1}{\alpha})$.

II: The time t value of the conditional fixed payment element is

$$L_T^G e^{-\int_t^T r(s)ds}\left\{N\left(d_{r_G}^-\left(\frac{A_t}{B_t},t\right)\right)-\left(\frac{A_t}{B_t}\right)\cdot N\left(d_{r_G}^-\left(\frac{B_t}{A_t},t\right)\right)\right\}.$$

III: The time t value of the shorted put option element is

$$-1_{\{\lambda<1\}}\left\{L_T^G e^{-\int_t^T r(s)ds}\left[N(-d_0^-\left(\frac{A_t}{L_T^G},t\right))-N\left(-d_{r_G}^-\left(\frac{A_t}{B_t},t\right)\right)\right]\right.$$

$$-A_t\left[N(-d_0^+\left(\frac{A_t}{L_T^G},t\right))-N\left(-d_{r_G}^+\left(\frac{A_t}{B_t},t\right)\right)\right]$$

$$-\left(\frac{A_t}{B_t}\right)\times$$

$$\left\{L_T^G e^{-\int_t^T r(s)ds}\left[N(-d_0^-\left(\frac{B_t^2}{A_tL_T^G},t\right))-N\left(-d_{r_G}^-\left(\frac{B_t}{A_t},t\right)\right)\right]\right.$$

$$\left.\left.-\frac{B_t^2}{A_t}\left[N(-d_0^+\left(\frac{B_t^2}{A_tL_T^G},t\right))-N\left(-d_{r_G}^+\left(\frac{B_t}{A_t},t\right)\right)\right]\right\}\right\}.$$

IV: The time t value of the rebate is

$$\frac{(\lambda\wedge 1)}{\lambda}A_t\left\{N\left(-d_{r_G}^+\left(\frac{A_t}{B_t},t\right)\right)-\left(\frac{A_t}{B_t}\right)^{-1}N\left(d_{r_G}^+\left(\frac{B_t}{A_t},t\right)\right)\right\}.$$

The proof relies on Black & Scholes formula and is presented in the Appendix.

Theorem 2.
We have the following value of the Equityholders' Claim $\Psi_E(A_T)$: One residual claim call option and another shorted bonus option. if a premature barrier is hit there will be a rebate payment specified by $\Theta_L(\tau)$ in (10).
$V_L(A_t, t)$ is the sum of the time t value of these three elements as given below.

I. The time t value of the long residual claim (call option) element is

$$A_t N\left(d_o^{+}\left(\frac{A_t}{Y}, t\right)\right) - \frac{L_T^G}{\alpha} e^{-\int_t^T r(s)ds} N(d_0^{-}(\frac{A_t}{Y}, t))$$
$$-\left(\frac{A_t}{B_t}\right) \cdot \left[\frac{B_t^2}{A_t} N\left(d_o^{+}\left(\frac{\frac{B_t^2}{A_t}}{Y}, t\right)\right) - L_T^G \cdot e^{-\int_t^T r(s)ds} N\left(d_o^{-}\left(\frac{\frac{B_t^2}{A_t}}{Y}, t\right)\right)\right],$$

where, $Y = (B_T \vee L_T^G)$.

II: The time t value of the shorted bonus option is given in part I of Theorem 1.

III The time t value of the equity holders' rebate is

$$\frac{((\lambda - 1) \vee 0)}{\lambda} A_t \left\{N\left(-d_{r_G}^{+}\left(\frac{A_t}{B_t}, t\right)\right) - \left(\frac{A_t}{B_t}\right)^{-1} N\left(d_{r_G}^{+}\left(\frac{B_t}{A_t}, t\right)\right)\right\}.$$

Proof of Theorem 2.
Part I is established by setting $\delta = \alpha = 1$ in Part I of Theorem 1. Part II is precisely Part I of Theorem 1 with the sign reversed. Part III rests on calculations similar to those that established Part IV of Theorem 1.

5 Numerical Examples and Comparisons

5.1 Fair Contracts

The formulas for the values of the liability and equity claims are closed formulas that can be calculated once the relevant parameters are given.

Just as discussed in the introduction part, it is clear that not every choice of parameters will represent fair contracts. So the first question to ask is which combinations of parameters will represent fair contracts. A fair contract should satisfy the initial equation:

$$L_0 = \alpha A_0 = V_L(A_0, 0; \alpha, \delta, \lambda, \sigma T, r, r_G),$$

The equation is obvious because the liability holders' initial contribution to the total assets $L_0,$ should equal to the initial market value of contingent claim.

Another equation is

$$E_0 = (1-\alpha)A_0 = V_E\left(A_0, 0; \alpha, \delta, \lambda, \sigma,\ T, r, r_G\right).$$

Next, we provide some selected representative plots to illustrate some typical relations between parameters of initially fair contracts, where r=r_G are simulated from different short interest rate models such as Vasicek, CIR and Ho-Lee. This extends the work by Grosen & Jorgensen(2001), who considered the case where r and r_G are held constant.

5.1.1 Using Vasicek model

Figures 5 illustrates the relation between fair values of the participation coefficient, δ, and the wealth distribution coefficient, α, for some fixed and representative values of the remaining parameters. It is noted that all these graphs are negatively sloped as a higher wealth distribution coefficient will be associated with a lower participation coefficient in order for the contract to be fair to both sides (note r and r_G are simulated from random process and valuations are averages in these examples).

Figure 5a

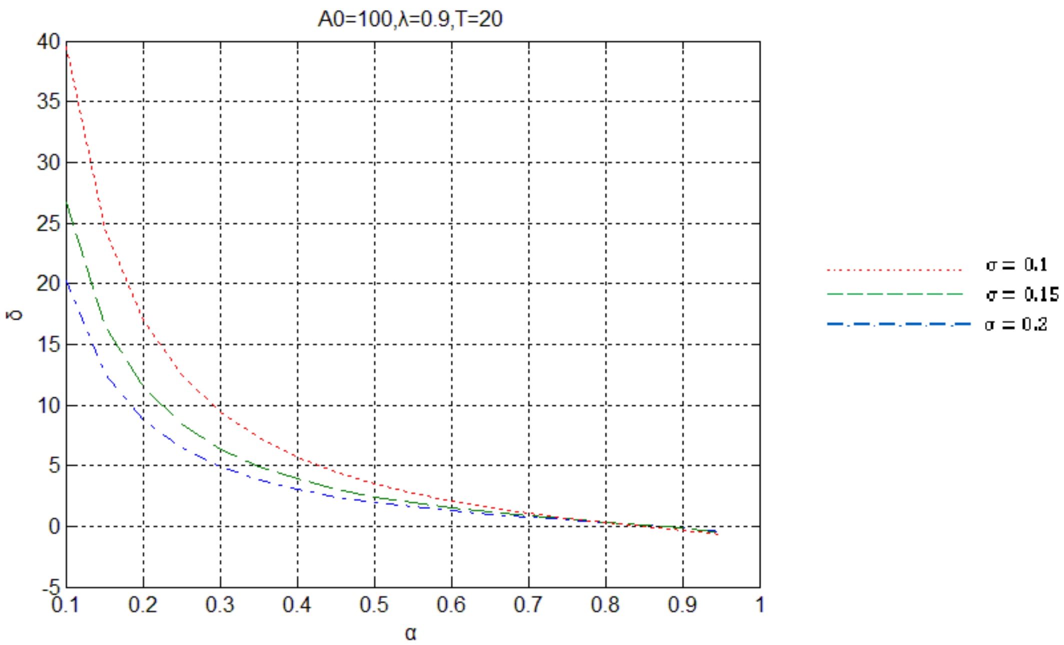
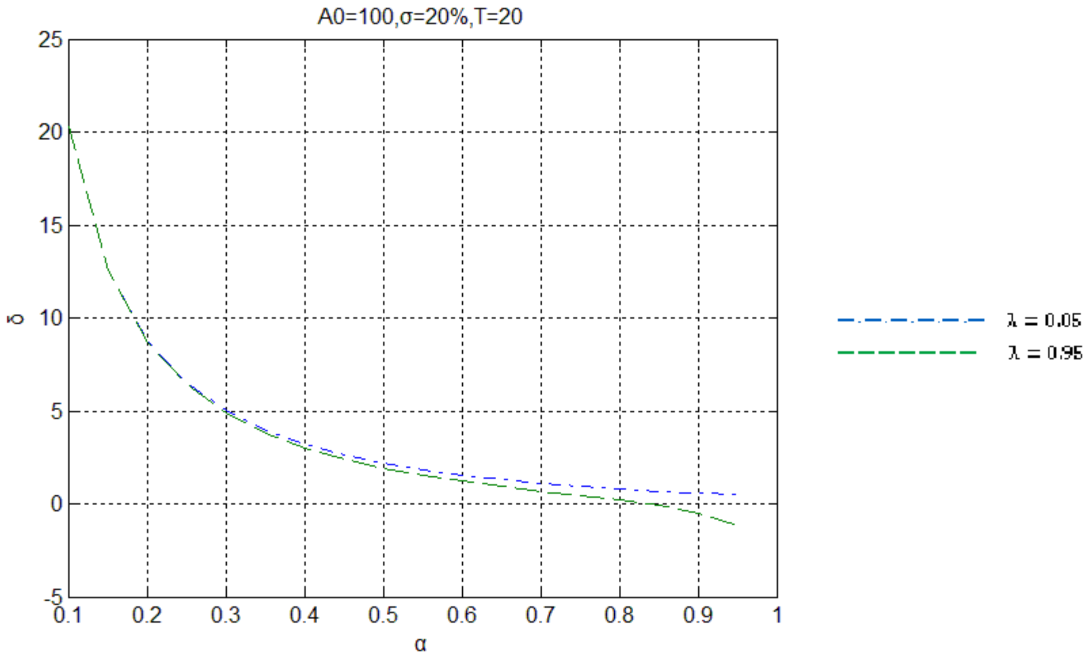


Figure 5b



5.1.2 Using CIR model

Figure 6 and Figure 7 illustrate similar comparison under CIR interest rate model and Ho-Lee model. We find the results are little different when we use three different interest rate models:

Cox_ingersoll_Ross short rate model:

$dr = a(b-r)dt+\sigma_r\sqrt{r}dW$

Here we set parameters as a=0.25, b=0.05, $\sigma_r = 0.02$ and r_0=0.1.

Under such a setting, the short rate will start from r_0=10% toward to 5% when time is infinite. The purpose of such an initial parameter setting is to make short rate under this model is much more different to Vasicek model. However, after careful comparison, we find that the difference of comparison among three models is negligible.

Another thing we want to mentioned here is that we will keep the above parameter setting in the CIR model when we compare the difference among models in section 5.2 and section 5.3

Figure 6a

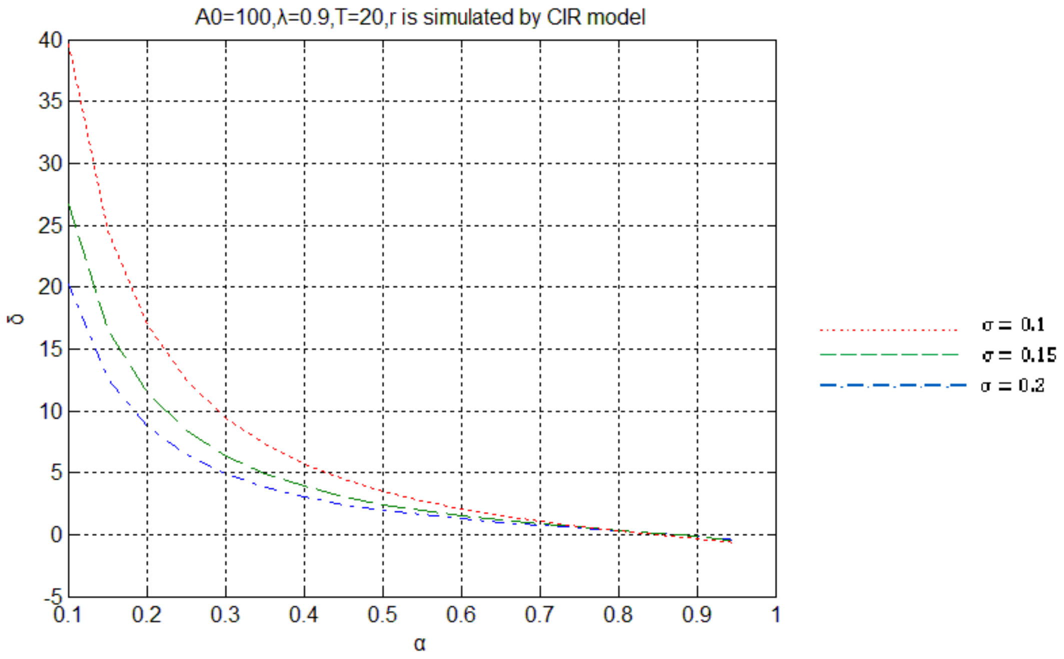
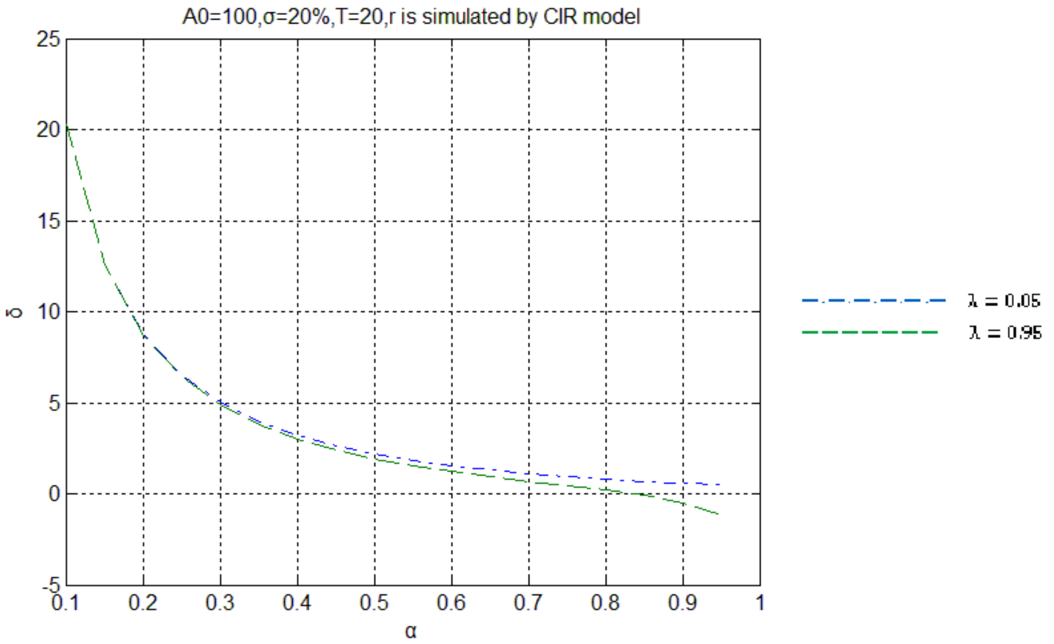


Figure 6b



5.1.3 Using Ho-Lee model

Ho-Lee interest rate model:

$dr = \Theta_t dt + \sigma_r dW$,

Here we set parameters as: Θ_t= -0.05e^{-t}, σ_r=0.02 and $r_0 = 0.1$

Under such a setting, the short rate will start from r_0=10% toward to 5% when time is infinite. The purpose of such an initial parameter setting is to make short rate under this model is much more different to Vasicek model but similar as Cox-Ingersoll-Ross model.

We want to mention is that we will keep the above parameter setting in the Ho-Lee model when we compare the difference among models in section 5.2 and section 5.3

Figure 7a

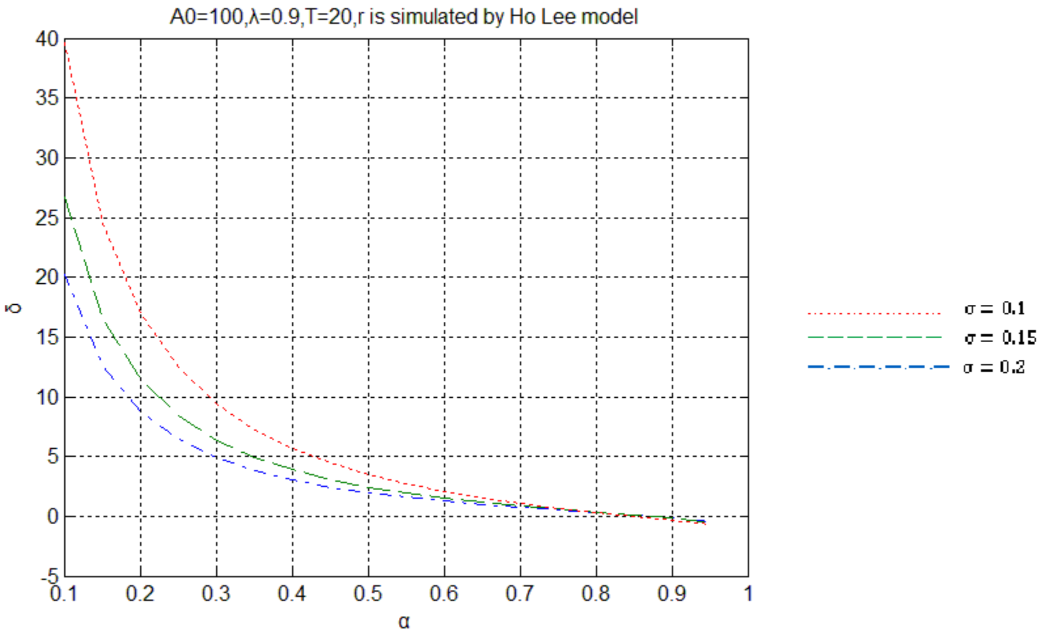
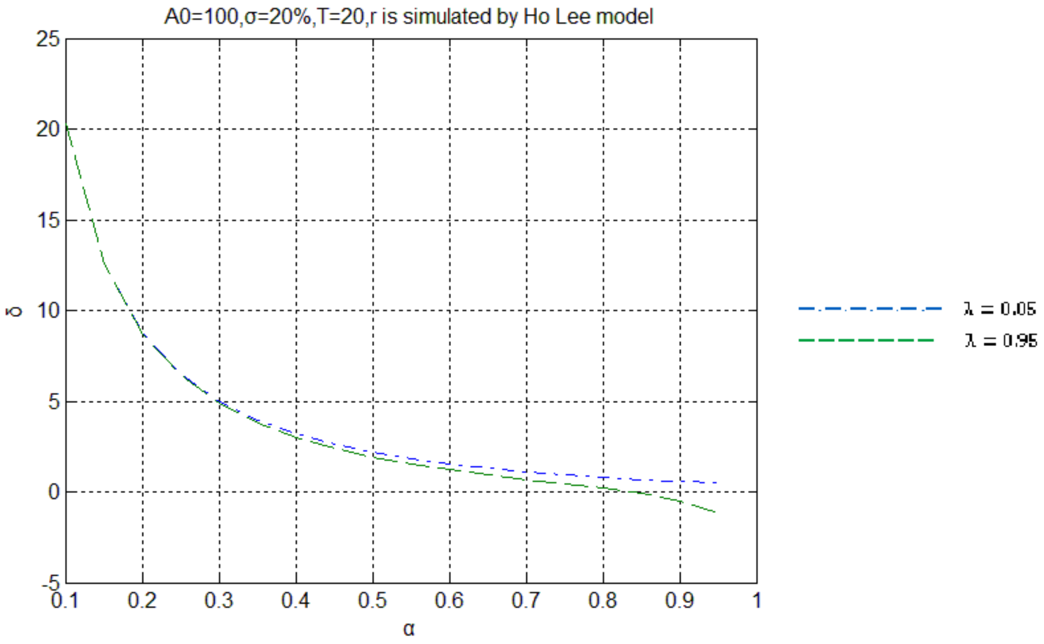


Figure 7b



5.2 Components of Fair contracts

In Section 4 we derived value formulas for each of the components. The following table shows some examples of how the total contract value decomposes into the separate elements.

We will find that the results are little different when we use three different interest rate models.

5.2.1 Using Vasicek model

Let $r=r_G$ be the values simulated from the following Vasicek short rate model:

$$dr = a\cdot(b-r)dt+\sigma_r dW,\ r=r_G.$$

Table 2:

Decompositions of Contract Values

$a=0.25,\ b=0.06,\ \sigma_r = 0.02,\ r0=0.04$

$A_0 = 100, r = r_G, \alpha = 0.8, T = 20$

σ	δ	BO	SP	CFP	RL	$V_L(A_0,0)$	RC	SBO	RE	$V_E(A_0,0)$

$\lambda \downarrow 0.00$										
0.10	0.553	7.83	-7.82	79.99	0	80.00	27.83	-7.83	0	20
0.15	0.708	14.88	-14.88	80	0	80.00	34.88	-14.88	0	20
0.2	0.795	21.98	-21.96	79.98	0	80.00	41.98	-21.98	0	20
0.25	0.85	28.82	-28.82	80	0	80.00	48.82	-28.82	0	20
$\lambda = 0.8$										
0.1	0.482	6.68	-0.38	48.56	25.14	80	26.68	-6.68	0	20
0.15	0.534	10.05	-0.15	30.52	39.58	80	30.05	-10.05	0	20
0.2	0.547	12.04	-0.06	20.14	47.88	80	32.04	-12.04	0	20
0.25	0.552	13.28	-0.03	13.79	52.96	80	33.28	-13.28	0	20
$\lambda = 0.9$										
0.1	0.334	4.36	-0.04	36.84	38.84	80	24.36	-4.36	0	20
0.15	0.351	5.8	-0.01	22.16	52.05	80	25.80	-5.8	0	20
0.2	0.355	6.57	0	14.37	59.06	80	26.57	-6.57	0	20
0.25	0.356	7.03	0	9.76	63.21	80	27.03	-7.03	0	20
$\lambda = 1$										
0.1	0	0	0	25.17	54.82	80	20	0	0	20
0.15	0	0	0	14.70	65.3	80	20	0	0	20
0.2	0	0	0	9.43	70.56	80	20	0	0	20
0.25	0	0	0	6.37	73.63	80	20	0	0	20
$\lambda = 1.1$										
0.1	0	0	0	14.24	65.76	80	13.42	0	6.58	20
0.15	0	0	0	8.17	71.83	80	12.81	0	7.18	20
0.2	0	0	0	5.2	74.80	80	12.52	0	7.48	20
0.25	0	0	0	3.5	76.5	80	12.35	0	7.65	20
$\lambda = 1.2$										
0.1	0	0	0	4.42	75.58	80	4.88	0	15.12	20.00
0.15	0	0	0	2.52	77.48	80	4.5	0	15.5	20.00
0.2	0	0	0	1.6	78.4	80	4.32	0	15.68	20
0.25	0	0	0	1.07	78.93	80	4.21	0	15.79	20
$\lambda = 1.25$										
All	All	0	0	0	80	80	0	0	20	20

From the table several interesting observations can be made. For example:

- If volatility is increased, δ must be increased to maintain a fair value distribution
- A larger volatility tends to increase the value of the rebate element and to decrease the value of the conditional fixed payment. This is, of course, explained by the fact that generally a larger volatility is associated with a larger probability of an early 'barrier hit'.
- As expected, the value of the equity rebate element is nil when $\lambda \leq 1$ and positive when $\lambda > 1$.

5.2.2 Using CIR model

Here, Let r=r_G be the values simulated from the following Cox_ingersoll_Ross short rate model:

$$dr = a(b-r)dt+\sigma_r\sqrt{r}dW$$

Table 3:

a=0.25, b=0.05, $\sigma_r = 0.02$ and r_0=0.1

$A_0 = 100, r = r_G, \alpha = 0.8, T = 20$

σ	δ	BO	SP	CFP	RL	$V_L(A_0,0)$	RC	SBO	RE	$V_E(A_0,0)$
$\lambda \downarrow 0.00$										
0.10	0.552	7.8	-7.84	80.04	0	80.00	27.8	-7.8	0	20
0.15	0.708	14.86	-14.9	80.04	0	80.00	34.86	-14.86	0	20
0.2	0.795	21.95	-21.88	79.92	0	80.00	41.95	-21.95	0	20
0.25	0.85	28.81	-27.51	78.63	0	80.00	48.81	-28.81	0	20
$\lambda = 0.8$										
0.1	0.48	6.64	-0.38	48.61	25.13	80	26.64	-6.64	0	20
0.15	0.533	10.03	-0.15	30.54	39.58	80	30.03	-10.03	0	20
0.2	0.547	12.03	-0.06	20.16	47.88	80	32.03	-12.03	0	20
0.25	0.551	13.27	-0.03	13.8	52.96	80	33.26	-13.26	0	20
λ = 0.9										
0.1	0.333	4.34	-0.04	36.87	38.84	80	24.34	-4.34	0	20
0.15	0.35	5.78	-0.01	22.18	52.05	80	25.78	-5.78	0	20
0.2	0.354	6.56	0	14.38	59.06	80	26.56	-6.56	0	20
0.25	0.356	7.02	0	9.76	63.22	80	27.02	-7.02	0	20
λ = 1										
0.1	0	-0.01	0	25.19	54.82	80	19.99	0.01	0	20
0.15	0	-0.01	0	14.72	65.29	80	19.99	0.01	0	20
0.2	0	-0.005	0	9.435	70.57	80	19.99	0.01	0	20
0.25	0	0	0	6.37	73.63	80	20	0	0	20
λ = 1.1										
0.1	0	0	0	14.24	65.76	80	13.42	0	6.58	20
0.15	0	0	0	8.17	71.83	80	12.81	0	7.18	20
0.2	0	0	0	5.207	74.80	80	12.52	0	7.48	20
0.25	0	0	0	3.5	76.5	80	12.35	0	7.65	20
λ = 1.2										
0.1	0	-0.002	0	4.42	75.58	80	4.88	0	15.12	20.00
0.15	0	-0.001	0	2.52	77.48	80	4.5	0	15.5	20.00
0.2	0	0	0	1.6	78.4	80	4.32	0	15.68	20
0.25	0	0	0	1.07	78.93	80	4.21	0	15.79	20
λ =1.25										
All	All	0	0	0	80	80	0	0	20	20

When Lambda is small the volatility is large, when Sigma is small the volatility is large.

5.2.3 Using Ho-Lee model

Here, Let $r=r_G$ be the values simulated from the following Ho-Lee short rate model:

$$dr=\Theta_t dt+\sigma_r dW$$

Table 4:

Θ_t= $-0.05e^{-t}$, σ_r=0.02 and $r_0 = 0.1$

$A_0 = 100, r = r_G, \alpha = 0.8, T = 20$

σ	δ	BO	SP	CFP	RL	$V_L(A_0, 0)$	RC	SBO	RE	$V_E(A_0, 0)$
$\lambda \downarrow 0.00$										
0.10	0.55	7.82	-7.83	80.01	0	80.00	27.82	-7.82	0	20
0.15	0.708	14.85	-14.9	80.04	0	80.00	34.85	-14.85	0	20
0.2	0.795	21.98	-21.84	79.85	0	80.00	41.98	-21.98	0	20
0.25	0.85	28.84	-27.46	78.56	0	80.00	48.84	-28.84	0	20
$\lambda = 0.8$										
0.1	0.48	6.61	-0.39	48.64	25.13	80	26.61	-6.61	0	20
0.15	0.533	10.04	-0.15	30.53	39.58	80	30.04	-10.04	0	20
0.2	0.547	12.02	-0.07	20.16	47.88	80	32.03	-12.02	0	20
0.25	0.551	13.26	-0.03	13.8	52.96	80	33.26	-13.26	0	20
λ = 0.9										
0.1	0.333	4.35	-0.04	36.86	38.84	80	24.35	-4.35	0	20
0.15	0.35	5.77	-0.01	22.19	52.05	80	25.78	-5.78	0	20
0.2	0.354	6.55	0	14.39	59.06	80	26.55	-6.55	0	20
0.25	0.355	7.01	0	9.78	63.22	80	27.01	-7.01	0	20
λ = 1										
0.1	0	0.0	0	25.16	54.81	80	20.02	-0.02	0	20
0.15	0	-0	0	14.70	65.29	80	20	0.00	0	20
0.2	0	-0.00	0	9.43	70.57	80	20	0.00	0	20
0.25	0	0.00	0	6.37	73.63	80	20	0	0	20
λ = 1.1										
0.1	0	0	0	14.25	65.75	80	13.42	0	6.58	20
0.15	0	0	0	8.17	71.83	80	12.82	0	7.18	20
0.2	0	0	0	5.2	74.80	80	12.52	0	7.48	20
0.25	0	0	0	3.5	76.5	80	12.35	0	7.65	20

λ = 1.2										
0.1	0	0.00	0	4.4	75.5	80	4.88	0	15.12	20.00
0.15	0	0.0	0	2.5	77.5	80	4.5	0	15.5	20.00
0.2	0	0	0	1.61	78.4	80	4.32	0	15.68	20
0.25	0	0	0	1.07	78.93	80	4.21	0	15.79	20
λ =1.25										
All	All	0	0	0	80	80	0	0	20	20

When Lambda is small the volatility is large, when Sigma is small the volatility is large

Ho-Lee model has a higher volatility

5.3 Contract component prices

It is an important property of our model that it can identify fair contracts for a given set of initial conditions. However, it is equally important that the model can price contracts and their constituting elements at any given point in time given the initially specified terms.

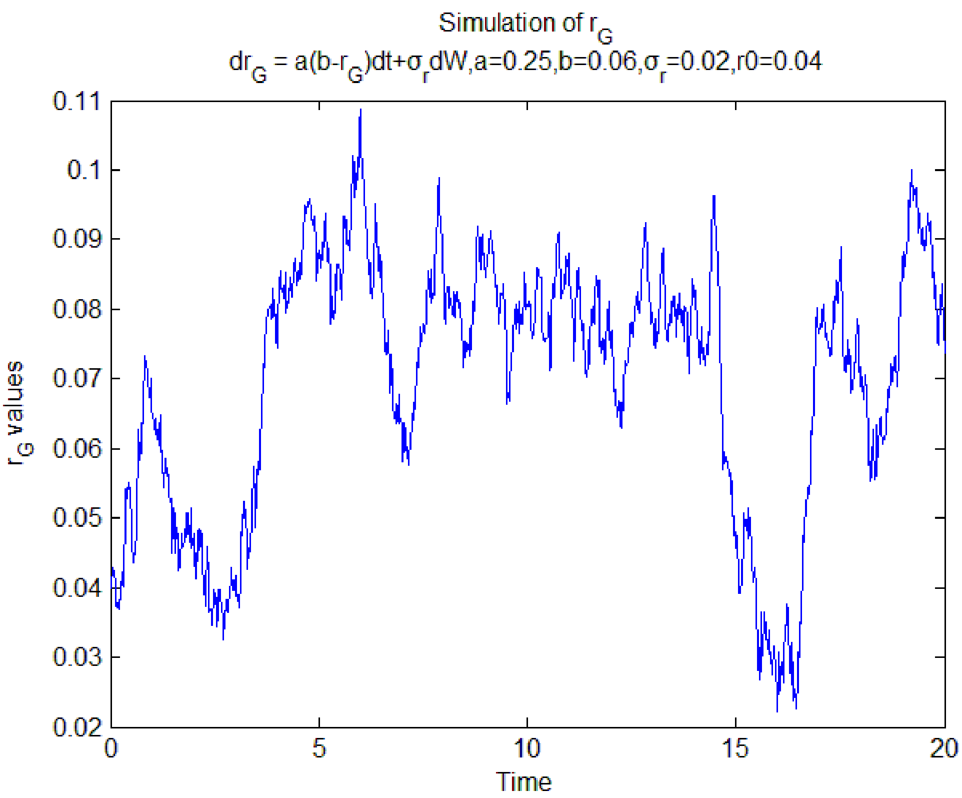
Now, we present the following numerical examples in which the values of the components of the liability holders' contracts as a function of the state variable, A_t, at different times during the life of the contract. The contract parameters have been set so that the contract was fair at t=0 and the contract elements values are plotted for t=0.1 (right after inception), t=10, t=19 and t=20(see Grosen & Jorgensen (2001)).

We will find that the results are little different when we use three different interest rate models.

5.3.1 Using the Vasicek model

The corresponding r_G:

Figure 8a



Total Liability Value

Bonus call option

Conditional Fixed Payment

Shorted put option

Rebate

Figure 8b

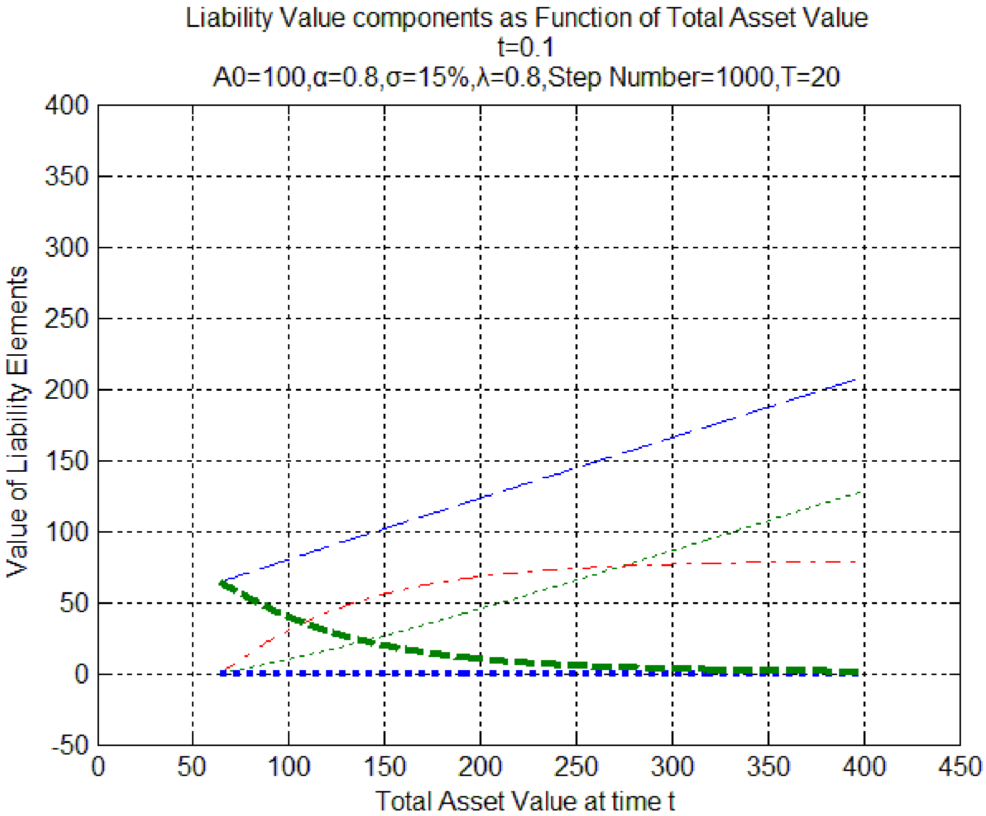


Figure 8c

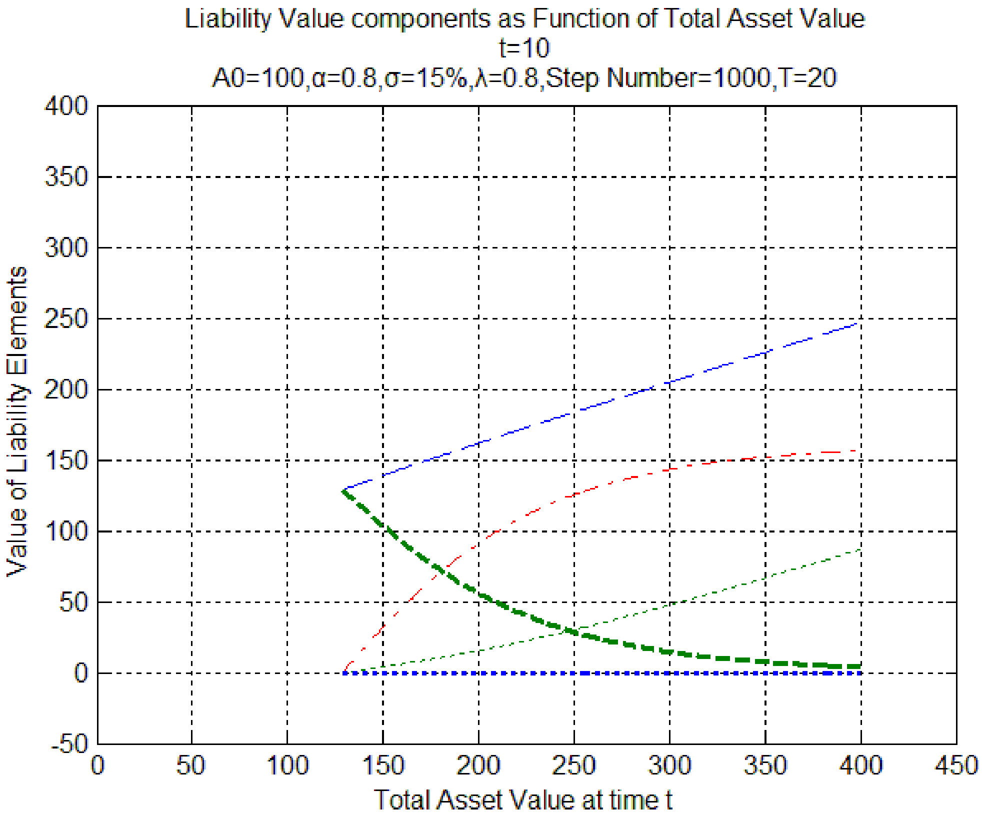


Figure 8d

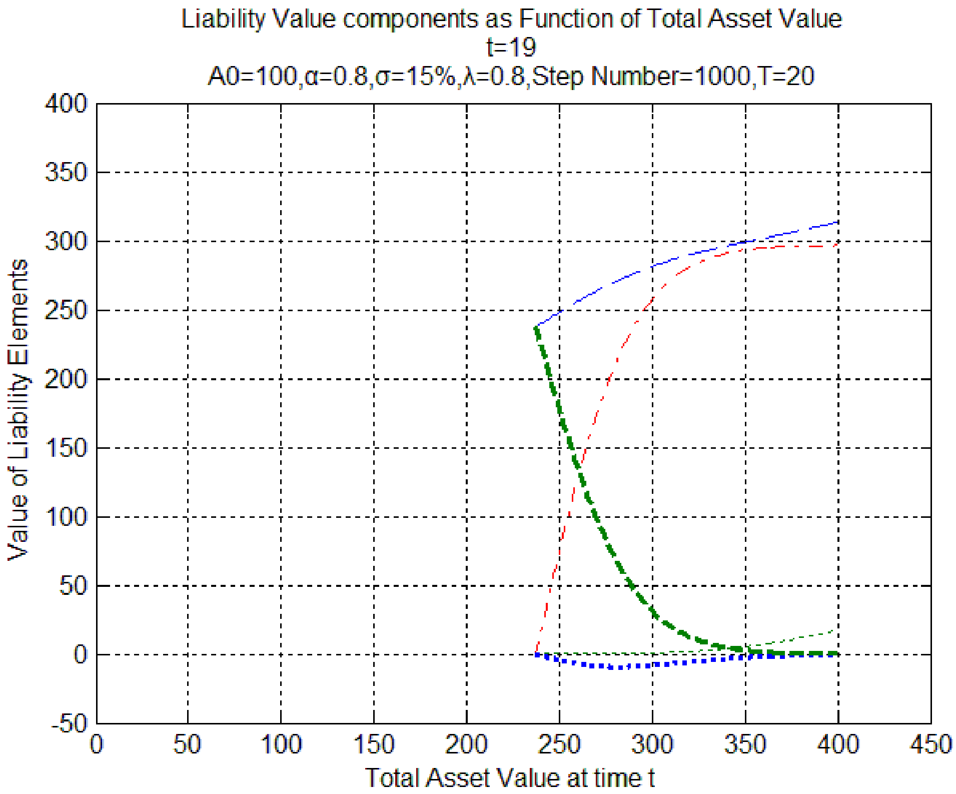
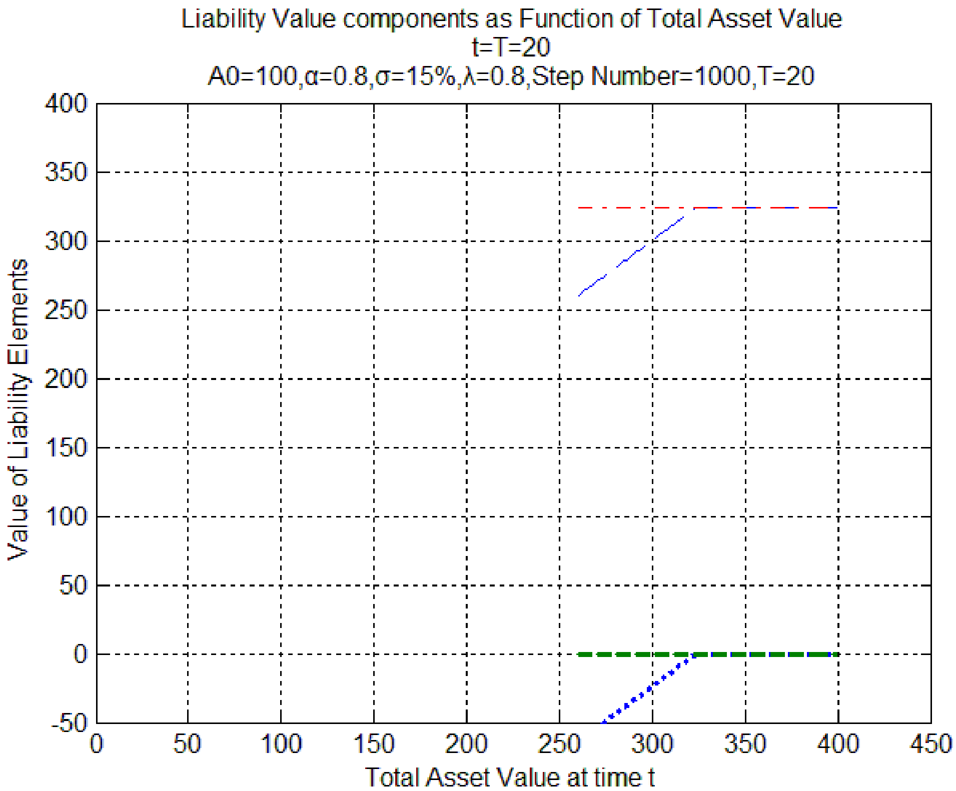


Figure 8e

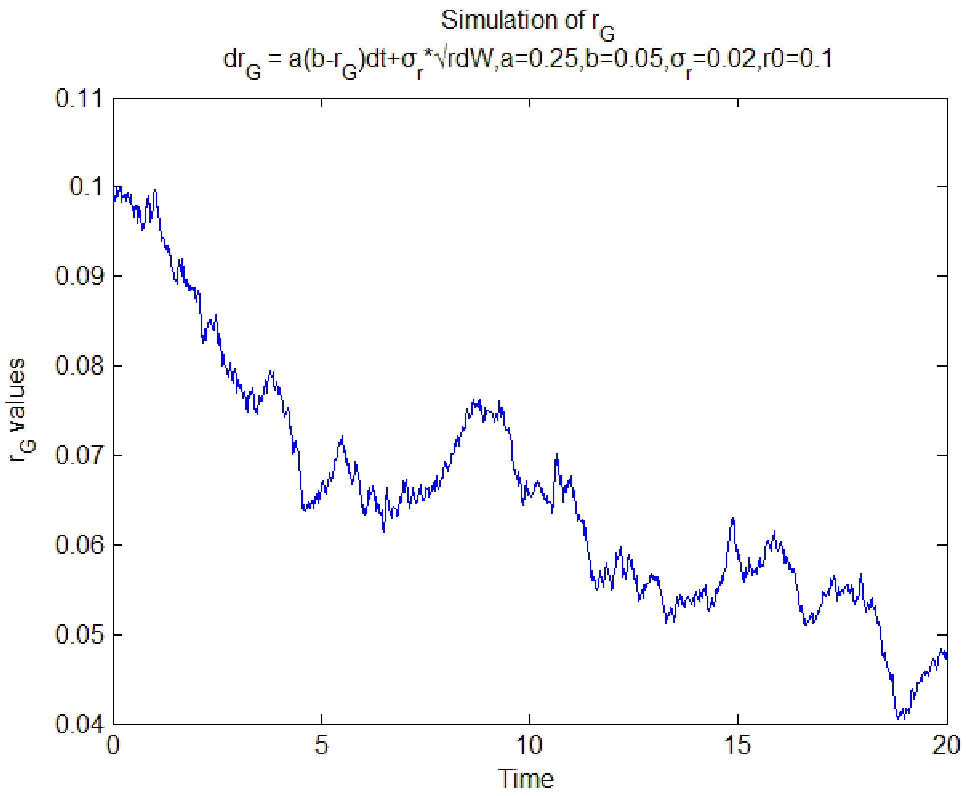


For each time t=0.1,10,19,20, we assume A_t start from corresponding B_t to 450.
For different simulated values of r_G, the corresponding prices are different at time t=0.1,10,19.20.
To obtain fair prices at different time, we should simulate r sufficient many times and choose averages.

5.3.2 Using the Cox_Ingersoll_Ross model

The corresponding r_G:

Figure 9a



Total Liability Value
Bonus call option
Conditional Fixed Payment

Shorted put option

Rebate

Figure 9b

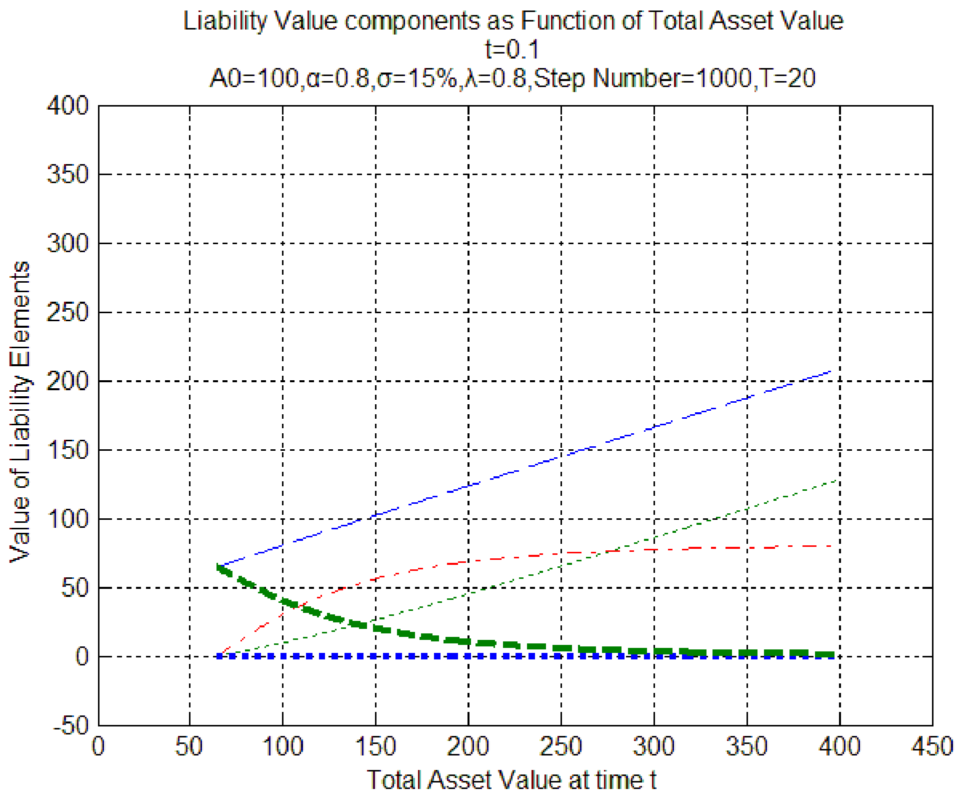


Figure 9c

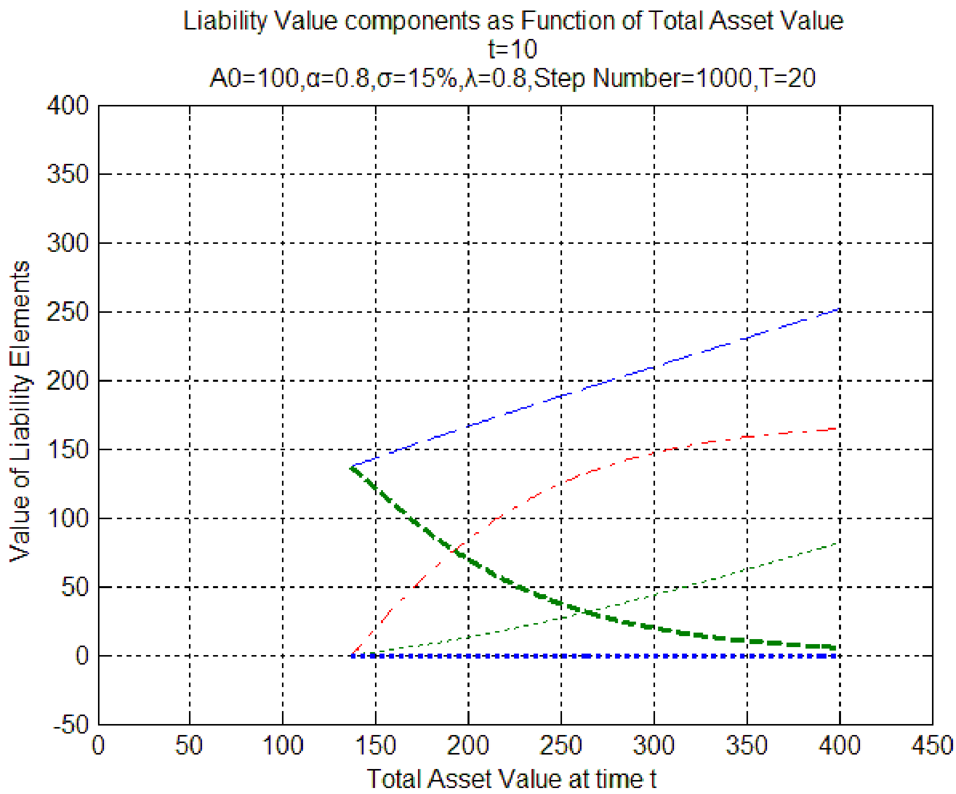


Figure 9d

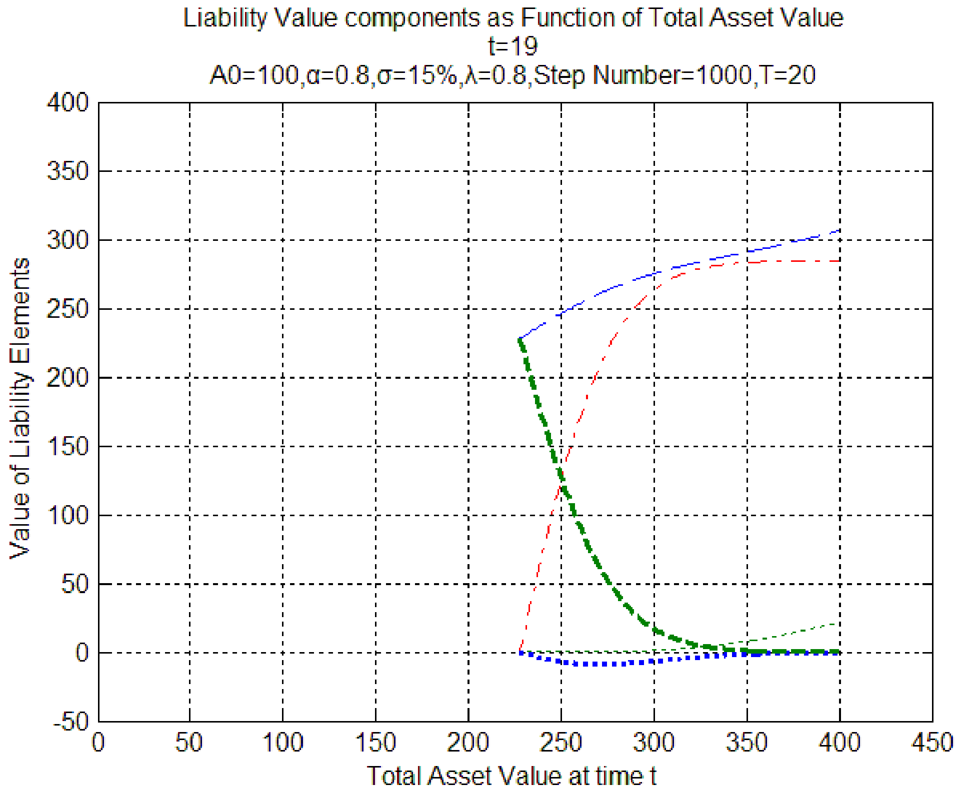
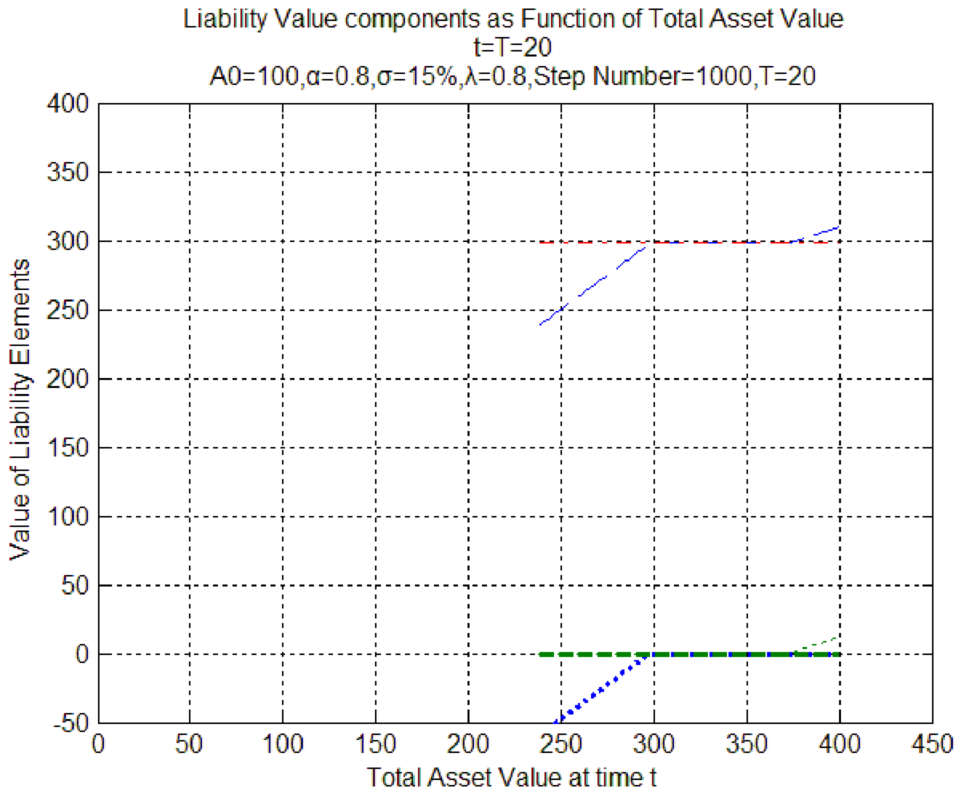


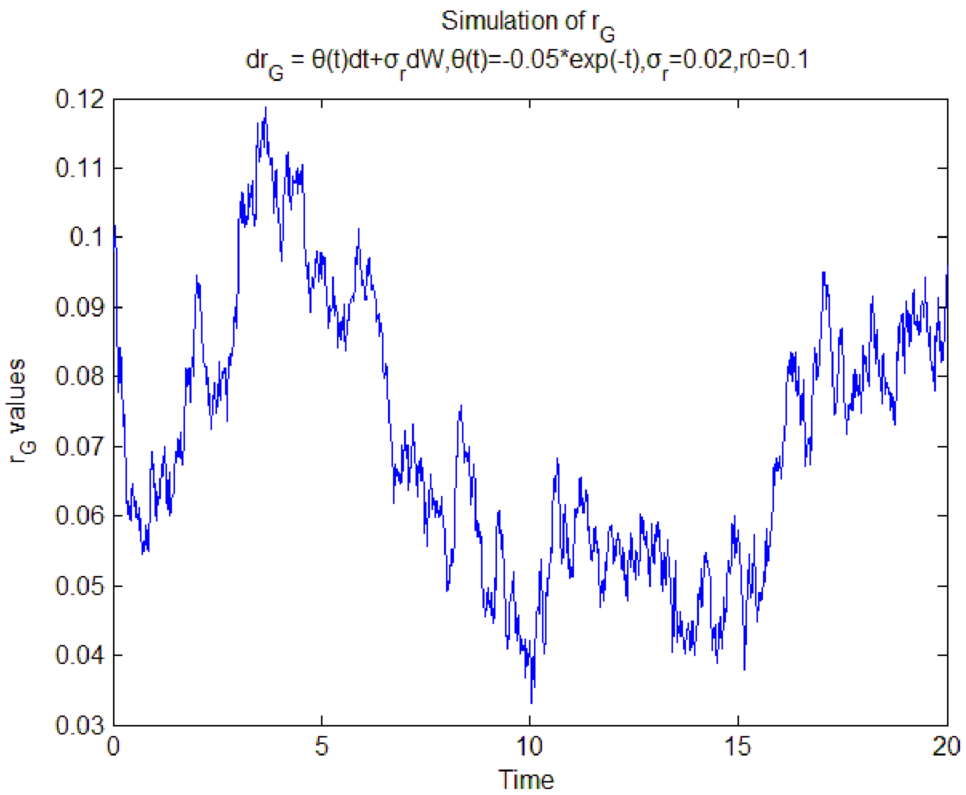
Figure 9e



5.3.3 Using the Ho-Lee model

The corresponding r_G:

Figure 10a



Total Liability Value

Bonus call option

Conditional Fixed Payment

Shorted put option

Rebate

Figure 10b

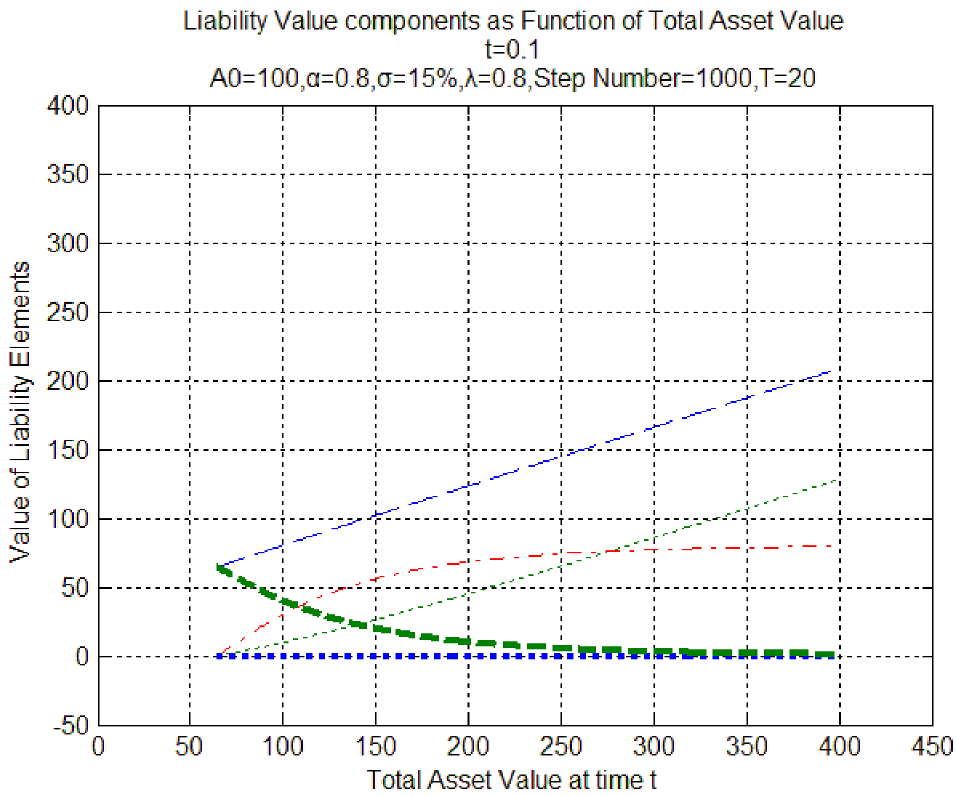


Figure 10c

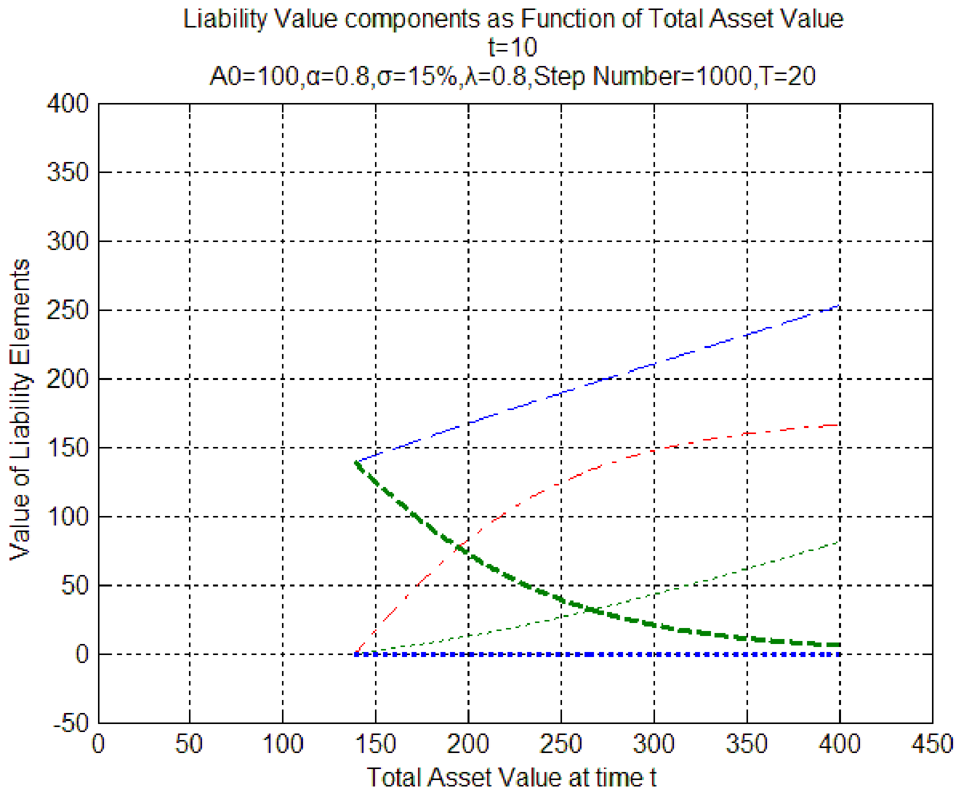


Figure 10d

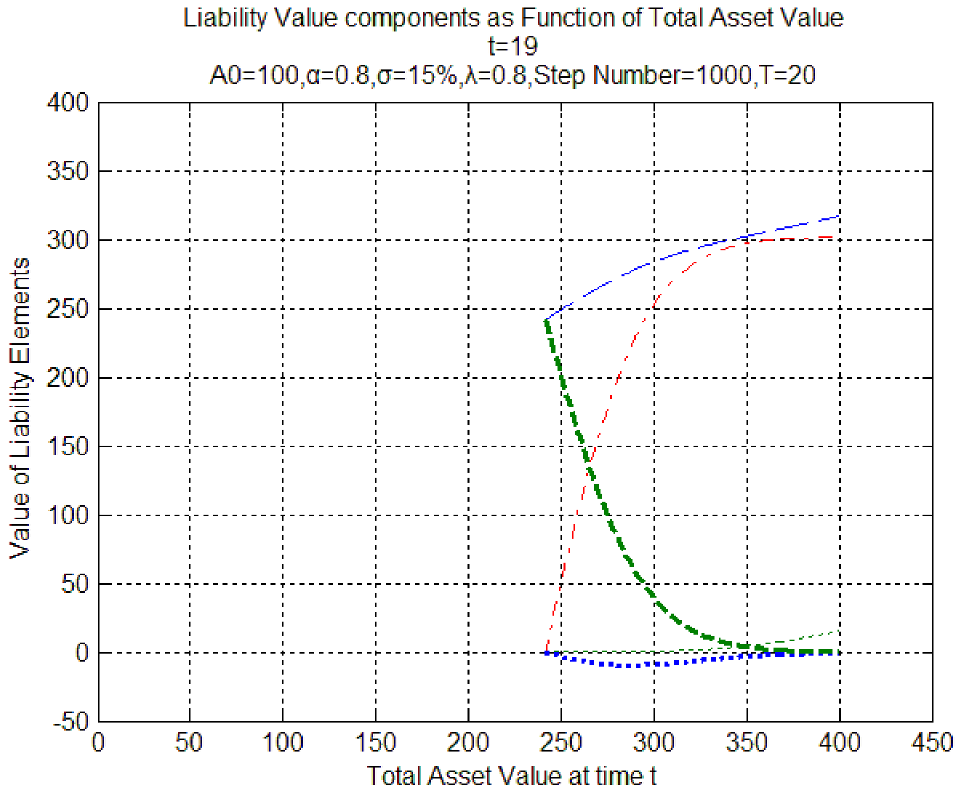
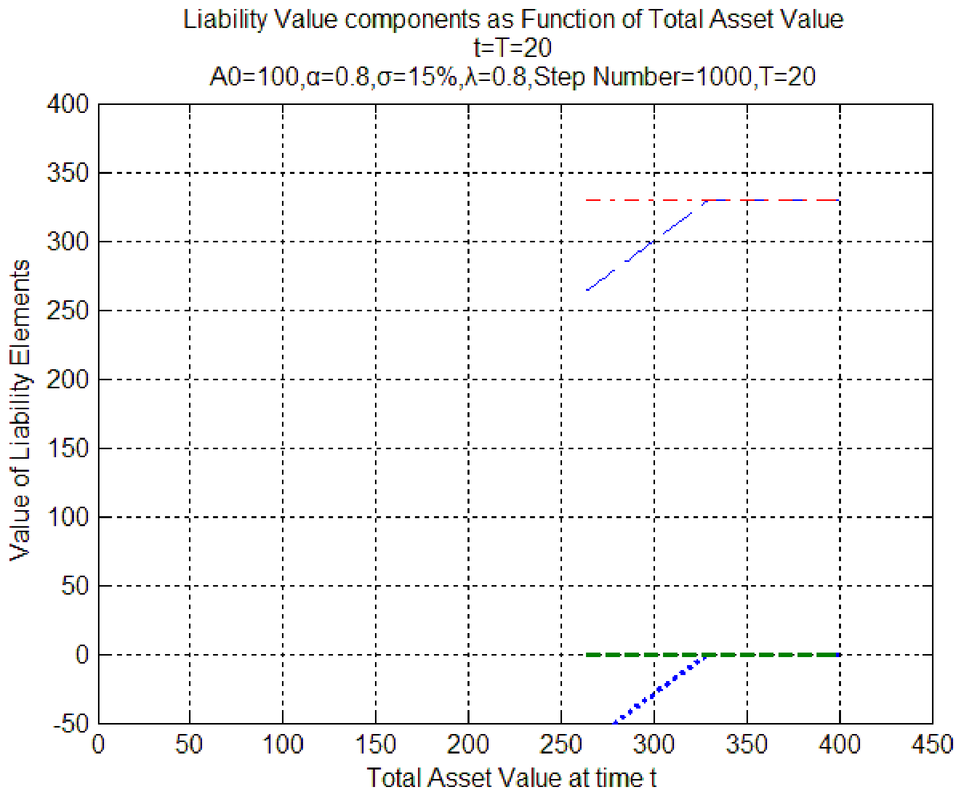


Figure 10e



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Appendix

The process of the liability holders' claim valuation

In this appendix we present a series of corollaries each of which can lead to the theorem which gives the closed-formula for fair value of every component of the liability holders' claim in main context. We begin to research on the time t valuation of the maturity payment elements. Here most methods are based on Grosen and Jørgensen (2002).

First refer to Grosen and Jørgensen (2002) who show the derivation of the defective transition density for Brownian motion with drift and an absorbing barrier at the origin in appendix A. Here we use their results directly. Namely:

$$f(Z,T;z_t,t) = \frac{1}{\sigma\sqrt{(T-t)}}\{N(\frac{Z-z_t-\mu(T-t)}{\sigma\sqrt{(T-t)}})\text{-}e^{-\frac{2z_t\mu}{\sigma^2}}\cdot N(\frac{Z+z_t-\mu(T-t)}{\sigma\sqrt{(T-t)}})\} \qquad \text{(C.1)}$$

Note that the above density function is for Brownian motion of the form like

$$z_u = \mu(u-t)+\sigma(W_u - W_t)+z_t$$

Recall that we have

$$\mathrm{A_u} = A_t \cdot e^{\int_t^u r_s ds - \frac{1}{2}\sigma^2(u-t)+\sigma(W_u - W_t)},$$

and the exponential barrier:

$$B_u = B_0 e^{\int_0^u r_G ds} = B_t e^{\int_t^u r_G ds} = \lambda L_0 e^{\int_0^u r_G ds} = \lambda L_t e^{\int_t^u r_G ds}, u \in [t,T] \qquad \text{(A.1)}$$

Through the whole article we assume that at the valuation date the barrier has not yet been reached i.e $A_t > B_t$.Further we let that $r = r_G$ are deterministic function.

Corollary 1 The bonus (call) option element:

$$E_Q^t\left[e^{-\int_t^T r_s ds}\left(\alpha A_T - L_T^G\right)^+ 1_{\tau \geq T}\right]$$

$$= e^{-\int_t^T r_s ds} E_Q^t\left[\left(\alpha A_T - L_T^G\right)^+ 1_{\tau \geq T}\right]$$

$$= e^{-\int_t^T r_s ds}\int_{B_T}^{\infty}\left[\alpha A_T - L_T^G\right]^+ f\left(A_T, T; A_t, t\right) dA_T$$

$$= e^{-\int_t^T r_s ds}\int_{(B_T \vee \frac{L_T^G}{\alpha})=X}^{\infty}(\alpha A_T - L_T^G) f(A_T, T; A_t, t) dA_T \qquad \text{(A.2)}$$

$$= \alpha\{A_t N\left(\frac{\ln\frac{A_t}{X} + \int_t^T r_s ds + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)$$

$$-\frac{L_T^G}{\alpha}e^{-\int_t^T r_s ds}N\left(\frac{\ln\frac{A_t}{X}+\int_t^T r_s ds-\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)$$

$$-\frac{A_t}{B_t}\left(\frac{B_t^2}{A_t}N\left(\frac{\ln\frac{B_t^2/A_t}{X}+\int_t^T r_s ds+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)-\frac{L_T^G}{\alpha}e^{-\int_t^T r_s ds}N\left(\frac{\ln\frac{B_t^2/A_t}{X}+\int_t^T r_s ds-\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)\right)\}$$

where $f\left(A_T,T;A_t,t\right)$ is the density of A_T with an absorbing barrier B_u as given in (A.1).

Proof of Corollary 1:

We first perform a change of variables in order to be able to use the result (C.1). Note first that

$$A_u > B_u$$

$$\Updownarrow$$

$$A_t \cdot e^{\int_t^u r_s ds-\frac{1}{2}\sigma^2(u-t)+\sigma(W_u-W_t)} > B_t \cdot e^{\int_t^u r_G ds}$$

$$\Updownarrow$$

$$\int_t^u r_s ds-\frac{1}{2}\sigma^2\left(u-t\right)+\sigma\left(W_u-W_t\right) > \ln\frac{B_t}{A_t}+\int_t^u r_G ds \text{, since we choose } r=r_G$$

$$\Updownarrow$$

$$-\frac{1}{2}\sigma^2\left(u-t\right)+\sigma\left(W_u-W_t\right)+\ln\frac{A_t}{B_t}>0$$

Hence, passage of $A(\cdot)$ through $B(\cdot)$ is equivalent to passage of the Brownian motion $Z_u \equiv -\frac{1}{2}\sigma^2\left(u-t\right)+\sigma\left(W_u-W_t\right)+\ln\frac{A_t}{B_t}$ through zero. Define now $z_t=\ln\frac{A_t}{B_t}$ and note that:

$$A_T > (B_T \vee \frac{L_T^G}{\alpha}) \equiv X$$

$$\Updownarrow$$

$$A_t \cdot e^{z_T+\int_t^T r_s ds - z_t} > X$$

$$\Updownarrow$$

$$z_T > \ln\frac{X}{A_t}+z_t-\int_t^T r_s ds$$

$$\Updownarrow$$

$$z_T > \ln\frac{X}{B_t}-\int_t^T r_s ds \equiv q$$

Therefore we can write (A.2) as

$$\alpha e^{-\int_t^T r_s d_s} \int_q^\infty \left(B_t e^{z_T + \int_t^T r_s d_s} - \frac{L_T^G}{\alpha}\right) f(z_T, T; z_t, t)\, d_{z_T}$$

where $f(z_T, T; z_t, t)$ is the defective density of Brownian motion with drift and absorbing barrier at zero. Substituting (C.1) for this, the rest is just tedious calculations. Note that the drift μ is $-\frac{1}{2}\sigma^2$ in our formula. So we get:

$$\alpha e^{-\int_t^T r_s d_s} \int_q^\infty \left(B_t e^{z_T + \int_t^T r_s d_s} - \frac{L_T^G}{\alpha}\right) f(z_T, T; z_t, t)\, d_{z_T}$$

$$= \alpha e^{-\int_t^T r_s ds} \int_q^\infty (B_t e^{z_T + \int_t^T r_s ds} - \frac{L_T^G}{\alpha}) \times$$

$$\left\{ \frac{1}{\sigma\sqrt{(T-t)}} n\left(\frac{Z - z_t + \frac{1}{2}\sigma^2 (T-t)}{\sigma\sqrt{(T-t)}} \right) - e^{z_t} \frac{1}{\sigma\sqrt{(T-t)}} n\left(\frac{Z + z_t + \frac{1}{2}\sigma^2 (T-t)}{\sigma\sqrt{(T-t)}} \right) \right\}$$

$$= \alpha e^{-\int_t^T r_s ds} \{ \int_q^\infty B_t e^{\int_t^T r_s ds + z_t} \frac{1}{\sigma\sqrt{2\pi(T-t)}} e^{-\frac{1}{2}\left(\frac{z_T - z_t - \frac{1}{2}\sigma^2 (T-t)}{\sigma\sqrt{(T-t)}}\right)^2} dz_T$$

$$- \frac{L_T^G}{\alpha} \int_q^\infty \frac{1}{\sigma\sqrt{2\pi(T-t)}} e^{-\frac{1}{2}\left(\frac{z_T - z_t + \frac{1}{2}\sigma^2 (T-t)}{\sigma\sqrt{(T-t)}}\right)^2} dz_T$$

$$- e^{z_t} \int_q^\infty B_t e^{\int_t^T r_s ds - z_t} \frac{1}{\sigma\sqrt{2\pi(T-t)}} e^{-\frac{1}{2}\left(\frac{z_T + z_t - \frac{1}{2}\sigma^2 (T-t)}{\sigma\sqrt{(T-t)}}\right)^2} dz_T$$

$$+ \frac{L_T^G}{\alpha} e^{z_t} \int_q^\infty \frac{1}{\sigma\sqrt{2\pi(T-t)}} e^{-\frac{1}{2}\left(\frac{z_T + z_t + \frac{1}{2}\sigma^2 (T-t)}{\sigma\sqrt{(T-t)}}\right)^2} dz_T \}$$

$$= \alpha e^{-\int_t^T r_s ds} \{ A_t e^{\int_t^T r_s ds} N\left(\frac{-q + z_t + \frac{1}{2}\sigma^2 (T-t)}{\sigma\sqrt{(T-t)}} \right)$$

$$- \frac{L_T^G}{\alpha} N\left(\frac{-q + z_t - \frac{1}{2}\sigma^2 (T-t)}{\sigma\sqrt{(T-t)}} \right)$$

$$- e^{z_t} \frac{B_t^2}{A_t} e^{\int_t^T r_s ds} N\left(\frac{-q - z_t + \frac{1}{2}\sigma^2 (T-t)}{\sigma\sqrt{(T-t)}} \right)$$

$$+\frac{L_T^G}{\alpha}e^{z_t}N\left(\frac{-q-z_t-\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)\}$$

$$=\alpha\{A_t N\left(\frac{\ln\frac{A_t}{X}+\int_t^T r_s ds+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)$$

$$-\frac{L_T^G}{\alpha}e^{-\int_t^T r_s ds}N\left(\frac{\ln\frac{A_t}{X}+\int_t^T r_s ds-\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)$$

$$-\frac{A_t}{B_t}\left(\frac{B_t^2}{A_t}N\left(\frac{\ln\frac{B_t^2/A_t}{X}+\int_t^T r_s ds+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)-\frac{L_T^G}{\alpha}e^{-\int_t^T r_s ds}N\left(\frac{\ln\frac{B_t^2/A_t}{X}+\int_t^T r_s ds-\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)\right)\}$$

□

Corollary 2 The conditional fixed payment element

As a part of the maturity payoff, the liability holders' receive L_T^G on the condition that the knock-out barrier has not been hit. At time t with $A_t > B_t$ and $r = r_G$ are deterministic this payoff element is valued as follows:

$$E_Q^t\left[e^{-\int_t^T r_s ds}L_T^G 1_{\tau\geq T}\right]$$

$$=e^{-\int_t^T r_s ds}E_Q^t\left[L_T^G 1_{\tau\geq T}\right]$$

$$=L_T^G e^{-\int_t^T r_s ds}\int_{B_T}^{\infty} f\left(A_T,T;A_t,t\right)dA_T$$

$$=L_T^G e^{-\int_t^T r_s ds}\times\left\{N\left(\frac{\ln\frac{A_t}{B_t}-\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)-\left(\frac{A_t}{B_t}\right)N\left(\frac{-\ln\frac{A_t}{B_t}-\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)\right\}$$

Proof of Corollary 2:
Changing variables as in Corollary 1 and using (C.1) yields

$$\int_{B_T}^{\infty} f\left(A_T,T;A_t,t\right)dA_T$$

$$=\int_0^{\infty} f\left(z_T,T;z_t,t\right)dz_T$$

$$= \int_0^{\infty} \frac{1}{\sigma\sqrt{(T-t)}} n\left(\frac{z_T - z_t + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}} \right) - e^{z_t} \frac{1}{\sigma\sqrt{(T-t)}} n\left(\frac{z_T + z_t + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}} \right) dz_T$$

$$= N\left(\frac{\ln\frac{A_t}{B_t} - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}} \right) - \left(\frac{A_t}{B_t}\right) N\left(\frac{-\ln\frac{A_t}{B_t} - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}} \right)$$

which establishes the result. □

–

Corollary 3 The put option element

Note first that this payoff element can only be strictly positive when $\lambda < 1$, i.e. when the barrier lies below the curve $L_0 e^{\int_0^u r_G ds}, u \in [0,T]$. The time t value of the put option element is given as follows:

$$1_{\{\lambda<1\}} E_Q^t \left[e^{-\int_t^T r_s ds} \left(L_T^G - A_T\right)^+ 1_{\{\tau \geq T\}} \right]$$

$$= 1_{\{\lambda<1\}} e^{-\int_t^T r_s ds} E_Q^t \left[\left(L_T^G - A_T\right)^+ 1_{\{\tau \geq T\}} \right]$$

$$= 1_{\{\lambda<1\}} e^{-\int_t^T r_s ds} \int_{B_T}^{\infty} \left(L_T^G - A_T\right)^+ f(A_T, T; A_t, t) dA_T$$

$$= 1_{\{\lambda<1\}} e^{-\int_t^T r_s ds} \int_{B_T}^{L_T^G} \left(L_T^G - A_T\right) f(A_T, T; A_t, t) dA_T \tag{A.3}$$

$$= 1_{\{\lambda<1\}} \{ L_T^G e^{-\int_t^T r_s ds} \left[N\left(\frac{\ln\frac{L_T^G}{A_t} - \int_t^T r_s ds + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}} \right) - N\left(\frac{\ln\frac{B_t}{A_t} + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}} \right) \right]$$

$$-A_t \left[N\left(\frac{\ln\frac{L_T^G}{A_t} - \int_t^T r_s ds - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}} \right) - N\left(\frac{\ln\frac{B_t}{A_t} - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}} \right) \right] - \left(\frac{A_t}{B_t}\right) \times$$

$$[L_T^G e^{-\int_t^T r_s ds} \left[N\left(\frac{\ln\frac{L_T^G A_t}{B_t^2} - \int_t^T r_s ds + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}} \right) - N\left(\frac{\ln\frac{A_t}{B_t} + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}} \right) \right] /$$

$$-\frac{B_t^2}{A_t}\left[N\left(\frac{\ln\frac{L_T^G A_t}{B_t^2}-\int_t^T r_s ds-\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)-N\left(\frac{\ln\frac{A_t}{B_t}-\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)\right]]\}$$

Proof of Corollary 3:

Change variables as before. First $A_T > B_T$ implies that $z_T > 0$ .Also we should have

$$A_T < L_T^G$$

$$\Updownarrow$$

$$A_t \cdot e^{z_T+\int_t^T r_s ds - z_t} < L_T^G$$

$$\Updownarrow$$

$$z_T < \ln\frac{L_T^G}{A_t} + z_t - \int_t^T r_s ds$$

$$\Updownarrow$$

$$z_T < \ln\frac{L_T^G}{B_t} - \int_t^T r_s ds$$

setting $q = \ln\frac{L_T^G}{B_t} - \int_t^T r_s ds$ we can rewrite (A.3) as

$$1_{\{\lambda<1\}}\cdot e^{-\int_t^T r_s ds}\int_0^q \left(L_T^G - A_t\cdot e^{z_T+\int_t^T r_s ds - z_t}\right) f\,(z_T,\ T;\ z_t,\ t)\,dz_T$$

$$=1_{\{\lambda<1\}} e^{-\int_t^T r_s ds}\int_0^q (L_T^G - B_t e^{z_T+\int_t^T r_s ds})\times$$

$$\left\{\frac{1}{\sigma\sqrt{(T-t)}}\, n\left(\frac{Z-z_t+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)-e^{z_t}\,\frac{1}{\sigma\sqrt{(T-t)}}\, n\left(\frac{Z+z_t+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)\right\}$$

=…….

$$=1_{\{\lambda<1\}} e^{-\int_t^T r_s ds}\{\int_q^\infty L_T^G\,\frac{1}{\sigma\sqrt{2\pi(T-t)}}\, e^{-\frac{1}{2}\left(\frac{z_T-z_t+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)^2} dz_T$$

$$-\int_0^q B_t e^{z_t+\int_t^T r_s ds}\,\frac{1}{\sigma\sqrt{2\pi(T-t)}}\, e^{-\frac{1}{2}\left(\frac{z_T-z_t-\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)^2} dz_T$$

$$-e^{z_t}(\int_0^q L_T^G \frac{1}{\sigma\sqrt{2\pi(T-t)}} e^{-\frac{1}{2}\left(\frac{z_T+z_t+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)^2} dz_T$$

$$-\int_0^q B_t e^{\int_t^T r_s ds - z_t} \frac{1}{\sigma\sqrt{2\pi(T-t)}} e^{-\frac{1}{2}\left(\frac{z_T+z_t-\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)^2} dz_T)\}$$

$$=1_{\{\lambda<1\}} e^{-\int_t^T r_s ds}$$

$$\{L_T^G e^{-\int_t^T r_s ds}\left[N\left(\frac{\ln\frac{L_T^G}{A_t}-\int_t^T r_s ds+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)-N\left(\frac{\ln\frac{B_t}{A_t}+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)\right]$$

$$-A_t e^{\int_t^T r_s ds}\left[N\left(\frac{q-z_t-\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)-N\left(\frac{-z_t-\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)\right]$$

$$-e^{z_t}(L_T^G\left[N\left(\frac{q+z_t+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)-N\left(\frac{z_t+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)\right]$$

$$-e^{\int_t^T r_s ds}\frac{B_t^2}{A_t}\left[N\left(\frac{q+z_t-\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)-N\left(\frac{z_t-\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)\right])\}$$

$$=1_{\{\lambda<1\}}\{L_T^G e^{-\int_t^T r_s ds}\left[N\left(\frac{\ln\frac{L_T^G}{A_t}-\int_t^T r_s ds+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)-N\left(\frac{\ln\frac{B_t}{A_t}+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)\right]$$

$$-A_t\left[N\left(\frac{\ln\frac{L_T^G}{A_t}-\int_t^T r_s ds-\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)-N\left(\frac{\ln\frac{B_t}{A_t}-\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)\right]-\left(\frac{A_t}{B_t}\right)\times$$

$$[L_T^G e^{-\int_t^T r_s ds}\left[N\left(\frac{\ln\frac{L_T^G A_t}{B_t^2}-\int_t^T r_s ds+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)-N\left(\frac{\ln\frac{A_t}{B_t}+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)\right]$$

$$
-\frac{B_t^2}{A_t}\left[N\left(\frac{\ln\frac{L_T^G A_t}{B_t^2}-\int_t^T r_s ds-\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)-N\left(\frac{\ln\frac{A_t}{B_t}-\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)\right]]\}
$$

□

Corollary 4 (The rebate)

The time t value of the rebate payment to the liability holders assuming $A_t > B_t$ is

$$
\begin{aligned}
&E_Q^t\left[e^{-\int_t^T r_s ds}\left(\lambda\wedge 1\right)L_\tau^G 1_{\tau<T}\right]\\
&= e^{-\int_t^T r_s ds}E_Q^t\left[\left(\lambda\wedge 1\right)L_\tau^G 1_{\tau<T}\right]\\
&= e^{-\int_t^T r_s ds}\int_t^T\left(\lambda\wedge 1\right)L_0 e^{\int_0^\tau r_G ds}\cdot g(\tau;z_t,t)d\tau \qquad \text{(A.4)}\\
&= \frac{\left(\lambda\wedge 1\right)}{\lambda}\times\left[A_t\cdot N\left(\frac{-\ln\frac{A_t}{B_t}-\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)+B_t\cdot N\left(\frac{-\ln\frac{A_t}{B_t}+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)\right]
\end{aligned}
$$

where $g(\tau;z_t,t)$ is the first hitting time density which will be derived below.

Proof of Corollary 4

First we would like to present the first passage time density of geometric Brownian motion through an exponential barrier:

Since $\mathrm{A_u} = A_t\cdot e^{\int_t^u r_s ds-\frac{1}{2}\sigma^2(u-t)+\sigma(W_u-W_t)}$,

and the exponential barrier:

$$
B_u = B_0 e^{\int_0^u r_G ds} = B_t e^{\int_t^u r_G ds} = \lambda L_0 e^{\int_0^u r_G ds} = \lambda L_t e^{\int_t^u r_G ds},\ u\in\left[t,T\right]
$$

Let τ denote the smallest u such that $A_u = B_u$ .Set $z_t = \ln\frac{A_t}{B_t}$ and $\mu = -\frac{1}{2}\sigma^2$ .Do exactly the same calculation as Grosen and Jørgensen (2002) did in appendix B. We can establish that

$$
g(\tau;z_t,t) = \frac{z_t}{\sigma\left(\tau-t\right)^{\frac{3}{2}}}n\left(\frac{z_t+\mu(\tau-t)}{\sigma\sqrt{(\tau-t)}}\right) \qquad \text{(C.2)}
$$

Now substituting for $g(\tau;z_t,t)$ in (A.4) we get

$$\left(\lambda\wedge 1\right)L_t^G\int_t^T e^{-\int_t^\tau r_s ds}e^{\int_t^\tau r_G ds}\cdot\frac{z_t}{\sigma\left(\tau-t\right)^{\frac{3}{2}}}\frac{1}{2\pi}e^{-\frac{1}{2}\left(\frac{z_t+(\mu+\sigma^2\ )\tau(-t}{\sigma\sqrt{(\tau-t)}}\right)^2)}d\tau$$

$$=\left(\lambda\wedge 1\right)L_t^G\int_t^T e^{-\int_t^\tau r_s ds}e^{\int_t^\tau r_G ds}\cdot\frac{z_t}{\sigma\left(\tau-t\right)^{\frac{3}{2}}}\frac{1}{2\pi}e^{-\frac{1}{2}\left(\frac{z_t+(\mu+\sigma^2)(\tau-t)}{\sigma\sqrt{(\tau-t)}}\right)^2}d\tau$$

$$=\left(\lambda\wedge 1\right)L_t^G\frac{A_t}{B_t}\int_t^T g^{\mu+\sigma^2}(\tau;z_t,t)d\tau$$

$$=\frac{\left(\lambda\wedge 1\right)}{\lambda}\times\left[A_t\cdot N\left(\frac{-\ln\frac{A_t}{B_t}-\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)+B_t\cdot N\left(\frac{-\ln\frac{A_t}{B_t}+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}\right)\right]$$

□