GARCH models with Generalized Hyperbolic innovations with application to carbon derivative pricing

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Abstract

The aim of this work is to use a new modelling technique for CO2 emission quotas (EUA), in order to calculate the price of structured products, in particular options on the CO2 allowances. After a short discussion about the specificities of this market, we investigate several GARCH-filtering processes for CO2 emission permits prices. We take interest, in particular, in GARCH models with fractional powers in the autoregressive process of the volatility (the APARCH), and in GARCH models with regime switching (RS-GARCH).
Calibration of GARCH, conducted using the CO2 European Union Allowances (EUA) daily prices from 2005 to 2009, is carried out maximizing likelihood as well as using Bayesian inference in models with too high complexity for relying upon numerical optimization.

We use these modellings under the historical measure to derive a model for options pricing under the risk neutral measure. We compare both approaches, one following the work of Gerber and Siu (1994) using a stochastic discount factor exponential affine, the other one using the recent method developed by Chorro, Guégan and Ielpo (2009) and considering an empirical martingale correction technique
Interestingly, we notice that the GARCH processes with fat-tailed distributions (such as Student or Normal Inverse Gaussian ones) fit better CO2 market data. Option prices calculation still gives evidence to the fact that normal innovations are not satisfactory, but may conduct to rather different conclusions from the fitting study, showing the importance to distinguish historic and risk neutral probabilities.

Keywords: Carbon, EUA, Generalized Hyperbolic Distribution , GARCH modelling, Markov Switching processes, pricing, Incomplete markets, Empirical Martingale Correction.

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Contents

1 Introduction

Human activities, in particular population growth and the development of industry over the last 200 years, have caused an increase in the emission and atmospheric concentration of certain gases, called "greenhouse gases" (primarily carbon dioxide and methane). These gases intensify the natural greenhouse effect that occurs on Earth, which in itself allows life to exist. The man-induced, enhanced greenhouse effect is leading to an increase in the average temperature of the planet that, will potentially cause increasingly severe and perhaps even more extreme disruptions to the Earth's climate, and consequently human activity. As a result, several governments, firms and individuals have taken steps to reduce their greenhouse gas (GHG) emissions either voluntarily, or, increasingly, because of current or expected regulatory constraints. According to Kyoto protocol's provisions, the industrialized countries have to reduce the greenhouse gas emissions by 5 percent in the period 2008-2012, with respect to levels in 1990. Protocol dictates the trading of emission allowances as one of the primary mechanisms through which greenhouse gas emission reduction should be achieved. Thus, the right to pollute is considered to be a tradable asset, with its price determined by market forces of supply and demand.

We define carbon transactions as contracts whereby one party pays another party in exchange for a given quantity of GHG emission permits that the buyer can use to meet its objectives vis-à-vis climate change mitigation.

In the present paper, these contract prices constitute data time series we want to develop models for : we consider a new class of models based on Generalized Hyperbolic innovation of different GARCH processes and we apply the results of price calibration to simple financial products, such as European options. We are particularly interested to find the "best" distribution which characterizes the data we consider.

- The first objective of this paper is to provide a "good" model for CO2 historic prices. We consider a new class of models based on Generalized Hyperbolic (GH) innovations and take special interest in Normal Inverse Gaussian (NIG), particular case in the GH class. Over the time period 2005-2009, by looking at likelihood (and in a second time at moments generated), we compare fitting modellings and observe significant differences in accordance with the models we consider, derived from three competitive classes.

    1. A probabilistic modelling class where the distribution directly fits the log-return time series of daily prices, and where Generalized Hyperbolic distributions outperform Black and Scholes models.
    2. A class of GARCH models with fractional powers in the regressive process of the volatility (the APARCH models). Here, that is the NIG distribution which brings about the best fitting among the APARCH class.
    3. A class derived from GARCH models, with Markov Switching distributions, where Student distributions turn out to catch much information about the market prices.

- Then, we want to understand the evolution of options built on the CO2 allowances. We compute option pricing using Monte-Carlo methods, we focus on year 2009 and calibrate the previous best models on that time period. We retain Generalized Error Distribution (GED), NIG and GH(λ=0.5) distributions that we compare to Gaussian models. Through observations of pricing errors between computed prices and market real prices, we see that risk neutralization turns out to be a challenging issue to pricing computations, and we suggest two strategies for moving toward the risk neutral measure.

  1. An analytic transformation, through a Stochastic Discount Factor.
  2. An empirical method, the Martingalisation.

The paper is organized as follows. We start from real data, and after some transformations due to the impact of the VAT fraud, we justify the importance of GARCH filters with adequate distributions. We deduce the importance of filtering data, and compare different GARCH-processes that we calibrate maximizing likelihood.
We will be interested, in particular, in two classes of GARCH models for calibrating the CO2 emission permits : GARCH models with fractional powers in the regressive process of the volatility (the APARCH), and GARCH models with regime switching (RS-GARCH).
We will notice that for the RS-GARCH process, the likelihood optimization is a priori infeasible, because of the path dependence of each regime at each time, that's what Cai, Hamilton and Susmel first pointed out [1994]. We suggest to use Bayesian inference for calibration of the most complex models, based on GIBBS methods associated with a rejection test so as to filter out simulations which make the likelihood fall out.

Finally, we will compute option pricing using Monte-Carlo methods, and through back-testing strategies and error comparisons from market prices, we will conclude on the efficiency of GARCH filters. We will also look at likelihood and moments generated and see that moving toward the risk neutral measure will turn out to be a challenging issue to pricing computations.

2 Data sets

Our dataset is composed with the CO2 daily prices from April 2005 to December 2009, it is to say around 1150 data. On Figure 1, EUA historic prices from 2005 to 2009 show a very high variability in prices. From this point, we will centre our study on the daily log-return time series $R_t = log(\frac{S_{t+1}}{S_t})$ based on the daily CO2 quotas prices S_t.

2.1 Non Gaussian behaviour

Let us compare how well adapted is a Gaussian distribution to the CO2 quotas exchange market. Therefore, we look at QQ plot and see how this distribution fits the histogram of real data (Figure 2). We add the first moments of the data distributions to highlight the market characteristics.

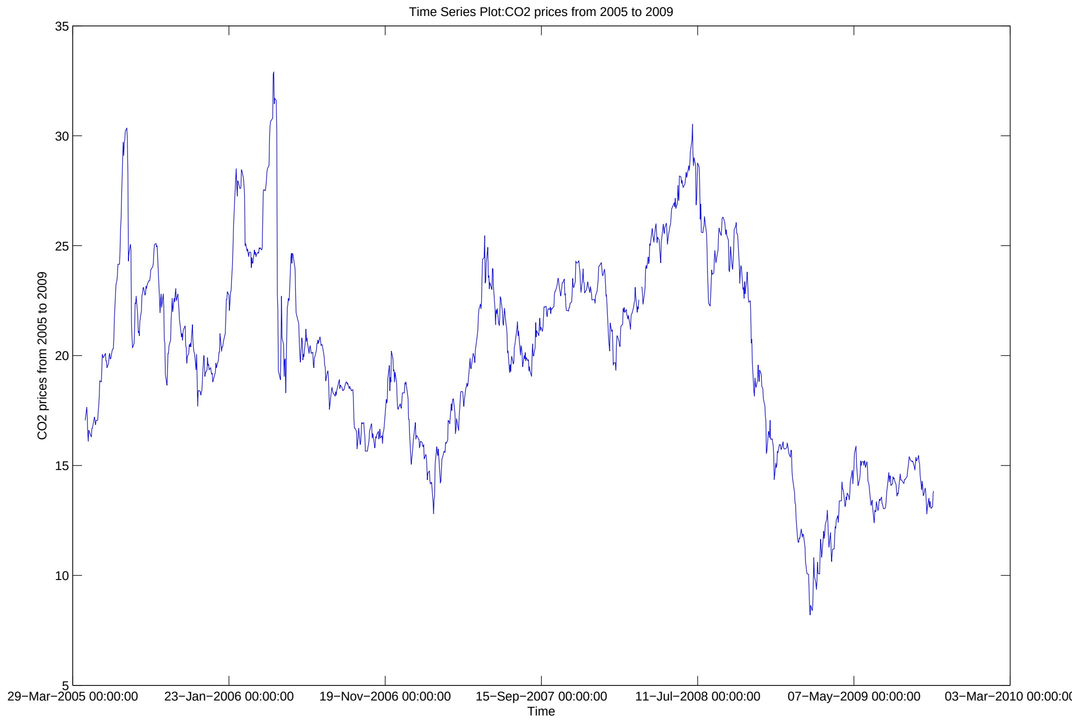

Figure 1: CO2 data from April 2005 to October 2009

	Variance	Skewness	Kurtosis
DATA	6.97×10^{-4}	-1.28	16.91

Table 1: Moments of the daily log-return time series

These plot and table give evidence to three main features of the CO2 market.

- Asymmetric distribution.
- Fat tails.
- High volatility.

The non-Gaussian behaviour of the market leads us to turn to different kinds of distributions in our modellings, such as Student, Generalized Error Distributions (GED) or Generalized Hyperbolic distributions (GH), whose particular class of Normal Inverse Gaussian (NIG) seems really well-adapted to the market (we will introduce and present these distributions later).

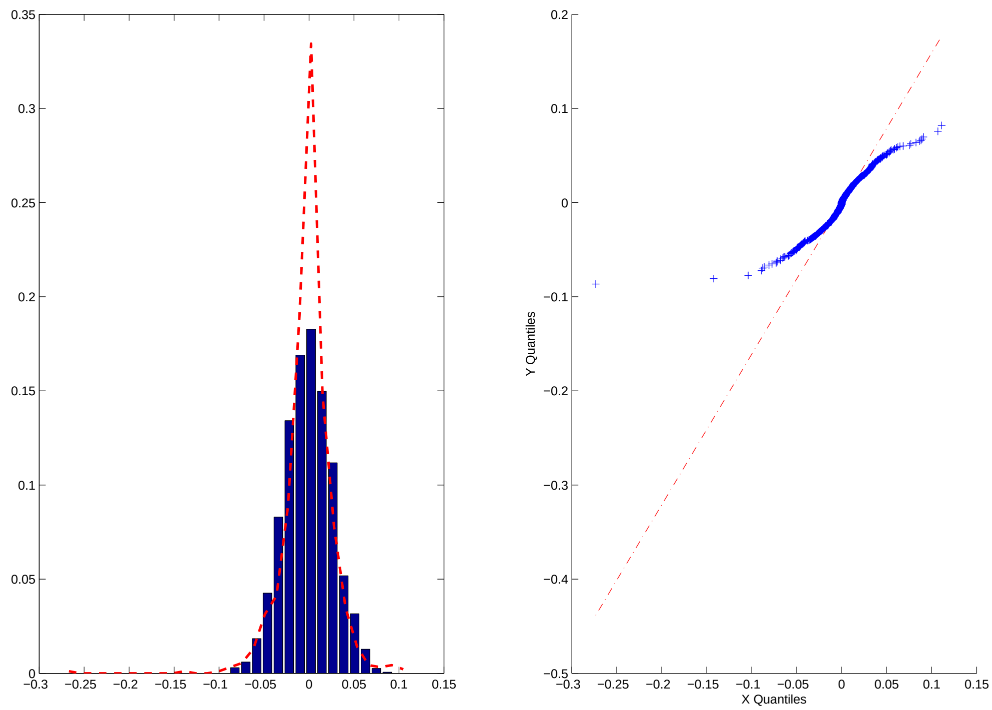

Figure 2: Geometric Brownian Motion (GBM) adapted on EUA. On the left yields histogram and the fitted GBM, on the right QQ plot

2.2 Serial correlation

We can notice some cyclic correlations on the market, mainly every 1 day, 7 days, 12 days and 15 days, as we can see on the Figure 3 representing the autocorrelation functions of the data. This phenomenon results from the VAT fraud which occurred on EUA market until 2009 and which might infer with our fitting modellings. We suggest one solution here to get rid of these disruptions.

We fix this issue by linearly regressing the daily log-return R_t on R_{t-1}, the residual Y_t on Y_{t-7}, and the second residual Z_t on Z_{t-12}, constituting new time series

After filtering out the three main autocorrelation peaks (lags 1,7 and 12), the residual is chiefly cleared out, even if other peaks still exist, but we will not take heed of them.

But Figure 4, representing the autocorrelation functions on the 960 most recent squared log-return data, still shows a high correlation in the prices, and a serial dependence.

Hence, we decide to turn to GARCH filters to adapt our distributions to the serial dependence of our dataset

So both non-Gaussian behaviours and serial correlation of the market lead us to introduce

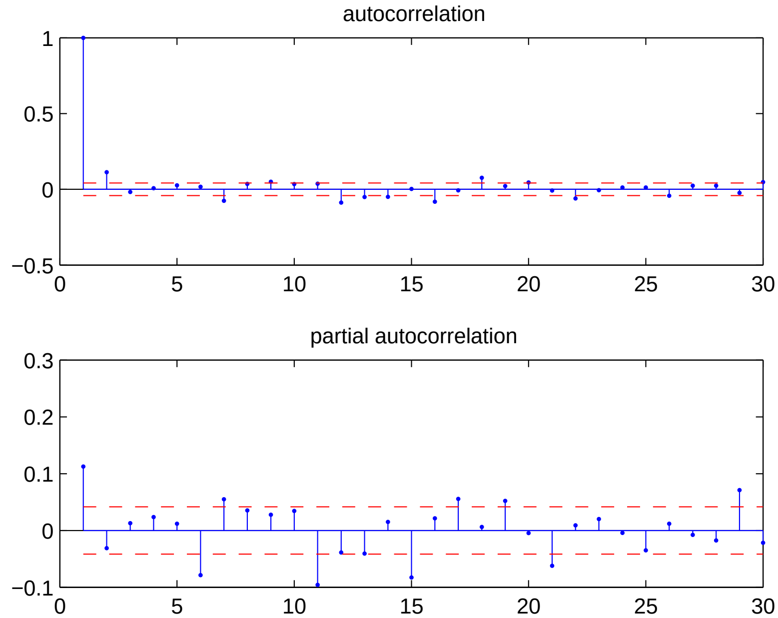

Figure 3: Autocorrelation functions of the daily log-return time series

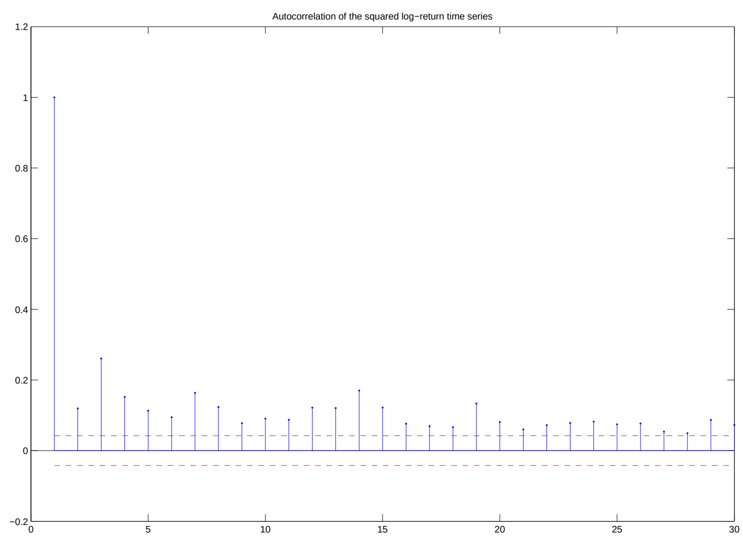

Figure 4: Autocorrelation of the squared daily log-return time series

GARCH models with, among other distributions, Normal Inverse Gaussian innovations (NIG), with the constant idea that the better our fitting will be under the historic probability, the closer to the market the pricing will be under the risk neutral measure in a second part.

3 Modelling on 2005 - 2009

As explained above, we separate our modellings under three classes.

3.1 Modelling with GH distributions

We consider the log-return time series R_t following a Generalized Hyperbolic distribution, whose general form of density is described here :

$$f(x;\lambda;\chi;\psi;\mu;\sigma;\gamma) = \frac{(\sqrt{\psi\chi})^{-\lambda}\psi^{\lambda}(\psi+\frac{\gamma^2}{\sigma^2})^{0.5-\lambda}}{\sqrt{2\pi}\sigma K_{\lambda}(\sqrt{\psi\chi})} \times \frac{K_{\lambda-0.5}(\sqrt{(\chi+\frac{(x-\mu)^2}{\sigma^2})(\psi+\frac{\gamma^2}{\sigma^2})})e^{\frac{\gamma(x-\mu)}{\sigma^2}}}{(\sqrt{(\chi+\frac{(x-\mu)^2}{\sigma^2})(\psi+\frac{\gamma^2}{\sigma^2})})^{\lambda-0.5}},$$

where $K_{\lambda}(x)$ is the modified Bessel function of the third kind:

$$K_{\lambda}(x) = \frac{1}{2}\int_0^{\infty} y^{\lambda-1}e^{-\frac{x}{2}(y+y^{-1})}dy.$$

We particularly focus on the NIG distribution, which corresponds to the specific case with $\lambda = -\frac{1}{2}$ in the previous description. Thus:

$$f(x;-\frac{1}{2};\chi;\psi;\mu;\sigma;\gamma) = \frac{\chi^{\frac{1}{2}}(\psi+\frac{\gamma^2}{\sigma^2})}{\pi\sigma e^{\sqrt{-\psi\chi}}} \times \frac{K_1(\sqrt{(\chi+\frac{(x-\mu)^2}{\sigma^2})(\psi+\frac{\gamma^2}{\sigma^2})})e^{\frac{\gamma(x-\mu)}{\sigma^2}}}{(\sqrt{(\chi+\frac{(x-\mu)^2}{\sigma^2})(\psi+\frac{\gamma^2}{\sigma^2})})}.$$

Through changes of variables that we will not develop here, the NIG distribution is equivalent to the following form of density :

$$f_{NIG}(x;\alpha;\beta;\mu;\delta) = \frac{\delta\alpha\cdot e^{(\delta\gamma+\beta(x-\mu))}}{\pi\cdot\sqrt{\delta^2+(x-\mu)^2}}K_1(\alpha\sqrt{\delta^2+(x-\mu)^2}).$$

What is important to notice is the form of the moments generated by this distribution, and particularly moments of orders 1 to 4 :

$$\mathtt{E}(X) = \mu + \delta\frac{\beta}{\gamma}$$

$$\mathtt{V}(X) = \delta\frac{\alpha^2}{\gamma^3}$$

$$\mathtt{S}(X) = 3\frac{\beta}{\alpha\cdot\sqrt{\delta\gamma}}$$

$$\mathtt{K}(X) = 3 + 3(1+4(\frac{\beta}{\alpha})^2)\frac{1}{\delta\gamma}.$$

So as we can see, the NIG distribution is liable to imply behaviours characterized by fat-tailed and high asymmetry, matching non null skewness and important kurtosis (which depend on the distribution parameters α, β and δ).

The class parameter λ is treated externally and not estimated in the optimisation algorithm.

We will display results for different values of λ, and compare them to the classic GBM model. This class will be denoted "Pure GH models" in the Comparison section.

Let's just see how adapted is the pure NIG distribution to our data. We compute like before the QQplot and fitting for this modelling, viewable on Figure 5. The features of GH and

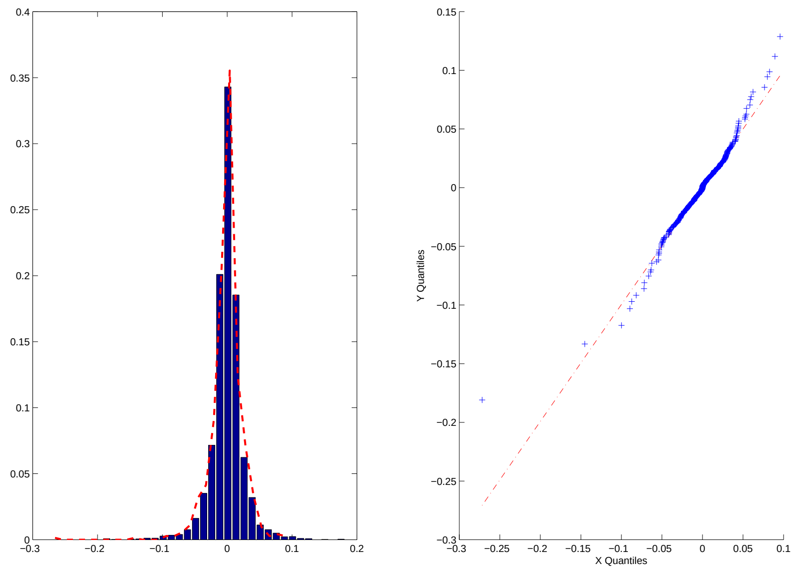

Figure 5: NIG distribution adapted on EUA : on the left yields histogram and the fitted NIG, on the right QQ plot

particularly NIG distributions obviously match the data ones better than GBM does.

3.2 Modelling with GARCH type models and several distributions for residuals

The APARCH processes we will use to filter out data derived from the classic GARCH class of models :

$$R_t = \sigma_t \epsilon_t$$
$$\sigma_t^2 = \omega + \alpha \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2$$

where, ϵ_t can be a Gaussian, GH or GED distribution in our study.

The APARCH models differ in the power we add to both volatility and innovation in the autoregressive process.
We consider this power as a new parameter δ of the model, taking real values.

$$\sigma_t^\delta = \omega + \alpha |\epsilon_{t-1}|^\delta + \beta \sigma_{t-1}^\delta$$

We can notice that in the APARCH equation, the special case with δ=2 matches the classic GARCH model. δ as λ is also manipulated externally, not estimated.

So we estimate 3 GARCH parameters for each distribution, plus 1 parameter for the GED and two for the GH distributions. Indeed, we normalize the GH distribution, so as to get mean and variance respectively equal to 0 and 1, which fixes μ_{GH} and δ_{GH}. Only the two first parameters α_{GH} and β_{GH} remain.

3.3 Modelling with Markov switching approaches

We now develop models constituted with a unique GARCH process, but whose innovation is switching between two distributions ϵ_1 and ϵ_2. The switching probability is considered to be constant, and constitutes a parameter of the model.
We will consider that the regime at time t is independent of the one at time $t-1$.
So the probability at each time to be set in state 1 is p, and in state 2 is $1-p$. We denote by s_t the state variable, taking values in (1,2).

$$R_t = \sigma_t^2 \epsilon_{t,s_t}$$
$$\sigma_t^2 = \omega + \alpha \epsilon_{t-1,s_{t-1}}^2 + \beta \sigma_{t-1}^2$$

In this model, it is much harder to express the likelihood at time t, as we don't know in which regime we are. One possibility, provided by Gray et al. (1996), is to express the global density function as the mean of the density along each state. At each time t,

$$f_{global,t}(x) = p.f_t(x/s_t = 1) + (1-p)f_t(x/s_t = 2)$$

We estimate our models with GIBBS algorithm (see appendix A). Practically, we tested the model where the innovations are switching between two NIG distributions, and two ending estimations are liable to occur. The first one is characterized by a transition probability equal to 0 or 1, and the other one by equal values of parameters of the two NIG distributions. That means that only one regime is detected by the algorithm.
Obviously, the regime detected is the same as the simple GARCH-NIG(1,1) process.
The conclusion here is that a two-steps regime switching GARCH doesn't provide so much information for the modelling of our data, when we consider NIG innovations.
Before applying it to our real data, we tested the model on an artificial GARCH model with a mixture of two different NIG distributions that we simulated, and the calculation always converges to an intermediate single GARCH-NIG model.
Indeed, this fact is quite explainable : the NIG distribution is a kind of generalized hyperbolic distribution, also called normal mean-variance mixture, whose expression is of the type

$$X = \alpha + \beta.A + \sigma.\sqrt{A}.Z$$

with $Z = N(0, 1)$
Consequently, X conditional on the matrix A has a normal distribution with mean $\alpha + \beta.A$ and variance $\sigma^2.A$.

The role of A is to model shocks which can change the volatility. It is to say that jumps are already contained in the NIG distribution, and that's why the likelihood function as it is defined (mean of the density of both regimes along the transition probability) performs as well as the likelihood of a single intermediate NIG distribution, characterizing alone pretty well jumps in the model. This remark obviously applies to Generalized Hyperbolic distributions.

The GED distribution is not suitable neither for regime switching, and turns out to bring about unstable fitting of data. So we only focus on Student and Gaussian innovations in this modelling, as viewable next section.

3.4 Comparison of the modellings

We present the table of results corresponding to the three main classes of models introduces previously: The pure GH models are displayed for λ varying from -1.5 to 1.5, and compared to GBM.

We show the estimation results for GARCH models with different families of innovations and for APARCH models with δ from 0.5 to 3, and λ from -1.5 to 1.5 (results can be visualised on the 3D surface on Figure 6, and the best ones in Table 5)

Finally, we consider GARCH with distributions switching respectively between two Student, and one Student and one Gaussian.

	Log-lik	drift	volatility
GBM	2041.31	-0.1811 [-0.6187 , 0.2565]	0.4242 [0.4059, 0.4447]

Table 2: Estimation results of the GBM model with 95% confidence intervals

The first information we can extract from these results, is that the optimal λ in pure GH models and APARCH-GH models don't match. Whereas the former is between 0.5 and 1 in the pure GH case, plots show that in the APARCH-GH case, the optimal λ ranges from -0.5 to 0.5, even if it globally depends on the choice of δ.

Then, we can see that in APARCH-GH, for each value of λ we still have the best fitting for the lowest values of δ. For higher δ, the estimation is better for λ closer to 0 than to -0.5, adding evidence to the fact that we can get better results than the NIG distribution provides.

	Log-likelihood	α	β	μ	δ
GH($\lambda = 0.5$)	2186.45	37.53 [33.77, 41.30]	0.000 [-0.005, 0.005]	0.0000 [0.0000, 0.0000]	0.0008 [0.0008, 0.0008]
GH($\lambda = 1$)	2176.24	57.39 [51.18, 63.59]	0.000 [-0.012, 0.012]	-0.0001 [-0.0001, -0.0001]	0.0008 [0.0008, 0.0008]
GH($\lambda = 0$)	2175.90	29.60 [26.92, 32.28]	0.000 [-0.010, 0.010]	0.0000 [0.0000, 0.0000]	0.0090 [0.0090, 0.0091]
GH($\lambda = -0.5$)	2170.95	21.75 [19.61, 23.89]	0.000 [-0.032, 0.032]	0.0000 [0.0000, 0.0000]	0.0156 [0.0156, 0.0156]
GH($\lambda = -1$)	2166.78	12.91 [11.37, 14.44]	0.000 [-0.037, 0.037]	0.0000 [0.0000, 0.0000]	0.0215 [0.0215, 0.0215]
GH($\lambda = -1.5$)	2162.69	2.89 [1.74, 4.04]	0.000 [-0.166, 0.166]	0.0000 [0.0000, 0.0000]	0.0278 [0.0278, 0.0279]
GH($\lambda = 1.5$)	2159.20	71.97 [56.67, 87.26]	0.000 [-0.038, 0.038]	-0.0001 [-0.0001, -0.0001]	0.0008 [0.0008, 0.0008]

Table 3: Estimation results of pure GH models model with 95% confidence intervals

	Log-lik	ω	α	β	$\alpha_{distribution}$	$\beta_{distribution}$
GARCH-NIG	2234.27	0.0000 [0.0000 , 0.0000]	0.1387 [0.1378, 0.1397]	0.8613 [0.8612, 0.8613]	0.7319 [0.7213, 0.7426]	0.0000 [0.0000 , 0.0000]
GARCH-GED	2234.02	0.0000 [0.0000 , 0.0000]	0.1305 [0.1299, 0.1311]	0.8499 [0.8497, 0.8501]	1.1000 [1.0925, 1.1075]	- -
GARCH-Student	2232.36	0.0000 [0.0000 , 0.0000]	0.1447 [0.1442, 0.1453]	0.8553 [0.8550, 0.8555]	3.4872 [3.3114,3.6629]	- -
GARCH-Gaussian	2120.00	0.0000 [0.0000 , 0.0000]	0.1381 [0.1363, 0.1398]	0.8306 [0.8285, 0.8328]	- -	- -

Table 4: Estimation results of GARCH models with 95% confidence intervals

For a better view of the model fitting, we plot in Figure 7 the evolution of the log-likelihood of APARCH with respect to δ, in the case of NIG innovations.

Furthermore, the GED distribution seems to be the only one likely to challenge the NIG distribution in GARCH processes.
The Geometric Brownian Motion is outperformed by the whole GARCH processes, and especially the GARCH-Gaussian, showing the importance of the GARCH filtering in data fitting.
Finally, for Regime Switching GARCH models, results always converge to a "classic" process, switching with a very volatile Student one (whose parameter driving the distribution is very closed, but not equal, to its bounds).

Of course, we get a high probability to stay in the non-volatile regime, (even higher than 0.93 in the RS-Student mode). So, assuming that we have about 1000 data, it means that around 50 will fall in the regime with high volatility for the first model (with more explosive Student).

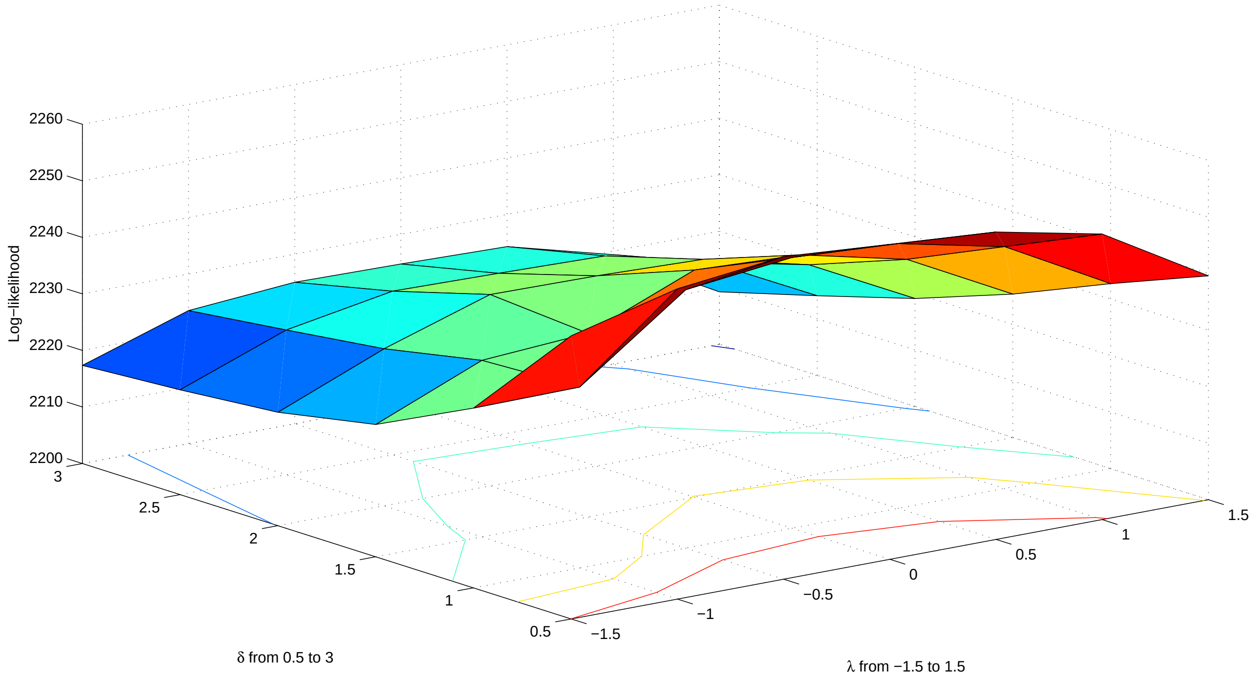

Figure 6: Evolution of the Log-likelihood with δ and λ in the APARCH modelling with GH innovations over time period 2005-2009

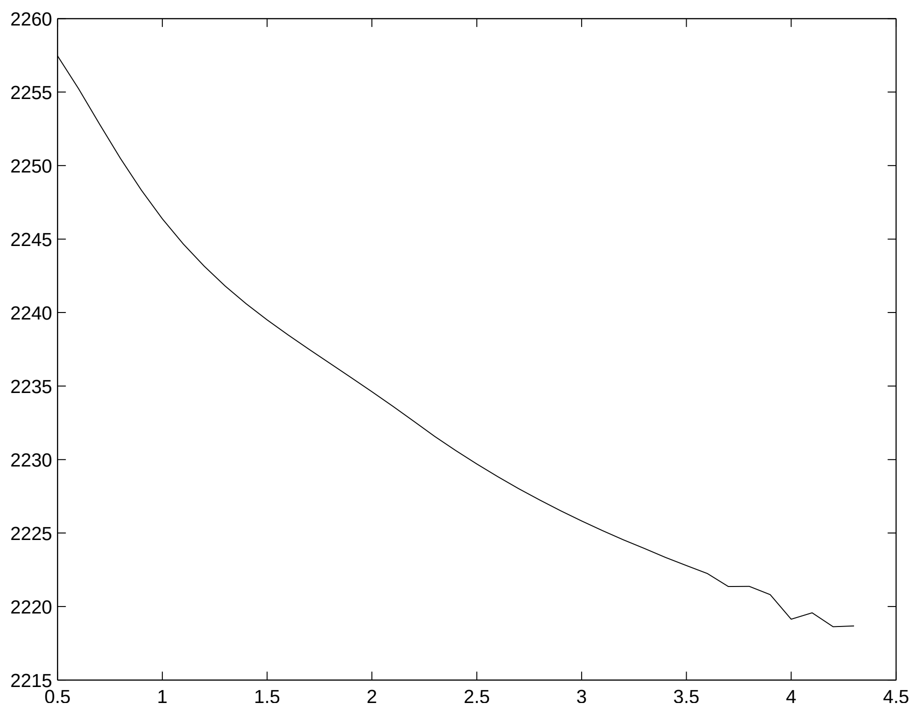

Figure 7: δ influence in the log-likelihood, NIG innovations

	Log-lik	ω	α	β	α_{GH}	β_{GH}
APARCH ($\delta = 0.5$, $\lambda = -0.5$)	2257.06	0.008 [0.008, 0.008]	0.180 [0.179, 0.181]	0.820 [0.819, 0.821]	0.652 [0.642, 0.662]	0.000 [-0.002, 0.002]
APARCH ($\delta = 0.5$, $\lambda = 0$)	2255.70	0.007 [0.007, 0.007]	0.159 [0.158, 0.159]	0.813 [0.812, 0.814]	0.629 [0.621, 0.638]	0.000 [-0.001, 0.001]
APARCH ($\delta = 0.5$, $\lambda = -1$)	2254.86	0.013 [0.013, 0.013]	0.202 [0.201, 0.203]	0.798 [0.797, 0.799]	0.692 [0.676, 0.708]	0.000 [-0.001, 0.001]
APARCH ($\delta = 0.5$, $\lambda = 0.5$)	2254.37	0.007 [0.007, 0.007]	0.134 [0.133, 0.134]	0.787 [0.785, 0.788]	0.491 [0.474, 0.508]	0.000 [-0.001, 0.001]
APARCH ($\delta = 0.5$, $\lambda = 1$)	2250.51	0.006 [0.006, 0.006]	0.096 [0.091, 0.101]	0.750 [0.749, 0.752]	0.288 [0.120, 0.457]	0.000 [-0.000, 0.000]
APARCH ($\delta = 0.5$, $\lambda = -1.5$)	2250.06	0.018 [0.018, 0.018]	0.216 [0.214, 0.217]	0.784 [0.783, 0.786]	0.741 [0.717, 0.766]	0.000 [-0.001, 0.001]
APARCH ($\delta = 1$, $\lambda = 0$)	2247.90	0.001 [0.001, 0.001]	0.144 [0.143, 0.145]	0.843 [0.841, 0.844]	0.648 [0.639, 0.657]	0.000 [-0.001, 0.001]
APARCH ($\delta = 1$, $\lambda = 0.5$)	2246.82	0.000 [0.000, 0.000]	0.098 [0.097, 0.099]	0.818 [0.816, 0.820]	0.509 [0.486, 0.532]	0.000 [-0.000, 0.000]
APARCH ($\delta = 1$, $\lambda = -0.5$)	2245.71	0.001 [0.001, 0.001]	0.176 [0.174, 0.178]	0.824 [0.821, 0.827]	0.746 [0.734, 0.758]	0.000 [-0.002, 0.002]
APARCH ($\delta = 1$, $\lambda = 1$)	2242.77	0.000 [0.000, 0.000]	0.057 [0.057, 0.057]	0.782 [0.780, 0.785]	0.352 [0.345, 0.360]	0.000 [-0.000, 0.000]
APARCH ($\delta = 1.5$, $\lambda = 0$)	2240.16	0.000 [0.000, 0.000]	0.113 [0.113, 0.113]	0.855 [0.854, 0.857]	0.662 [0.653, 0.670]	0.000 [-0.000, 0.000]

Table 5: Estimation results of APARCH models with 95% confidence intervals

	Log-lik	ω_1	α_1	β_1	ν_1	ν_2	Frequency
RS Student	2257.43	0.000 [0.0,0.0]	0.150 [0.150, 0.151]	0.850 [0.849, 0.850]	2.001 [2.001, 2.001]	3.986 [3.587, 4.385]	0.065 [0.064, 0.065]
RS Stud/Norm	2238.03	0.000 [0.0, 0.0]	0.158 [0.156, 0.160]	0.840 [0.840, 0.841]	2.138 [2.118, 2.158]	- -	0.406 [0.378, 0.435]

Table 6: Estimation results for GARCH with innovations switching, with 95% confidence intervals

In appendix B, we develop for the most important models, the main moments corresponding to these estimated parameters, that we compare to data moments.

4 Calibration over the period 2009

The aim of our study is to use the best modellings under the historic probability to price options and compare them to market prices. We have at our disposal a set of option prices issued in 2009, and that is why we compute our calibration only over the year 2009.

The previous study enables us to select in each class the modellings used to price options. Among the GH distributions, depending on the models, the cases λ=-0.5 (NIG) and λ=0.5 outperform the others, and we will select them.
Furthermore, if we look more accurately at the evolution of the likelihood in APARCH-NIG models with respect to δ over the only year 2009 (Figure 8), it is obviously for δ around 1.5 that the fitting is the most realistic. GED innovations are also legitimate for being chosen according to the previous study.

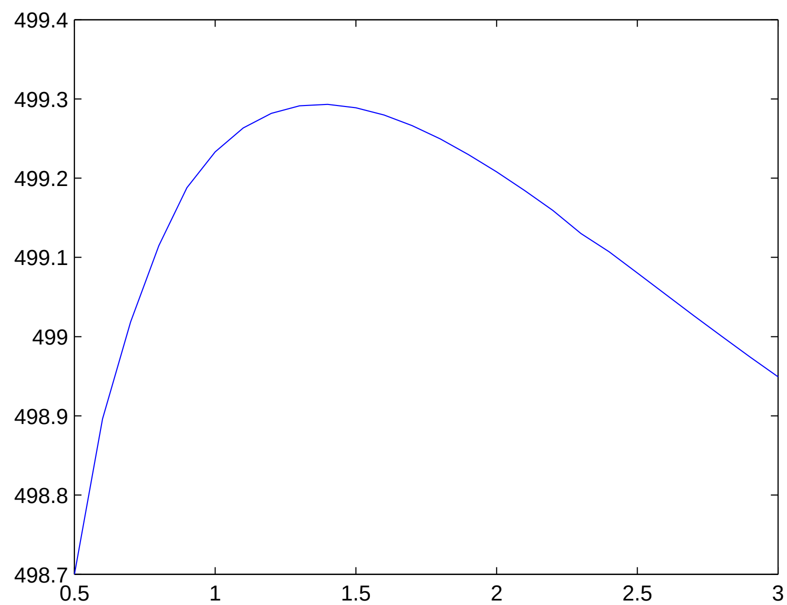

Figure 8: δ influence in the log-likelihood, NIG innovations, over year 2009

In addition, we will compare these models to two Gaussian ones : GBM and the GARCH-Gaussian process.

Finally, Student distributions are unworkable because they present too fat tails that make the underlying path take really high values in the simulation processes, and simulated options prices become unrealistic. That is why we will not take Regime Switching GARCH into account in the next pricing part.

So we summarize the seven models selected for the next section :

- APARCH(δ=1.5 ; λ=0.5) and APARCH-NIG(δ=1.5)
- pure GH (λ=0.5) and pure NIG
- APARCH-GED(δ=1.5)

- GARCH-Gaussian

- GBM, equivalent to Black-Scholes

5 Pricing

The purpose of our study resides in the calculation of option prices, based on new modellings of the CO2 allowances, what has been carried out in the previous sections.
We look at European options, calls as well as puts, whose pricing formula at initial time are recalled for specific strike K and maturity T :

$$C(K,T) = e^{-r.T}.max(S_T - K, 0)$$

for a call option

$$P(K,T) = e^{-r.T}.max(K - S_T, 0)$$

for a put option,

where r is the risk-free annual rate, and S_t the spot price at time t.

This spot price is calculated using Monte-Carlo methods, through which we simulate the daily log-returns from the different selected models. We compute 10000 path samples for each price calculation.
After the fitting study under the historic probability, the stake of this paper is now to price options under the martingale measure. We will present two ways for neutralizing risk in option pricing. The first one is the Stochastic Discount Factor (SDF), based on a specific form inspired from the Randon-Nikodyn form.
The second one is the Empirical Martingale Correction (EMC) and consists in risk neutral constraints on expectation of simulated paths of prices, for avoiding arbitrage opportunities. Both methods have been presented by Chorro, Guegan and Ielpo (2009). Then, we will apply the EMC methods to compare its efficiency to real market data.

5.1 The Stochastic Discount Factor

Let's recall that the Pricing Kernel PK is defined from the ratio of the risk neutral distribution to the historical one. If we denote Q the martingale measure and P the historic one :

$$PK = \frac{dQ}{dP}$$

From this expression, we define the quantity $e^{-r}PK$ as the Stochastic Discount Factor SDF) that we denote $M_{t,t+1}$ between times t and t+1. We start from an hypothesis on the form of the SDF:

$$M_{t,t+1} = e^{\theta_{t+1}R_{t+1}+\zeta_{t+1}}$$

at each time t. By definition of dQ, we have

$$E^P(e^r M_{t,t+1}|F_t) = 1$$

$$E^P(e^{R_{t+1}} M_{t,t+1}) = 1$$

So we get:

$$log(\frac{E^P(e^{(1+\theta_{t+1})R_{t+1}}|F_t)}{E^P(e^{\theta_{t+1}}R_{t+1}|F_t)}) = r$$

In our modellings, we have an expression of R_t under the form $\sqrt{h_t}\epsilon_t$, with ϵ_t, for example, a $NIG(\alpha, \beta, \mu, \delta)$ distribution. In that case we know explicitly the moment-generating function

$$E(e^{z\epsilon_t}) = e^{\mu z + \delta(\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+z)^2})}$$

Hence the previous expression implies at each time t:

$$\sqrt{\alpha^2 - (\beta + \sqrt{h_{t+1}}\theta_{t+1})^2} - \sqrt{\alpha^2 - (\beta + \sqrt{h_{t+1}}(\theta_{t+1}+1))^2} = \frac{r - \mu\sqrt{h_{t+1}}}{\delta}$$

Chorro, Guegan and Ielpo (2009) proved that if there is a solution to this problem, the distribution of ϵ_t under the risk neutral measure becomes at time t $NIG(\alpha, \beta + \sqrt{h_t}\theta_t, \delta, \mu)$. The transformation will occur on the second parameter of the distribution β, bringing about changes in the mean, variance, skewness and kurtosis. So the condition $\beta + \sqrt{h_t}\theta_t < \alpha$ must be respected, involving that

$$\frac{\beta + \sqrt{h_{t+1}}\theta_{t+1}}{\alpha} < 1$$

Thus, the previous equation in θ_{t+1} can be simplified as

$$\alpha(1 - \frac{\beta + \sqrt{h_{t+1}}\theta_{t+1}}{2\alpha^2}) - \alpha(1 - \frac{\beta + \sqrt{h_{t+1}}(\theta_{t+1}+1)}{2\alpha^2}) = \frac{r - \mu\sqrt{h_{t+1}}}{\delta}$$

which is equivalent to

$$\theta_{t+1} = (\frac{2\alpha}{\delta}(r - \mu\sqrt{h_{t+1}}) - 2\beta\sqrt{h_{t+1}})\frac{1}{2h_{t+1}} - \frac{1}{2}$$

So the SDF correction we consider consists in a change into the second parameter of distribution (here it is a NIG, but the result is generalizable to GH distributions) to move toward the risk neutral distribution, but this change is specific to the hypothesis on the form of the Stochastic Discount Factor we did at the beginning. Furthermore, as the CO2 market is incomplete, several different SDF might drive to different changes toward the martingale measure, and this fact brings about comprehensible difficulties in the choice of the SDF. That's why we prefer looking to another way to neutralize risk, based on empirical correction.

5.2 The Empirical Martingale Correction

The Empirical Martingale Correction, also presented by Chorro, Guegan and Ielpo (2009), is a correction in sampled prices so as to assure that the relation $E(S_t) = S_0 e^{-rt}$ is still valuable. Practically, if we compute simulations for the price of the underlying at time T, we will get a sample of N simulations $S_{T,i}$. The EMC resides in the fact to replace the i^{th} path $S_{T,i}$ by

$$S'_{T,i} = \frac{S_{T,i}}{\frac{1}{N}\sum_{k=1}^{N} S_{T,k}} S_0 e^{-rT}$$

So the average of the sample $(S_{T,i})$ is equal to the expected value $S_0 e^{-rT}$, and so the sample respects the risk neutral conditional expectation. This sample constraint is equivalent to a shift in the historic distribution in order to move under the martingale measure.

5.3 Results

We have at our disposal 925 options issued in 2009, with different starting dates, strikes and maturities (December 2009, 2010, 2011 and 2012). They are calls as well as puts.
We applied the modellings (calibrated over 2009) to calculate the error between computed and real prices over the whole 925 market options, for different modellings.

We consider two statistics for comparing models. The Absolute Pricing Error (APE) as well as the Relative Pricing Error (RPE).

$$APE = \sum_{j=1}^{925} \frac{\mid C_{computed}(T_j, K_j) - C_{market} \mid}{925}$$

$$RPE = \frac{1}{925} \sum_{j=1}^{925} \frac{\mid C_{computed}(T_j, K_j) - C_{market} \mid}{C_{market}}$$

Practically, the SDF method might be unstable and bring about drift corrections which would make prices paths diverge.
Nevertheless, we computed statistics for several models under the historic distribution and with an EMC method. We compared them to the ones obtained with Black and Scholes (BS). We separated call and put results in Table 19.
The results are ranked from the best model according to the absolute error to the worse one.

Contrary to the fitting results, the pricing with GH models performs better than the one with APARCH processes.
Furthermore, the Black-Scholes model clearly challenges the other models.

	APE			RPE		
	calls	puts	all options	calls	puts	all options
GHpure ($\lambda = 0.5$)	0.3435	0.2281	0.2858	0.3216	0.2317	0.2766
GHpure ($\lambda = -0.5$)	0.3449	0.2311	0.2881	0.3187	0.236	0.2773
GARCH-Gaussian	0.3454	0.2326	0.2890	0.5104	0.3801	0.4453
GARCH-GED	0.3438	0.2381	0.2909	0.5101	0.3725	0.4413
APARCH ($\delta = 1.5$, $\lambda = -0.5$)	0.353	0.2326	0.2928	0.5558	0.3833	0.4696
BS	0.3446	0.2896	0.3171	0.3238	0.2525	0.2882
APARCH ($\delta = 1.5$, $\lambda = 0.5$)	0.6902	0.7147	0.7024	0.6153	0.5423	0.5788

Table 7: Pricing statistics under EMC correction

Finally, it is very clear and distinct, according to the relative errors (RPE), that APARCH models evolve in a really different way from the "pure" modellings. That implies that APARCH don't perform on the same kind of options as the other models. That point of view prompted us to separate the dataset of options along different moyeness and maturities, and to compare more accurately which model is the most efficient on each set of options.

5.4 In-the-money vs out-of-the-money

We compare the efficiency of the models by distinguishing pricing results between long maturities (more than one year) and short maturities (less than one year) on one side, and options in the money (itm) and out of the money(otm) on the other side (options are considered to be out of the money when the underlying starts at more than two euros from the strike). We compare on Figure 9 the absolute pricing errors for the different models, excepting the APARCH ($\delta = 1.5$, $\lambda = 0.5$), whose behaviour is too far from the other ones.

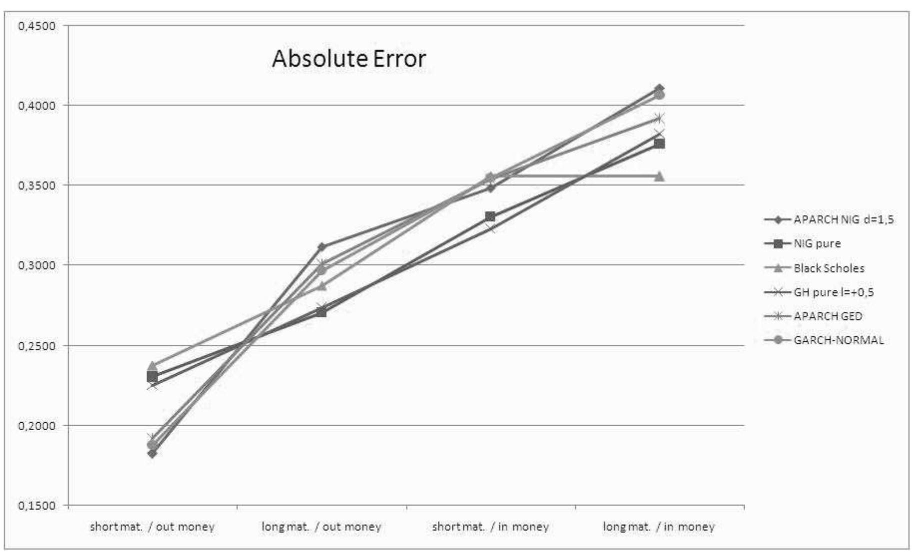

Figure 9: Comparison of pricing efficiency along different moyeness and maturities

Figure 9 points out the strengths and weaknesses of each model :

- Black-Scholes model works the best for long-maturity options in the money, where the fat-tailed property of the market is not so relevant for making NIG distributions be more efficient.
- At the contrary, the APARCH models with GH innovations catch pretty well the extreme behaviours of the market which predominate for short-maturity options out of the money.
- Finally, the flexible feature of the pure GH models make these ones be really satisfying for "intermediate" options (short maturity in the money and long maturity out of the money).

6 Conclusion

Understanding the emission allowances market goes beyond the classic stochastic apprehension of financial assets like commodities, and enters in a more subjective area of behavioural finance.

The main topic of this paper is to propose a modelling that could fit best the historic time series, using the likelihood function as a discriminating factor to rank models' relevance. The CO2 allowance prices show pronounced non-Gaussian behaviour with fat tails and negative skewness. The GARCH models with Generalized Hyperbolic distributions outperform the classic Gaussian models in terms of quantity of information. GH distributions capture far more information than the classic Black-Scholes model because of their ability to be customized, to different skews and tails forms simultaneously. Other rather flexible distributions such as Student or GED ones are still competitive according to the likelihood. But we could observe that the fitting seems to depend on both the model chosen and the distribution selected.

We applied the results of the model calibration on European options pricing.
Even if the Regime Switching GARCH fit pretty well the market, they are unusable for the pricing of European options, too volatile for giving good results. At the contrary, it seems that over recent data (year 2009), an optimal APARCH model exists for δ around 1.5, result that confirm likelihood study, back-testing (see appendix C and D), and pricing errors.

In our case, the carbon market is far from being Gaussian, (see details in section 2), but the pricing results show that Black-Scholes is still very competitive. Indeed, as the CO2 market is rather new, the former Black-Scholes model is still very used by its actors for options pricing so that it represents a reality in the prices (and besides, smile effects are rather low on the CO2 market). That proves that the historic probability differs from the risk neutral one. The best fitting under historic probability may not be the best one looking to comparison with the market, and the incompleteness of the former brings about difficulties to move toward a martingale measure. It is the reason why we used an empirical correction to neutralize risk.

Then, looking at pricing results, we could identify the particular performing of GARCH models with NIG distributions for options out of the money and with short maturities, where the explosive and fat-tailed features of these models are well-adapted. Globally, once again, the pricing performing is likely to be much dependent on both the model and the distribution chosen.

Eventually, I could observe that the CO2 market, in spite of its 5 years of existence, is really new and not liquid enough for basing our modelling judgement on the only pricing errors. More liquidity would allow us to calibrate dynamically complex models, and not over the whole year 2009 as we did in this study, and that may be possible in the next years, as the trading volume keeps increasing each day.
Furthermore, more than a comparison to real market option prices, the pricing from the modellings presented in this paper would be really interesting to see if options on the market are overvalued or undervalued, and if hedging strategies based on market options prices would make us earn or lose money.
Finally, there is a real challenge about building a coherent risk neutral measure, and as long as the illiquidity issue will exist, this question will really be at stake. And I am still convinced that the better our fitting is under the historic measure, the closer to the market it should be after perfectly moving toward the risk neutral measure.

A The GIBBS method

We want to estimate the parameters of a RS-GARCH, with for example NIG innovations.

$$R_t = \sigma_t^2 \epsilon_{t,s_t}$$

$$\sigma_t^2 = \omega + \alpha \epsilon_{t-1,s_{t-1}}^2 + \beta \sigma_{t-1}^2$$

where $\epsilon_{1,t}$ and $\epsilon_{2,t}$ have NIG($\alpha_{NIG1}, \beta_{NIG1}$) and NIG($\alpha_{NIG2}, \beta_{NIG2}$) distributions. So we have 8 parameters in that case (counting the state probability), too many for relying upon numerical optimisation

We start with random values for each parameter (from their existence interval), and run iterations as following:
At the iteration i, we express the posterior density of each parameter conditionally on the values of the others. We simulate from this conditional density, one value of the parameter considered. For example, from the conditional density of ω, we simulate a value ω_i. If this value makes the log-likelihood function increase, we keep it as the value of ω. Otherwise, we keep the previous value.
Then we repeat these operations over the remaining parameters. So we get Markov chains, representing draws of each parameter, conditional on the remaining ones.
If we cannot directly simulate under the density of the parameters (and that's our case here), we evaluate the cumulative function of the density kernel over a grid of points from the definition interval of each parameter.
We can numerically inverse this cumulative function and by simulating a uniform distribution, we extract a simulation of our complex initial distribution.
At the iteration i, if we have a grid of points $a_1, ..., a_n$ in the interval of definition of ω, we get a sample $(\phi_j)_{j=1..n}$ of the cumulative function :

$$\phi_j \approx \int_{a_1}^{a_j} \kappa(\omega | \alpha_{i-1}, \beta_{i-1}, \alpha_{NIG1,i-1}, \beta_{NIG1,i-1}, \alpha_{NIG2,i-1}, \beta_{NIG2,i-1}, p_{i-1}) d\omega_1$$

And we get an estimation of ω_i by a simulation of the uniform distribution between a_1 and a_n.

$$u = U([a_1, a_n])$$

$$\omega_i = \phi_n^{-1}(u)$$

B Comparison of moments

In order to add evidences to the analysis of models, we compute the comparison of moments of order 3 and 4, it's to say skewness and kurtosis. From the parameters estimated, we can rebuild random time series from which we extract these moments, with Monte-Carlo methods, from 100000 samples of time series.
The results are in Table 11. We used the kurtosis as a criterion in order to select APARCH models for which we display moments. We do not display APARCH models whose kurtosis is further than 2 from data kurtosis. The study of kurtosis might provide much more information

	Skewness	Kurtosis
DATA	-1.28	16.91

Table 8: Skewness and Kurtosis of real DATA

than skewness and actually, in each model, skewness is really weak, and logically null in some cases (Normal and GED distributions). Indeed, in the estimation of GARCH models with NIG distributions, we usually hit upon parameters β_{NIG} equal to 0, involving null skewness. So this implies that the fat-tailed feature of data is much more important that the asymmetric characteristic in the modelling, and the fitting of skewness doesn't matter so much, in comparison with the kurtosis.

Then, we can compare models. In pure GH models, kurtosis decreases as λ increases, giving a kurtosis closed to the data one for $\lambda = -1$.
With the add of GARCH filters, results change, and seem once again to be better for lower δ. What is not shown on these tables is that for δ higher than 2, estimation of parameters shows less stability, and kurtosis is liable to take more extreme values, around 30.

Eventually, we deduce from Table 11 that Normal distributions have too small tails, whereas at the contrary Student distributions provide tails really too fat for our data. That's why the RS-GARCH composed with both is not so bad, and generates intermediate kurtosis.
Furthermore, the far values of GARCH-Student kurtosis may involve a divergence of results in pricing based on models with these distributions, what we indeed observed.

The GED distribution seems to be the most satisfying distribution among the non-NIG ones according to these moments.

	Skewness	Kurtosis
APARCH($\delta = 0.5$, $\lambda = -1$)	-0.02	15.46
APARCH($\delta = 0.5$, $\lambda = -0.5$)	0.00	17,84
APARCH($\delta = 0.5$, $\lambda = 0.5$)	-0.04	17.05
APARCH($\delta = 0.5$, $\lambda = 1$)	0.03	16.11
APARCH($\delta = 1$, $\lambda = 1$)	-0.07	17.41
APARCH($\delta = 1.5$, $\lambda = 1$)	0.02	17.40
APARCH($\delta = 2$, $\lambda = -0.5$)	-0.01	17.97
APARCH($\delta = 2$, $\lambda = 0$)	0.00	18.42
APARCH($\delta = 2$, $\lambda = 1$)	-0.02	17.44
APARCH($\delta = 2.5$, $\lambda = -1.5$)	0.01	15.16
APARCH($\delta = 2.5$, $\lambda = -1$)	0.06	16.78
APARCH($\delta = 2.5$, $\lambda = 0$)	-0.06	18.63
APARCH($\delta = 3$, $\lambda = 0$)	0.06	18.26
APARCH($\delta = 3$, $\lambda = 1.5$)	-0.01	16.82

Table 9: Skewness and Kurtosis of best APARCH models

	Skewness	Kurtosis
GH($\lambda = -1.5$)	-0.15	22.94
GH($\lambda = -1$)	-0.02	14.28
GH($\lambda = -0.5$)	-0.01	11.62
GH($\lambda = 0$)	-0.00	9.75
GH($\lambda = 0.5$)	0.02	8.48
GH($\lambda = 1$)	0.01	5.87
GH($\lambda = 1.5$)	0.01	4.9

Table 10: Skewness and Kurtosis of pure GH models

	Skewness	Kurtosis
GARCH - Normal	0.00	4.29
GARCH - GED	0.00	12.27
GARCH - Student	-0.15	100.42
RS (GARCH switching) - Normal	0.00	6.31
RS (GARCH switching) - Student/Normal	0.00	14,34
RS (innovations switching - Student)	0.00	95.61
RS (innovations switching - Student/Normal)	-0.11	90.04

Table 11: Skewness and Kurtosis of models with non-NIG distributions

C Back-testing options pricing under historical probability

We present back-testing strategies for comparing pricing under the historic probability.
We will take interest in hedging strategies, so as to see which models fit the best the dynamics of the underlying constituted by the daily CO2 allowance's price.
We consider a strategy through which we sell a European call option C, with certain strike and maturity, and cover our position with the buying of underlying S.

To avoid all ambiguities, we now denote by Δ the Greek, corresponding to the differential of our portfolio's value with respect to the underlying price, different from δ the parameter corresponding to the power of the autoregressive process in the APARCH model.

So, in order to make this portfolio delta neutral, we buy Δ underlying for each option we sell.
Our portfolio can then be written at each time as :

$$V_t = -C_t + \Delta_t . S_t$$

The initial value of the CO2 allowance at the beginning of 2008 is 23.46, and we consider the option expiring at the end of December 2008, and being at the money the 1st of January 2008. We hedge the strategy every 3-days, starting in January 2008 until the end of December 2008 (around 85 hedging days). At each hedging time, we need to simulate the new option price, as well as Δ. For that hedging strategy, we compute the parameters estimations over 2008.

The purpose is to look at the Profit and Loss (PL), between two hedging times in a row, so as to compare the cumulative PL at the expiration date.

Knowing that at the maturity, the option will end out of the money, we also look at the symmetric portfolio, consisting of a short position in a European put option P, and the buying of Δ underlying (but Δ is negative in the case of a put).

$$V_t = -P_t + \Delta_t . S_t$$

So in these cases, the PL between time t and t+1 are respectively:

$$PL_{t,t+1} = -(C_{t+1} - C_t) + \Delta_t (S_{t+1} - S_t)$$
$$PL_{t,t+1} = -(P_{t+1} - P_t) + \Delta_t (S_{t+1} - S_t)$$

C.1 Back-testing calls

We represent on the figures 10 and 11 the simulated Δ and cumulative PL over the whole year 2008, for a Black-Scholes model (green), a Switching regime between two Black-Scholes models (yellow), an APARCH-NIG process($\delta = 1.5$) (cyan), GARCH models with GED innovations (black) and Normal innovations (red), and finally for a pure NIG model (blue). Each calibration has been done over the whole year 2008.

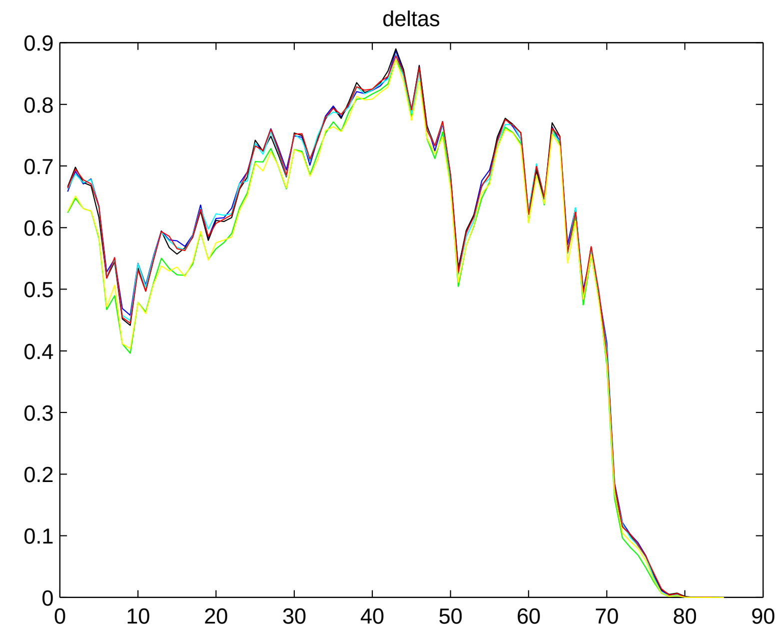

Figure 10: Δ simulated over 2008 for call options

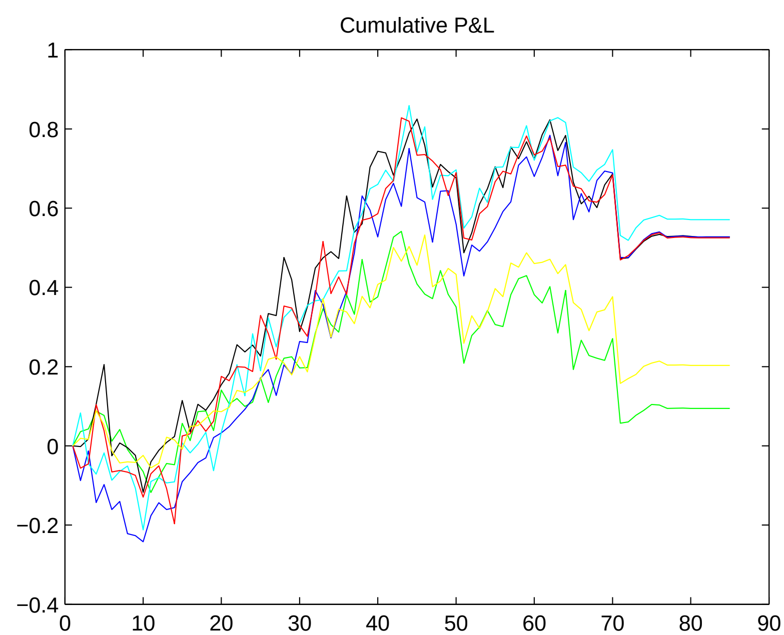

Figure 11: Cumulative PL over 2008 for a call option hedging

We can clearly notice from these figures, that there is a small gap between the 2 Black-Scholes models (simple and switching) on one side, and the GARCH models plus the pure NIG model on the other side. So once more we can distinct that Black-Scholes models behave in another way as other models.

Δ are really closed among these different models, even if once againg we can distinct Black-Scholes from the other models.

Results are also summarized in Table 12.

Model	Cumulative PL
APARCH-NIG (1.5)	0.5709
Pure NIG	0.5276
GARCH-GED	0.5273
GARCH-NORMAL	0.5253
Switching Black-Scholes	0.2029
Black-Scholes	0.0943

Table 12: Final PL for the call strategy over 2008

C.2 Back-testing puts

As we did for the call options, we compute the Δ time series and the cumulative PL, viewable on Figure 12 and Figure 13 with the same colour code as before. The results are also summarized in Table 13. Apparently the Black-Scholes models, which seemed to constitute the worse ones

Model	Cumulative PL
Switching Black-Scholes	-1.0060
Black-Scholes	-1.0657
APARCH-NIG (1.5)	-1.1949
GARCH-NORMAL	-1.1953
GARCH-GED	-1.2024
Pure NIG	-1.2103

Table 13: Final PL for the put strategy over 2008

in a call option hedging strategies, are more efficient concerning a strategy with put options. The most logical idea to have a final opinion about models efficiency is to consider a portfolio consisting of selling both a call and a put option. That is a straddle.

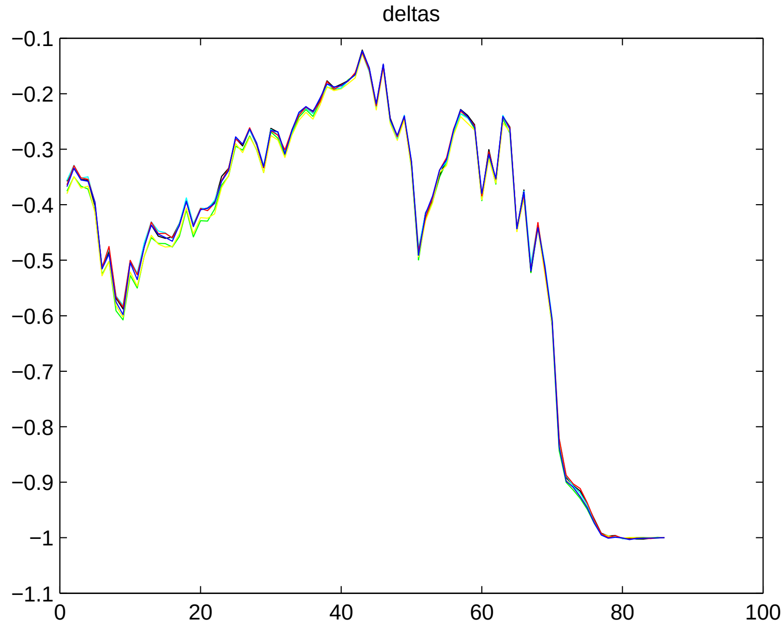

Figure 12: Δ simulated over 2008 for put options

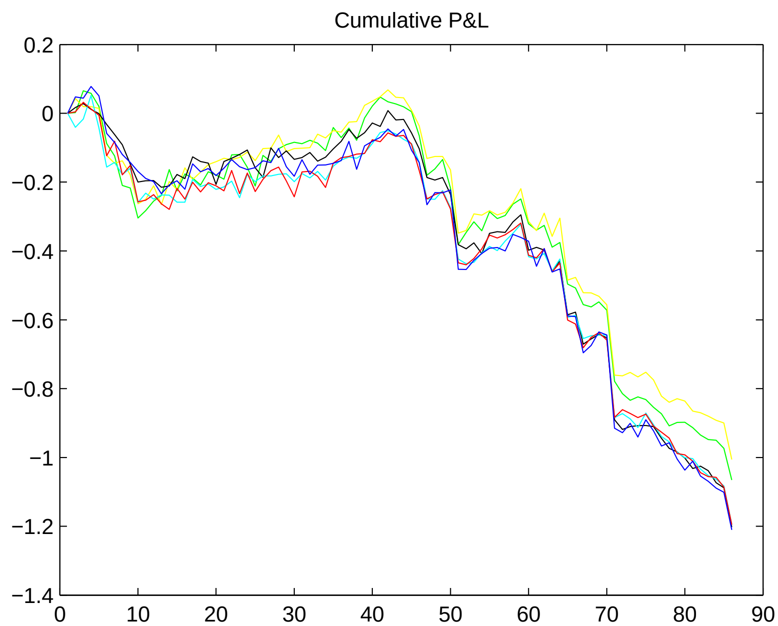

Figure 13: Cumulative PL over 2008 for a put option hedging

C.3 Back-testing straddles

The delta neutral portfolio, composed with straddle and underlying, becomes at a time t :

$$V_t = -C_t - P_t + (\Delta_{t,call} + \Delta_{t,put}).S_t$$

So the corresponding PL are :

$$PL_{t,t+1} = -(C_{t+1} + P_{t+1} - C_t - P_t) + (\Delta_{t,call} + \Delta_{t,put}).(S_{t+1} - S_t)$$

We obtain from Monte-Carlo simulations the simulated Δ and cumulative PL of the straddle strategy :

We add in Table 14 the results for the GARCH-NIG model

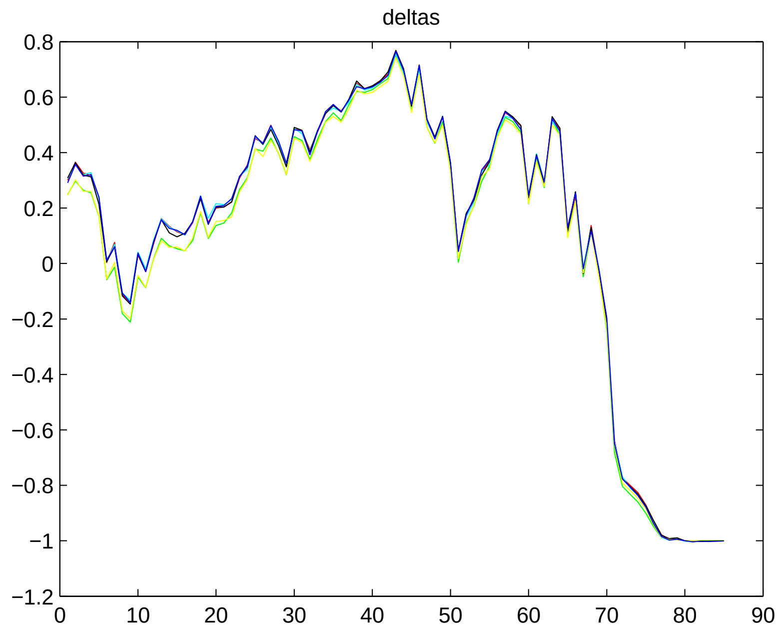

Figure 14: Δ simulated over 2008 for a straddle strategy

We can notice that the differences between models mainly come from differences of pricing, more

Model	Cumulative PL
APARCH-NIG (1.5)	-0.6239
GARCH-NORMAL	-0.6700
GARCH-GED	-0.6751
Pure NIG	-0.6827
GARCH-NIG	-0.6849
Switching Black-Scholes	-0.8030
Black-Scholes	-0.9714

Table 14: Final PL for the straddle strategy over 2008

than from the contribution of hedging.

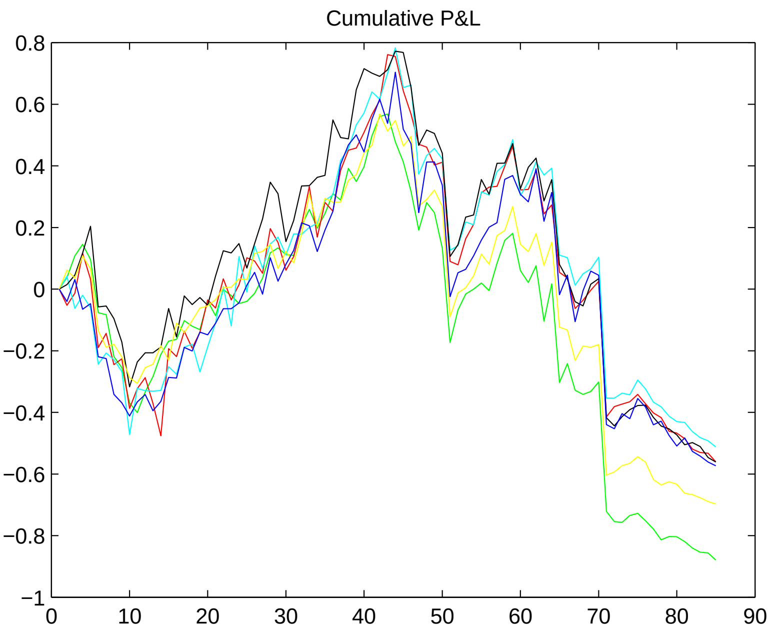

Figure 15: Cumulative PL over 2008 for a straddle hedging

Nevertheless, the use of GARCH filtering processes brings a real change in this covering strategy, as we can see on the previous Δ plots, which differ from GARCH models to non-GARCH models.

The GARCH efficiency doesn't really depend on the distribution we choose for the innovations. The final PL is almost the same for Normal and GED distributions, and slightly higher than the one provided by the pure NIG and the GARCH-NIG models.

Finally, the best APARCH model in the back-testing is the one whose parameter δ is set around 1.5. For too low values of δ, the option prices diverge (lower than 0.3, they become totally irrelevant). For more results about the optimal δ, see appendix D.

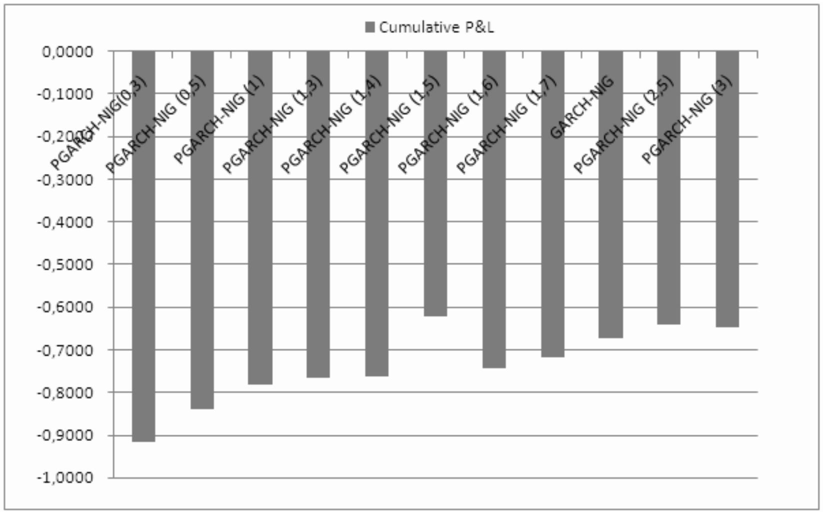

Figure 16: Final PL for the straddle strategy with APARCH processes over 2008

D Optimal APARCH process in back-testing

We now try to set more accurately the optimal value of δ in the back-testing results.
We repeat the previous back-testing process with a straddle strategy, for several values of δ from 0.3 to 3, and summarize the results in Table 15 and Figure 16. We focus on values of δ around 1.5.

The back-testing over 2008 is optimal for δ equal to 1.5. To check if this optimal value is global

Model	Cumulative PL
APARCH-NIG (1.5)	-0.6223
APARCH-NIG (2.5)	-0.6418
APARCH-NIG (3)	-0.6479
APARCH-NIG (2)	-0.6736
APARCH-NIG (1.7)	-0.7191
APARCH-NIG (1.6)	-0.7441
APARCH-NIG (1.4)	-0.7627
APARCH-NIG (1.3)	-0.7659
APARCH-NIG (1)	-0.7806
APARCH-NIG (0.5)	-0.8393
APARCH-NIG (0.3)	-0.9162

Table 15: Final PL for the straddle strategy with APARCH processes over 2008

or specific to 2008, we back-test over 2009, from January to November, and compute the same comparison along the different APARCH processes.

Our results are in Table 16 and Figure 17.

Model	Cumulative PL
APARCH-NIG (0.5)	-0.5298
APARCH-NIG (0.3)	-0.5384
APARCH-NIG (1)	-0.5788
APARCH-NIG (1.3)	-0.5794
APARCH-NIG (1.4)	-0.6134
APARCH-NIG (1.6)	-0.6136
APARCH-NIG (1.5)	-0.6231
APARCH-NIG (1.7)	-0.7157
GARCH-NIG	-0.8156
APARCH-NIG (2.5)	-0.8429
APARCH-NIG (3)	-1.0329

Table 16: Final PL for the straddle strategy with APARCH processes over 2009

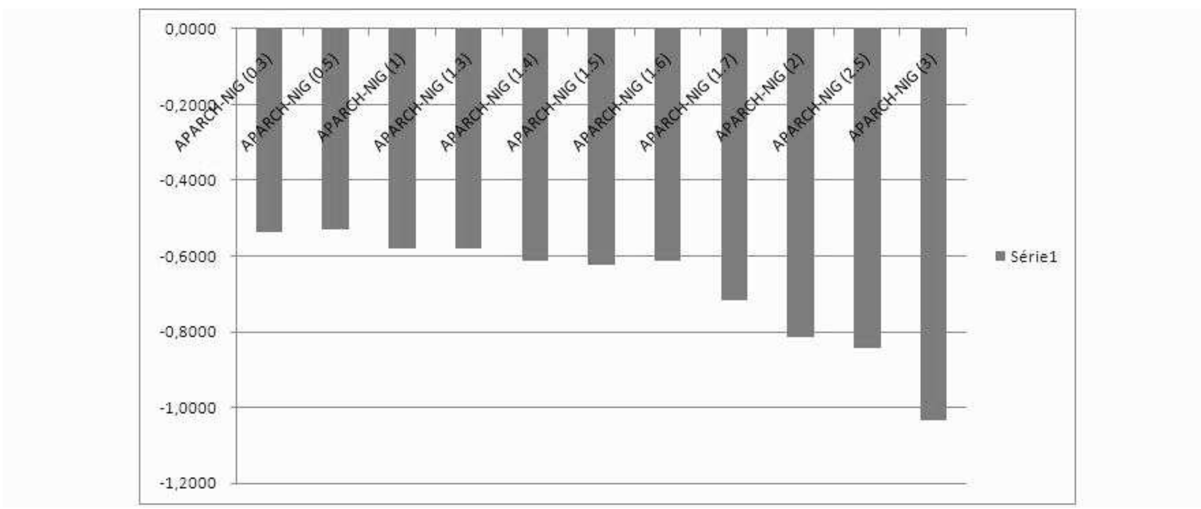

Figure 17: Final PL for the straddle strategy with APARCH processes over 2009

It's interesting to see that the back-testing efficiency becomes globally a decreasing function of δ over the year 2009. Nevertheless, decreasing from high values, the optimality starts when δ borders 1.6. Lower than 1.6, the back-testing remains more or less constant, excepting for very low values of δ. As these extremely low values of δ match a highly inefficient back-testing over 2008, we will not take them into account, and we can conclude that the optimal parameter δ of the APARCH model with NIG innovations is between 1.5 and 1.6.

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