Implementing and Testing Replicating Portfolios for Life Insurance Contracts

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Abstract

Due to the new Solvency II regulation, European insurance companies need to stress-test their balance sheet under various risks. These tests may need Monte Carlo methods, which can be very time-consuming when used for simulations on the entire liability portfolio. Using instead a replicating portfolio of financial assets that matches the company's liability increases computational efficiency.

In this thesis, we study different methods to compute these replicating portfolios, and test their robustness. We show how they can be implemented for some types of contracts, but may be inefficient for others.

Acknowledgements

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Chapter 1

Introduction

Estimating the balance sheet of a life insurance company is not an easy thing, as such companies deal with contracts involving payments on the very long run (40, 50 or 60 years from now). But still it is important for them to understand how the value of their contracts will evolve in time, how much they will be worth in 5 or 10 years from now.

A new legislation within the European Union forces insurance companies to stress test their balance sheet for various risks in order to reduce them and ensure solvency. Unfortunately, performing these stress tests requires a large amount of calculations involving stochastic simulations which can be very heavy for companies that have many different policyholders and policies.

One method to speed up calculations is the use of a **replicating portfolio** based on various types of assets such as bonds, equities, derivatives and real estate to match as closely as possible the premiums and payment obligations the insurance company will have. As this replicating portfolio is just a means of running calculations, it can be fictive in the sense that we do not need to invest a part of our capital in it. It is sufficient to be able to valuate it under different scenarios to match our requirements. Thus in this portfolio it is common to find zero-coupon bonds with maturity up to 50 years, or other assets that do not exist in the market.

The need to have good models for these fictive assets is therefore overriding, so in this thesis we will also take the time to test the replicating portfolios under different models.

Here is a brief description of the type of contract used in the whole thesis. A more detailed explanation of this contract will follow later in Section 3.1. We consider a **retirement pension** (*ålderspension*) where the policyholder pays yearly premiums until he or she is 65, and then gets back a yearly

random amount called **benefit** from 65 to 85. This random benefit is proportional to the company's yearly return but cannot be less than 3% a year, the **guaranteed rate**.
If the company's return is less than the guaranteed rate, the company might need to make **capital injections**, that is to say to take money from the buffer capital to pay the policyholders back.
Moreover the policyholder can surrender the contract at any moment. If for example the benefit is not likely to exceed the guaranteed benefit of 3%, the policyholder may think of investing his or her money somewhere else. If the policyholder surrenders, he or she gets the money back at the guaranteed rate.
If the policyholder dies, he or she will stop paying premiums (if retirement age is not reached) and his or her family will get the benefits.

The aim of this thesis is to replicate these capital injections by the means of bonds and financial products based on the company's return, such as options. Basically the whole work is to simulate the yearly returns of the company under different scenarios, compute the capital injections for each scenario, and then try to find a portfolio of financial products that matches as closely as possible the capital injections. To somewhat enlarge the scope of the thesis, we will also try to see how the replicating portfolio is affected by the choice of the scenarios.

Chapter 2

Generating Scenarios

As capital injections depend strongly on the company's return, we will have to simulate this return over several years and for different scenarios. One problem is that the real asset portfolio owned by the insurance company is very complex (bonds, stocks, funds, options, swaps, swaptions, property, as well as foreign assets, currency derivatives etc.) and proprietary. So we will simplify it and assume it is only composed of bonds and a stock index. To simulate the asset portfolio we therefore need to choose good interest rate and stock models.

2.1 Modeling Bonds

We first give some definitions about bonds and short rate (see [1]) that are important to understand, and then we present the Vasiček model.

2.1.1 Definitions

Definition 2.1. *A* **zero coupon bond** *with* **maturity date** *T, also called a* **T-bond***, is a contract which guarantees the holder 1 unit of currency to be paid on the date T. The price at time t of such a bond with maturity T is denoted by $p(t,T)$.*

Definition 2.2. *The* **yield** *$y(t,T)$ of a T-bond with market price $p(t,T)$ is given by*

$$y(t,T) = -\frac{\log p(t,T)}{T-t}.$$

For a fixed t, the function $T \mapsto y(t,T)$ is called the **yield curve***.*

Definition 2.3. *The* **short rate** *or* **instantaneous risk free spot rate** *r_t is the interest rate at which one can borrow money for an infinitesimally short period of time from time t.*

Bonds found in the market can be delivered either by the Government or by companies. They do not have the same price, as it may be more risky to buy a bond from a random company than from the Government. In this thesis we will only take Swedish Government bonds, considered as riskless. As a short rate we will take the yield of the one-month Swedish Government bond. That is to say $r_t = y(t, t + \frac{1}{12})$.

2.1.2 The Vasiček Model

According to the Vasiček model, the short rate solves the following stochastic differential equation, under the objective probability measure P:

$$dr_t = \kappa(\theta - r_t)dt + \sigma_r dW_t^r \tag{2.1}$$

where r_t is the short rate process, κ, θ and σ_r are constants, and W^r is a P-Wiener process.
Under the risk neutral measure Q, the SDE is written as follow:

$$dr_t = \kappa(\tilde{\theta} - r_t)dt + \sigma_r d\tilde{W}_t^r$$

where $\tilde{\theta}$ is another constant and $\tilde{W}^r$ is a Q-Wiener process.
With these parameters, the price at time t of a zero-coupon bond with maturity T is (see [1] for proof):

$$P(t,T) = A(t,T)e^{-B(t,T)r_t} \tag{2.2}$$

where

$$\begin{aligned} A(t,T) &= e^{(\tilde{\theta} - \frac{\sigma_r^2}{2\kappa^2})(B(t,T)-T+t) - \frac{\sigma_r^2}{4\kappa}B(t,T)^2}, \\ B(t,T) &= \frac{1}{\kappa}(1 - e^{-\kappa(T-t)}), \end{aligned}$$

and r_t is the short rate at time t generated under the Q-measure.

Estimating Parameters

In order to get a good estimate of this future bond price, we need to find the parameters $\tilde{\theta}$, κ and σ_r that correspond to the current market (θ is only useful for simulating the short rate under the objective P-measure). The most logical method to obtain κ, θ and σ_r would be to use short rate historical data, and for $\tilde{\theta}$ use the current yield curve of the zero-coupon bond (Method 1).
First we consider a short time period Δt (as we have monthly data, we will take Δt equal to 1 month, that is to say $\Delta t = \frac{1}{12}$), we would like to integrate

Equation (2.1) between times $t_{k-1} = (k-1)\Delta t$ and $t_k = k\Delta t$ (please note that k is an integer that is different to the parameter κ). Let us define

$$f(t, r_t) = r_t e^{\kappa t}$$

$$\begin{aligned} df(t, r_t) &= \kappa r_t e^{\kappa t} dt + e^{\kappa t} dr_t \\ &= \big(\kappa r_t e^{\kappa t} + \kappa(\theta - r_t) e^{\kappa t}\big) dt + \sigma_r e^{\kappa t} dW_t^r \\ &= \kappa\theta e^{\kappa t} dt + \sigma_r e^{\kappa t} dW_t^r. \end{aligned}$$

When integrating this between t_{k-1} and t_k, we obtain

$$r_k e^{\kappa k \Delta t} - r_{k-1} e^{\kappa(k-1)\Delta t} = \kappa\theta \frac{1}{\kappa}(e^{\kappa k \Delta t} - e^{\kappa(k-1)\Delta t}) + \sigma_r \int_{t_{k-1}}^{t_k} e^{\kappa u} dW_u^r.$$

So

$$r_k = r_{k-1} e^{-\kappa\Delta t} + \theta(1 - e^{-\kappa\Delta t}) + \sigma_r \int_{t_{k-1}}^{t_k} e^{-\kappa(t_k - u)} dW_u^r, \tag{2.3}$$

which we can rewrite in a simpler way:

$$r_k = \phi + \psi r_{k-1} + \epsilon_k \tag{2.4}$$

where

$$\begin{aligned} \phi &= \theta(1 - e^{-\kappa\Delta t}), \\ \psi &= e^{-\kappa\Delta t}, \\ \epsilon_k &= \sigma_r \int_{t_{k-1}}^{t_k} e^{-\kappa(t_k - u)} dW_u^r. \end{aligned}$$

The ϵ_ks, $k = 1, 2, ..., n$, are i.i.d normal random variables with mean

$$E[\epsilon_k] = 0$$

and variance

$$\begin{aligned} \mathrm{Var}[\epsilon_k] &= \sigma_r^2 E\Big[\Big(\int_{t_{k-1}}^{t_k} e^{-\kappa(t_k - u)} dW_u^r\Big)^2\Big] \\ &= \sigma_r^2 E\Big[\int_{t_{k-1}}^{t_k} e^{-2\kappa(t_k - u)} du\Big] \\ &= \sigma_r^2 \int_{t_{k-1}}^{t_k} e^{-2\kappa(t_k - u)} du \\ &= \frac{\sigma_r^2}{2\kappa}(1 - e^{-2\kappa\Delta t}) \\ &= \frac{\sigma_r^2(1 - \psi^2)}{2\kappa}. \end{aligned}$$

Based on the one-month Swedish treasury bill as the short rate, monthly data from January 1983 to October 2009 gives us the following estimates:

$$\kappa = 0.0883,$$
$$\theta = 0.0677,$$
$$\sigma_r = 0.0201.$$

The parameter $\tilde{\theta}$ is chosen as the number that makes the theoretical zero coupon yield curve (see Definition 2.2) match the current market yield curve. We denote by $y^M(T)$ the zero-coupon yield given by a Swedish Government bond for the maturity T and $y(T)$ the theoretical yield given by our model. The expression of $y(T)$ is derived from Equation (2.2):

$$y(T) = -(\tilde{\theta} - \frac{\sigma_r^2}{2\kappa^2})\frac{(B(0,T) - T)}{T} + \frac{\sigma_r^2}{4\kappa}\frac{B(0,T)^2}{T} + \frac{B(0,T)}{T}r_0 \tag{2.5}$$

where B is the function defined in Equation (2.2).
Therefore,

$$\tilde{\theta} = \underset{\vartheta}{\operatorname{argmin}} \sum_k \left(-(\vartheta - \frac{\sigma_r^2}{2\kappa^2})\frac{(B(0,\tau_k) - \tau_k)}{\tau_k} + \frac{\sigma_r^2}{4\kappa}\frac{B(0,\tau_k)^2}{\tau_k} + \frac{B(0,\tau_k)}{\tau_k}r_0 - y^M(\tau_k)\right)^2 \tag{2.6}$$

where τ_k are the maturities corresponding to the government bond yields existing in the market.
We find that the $\tilde{\theta}$ corresponding to our data is:

$$\tilde{\theta} = 0.0802.$$

Having found κ, σ_r and $\tilde{\theta}$, we construct the theoretical yield curve thanks to Equation (2.5), which is done in figure 2.1. We can actually see that the estimation of the parameters is quite bad as the theoretical yield curve does not match the market one.

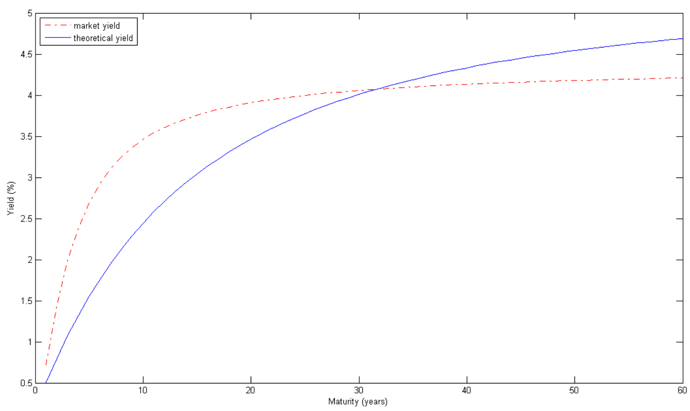

Figure 2.1: Market and theoretical yield curve with Method 1.

Other methods to estimate the parameters lead to different results. For instance we could look for κ, σ_r and $\tilde{\theta}$ that make the theoretical yield curve match the market one, without estimating κ and σ_r from the short rate data (Method 2). This method gives the following results:

$$\kappa = 0.4334,$$
$$\sigma_r = 0.0021,$$
$$\tilde{\theta} = 0.0438.$$

The theoretical yield curve obtained with these estimates is logically much closer to the market yield curve, as we can see in figure 2.2.

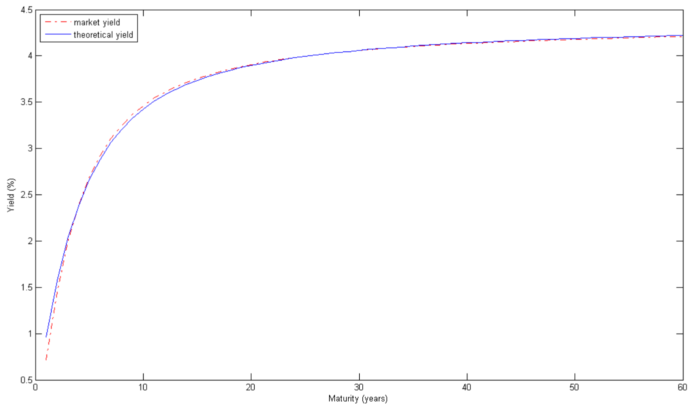

Figure 2.2: Market and theoretical yield curve with Method 2.

The difference between the results obtained with the different methods is due to the fact that the Vasiček model is probably not adequate to model Swedish Government bonds with one-month treasury bill as a short rate. Anyway, we can try to see if the choice of the value of these parameters affects the replicating portfolio's performance.

Generating Short Rate, Bond Price and Bond Portfolio

Having found our parameters, it is now easy to generate a short rate path thanks to Equation (2.4). We can actually generate the short rate under either P or Q measures by using either θ or $\tilde{\theta}$.

The estimation of the price of a T-bond at time t becomes straightforward with Equation (2.2), basically we just need to generate the short rate at time t under the Q-measure.

For the bond portfolio, let us explain a few concepts first. We consider a portfolio of several bonds of different maturities. We denote by B_t the value of the portfolio at time t, that is to say the sum of the prices of the different bonds at time t.

Definition 2.4. *The* **duration** *at time t of a bond portfolio is defined as*

$$d_t = \frac{1}{B_t} \sum_k (\tau_k - t) P(t, \tau_k)$$

where τ_k is the maturity of bond k and $P(t, \tau_k)$ its price at time t. The τ_ks are not necessarily different.

The duration is broadly the average length of time before payments are done. Some strategies consist in rebalancing every time step the portfolio weights in order to keep the duration constant.
Considering a bond portfolio with constant duration d, the dynamics of its value is given by (see [3] for proof)

$$dB_t = (r_t + \mu_B) B_t dt - \frac{\sigma_r}{\kappa}(1 - e^{-\kappa d}) B_t dW_t^r$$

where

$$\mu_B = (\tilde{\theta} - \theta)(1 - e^{-\kappa d}) - \frac{\sigma_r^2}{2\kappa^2}(1 - e^{-\kappa d})^2.$$

These dynamics are like the ones for stock price under Black & Scholes model (see Section 2.2), so it will also be easy to generate the path of a bond portfolio.

2.2 Modeling Stocks with Black & Scholes Model

According to the Black & Scholes model, the P-dynamics of the price of a stock S is given by:

$$dS_t = (r_t + \mu) S_t dt + \sigma_S S_t dW_t^S \tag{2.7}$$

where μ (the risk premium) and σ_S are constants, and W^S is a brownian motion under P. As in Section 2.1.2, the parameters μ and σ_S are found thanks to market data. In order to be consistent with the Swedish short rate modeled previously, we use monthly Swedish SAX index from January 1983 to October 2009, and we get the following estimates:

$$\mu = 0.0679,$$
$$\sigma_S = 0.2247.$$

Correlation between Stocks and Interest Rate

In real life, it is observed that the processes W^r from Section 2.1.2 and W^S are negatively correlated: if the risk free rate increases, people will rather invest in riskless assets, so stock prices will decrease and vice versa. Our data gives indeed a negative correlation factor γ_{rS}:

$$\gamma_{rS} = -0.1851.$$

Generating Stock Price

By integrating Equation (2.7) between time $t_k = k\Delta t$ and $t_{k+1} = (k+1)\Delta t$, we obtain

$$S(t_{k+1}) = S(t_k)e^{\int_{t_k}^{t_{k+1}} r_s ds + (\mu - \frac{\sigma_S^2}{2})\Delta t + \sigma(W_{t_{k+1}}^S - W_{t_k}^S)}.$$

We assume that the short rate is constant over time interval $[t_k, t_{k+1})$. If we write S_k instead of $S(t_k)$ and r_k instead of r_{t_k} we get

$$S_{k+1} = S_k e^{(r_k + \mu - \frac{\sigma_S^2}{2})\Delta t + \sigma\sqrt{\Delta t}\epsilon_{k+1}} \tag{2.8}$$

where ϵ_k, $k = 1, 2, ..., n$, are i.i.d standard normal random variables.
If we know S_0, μ, σ_S and we are able to generate the short rate path (see Section 2.1.2), then we can simulate the stock price path.

2.3 Generating the Company's Yearly Return

We suppose that the company's asset portfolio is constituted by 60% of 5-year duration Swedish government bonds and 40% of Swedish SAX index stocks. We denote by R_t^B and R_t^S the return at time t of B_t and S_t, the value of respectively the bond and the stock portfolio.

$$R_t^B = \frac{B_t}{B_{t-1}} - 1,$$
$$R_t^S = \frac{S_t}{S_{t-1}} - 1.$$

The global portfolio return R_t is given by

$$R_t = 0.6R_t^B + 0.4R_t^S.$$

Chapter 3

Simulating the Capital Injections

In this chapter we will explain how to simulate the capital injections needed by the insurance company to pay the policyholders when the return is too low.

3.1 The Basic Insurance Policy

Here is a more detailed explanation of the insurance policy considered throughout this thesis.

Policyholder A enters a contract with the insurance company at age b. A will have to pay every year a constant premium P to the company, until he or she is $z-1$ years old (z is specified at the beginning of the contract, it is known as the **retirement age**).

Every year until age $z-1$, three numbers are calculated:
- the **guaranteed benefit** G_t is the amount already paid by the policyholder with premiums grown at the rate $g = 3\%$ (the **guaranteed rate**). G_t is computed at the beginning of year t (when A is $b+t$ years old), just after the premium is paid:

$$G_t = G_{t-1}(1+g) + P, \tag{3.1}$$

$$G_0 = P, \tag{3.2}$$

- the **retrospective reserve** V_t is calculated exactly like the guaranteed benefit, but at the random rate R_t (the **company's yearly return** between year $t-1$ and year t):

$$V_t = V_{t-1}(1+R_t) + P, \tag{3.3}$$

$$V_0 = P, \tag{3.4}$$

- the **quotient** q_t is the ratio of the retrospective reserve to the guaranteed benefit:

$$q_t = \frac{V_t}{G_t}.$$

From age z (i.e. year $t_z = z - b$) to age m (called **maturity age**), A does not pay any premium but earns a yearly random benefit B_t which is a function of G_t and V_t. To be more precise, $B_t = \max(G^y, V_t^y)$ where G^y and V_t^y are fractions of respectively the guaranteed benefit and the retrospective reserve. We will see in the next paragraph how these fractions are computed, just note that G^y is a constant whereas V_t^y depends on time.

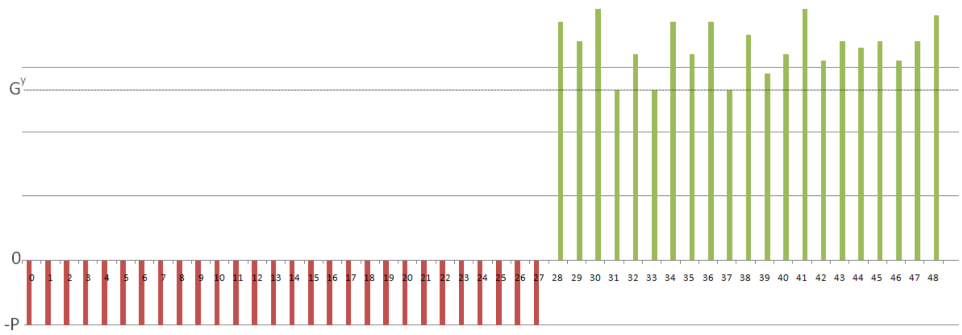

Figure 3.1: Example of a policy cash flow.

From year $t_z = z - b$ to $t_m = m - b$, the guaranteed benefit and retrospective reserve are computed as follow:

$$G_t = G_{t-1}(1 + g) - G^y, \tag{3.5}$$
$$V_t = V_{t-1}(1 + R_t) - B_t. \tag{3.6}$$

At the end of year t_m when the contract ends, all the money must have been given back to the policyholder. So G_{t_m} and V_{t_m} must be equal to zero, this is how G^y is computed.

$$\begin{aligned} G_{t_m} &= G_{t_{z-1}}(1+g)^{m-z+1} - (G^y + G^y(1+g) + \ldots + G^y(1+g)^{m-z}) \\ &= 0 \end{aligned}$$

So

$$G^y = G_{t_{z-1}} \frac{g}{1 - (\frac{1}{1+g})^{m-z+1}}.$$

The calculation of V_t^y is somewhat more complicated because it is based on the returns $R_{t+1}, \ldots, R_{t_m}$ which are unknown at time t. A reasonable assumption is to approximate these returns with an *expected* return $\bar{R} = 5\%$. So if we suppose that the return for the next $t_m - t$ years is $\bar{R}$, we have

$$\begin{aligned} V_{t_m} &= V_{t-1}(1+R_t)(1+\bar{R})^{t_m-t} - (V_t^y + V_t^y(1+\bar{R}) + \ldots + V_t^y(1+\bar{R})^{t_m-t}) \\ &= 0 \end{aligned}$$

And we obtain

$$V_t^y = V_{t-1} \frac{1+R_t}{1+\bar{R}} \frac{\bar{R}}{1-(\frac{1}{1+\bar{R}})^{m-b-t+1}}. \tag{3.7}$$

For t between t_z and t_m, we recall Equation (3.6):

$$V_t = V_{t-1}(1+R_t) - \max(G^y, V_t^y).$$

So basically if $V_t^y < G^y$, which means that the company's return was too low to give the policyholder a benefit above the guaranteed rate, the retrospective reserve can become negative before the end of the contract, so we can have $V_t \leq 0$ for $t < t_m$. In that case the insurance company will need to make a **capital injection** to be able to pay G^y to the policyholder. It means that the company takes money from its buffer capital (this sum of money kept away from risky investments for solvency purposes) and gives it to the policyholder.
If we manage to have a good estimate of the future capital injections, the insurance company will be able to ensure solvency by dedicating the right amount of money to the buffer capital.
So to be more precise, the expression of C_t, the capital injection at time t, is:

$$C_t = \begin{cases} 0 & \text{if } V_t \geq 0, \\ V_t & \text{if } -G^y \leq V_t < 0, \\ G^y & \text{otherwise.} \end{cases}$$

3.2 The Liability Data

Figure 3.1 is an overview of an insurance file. For each of the N policies, we know the quotient q at time $t = t_0$ (now), the premium, the age of the policyholder, the age at issue of the contract, the retirement age, the maturity age, and whether it is a male or a female.

In this thesis we consider only policies with same retirement age and same maturity age. That is to say for every policy, $z = 65$ and $m = 85$.

3.3 Mortality, Fees, Surrender and the Allan Rule

The policies explained so far can be made more complex by adding fees, taxes, clauses in case of death, a possibility for the policyholder to surrender the contract, and/or the so-called Allan Rule.

Policy	q	Premium	x	b	z	m	sex
1	103.23	3000	43	29	65	85	1
2	154.11	2400	25	22	65	85	1
3	94.01	6200	67	53	65	85	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	25.97	3200	73	44	65	85	1

Table 3.1: Preview of an insurance file.

Variable	Definition
q	Quotient
x	Policyholder age
b	Age at issue
z	Retirement age
m	Maturity age
sex	$= 0$ if male, $= 1$ if female

Table 3.2: Explanation of the variables.

3.3.1 Mortality Risk

In this section we explain what happens to the policy if policyholder A dies before the end of the contract. If A dies during benefits (between t_z and t_m), they will go to A's family. In this case, the policy is not affected by death. On the other hand, if A dies during premium payments (before t_z), A will of course stop paying premiums, and at time t_z A's family will start getting the benefits (lower than if A had survived, as less premiums were paid). To have a model as accurate as possible, it will therefore be necessary to catch this mortality risk in our calculations.

The sex of the policyholder is important to model the mortality risk: according to statistics, women live longer than men in most countries (like in Sweden), so the probability to be alive at age x is higher for women than for men.

In order to catch mortality risk for policyholder A, we should adjust Equations (3.1) and (3.3) by inserting a dummy variable d_t equal to 1 if A is alive at time t, and 0 if A is dead:

$$G_t = G_{t-1}(1+g) + Pd_t, \tag{3.8}$$

$$V_t = V_{t-1}(1+R_t) + Pd_t. \tag{3.9}$$

But of course it is impossible for us to know when A will die. According to the law of large numbers, if the number of policies N is high enough, we can replace d_t by $E[d_t] = Pr(A\ \textit{is alive at time}\ t)$, the probability that A is still

alive at time t.
Even if loads of models exist, here we decided to use the M90 table derived from Makeham's distribution to estimate this probability because it is supposed to be particularly adequate for Swedish mortality. The estimate of the probability is given by:

$$F(x) = Pr(alive\ at\ age\ x) = e^{-\alpha x - \frac{\beta}{\gamma \log(10)} 10^{-\gamma f}(10^{\gamma x}-1)} \tag{3.10}$$

where the parameters for Swedish tables are $\alpha = 0.001$, $\beta = 0.000012$, $\gamma = 0.044$, $f = 0$ for men and $f = 6$ for women.
And Equations (3.8) and (3.9) become:

$$G_t = G_{t-1}(1+g) + PF(x_t), \tag{3.11}$$
$$V_t = V_{t-1}(1+R_t) + PF(x_t) \tag{3.12}$$

where x_t is the policyholder age at time t.

3.3.2 Fees, Taxes & Profit Sharing

Policyholders actually must pay fees and taxes on their policy. These extra costs are withdrawn from the guaranteed rate and from the company's yearly return. Fees are 0.7% and taxes 0.45%. The actual guaranteed rate becomes $3\% - 0.7\% - 0.45\% = 1.85\%$.
On the yearly return, an extra fee called profit sharing fee is implemented: if the company's return R_t is above 4%, then 5% of the return goes to shareholders, and the rest to the policyholders.
So the actual return for the policyholder is $R_t - 0.7\% - 0.45\%$ if $R_t \leq 4\%$, and $0.95R_t - 0.7\% - 0.45\%$ otherwise.

3.3.3 Surrender Risk

If it is specified in the contract, the policyholder may have the right to surrender. If he or she surrenders at time t (necessarily before retirement age), premium payments are stopped, the guaranteed benefit G_t is given back to the policyholder, and the contract ends immediately.
Equations (3.11) and (3.12) become

$$G_t = (G_{t-1}(1+g) + PF(x_t))s_t, \tag{3.13}$$
$$V_t = (V_{t-1}(1+R_t) + PF(x_t))s_t, \tag{3.14}$$

where s_t is a dummy variable equal to 0 if the policyholder decided to surrender before or at time t, and equal to 1 otherwise.
For the same reason as when modeling mortality risk, we can replace s_t by $Pr(s_t = 1)$, the probability that the policyholder has not surrendered at time t. To estimate this probability, let us try to analyze what happens

every time step.
At the beginning of year t, policyholder A has not paid the premium yet so the guaranteed benefit is $G_{t-1}(1+g)$ and the retrospective reserve $V_{t-1}(1+R_t)$. If the retrospective reserve is higher than the guaranteed benefit, A should not worry because he or she will probably earn more money than the guaranteed benefit at retirement age. But if the retrospective reserve is lower than the guaranteed benefit, A will not expect to get much more than the guaranteed benefit, so it could be a good idea to surrender the contract. So if A were rational, the probability *not* to surrender at time t would look like this:

$$Pr(s'_t = 1) = \min\Big(\frac{V_{t-1}(1+R_t)}{G_{t-1}(1+g)}, 1\Big)$$

where $s'_t = 1$ is the event "A did not decide to surrender at time t" and $s_t = 1$ is "A has not surrendered at time t or before". The relation between the two variables is given by

$$s_t = \prod_{i=1}^{t} s'_i.$$

If we assume that the s'_is are independent, we can write

$$Pr(s_t = 1) = \prod_{i=1}^{t} \min\Big(\frac{V_{i-1}(1+R_i)}{G_{i-1}(1+g)}, 1\Big) \tag{3.15}$$

with $Pr(s_0 = 1) = 1$. When rewriting Equation (3.13) we cannot just replace s_t with $Pr(s_t = 1)$ because G_{t-1} and s_t are not independent:

$$\begin{aligned} E^s[G_t] &= E^s[(G_{t-1}(1+g) + PF(x_t))s_t] \\ &\neq (E^s[G_{t-1}](1+g) + PF(x_t))Pr(s_t = 1) \end{aligned}$$

where E^s denotes the expected value with respect to the random variables s.
If we expand expression (3.13) and notice that $\forall t \geq i \geq 1, s_t \times s_i = s_t$, we obtain

$$\begin{aligned} G_t &= G_0(1+g)^t \prod_{i=1}^{t} s_i + \sum_{j=1}^{t} PF(x_j)(1+g)^{t-j} \prod_{i=j}^{t} s_i \\ &= G_0(1+g)^t s_t + \sum_{j=1}^{t} PF(x_j)(1+g)^{t-j} s_t \end{aligned}$$

so

$$E^s[G_t] = (G_0(1+g)^t + \sum_{j=1}^{t} PF(x_j)(1+g)^{t-j})Pr(s_t = 1)$$

and

$$E^s[G_{t-1}] = (G_0(1+g)^{t-1} + \sum_{j=1}^{t-1} PF(x_j)(1+g)^{t-j-1})Pr(s_{t-1}=1)$$

so

$$E^s[G_t] = \Big(\frac{E^s[G_{t-1}](1+g)}{Pr(s_{t-1}=1)} + PF(x_t)\Big)Pr(s_t=1)$$

and we have a similar result with V_t.
To make notations simpler we replace $E^s[G_t]$ by G_t. Let us define now

$$G_t^{Pr} = \frac{G_t}{Pr(s_t=1)},$$
$$V_t^{Pr} = \frac{V_t}{Pr(s_t=1)}.$$

We notice that

$$\frac{V_t}{G_t} = \frac{V_t^{Pr}}{G_t^{Pr}}$$

and that

$$G_t^{Pr} = G_{t-1}^{Pr}(1+g) + PF(x_t),$$
$$V_t^{Pr} = V_{t-1}^{Pr}(1+R_t) + PF(x_t)$$

Thanks to Equation (3.15) we have

$$G_t = G_t^{Pr} \prod_{i=1}^{t} \min\Big(\frac{V_{i-1}^{Pr}(1+R_i)}{G_{i-1}^{Pr}(1+g)}, 1\Big)$$
$$V_t = V_t^{Pr} \prod_{i=1}^{t} \min\Big(\frac{V_{i-1}^{Pr}(1+R_i)}{G_{i-1}^{Pr}(1+g)}, 1\Big)$$

In case of surrender, capital must be injected to be able to pay the policyholder. The expression of this capital injection is

$$C_t = (G_{t-1}(1+g) - V_{t-1}(1+R_t)) \max\Big(1 - \frac{V_{t-1}(1+R_t)}{G_{t-1}(1+g)}, 0\Big)$$

3.3.4 The Allan Rule

The Allan rule stipulates that the benefit payments cannot decrease, even if the company's return drops. Some insurance companies have this rule on their policies, but although it is very profitable for the policyholder, the cost of this clause may be tremendous for the company.
This rule only affects the expression of V_t for $t \in [t_z, t_m]$: Equation (3.6) becomes

$$V_t = V_{t-1}(1+R_t) - \max(G^y, V_{t_z}^y, V_{t_z+1}^y, \dots, V_t^y).$$

Chapter 4

Replicating the Capital Injections

In this chapter we are going to see how we can replicate the capital injections with a set of financial products (mainly options). As we saw previously, the capital injection at year T depends essentially on the company's yearly return at year T, but also on this return at years $t < T$. It is therefore relevant to replicate the capital injections with options on the cumulated return:

$$I_T = \prod_{t=1}^{T}(1 + R_t).$$

4.1 Main Idea to Solve the Problem

In order to find the best set of replicating assets, the method we use is the following:
- We generate n different scenarios ($n \approx 1\,000$) on the company's return for every year, that actually create n scenarios on the capital injections for each time step.
- For each time step, we select a portfolio of financial products and we calculate their payoff for each scenario.
- By least-square method we compute the portfolio weights that match most closely the capital injection at time t.

We denote $T = m - \min_{policies}(x)$ the maximum time to maturity of all policies. For $t = 1, 2, \ldots, T$, we define

$$C_t = \begin{pmatrix} C_t^1 \\ C_t^2 \\ \vdots \\ C_t^n \end{pmatrix}$$

the vector of capital injections at time t for scenarios 1 to n.
We also define

$$A_t = \begin{pmatrix} a_{1,t}^1 & a_{2,t}^1 & \cdots & a_{s,t}^1 \\ a_{1,t}^2 & a_{2,t}^2 & \cdots & a_{s,t}^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,t}^n & a_{2,t}^n & \cdots & a_{s,t}^n \end{pmatrix}$$

the matrix of payoffs of the s replicating assets at time t for the n scenarios.
For each time step the optimization problem to solve is:

$$\min_{w_t} \|A_t w_t - C_t\|^2 \tag{4.1}$$

where w_t is the vector of portfolio weights at time t.
Then we can compare the sum of the discounted capital injections

$$\sum_{t=1}^{T} p(0,t) C_t$$

with the sum of the discounted replicating portfolio values

$$\sum_{t=1}^{T} p(0,t) A_t w_t$$

where $p(0,t)$ is the current price of a zero coupon bond maturing in t years.

4.2 The Basic Policy

Here we consider the policy as described in Chapter 3, without possibility to surrender nor Allan rule.

4.2.1 Selecting the replicating assets

To have a good solution to the minimization problem (4.1), we first need to select which assets are most suitable to replicate the capital injections. The idea here is to replicate C_t with t-bonds and several put options on I_i, $i = 1, \ldots, t$. The tricky part is to choose the number of put options and their strike price.
If we select too many options, the payoff matrix's rank does not turn out to be maximal and the optimization problem is not well solved. The same problem arises when the strike prices are out of the range of the I_is' realizations.
On the other hand if we do not select enough options, the subspace spanned by the options is too small and the minimization result is not optimal.
Moreover, the capital injection C_t at time $t \gg i$ is influenced in a lesser

extent by I_i than by I_t. So to replicate C_t we should pick up more options on I_t than on I_i.
In this thesis, we tried two different methods to determine these number of options and strikes, we call them Method A and Method B.

Method A

The idea of Method A is to set the number of options on I_t at n^K (here we choose $n^K = 10$), take $n^K - 1$ options on I_{t-1}, $n^K - 2$ options on I_{t-2} and so on. So C_t is replicated with exactly $\frac{1}{2}n^K(n^K + 1)$ options (or less when $t < n^K$).
Capital injections are performed when the cumulated returns I_is are too low. Of course "too low" does not mean anything quantitatively, but simulations show that results are better if we pick up the strikes among possible realisations of I_i when I_i is below $F_i^{-1}(\frac{1}{3})$, where F_i^{-1} denotes the inverse of the cumulative distribution function of I_i. In fact, the n^K strikes for the I_t-options are given by

$$K_t^j = F_t^{-1}\Big(\frac{1}{3} - \frac{j-1}{3n^K}\Big) \text{ for } j = 1, \ldots, n^K,$$

the $n^K - 1$ strikes for the I_{t-1}-options are

$$K_{t-1}^j = F_{t-1}^{-1}\Big(\frac{1}{3} - \frac{j-1}{3n^K}\Big) \text{ for } j = 1, \ldots, n^K - 1,$$

and so on.

Method B

In Method B we use a different approach to select the number of options on each I_i. In this method, we ask ourselves how relevant it is to use I_i-options, $i = 1, \ldots, t$ to replicate C_t.
We believe that put options on I_i will be a good means of replicating C_t if and only if the correlation coefficient $\rho_{i,t}$ between I_i and C_t is below an arbitrary threshold $\rho_{max} < 0$. Why this?
First of all it makes sense that I_is and C_t are negatively correlated: capital injections are higher when the return is low. So if $\rho_{i,t}$ is close to 0 or positive, I_i might not be a good explanatory variable for C_t. In figure 4.1 we plotted I_i and C_t for $i = t = 4$ to show that when correlation is highly negative (here $\rho_{4,4} \approx -0.8$), it makes sense to replicate the capital injection with put options, as the shape of this plot looks like the payoff function of a put option.
The larger in absolute value the correlation coefficient, the tighter the point cloud. And the tighter the point cloud, the fewer options needed to replicate. Figure 4.2 illustrates this idea. On this figure we plotted I_7 and C_{20}, and the

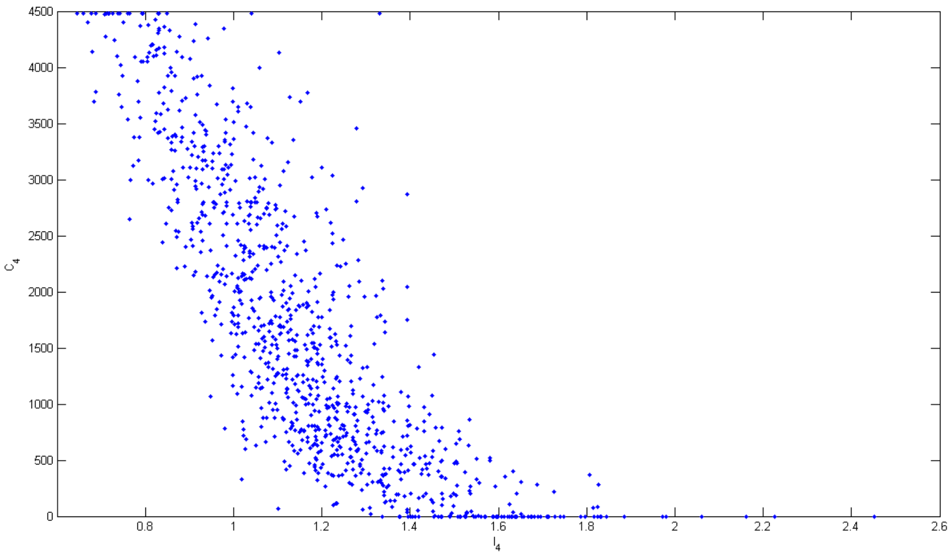

Figure 4.1: 1 000 points (I_4,C_4) highly negatively correlated ($\rho_{4,4} \approx -0.8$).

correlation of the non-constant part (when we eliminate the points where I_i is high and the capital injection is close to 0) is around -0.3. It seems to be more difficult to replicate C_{20} with options on I_7 than before (C_4 with options on I_4). This is why we will need more options with different strikes.

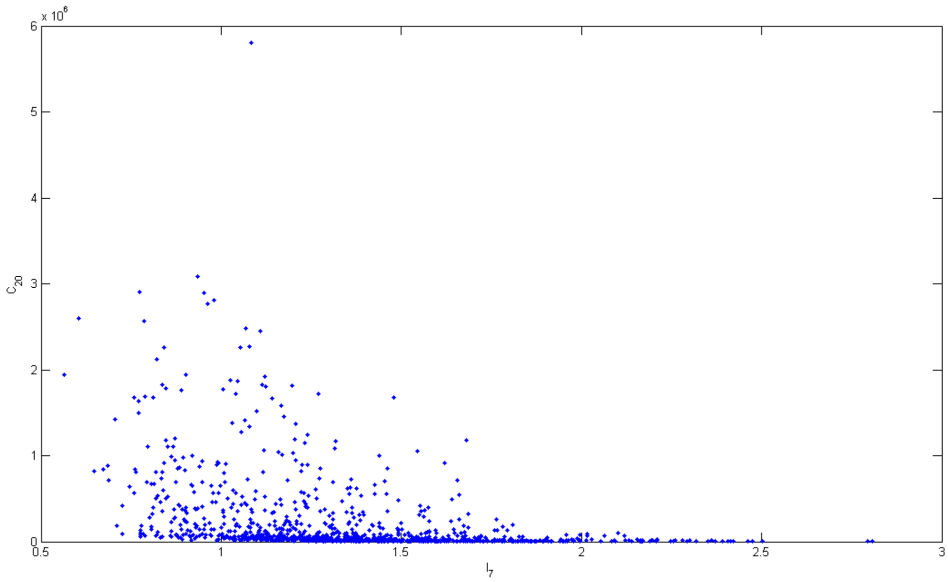

Figure 4.2: 1 000 points (I_7,C_{20}) poorly negatively correlated ($\rho_{7,20} \approx -0.3$).

We decided in this thesis to take $\rho_{max} = -0.3$, and the number of I_i-options to replicate C_t is given by

$$\begin{cases} 0 & \text{if } \rho_{i,t} \geq \rho_{max} \\ \lfloor n^K \frac{1+\rho_{i,t}}{1+\rho_{max}} \rfloor + 1 & \text{otherwise} \end{cases}$$

where $\lfloor x \rfloor$ denotes the floor of x and n^K is an arbitrary constant set at 7 here. Basically, this formula means that when $\rho_{i,t}$ is close to ρ_{max}, we select n^K options, and when it is close to -1 we just select one.
Of course, it could be very interesting to study the effects of changing these parameters, or improve this method with other ideas, but we will not discuss this in this thesis.

The strikes are then selected as in Method A, among possible realisations of I_i when I_i is low, particularly in the non-constant part of the graphs.

4.2.2 Results

No matter which method we choose, we can now simulate the payoff of the put options for each scenario and create our matrix A_t for each time step with the selected put options and the t-bond.
Solving the minimization problem (4.1) gives the portfolio weights. In figures 4.3 and 4.4 we plotted the capital injections and replicating portfolio value for 1 000 scenarios, the replicating assets being found respectively with Methods A and B. The R^2 coefficient of the Method A graph is 97.76% whereas for Method B it is 98.33%.

Testing Robustness

As the results for Methods A and B are approximately the same, it is interesting to see how robust the portfolio weights are. To test robustness, we test these weights on another set of 1 000 scenarios taken from another distribution. This new distribution is created by changing parameters for interest rate, bond and stock obtained in Chapter 2. Here we use parameters found in Section 2.1.2 with Method 2 ("new" distribution) instead of Method 1 ("old" distribution). In addition, we increase the stock volatility by 20%.
In figures 4.5 and 4.6 we plotted the capital injections and replicating portfolio value for 1 000 scenarios from the new distribution, the portfolio weights being found thanks to 1 000 previous scenarios from the old distribution respectively with Methods A and B. With Method A we get $R^2 = 92.45\%$ and with Method B, $R^2 = 94.62\%$.

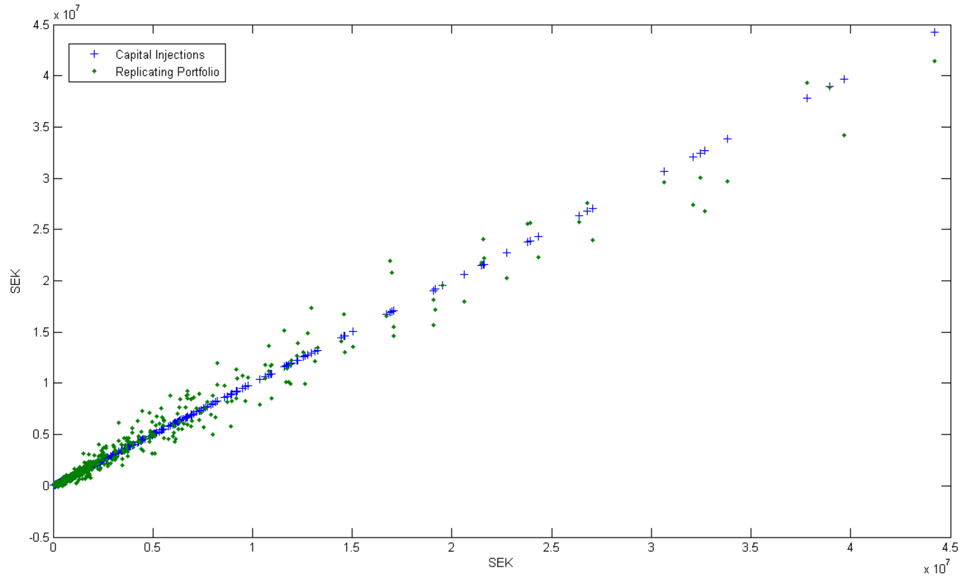

Figure 4.3: Capital injections and replicating portfolio value for 1 000 scenarios with Method A.

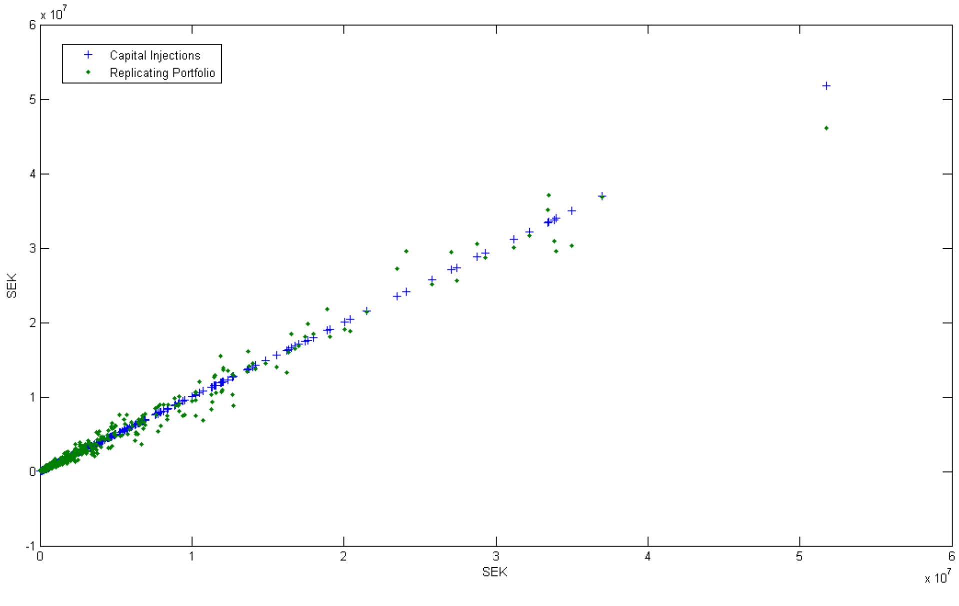

Figure 4.4: Capital injections and replicating portfolio value for 1 000 scenarios with Method B.

More simulations can be performed to test robustness of the replicating portfolio by changing the distribution to a greater extent. This can be done by modifying the bond and stock parameters even more than here, or adding other types of assets in the company's return.

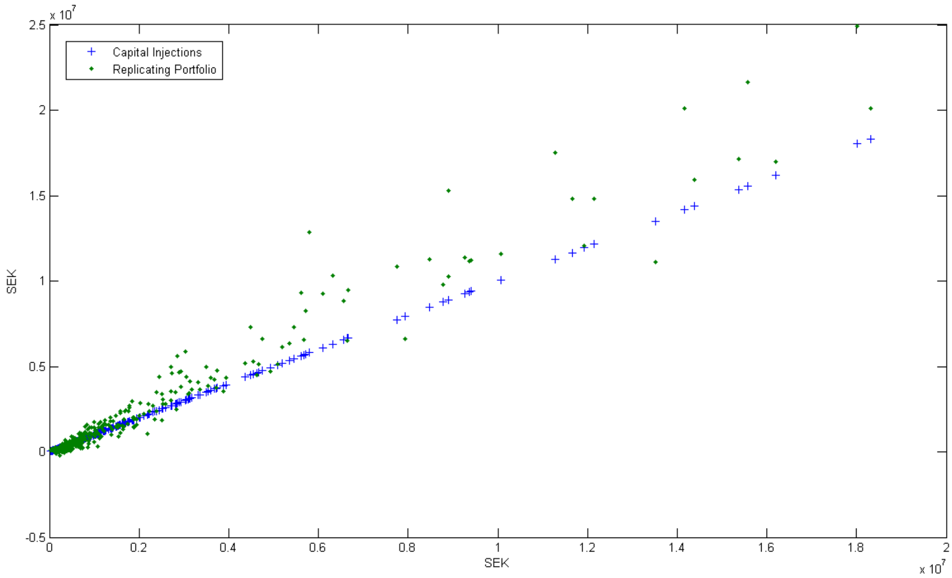

Figure 4.5: Testing the replicating portfolio on 1 000 scenarios from another distribution with Method A.

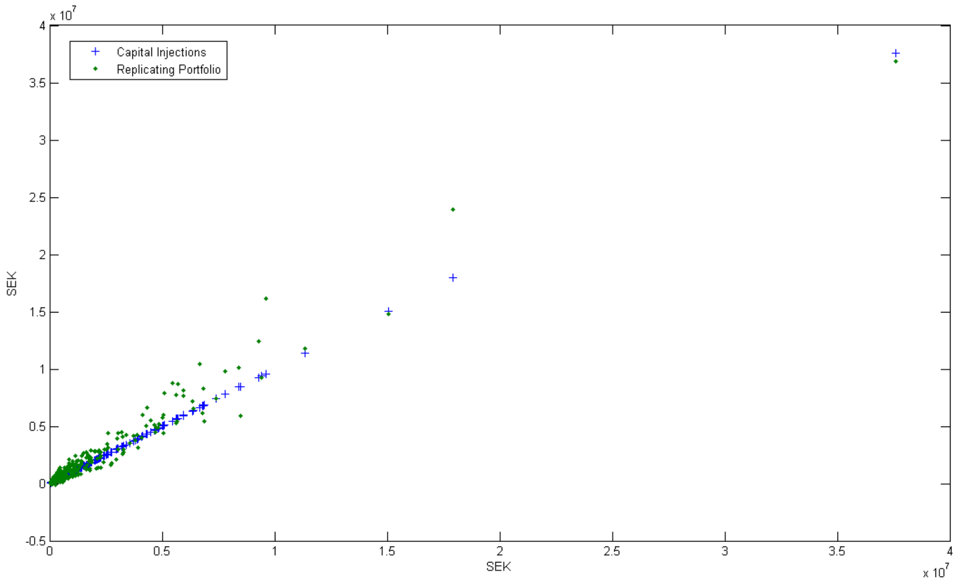

Figure 4.6: Testing the replicating portfolio on 1 000 scenarios from another distribution with Method B.

Anyway we saw that the results obtained with Methods A and B are quite similar, but it could be interesting to improve Method B somehow in order to have more robust portfolio weights.

4.3 Adding the Possibility to Surrender

Now we consider the policy with possibility to surrender before retirement age, as explained is section 3.3.3.

4.3.1 Selecting the replicating assets

The replicating assets are chosen as before: to replicate C_t we still take t-bonds and several put options on the cumulated returns I_is, $i = 1, \ldots, t$. Here again we use Methods A and B as explained in section 4.2.1.

4.3.2 Results

In figures 4.7 and 4.8 we plotted the capital injections and replicating portfolio value for 1 000 scenarios, the replicating assets being found repectively with Methods A and B. To have better results, we changed parameters in Method B: we took $\rho_{max} = -0.1$ and $n^K = 8$. The R^2 coefficient of the Method A graph is 92.52% and for Method B, 94.00%.

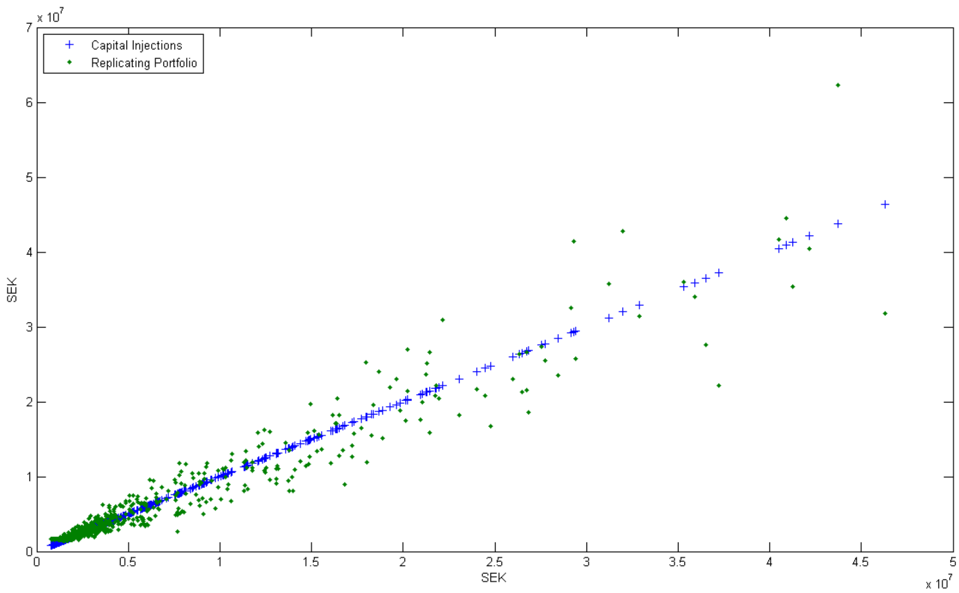

Figure 4.7: Capital injections and replicating portfolio value for 1 000 scenarios with Method A.

Testing Robustness

In the same way as we did in the previous section, we change distribution of the company's return.

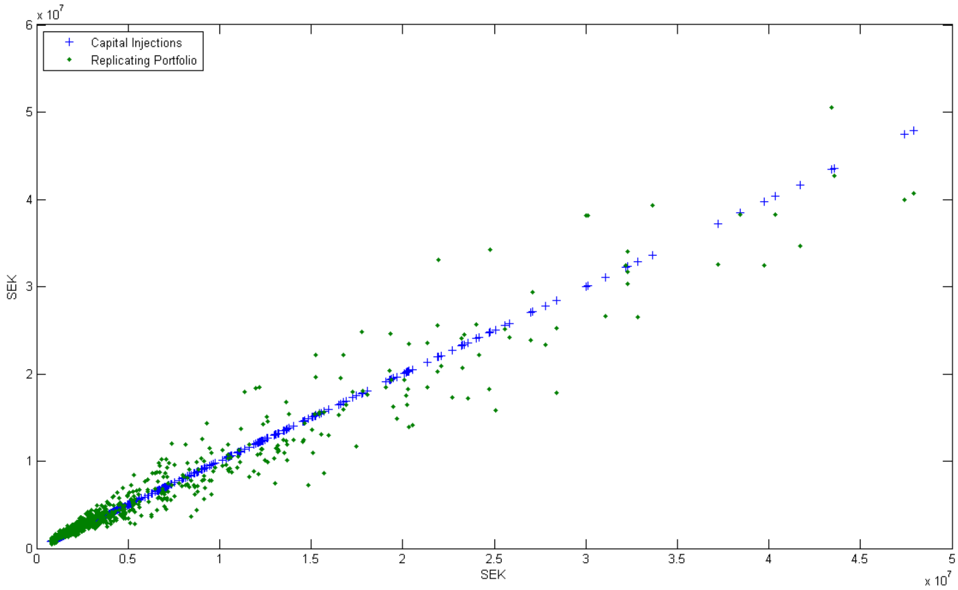

Figure 4.8: Capital injections and replicating portfolio value for 1 000 scenarios with Method B.

In figures 4.9 and 4.10 we plotted the capital injections and replicating portfolio value for 1000 scenarios from the new distribution, the portfolio weights being found thanks to 1 000 previous scenarios from the old distribution respectively with Methods A and B. With Method A we get $R^2 = 90.77\%$ and with Method B, $R^2 = 91.40\%$.

Here again more simulations can be performed to test robustness of the replicating portfolio. We can notice however that the results obtained with both methods are worse than before (without possibility to surrender). Moreover we made strong assumptions when modeling surrender risk, so replicating portfolio weights might be even less efficient as there is too much uncertainty.

4.4 The Allan Rule

We recall the expression of the retrospective reserve in section 3.3.4:

$$V_t = V_{t-1}(1 + R_t) - \max(G^y, V^y_{t_z}, V^y_{t_z+1}, \dots, V^y_t) \text{ for } t \geq t_z.$$

We can see that this expression depends strongly on the past so we cannot use simple options as before to replicate the capital injections. One solution could be to use path-dependent options but it is very complicated to select the most relevant ones.

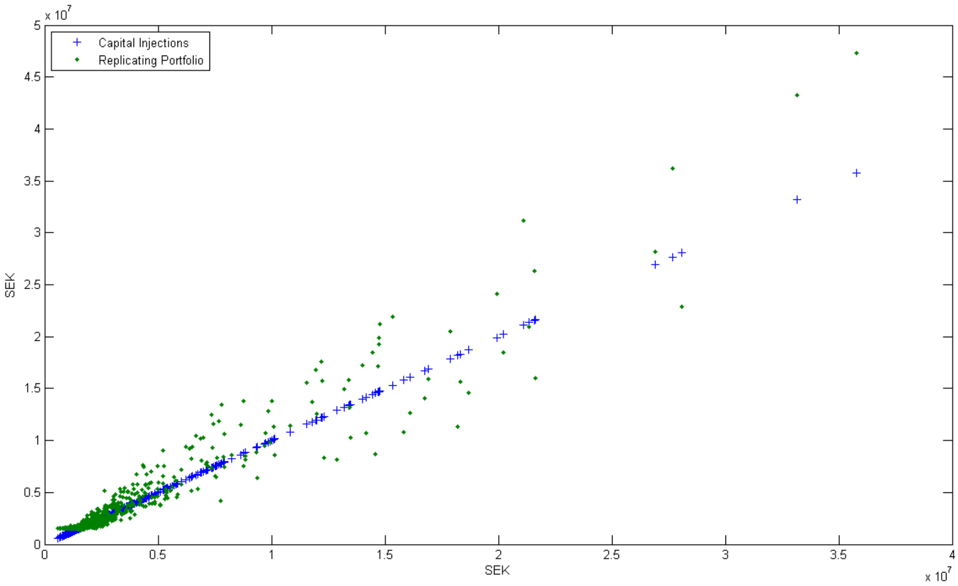

Figure 4.9: Testing the replicating portfolio on 1 000 scenarios from another distribution with Method A.

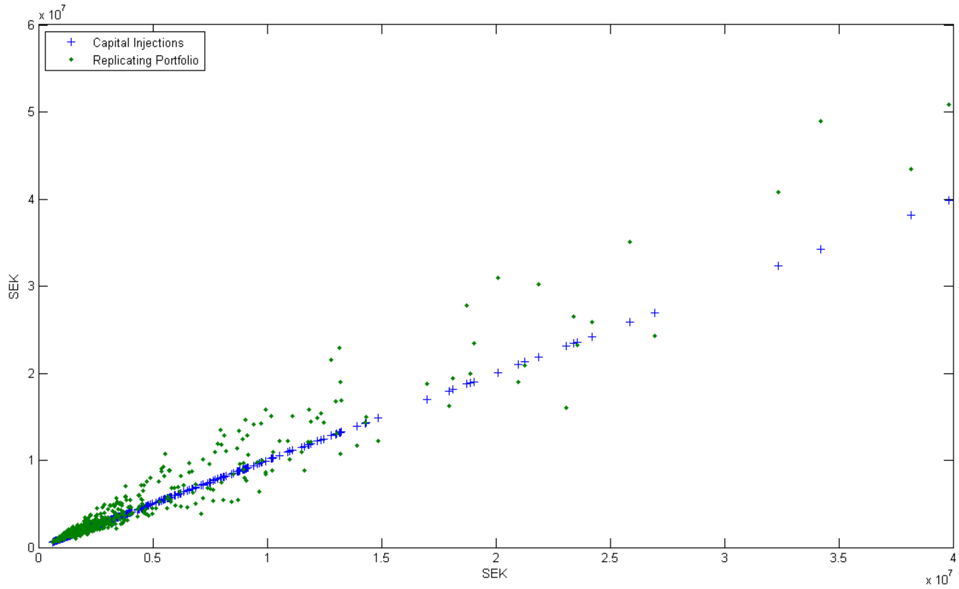

Figure 4.10: Testing the replicating portfolio on 1 000 scenarios from another distribution with Method B.

Here we choose a new approach. As it is not possible to select at time 0 the assets that will replicate the capital injection at time t, we will not select these assets before time $t-1$. Here is how we do:

Let us say we are at time $t-1$. As we do not allow to surrender, the only policies likely to induce capital injections are of course those who have reached retirement age and are not yet at maturity age. At time $t-1$, we know the value of the retrospective reserve V_{t-1} for every policy, as well as the benefit V^*_{t-1} given by

$$V^*_{t-1} = \max(G^y, V^y_{t_z}, V^y_{t_z+1}, \dots, V^y_{t-1}) \tag{4.2}$$

where V^y_i is defined as in Equation (3.7):

$$V^y_i = V_{i-1}\frac{1+R_i}{1+\bar{R}}\frac{\bar{R}}{1-(\frac{1}{1+\bar{R}})^{m-b-i+1}}.$$

Proposition 4.1. *The capital injection at time t induced by a given policy is exactly*

$$C_t = \begin{cases} V^*_{t-1} & \textit{if } V_{t-1} \leq 0 \\ V_{t-1}\max\left(\frac{V^*_{t-1}}{V_{t-1}} - 1 - R_t, 0\right) & \textit{otherwise} \end{cases}$$

Proof. First, let us rewrite Equations (4.2) and (3.7) in a simpler way:

$$V^*_t = \max(V^*_{t-1}, V^y_t), \tag{4.3}$$

$$V^y_t = V_{t-1}(1+R_t)f_t \tag{4.4}$$

where the expression of f_t is of no interest, just note that $\forall t, 0 < f_t \leq 1$. Suppose that $V_{t-1} \leq 0$, there will obviously be a capital injection at time t and we will need to pay the benefit V^*_t. But

$$V_{t-1} \leq 0 \Rightarrow V^y_t \leq 0$$

so according to Equation (4.3),

$$V^*_t = V^*_{t-1}.$$

Therefore

$$C_t = V^*_{t-1}.$$

Now we assume that $V_{t-1} > 0$. There will be a capital injection if and only if $V_{t-1}(1+R_t) < V^*_t$ and we will need to pay $V^*_t - V_{t-1}(1+R_t)$. So the capital injection is exactly

$$C_t = \max\left(V^*_t - V_{t-1}(1+R_t), 0\right).$$

Let us define

$$A = \max\left(V^*_t - V_{t-1}(1+R_t), 0\right)$$
$$B = \max\left(V^*_{t-1} - V_{t-1}(1+R_t), 0\right)$$

we want to prove that $A = B$.
If $V_{t-1}^* - V_{t-1}(1+R_t) \geq 0$, then $V_t^* - V_{t-1}(1+R_t) \geq 0$ because $V_t^* \geq V_{t-1}^*$.
So

$$\begin{aligned} A - B &= V_t^* - V_{t-1}^* \\ &= \max(V_{t-1}^*, V_t^y) - V_{t-1}^* \\ &= \max(V_t^y - V_{t-1}^*, 0) \\ &= \max\big(V_{t-1}(1+R_t)f_t - V_{t-1}^*, 0\big) \\ &= 0 \end{aligned}$$

If $V_{t-1}^* - V_{t-1}(1+R_t) < 0$, then $B = 0$ and either $V_t^* = V_{t-1}^*$ or $V_t^* > V_{t-1}^*$.

$$\begin{aligned} V_t^* = V_{t-1}^* &\Rightarrow \max\big(V_t^* - V_{t-1}(1+R_t), 0\big) = \max\big(V_{t-1}^* - V_{t-1}(1+R_t), 0\big) \\ &\Rightarrow A = B \end{aligned}$$

Otherwise,

$$\begin{aligned} V^*t > V_{t-1}^* &\Rightarrow V_t^y > V_{t-1}^* \\ &\Rightarrow V_{t-1}(1+R_t)f_t > V_{t-1}^* \end{aligned}$$

So

$$\begin{aligned} A &= \max\big(V_t^* - V_{t-1}(1+R_t), 0\big) \\ &= \max\big(V_t^y - V_{t-1}(1+R_t), 0\big) \\ &= \max\big(V_{t-1}(1+R_t)(f_t - 1), 0\big) \\ &= 0 \\ &= B \end{aligned}$$

So in any case, $A = B$.
Therefore,

$$C_t = \begin{cases} V_{t-1}^* & \text{if } V_{t-1} \leq 0 \\ V_{t-1} \max\left(\frac{V_{t-1}^*}{V_{t-1}} - 1 - R_t, 0\right) & \text{otherwise} \quad \square \end{cases}$$

So at each time step, we can replicate the capital injection induced the next year by each policy with either V_{t-1}^* bonds maturing in one year, or V_{t-1} put options with underlying asset the company's yearly return R_t and strike $\frac{V_{t-1}^*}{V_{t-1}} - 1$. The result will be far more accurate than when replicating the two previous contracts, as the replicating portfolio is now analytically equal to the capital injections. And no need to test the robustness of the portfolio, as we have not calibrated it in any way. We just defined the strategy to adopt at each time step to replicate the capital injections for the next year.
In figure 4.11 we plotted the capital injections and replicating portfolio value for 1 000 scenarios and we get $R^2 = 99.95\%$.

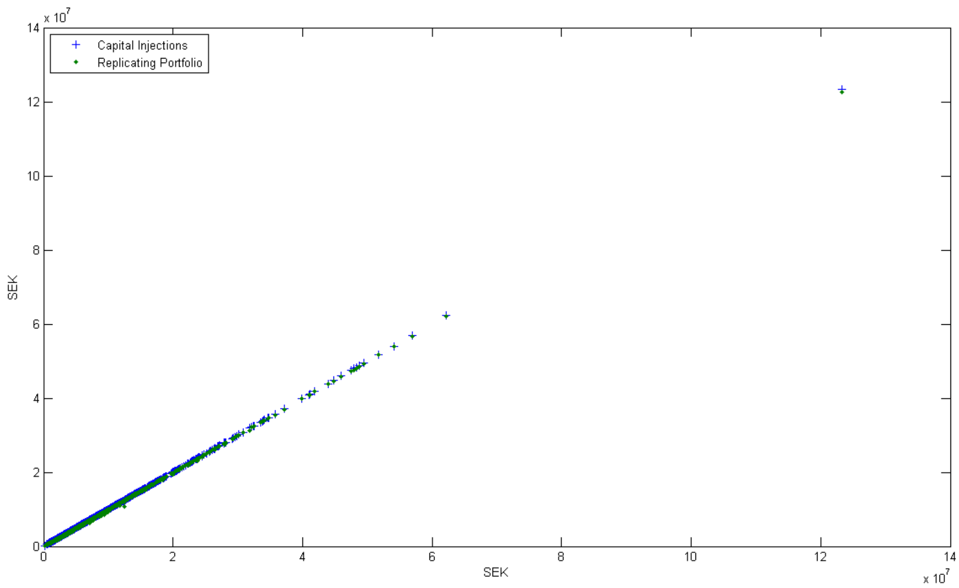

Figure 4.11: Capital injections and replicating portfolio value for 1 000 scenarios.

Why is not R^2 equal to 100% although the replicating portfolio is supposed to be analytically equal to the capital injections?
Actually the underlying asset of the options is the company's return at year t and not the actual return for the policyholder (see section 3.3.2) used to compute the retrospective reserve and the capital injections. So the difference between these two returns (due to taxes, fees and profit sharing) cause a small mismatch. But it is possible to reduce this mismatch without changing the underlying asset by adding taxes and fees to the strike. When doing this, we get $R^2 = 99.999\%$ (it becomes of no interest to show the plot).
The remaining 0.001% are due to profit sharing, which would require to be matched with more complicated options.

If this method gives such good results, why do not we use it with the two previous contracts as well?
Depending on the use we want to make of replicating portfolios, this method may be of poor interest. Länsförsäkringar would like to use replicating portfolios to speed up simulations of the future capital injections: instead of running simulations for hundreds of thousands of policies, they would just run simulations for a few simple assets like bonds and options, which is way faster. But with this new method, we have to simulate the performance of each policy to be able to simulate the replicating portfolio performance, so there is no time saving in this case.

Chapter 5

Conclusion

In this report we presented a few ideas on how to implement and test replicating portfolios in life insurance. This study was performed on three types of contracts:

- the "basic policy" in which the policyholder pays premiums until retirement age and then receives benefits until maturity age,
- the policy with possibility to surrender in which there is a clause allowing the policyholder to give up the contract before retirement age,
- the Allan rule contract in which benefits cannot decrease with time.

For each one of these contracts we tried to define at time $t = 0$ the best portfolio of financial assets that matches the capital injections by using several methods, and then we stress-tested this portfolio by doing out-of-sample tests under another distribution.
The results for the first two types of contracts were not perfect but very encouraging as the replicating portfolio value matched the capital injections by more than 90% in the out-of-sample test. It would be very interesting to try to improve the methods implemented here to reduce this mismatch. Then, use of the replicating portfolio could speed up simulations considerably, without substantial loss of accuracy.
For the third type of contract, definition at time $t = 0$ of a replicating portfolio could not give good results, so we had to redefine it at every time step (annually). Unfortunately, although this method gives good results, it does not improve the speed of calculations as in the first two types of contracts.

An alternate way to speed up computations could also be the use of model points (aggregating hundreds of thousands of policies into small groups of similar ones), possibly in combination with annually defined replicating portfolios.
To this end, we implemented the combination of model points and replicating portfolio for the Allan rule contract. We first simulated the capital

injections of 2 848 different policies for 1 000 different scenarios, and then computed the replicating portfolio value for the same 1000 scenarios, having aggregated the policies into N model points. In table 5.1, we present the number of model points "N", the time (in seconds) taken to compute the replicating portfolio "RP time", and the accuracy of the replicating portfolio characterised by the R^2 coefficient. These calculation times are to be compared with the time taken to compute the capital injections of 2848 policies: 3.9 seconds.

N	RP time	R^2
2848	17.1	100%
681	4.2	100%
356	2.2	99.96%
210	1.3	99.90%
49	0.3	99.30%

Table 5.1: Calculation time and accuracy of the model points and replicating portfolio combination method.

Thus, integrating the model points method and the replicating portfolio gives remarkable time savings for the Allan rule contract, without significant loss of accuracy.

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