Solvency II / SST and Modeling of Risk Aggregation

Malte Obbel Forsberg
malteof@kth.se

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Abstract

SST and Solvency II, while sharing many similarities, differ in a few important areas. In this paper we will explore some standard copulas used for aggregating loss distributions per risk type. Standard practice in the insurance industry is to use the Gaussian copula but there are reasons to believe that this copula is not really suitable in some aspects. The choice of copula has a large impact on the resulting solvency ratio, unfortunately there is often in the insurance industry a problem with fitting real data to a given model. We will also analyse the diversification of risk between companies within the insurance group studied.

Finally and perhaps the largest part of this thesis, the impact of CRTI defaults (i.e., inter-company loans, guarantees, reinsurance etc.) between companies has been studied.

Contents

1 Introduction

The solvency ratio for a company is defined as

$$\text{Solvency} = \frac{\text{Available Financial Resources}}{\text{Target Capital}} \tag{1.1}$$

where Available Financial Resources or AFR are the assets that are available as cash within a one-year horizon and Target Capital or TC is how much capital the company is exposing to risk. TC may be based on the risk measures VaR or ES, both of which are described in Section 3.

The Solvency ratio in itself is then basically measuring how well a company is able to cover its potential losses. There are different quantitative requirements in different solvency regimes but the ratio described above should be above 100% and usually authorities may start looking closely at a company if the ratio is below some higher value.

In Section 2 a short introduction to Solvency II and SST is presented, as well as a discussion about the differences between them. Due to the limited scope of this thesis there will not be a comparison to other solvency regimes. The main interests of this paper are risk aggregation modeling (copulas) and CRTI modeling. The former concept is described in Section 4 and the latter in 5.4.

Except for describing it briefly we will not deal with the AFR part of solvency (which is more about financial statements and less about risk modeling), but instead look at the quantitative requirements of Solvency II and SST. We will examine seven types of risk distributions that are modeled with dependency given by a Gaussian copula. We will also examine different copulas impact on the solvency ratio given real data as well as analyse the reasons for choosing one copula over another. Furthermore an impact study of accounting for CRTIs is presented.

Sections 6 and forward contain the results and some further discussion.

2 Economic Background

2.1 Solvency II

In general solvency rules stipulate the minimum amount of available financial resources that insurers/reinsurers need to have in order to cover their risks so they are able to pay policyholders even when things go really wrong.

Solvency II is an EU solvency regime under development (it is about half done at the moment) that aims to ensure the financial soundness of insurance undertakings. The project was started in late 2001 and really took off with the Sharma report [10]. Its primary target is to protect policyholders from insurance companies' defaults. These new solvency requirements are more sophisticated than the old ones, especially in that they move away from a "one-mode-fits-all way" of estimating capital requirements in favour of more entity-specific requirements (see [4]). Another difference between Solvency I and Solvency II is that the latter not only looks at a company's historical performance, but also takes future scenarios based on historical events into account. What this means is that for example the impact of the financial crisis 2008-2009 and other crises or catastrophes is taken into account as if they could happen again in the future.

Solvency II demands that a group of companies does its risk assessment on a legal entity level. This means that risk measures etc. need to be calculated for companies split by e.g. geometrical boundaries rather than management boundaries.

Pillar I	**Pillar II**	**Pillar III**
Quantitative capital requirements	Qualitative supervisory review	Market discipline

Figure 2.1: The three pillars of Solvency II.

Solvency II consists of three so called pillars, see Figure 2.1. The quantitative requirements, i.e. the first pillar, are what this paper will examine in greater detail. The other pillars mainly deal with what risk apprehension techniques companies must implement and how supervisory reporting is carried out. Pillar II is for example about governance, rules about risk management frameworks and how the qualitative supervisory review process is carried out. Pillar III states rules about enhancing market transparency, how to do solvency and financial condition reporting and how to report to supervisors.

Since regulators cannot force companies to put aside financial resources to cover every accident with 100 percent probability (in doing so, regulators would also remove much of insurance companies financial incentives) the Solvency II rules will not try to protect policyholders in every possible case, but rather try to strike a balance between the two extremes.

2.2 Swiss Solvency Test (SST)

SST is the Swiss solvency rule framework. Just like for other frameworks, the goal is to obtain a picture of insurance companies' risks and their ability to manage them. This is done by means of calculating target capital and risk-bearing capital, where the latter is defined as the capacity to bear risks. [5]

SST encourages the use of an internal risk model which is to be developed internally but with guidance from authorities. In this risk model companies are grouped together in so called SST entities. These can either be one company of significant size, or several smaller ones within e.g. the same geographic area. The main principles influencing the clustering structure are:

- ownership structure,
- fungibility (i.e. ability to transfer funds between companies should be good),
- large legal entities become their own SST entitites,
- regional regulations.

2.3 Differences Between Solvency II and SST

One of the main differences between SST and Solvency II is that the former bases its risk measure on Expected Shortfall, ES instead of Value at Risk, VaR. The reason for this is that while VaR adequately provides an upper limit for the losses α percent of the time, it does not reveal any information about how bad things might go if that limit is indeed crossed. In Section 3 these risk measures are described in greater detail. In general ES is more important for reinsurers since they generally deal more with losses in the tail. Since ES is the average of the tail, it requires us to establish knowledge about the actual loss distribution and more specifically the tail of the distribution. It is for this purpose we shall look closer at copulas later.

In Solvency II developing one's own internal risk model is not compulsory since there is also a standard model that companies may use. However, in general this standard model gives a much higher capital requirement for the company in question. Part of the SST requirements (unlike the Solvency II) is the modeling of how CRTIs impact defaults within a group of companies.

There are of course many other differences as well – e.g. how to calculate a company's own funds, how to treat hybrid contracts such as convertibles etc. – but since they lie outside the scope of this thesis, from now on they will not be discussed further.

3 Risk Measures

3.1 VaR

Value at Risk is the α-quantile of a distribution. That is the probability that one would sample from the distribution in question a value below VaR is α.

$$VaR_\alpha(X) = inf\{x \in \mathbb{R} : P(X > x) \leq 1 - \alpha\} = inf\{x \in \mathbb{R} : F_X(x) \geq \alpha\} \tag{3.1}$$

We may also write

Definition 3.1.

$$VaR_\alpha(X) = F_X^{-1}(1 - \alpha). \tag{3.2}$$

If X corresponds to some loss distribution, then the VaR at confidence level α is the smallest number x such that the probability that the loss X exceeds x is at most $1 - \alpha$.

In our case losses are defined as positive. One problem with this measure is that it gives no information about what happens if we indeed do exceed VaR; therefore ES (see 3.2) is of interest in some cases and some regulators do indeed require companies to base their risk calculations on this instead. In other words ES gives us some information about the tail behavior, whereas VaR does not.

Furthermore, VaR is not a coherent risk-measure except for elliptical distributions. This means it is not sub additive which may cause problems when looking at combined loss distributions, an example looked at in the later parts of this paper.

3.2 ES

Expected Shortfall, or ES is defined as follows

Definition 3.2.

$$ES_\alpha(X) = E[X \mid X \geq VaR_\alpha]. \tag{3.3}$$

In words ES can be described as the expected value of our loss if we do exceed VaR, or the expected value of the tail to the right of VaR.

It is, as opposed to VaR a coherent risk measure and therefore sometimes the preferred choice when measuring risk.

3.3 Tail Dependency

Tail dependency is an important concept in insurance, and especially reinsurance since there the contracts mostly deal with really large losses that have a low probability of occurring.

Definition 3.3. *Upper tail dependency.*

$$\lambda_U(X_1, X_2) = \lim_{u \to 1^-} P(X_2 > F_2^{\leftarrow}(u) \mid X_1 > F_1^{\leftarrow}(u)) \tag{3.4}$$

Lower tail dependence is defined similarly, and for elliptical distributions they are the same because of symmetry.

4 Copulas

4.1 Definition

Copulas can be described as "functions that join or couple multivariate distribution functions to their one-dimensional marginal distribution functions". [7] They are used as a way to model various types of dependence of marginal distributions and rely on the fact that each marginal distribution can easily be transformed into a uniform distribution using the fact that if $X \sim F(x)$, then $F(X) \sim U(x)$.

A very useful feature of copulas is the fact that they are invariant under strictly increasing transformations:

Proposition 4.1. *Suppose $(X_1, \ldots X_d)$ has continuous marginal distribution functions and copula C. Then also the random vector $(T_1(X_1), \ldots T_d(X_d))$ has copula C for a function T_k that is strictly increasing.*

which then for example means that if we draw a sample of normal marginal distribution with a Gaussian copula, then the transformed sample of log-normal marginal distributions has the same copula (i.e. $X = e^{\mu + \sigma N}$ where $N \sim N(0,1)$). A proof of proposition 4.1 can be found in [7].

Theorem 4.2. *Let F be a joint distribution function with margins F_1 and F_2, then there exists a function called a copula C such that for all x, y in $\mathbb{R}$*

$$F(x_1, x_2) = C(F_1(x_1), F_2(x_2)) \tag{4.1}$$

C is unique if F_1 and F_2 are continuous.

Theorem 4.2 is called Sklar's theorem and is the most fundamental theorem in copula theory. Sklar presented it in 1959, even though most development of copula theory has been carried out only in the last ten years.

4.2 Basic Examples

Figure 4.1 shows a sample from two normal marginal distributions with a Gumbel copula. If the dependence structure between two marginal normal distributions was instead the Gaussian copula, then the sample would have been equivalent to a sample from a bivariate normal distribution. It is important to realize that the copula and the marginal distributions are two separate things; this is indeed what makes the concept so powerful.

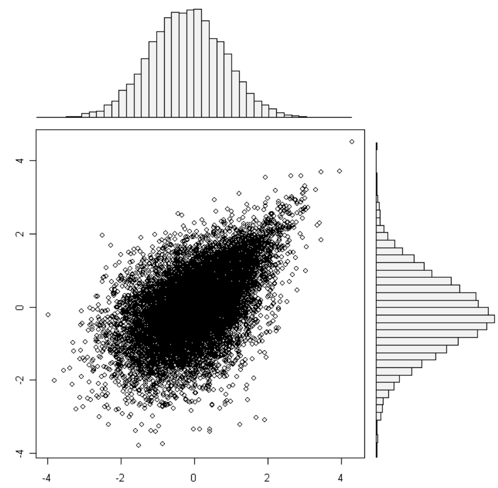

Figure 4.1: Two normal marginal distributions with a Gumbel copula.

4.3 Families of Copulas

4.3.1 Elliptical Copulas

One example of an elliptical copula often used for modeling dependence in finance is the Gaussian copula, which is constructed from the multivariate normal distribution via Sklar's theorem. [7] With Φ_R being the standard d-dimensional multivariate normal cumulative distribution function with correlation matrix R, the Gaussian copula function is defined as

Definition 4.3. *Gaussian copula*

$$C_R^{Ga}(\mathbf{u}) = \Phi_R^d(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d)) \tag{4.2}$$

Differentiating the copula function yields the copula density function

$$c_R^{Ga}(\mathbf{u}) = \frac{\varphi_R^d(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d))}{\prod_{i=1}^d \varphi(\Phi^{-1}(u_i))} \tag{4.3}$$

where φ is the normal density function.

The Gaussian copula can be a popular choice because of its ease of use. However, and this is a necessary point to make, it is quite important to be able to use it properly. It might for example heavily underestimate losses because of its lack of tail dependency (see Section 3.3). This might be a problem when dealing with dependent risks in insurance where the presence of tail dependency would be a natural feature. In such situations the other copulas, for example the t-copula or one of the Archimedean copulas are often better suited.

While one should not underestimate the power of ease of use there have been articles about how the Gaussian copula had been used improperly for valuing some financial contracts, and therefore contributing indirectly to the financial crisis 2008-2009. [8] It is definitely important to realize and acknowledge the shortcomings of this copula in some cases, even if one decides to use it. [3]

On more about how to simulate a sample from a given copula, and specifically the Gaussian copula, see part of Section 5.2.

The t-copula is defined similarly, but instead of using the normal distribution it is based on the Student's t-distribution. [2]

Definition 4.4. *t-copula*

$$C_{\nu,R}^t(\mathbf{u}) = t_{\nu,R}^d(t_\nu^{-1}(u_1), \ldots, t_\nu^{-1}(u_d)) \tag{4.4}$$

Differentiating yields

$$c_{\nu,R}^t(\mathbf{u}) = \frac{f_{\nu,R}(t_\nu^{-1}(u_1), \ldots, t_\nu^{-1}(u_d))}{\prod_{i=1}^d f_\nu(t_\nu^{-1}(u_i))} \tag{4.5}$$

where f is the Student's t-distribution's density function. While the Gaussian copula exhibits zero tail dependency, the t-copula does not. This means that for equal correlation parameters and marginal distributions the t-copula will always give higher risk measures such as ES or VaR. However, for increasing degrees of freedom the t-distribution will approach the normal distribution (and hence the same for the corresponding copulas).

4.3.2 Archimedean Copulas

Archimedean copulas have a simple form with properties such as associativity and have a variety of dependence structures. Unlike elliptical copulas (eg.

Gaussian), most of the Archimedean copulas have closed-form solutions and are not derived from the multivariate distribution functions using Sklar's Theorem.

The copulas within this family only use one parameter θ. This means that even in higher dimensions all the marginal distributions are assumed to be correlated in the same way. The d-dimensional Archimedean copula is defined as

Definition 4.5. *Archimedan copula*

$$C(\mathbf{u}) = \phi^{-1}(\phi(u_1) + \phi(u_2) + \ldots + \phi(u_d)) \tag{4.6}$$

See [7, pp. 121–124] for more information.

The density of an Archimedean copula is given by [9] as

$$c(u_1, \ldots, u_d) = \phi^{-1(d)}\left(\phi(u_1) + \ldots + \phi(u_d)\right) \prod_{i=1}^{d} \phi'(u_i) \tag{4.7}$$

where $\phi^{-1(d)}$ is the dth derivative of the inverse of the generator function ϕ_θ.

Table 4.1 introduces some important one-parameter Archimedean copulas. Cook-Johnson is the multivariate version of the Clayton family.

Family	$\phi_\theta(t)$	Range	$C_\theta(u_1, \ldots, u_d)$
Gumbel-Hougaard	$(-\ln t)^\theta$	$\theta \geq 1$	$\exp\left(-\left[\sum_{i=1}^{d}(-\ln u_i)^\theta\right]^{\frac{1}{\theta}}\right)$
Cook-Johnson	$\frac{1}{\theta}(t^{-\theta} - 1)$	$\theta > 0$	$\left[\left(\sum_{i=1}^{d} u_i^{-\theta}\right) - n + 1\right]^{-\frac{1}{\theta}}$
Frank	$-\ln\frac{e^{-\theta t}-1}{e^{-\theta}-1}$	$\mathbb{R} \setminus \{0\}$	$\frac{\prod_{i=1}^{d} u_i}{1-\theta\prod_{i=1}^{d}(1-u_i)}$

Table 4.1: Families of Archimedean copulas, see [9] for more information.

4.3.3 Product Copula

One of the simplest but also most important (albeit arguable not very useful in risk management of financial portfolios) copulas is the product copula, also called the independent copula. It is given by

$$\Pi(u, v) = uv \tag{4.8}$$

As is easily seen, if two stochastic variables' copula is the product copula then they are independent since Sklar's theorem (4.2) states that $F(x_1, x_2) = C(F_1(x_1), F_2(x_2)) = \Pi(F_1(x_1), F_2(y_2)) = F_1(x_1)F_2(x_2)$.

5 Internal Risk Model

5.1 Introduction

The insurance company that has been analysed uses an internal risk model for assessing risk and calculating risk based return measures (to be able to compare investments that carry different amount of risks). This model is however not meant for producing reporting data for authorities (FINMA, i.e. the Swiss authorities and the EU in this case), but the company has developed two more models (one for SST, one for Solvency II) on top of the existing one for that purpose. Conceptually they borrow a lot from each other but still differ in some key areas.

One of the differences between the internal SST model and the internal Solvency II model is that the former must take into account how CRTIs between companies within the group impact defaults (see Section 5.4). Another difference is that in the latter, risk types are aggregated at group level just using a correlation matrix and assuming normality everywhere. In the case of SST however, a Gaussian copula is simulated from to get a dependence structure that might be considered a bit more realistic. Of course there are problems with this approach as well, as discussed in Section 7.

In the internal SST model for the insurance company in question, a Gaussian copula is used to simulate from the combined loss distribution for seven different risk types. The approach used is to simulate the risk type loss distributions separately, using different proprietary systems, and then afterwards assuming a Gaussian copula as well as a correlation matrix. The correlation matrix was created by experts, which is why I in this thesis will assume it to be constant and correct. This matter is of course debatable but not within the scope of this study. An interesting discussion is the choice of a Gaussian copula. One of the main reasons for going with it in the first place was ease of use and industry practice, but several problems with this approach may be considered. For example the Gaussian copula has no tail dependency, as discussed earlier, which may give artificially low risk measures. FINMA does however for the time being seem satisfied with this choice.

One of the main problems with the approach taken by the insurance company in question is that there is no way to assess how well any copula models the true dependence between the different risk types. This is because no such information is present in the loss distributions simulated, for that we would need some kind of time series – e.g. historical data. While we acknowledge that other problems may arise when trying to fit a copula to historical data (maybe the dependence structure has changed over time?) it would nevertheless be a very interesting subject for further research.

As for now the choice of copula is pretty much arbitrary, even though one can argue for choosing one copula over another. For example the Archimedean copulas (Section 4.3.2) only use one parameter, which would mean that we assume the same rank correlation between each pair of risk types. This assumption would be rather questionable.

In summary two major aspects of the SST and Solvency II models that differ will be examined; first of all we will analyse the impact of using different copulas in aggregating risk type loss distributions, thereafter look at the impact of defaults within the group and how that affects the companies ability to fulfill their obligations (CRTIs) against other companies within the group.

5.2 Aggregating Different Risk Types Using Copulas

The following algorithm is used for aggregating risk type losses within the insurance company's internal SST model.

1. First the positive definite correlation matrix Σ is Cholesky decomposed so that L is found and $\Sigma = LL'$. Then L is applied to a vector of standard normal variates $\mathbf{Z} = (Z_1, \ldots, Z_d)$ where $\mathbf{Z} \sim N_d(\mathbf{0}, \mathbf{I})$, to get $\mathbf{N} = L\mathbf{Z} \sim N_d(\mathbf{0}, \Sigma)$.

2. The next step is to map these correlated normal marginal distributions to uniform variates, $\mathbf{U} = (U_1, \ldots, U_d) = (\Phi(N_1), \ldots, \Phi(N_d))$ where Φ denotes the cumulative distribution function of the standard normal distribution.

3. The third step is to apply the empirical quantile function or inverse distribution function, i.e. $F_k^{\leftarrow}(\cdot)$ to construct $\mathbf{Q} = (F_1^{\leftarrow}(U_1), \ldots, F_d^{\leftarrow}(U_d))$. Q can then be considered a sample from d marginal distributions with Gaussian copula.

4. Finally to get the desired empirical combined loss distribution, we merely construct $\mathbf{A} = (A_1, \ldots, A_n)$ where $A_i = \sum\limits_{k=1}^{d} Q_k$. Empirical VaR and ES can then be calculated from the empirical distribution made up by $\mathbf{A}$.

Of course, using another copula is merely a case of generating the vector $\mathbf{U} = (U_1, \ldots, U_d)$ in steps (1) – (2) differently; with copula C as its joint probability function. Then the steps (3) – (4) are the same for every copula we may choose.

In this case because of the approach taken by the insurance company, $F_k^{\leftarrow}(\cdot)$ is calculated from each of the empirical marginal distributions. How these are simulated will however not be covered.

5.3 Numerical Approach to Quantiles

Numerically computing quantiles, as needed in the third step in Section 5.2, is heavy on computational resources. Since one is generally dealing with a very large number of simulations, there's a numerical trick to achieve this more efficiently. The idea is basically to order the sample and choosing the correct index, noting that the last one corresponds to the 100% quantile and so on. That is

$$\hat{F}^{\leftarrow}(\alpha) = L_{[n\alpha+1],n} \tag{5.1}$$

where $L_{1,n} \geq \cdots \geq L_{n,n}$ is the ordered sample of loss amounts and $\hat{F}^{\leftarrow}(\alpha)$ is the empirical α-quantile. [6]

See real life example code in appendix A.1.

5.4 CRTI Model

CRTI stands for Capital and Risk Transfer Instrument and may be inter-company loans, reinsurance, guarantees et cetera.

The original risk models calculate loss amounts per RU (Reporting Unit), the finest granularity at which we may look at the company structure, and then aggregated for each SST entity. In this case, the losses are conditioned on the ability of all CRTI-counterparties to fulfill their obligations. However, in practice each CRTI-counterparty's ability to fulfill its obligations is a function of its capital position.

A CRTI model may use an iterative approach to determine a fixed point with respect to the capital position of all SST entities within the system. That is, an equilibrium point where each entity's capital position does not have any further impact on any other SST entity. In this way we may asses the impact of CRTI defaults.

Described in this chapter is a simplified version of the model.

5.4.1 Key Assumptions

1. **Default triggers based on Market Consistent Balance Sheet (MCBS)**. The purpose of the CRTI model is to determine the impact of SST entities becoming insolvent (or defaulting on CRTIs) on other SST entities within the group. The impact of default / insolvency is transmitted to other SST entities through a change in CRTI value on either the asset or liability side. In practice, insolvency is assessed using the statutory balance sheet of a legal entity. However, with the model the statutory balance sheet is approximated by the core capital of an entity.

   An SST entity will default with respect to a given CRTI held as a liability wherever the core capital of that entity does not cover its losses.

2. **External and internal debt is modeled to be of equal rank**. The valuation of CRTIs is carried out within successive seniority classes with CRTIs having the highest seniority having highest claim on core capital, i.e. the impact of a default is applied to CRTIs of the lowest seniority first.

   Where multiple contracts share the a seniority class their values are calculated in parallel and the impact of any default shared proportionately between those CRTIs. Table 5.1 lists the different seniority classes.

Seniority class	Instruments
4	Policy holder liabilities
3	Internal reinsurance, external and internal loans and derivatives
2	External and internal hybrids
1	Internal guarantees

   Table 5.1: Seniority classes.

3. **CRTIs will be re-valued symmetrically using an intrinsic value approach reflecting the core capital position of the issuing entity**. The default of an SST entity with respect to a given CRTI held as a liability, will impact the value of that CRTI on the asset side and the liability side.

4. **Only the net asset position of CRTIs between any two SST entities is modeled as we assume that CRTIs within SST entities can be netted**. Where several CRTIs of the same seniority are in

place between any two given SST entities, those CRTIs will be modeled by a single value representing the net asset or liability between those entities.

5. **No circularities in the participation structure**. The group structure is defined in terms of percentage participations for SST entities owning other SST entities; it is assumed that there are no circularities in the structure, i.e. if A participates in B and B in C, then C cannot have a participation in A (see Figure 5.1).

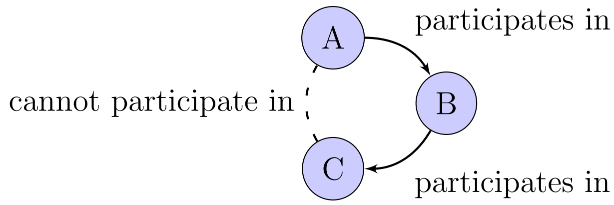

Figure 5.1: No circularities permitted in the ownership structure.

5.4.2 Input

- MCBS core capital per SST entity
- Net-net loss amounts per risk type, per global simulation m and RU
- CRTI base values per CRTI id

5.4.3 Calculation Steps

Figure 5.2 demonstrates the iterative calculation steps. We will focus on explaining the general idea behind each step in the algorithm. First of all the modeling of guarantees is carried out before the rest of the CRTIs. The guarantee calculations are encapsulated by the function $G(\cdot)$, and the rest by $F(\cdot)$ which outputs the vector of core capital amounts with net CRTI values and defaults in respect of those CRTIs reflected.

A readable algorithmic presentation is given in the rest of this section. Every step below will have to comply with the key assumptions presented in Section 5.4.1.

1. first read in: simulated losses, core capital and CRTI values;

2. calculate values of CRTIs on SST entity level;

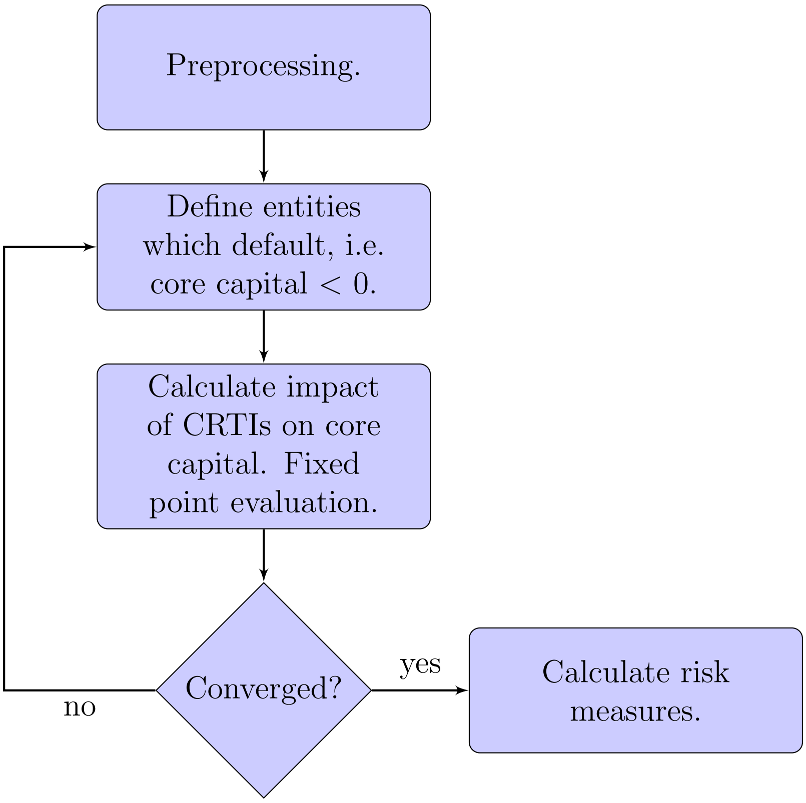

Figure 5.2: The calculations involved in examining impact of defaults on CRTIs for one global simulation.

3. subtract simulated losses from the MCBS based core capital values;
4. loop over global simulation id m and for each m:
   (a) calculate shortfall for each SST entity with regards to last iterations guarantee payments;
   (b) then based on shortfall, calculate new net guarantee positions;
   (c) now adjust the core capital for each SST entity including guarantee payments and defaults with respect to those payments;
   (d) then calculate shortfall with respect to each CRTI seniority class;
   (e) add the change in CRTI values to core capital;
   (f) repeat the above steps (a) – (e) until no change in core capital can be seen between two iterations.

Together we may use all of these m simulations to calculate risk measures. There has been a question regarding the convergence of this iterative calculation procedure and whether it can be proved. A major consulting firm which was involved in developing the methodology for the CRTI model has presented a logical argumentation in about four pages that proves the convergence, albeit not in mathematical form.

5.4.4 Example

This section contains a simple example of how one iteration of one simulation of guarantee payments may occur.

We consider first three SST entities; A, B and C. Each corresponds to for example a company and has some amount of core capital. Furthermore we shall look at some guarantee contracts (one form of CRTIs) between them.

SE	Core capital
A	140
B	120
C	40

Table 5.2: Core capital per RU

Table 5.3 shows the limits of the guarantee contracts between A and C and B and C. A and B are the only two guarantors in this example.

	C
A	20
B	50

Table 5.3: Guarantee limits

If now inside the loop described in Section 5.4.3 with for example a simulated loss for SST entity C of 80, then entity C will experience a shortfall of $40 - 80 = -40$. Because A has the highest core capital, it will be the first guarantor to pay out money. In doing so, its core capital is reduced by C's shortfall amount – but only up to the limit specified in the guarantee contract. In this case A will pay C 20, and hence reduce its core capital to 120.

Since C is still in shortfall (has a core capital of $-20 < 0$), the contract between B and C will be activated. B will pay C the remaining 20 so that C's core capital is 0. The guarantee contract between B and C will be reduced to $50 - 20 = 30$ and B's core capital to 100.

In this very simplistic case the iterations will stop, because there are no other SST entities involved. However if there were and for example C had defaulted, then we would have calculated that event's impact on the rest of the entities that it had a CRTI (for example a loan) against. Then when an equilibrium point had been reached, a new simulation would have begun.

6 Examining Marginal Distributions

Losses in this section are given in USD but rescaled with a common factor. We will briefly explain each risk type and then go into more detail for the credit risks.

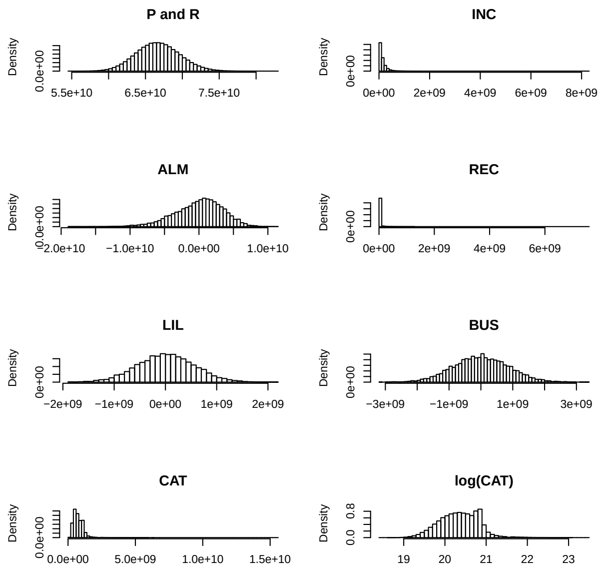

Figure 6.1: Histograms of the 7 marginal distributions. The lower-right plot is the log-transformed CAT distribution.

6.1 Assets and Liability Matching Risk (ALM)

ALM risk is the Asset and Liability Matching risk also called Market Risk. This is the risk that assets or liabilities change in value and the losses associated with this.

We find that in our case the simulated losses closely resembles the Skew-normal distribution which is defined as

$$f(x) = \frac{2}{\omega}\phi\left(\frac{x-\xi}{\omega}\right)\Phi\left(\alpha\frac{x-\xi}{\omega}\right) \tag{6.1}$$

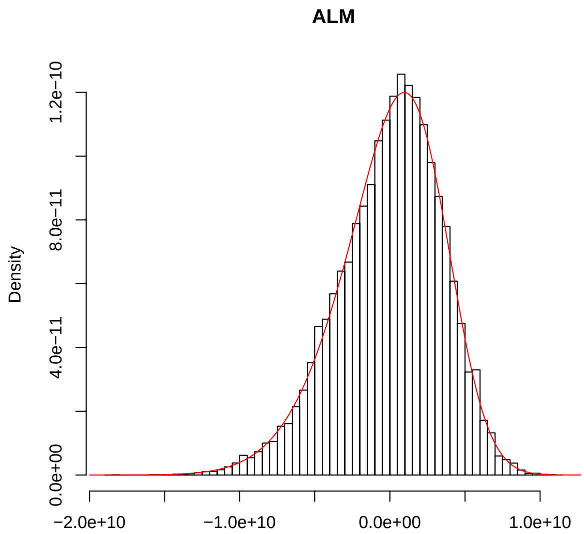

Figure 6.2: ALM risk, with a fitted skew-normal distribution.

where ϕ is the standard normal probability density function, and Φ is the cumulative distribution function. [1] Parameters ξ, ω and α define respectively: location, scale and shape.

Figure 6.2 presents a histogram of the simulated losses and a fitted skew-normal distribution.

Location ξ	Scale ω	Shape α
$3.591748 \cdot 10^9$	$4.996743 \cdot 10^9$	-2.076524

Table 6.1: ALM parameters.

6.2 Natural Catastrophes Risk (CAT)

CAT risk is the risk associated with natural disasters, for example typhoons and floods. In our case the distribution of losses is well approximated by a lognormal distribution with parameters $\mu = 20.34742$ and $\sigma = 0.4747446$. See Figure 6.3.

6.3 Investment Credit Risk (INC)

Investment Credit Risk is the risk that a company invested in defaults.

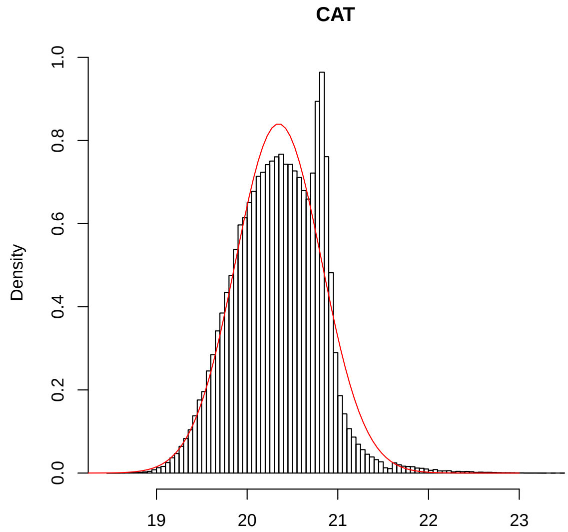

Figure 6.3: CAT risk log-transformed and a fitted normal distribution.

Both the credit risk types (see next section about Reinsurance Credit Risk as well) are simulated by means of Brownian motion. So each investment's future net worth is modeled by a process which depends on various micro and macro economic factors but also includes a stochastic part called a Wiener process. This stochastic part has independent increments with $W_t - W_s \sim \mathcal{N}(0, t-s)$ for $0 \leq s < t$.

The process describing the net worth is then simulated for a year, after which it is compared to a threshold value. If the net worth is below this threshold, then the investment is considered in default and a loss.

In our case the simulated loss distribution seems to closely resemble a generalized Pareto distribution with parameters given in Table 6.2.

Shape	Scale
$-1.714080 \cdot 10^{-2}$	$1.422129 \cdot 10^{8}$

Table 6.2: INC parameters.

6.4 Reinsurance Credit Risk (REC)

Reinsurance Credit Risk is the risk that reinsurers will default. This is simulated in the same way as Investment Credit Risk above. However, while INC

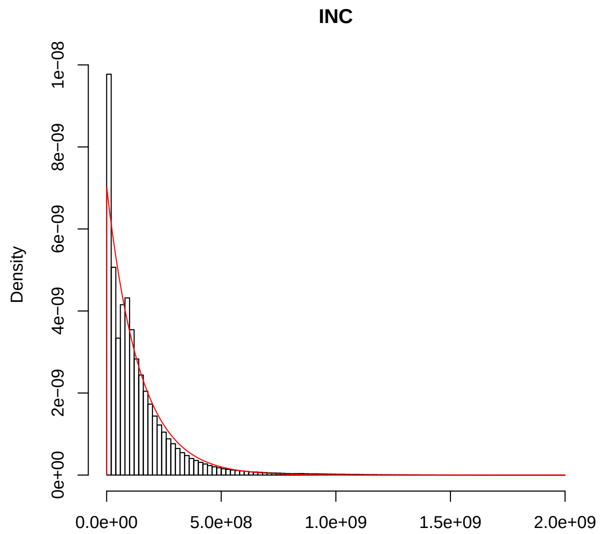

Figure 6.4: INC risk, with a fitted GPD distribution curve.

has a lot of historical data to rely on this is not the case with reinsurers.

Unfortunately we have not been able to fit a distribution (choosing among the usual suspects: generalized Pareto, Chi-square, Exponential, log-Normal) to this sample. Figure 6.5 shows a histogram of the simulated data.

6.5 Premiums and Reserves Risk (P&R)

P&R Risk is the risk that our company either charges the wrong premium for an insurance contract, or reserves the wrong amount of money for future payments. The losses associated with this can be quite large. In our case the simulated loss distribution is very well fitted by a normal distribution (see Figure 6.6).

6.6 Life Business Risk (BUS) and Life Liability Risk (LIL)

Both BUS and LIL are parametric to begin with and assumed to have normal distributions. These risk types are associated with the risk of wrongly projected mortality rates for example, resulting in premiums set too high or too low. Given parameters are presented in Table 6.3.

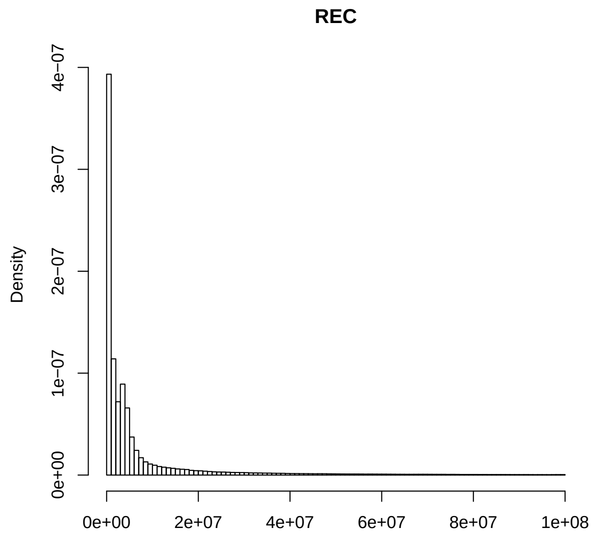

Figure 6.5: The REC distribution.

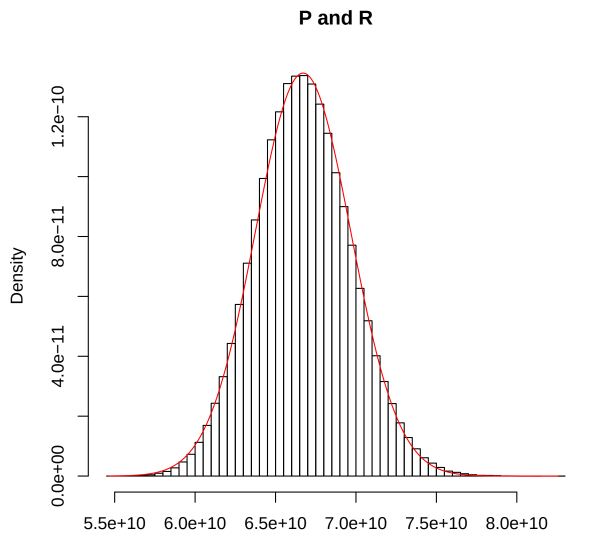

Figure 6.6: P&R risk with a fitted normal distribution.

7 Copulas Impact on VaR and ES

In choosing different copulas, one can get quite different results. Since the Gaussian copula does not have any tail dependency, but otherwise looks

Risk type	μ	σ
BUS	0	875'840'094
LIL	0	552'944'833

Table 6.3: BUS and LIL parameters.

similar to the t-copula the former will always give a lower value for VaR and ES as can be noted in Table 7.1 (using the same correlation matrix in both cases). As seen in the same table, a higher degree of freedom for the t-copula will give lower values for the risk measures; this is because the t-distribution approaches the normal distribution for increasing degrees of freedom. The Gumbel and Frank copulas are included even though they are intuitively not considered very useful in the present case (see discussion in 5.1). The independent copula is included for measure.

Family	$\text{VaR}_{99.95\%}$	$\text{ES}_{99\%}$
Gaussian copula	1.00	0.79
t_3-copula	1.25	0.90
s t_4-copula	1.22	0.88
t_5-copula	1.19	0.87
Frank copula	0.99	0.83
Gumbel copula	1.60	1.14
Independent copula	0.75	0.61

Table 7.1: The result on VaR and ES of choosing different copulas. $\text{VaR}_{99.95\%}$ for the Gaussian copula is set to 1, as base value all others are compared to.

Obviously, since the limit of the t-distribution approaches the normal distribution as the degrees of freedom increase, the values for VaR and ES using a t-copula should approach the Gaussian copula's values. Unfortunately, while the t-copula probably is better suited for modeling this kind of data, there is no way to assess how well it does compared to the Gaussian copula in our case. There is also no way of choosing a proper value for the degrees of freedom, so deciding between them would be arbitrary. Of course, just sticking with the Gaussian copula instead is also arbitrary but at least it can be considered industry practice.

There is also the question of which purpose the numbers even serve – if FINMA is fine with using the Gaussian copula instead of something that would give a higher risk measure, then the capital requirement based on the solvency ratio will be lower hence giving the company more room for investments.

8 Diversification Effect

We are interested in examining the difference between using the copula on group level, i.e. sorting across all SST entities compared to a copula on each SST entity separately. We would then see the diversification effect between entities. Compare the following algorithm with the one described in Section 5.2.

1. Generate a uniform sample from a Gaussian copula.
2. For each SST entity:
   (a) work with the marginal distributions for each risk type on the level of the the SST entity in question;
   (b) order the samples according to the copula indices generated above (see section 5.3);
   (c) sum over the samples and extract ES and VaR risk measures.

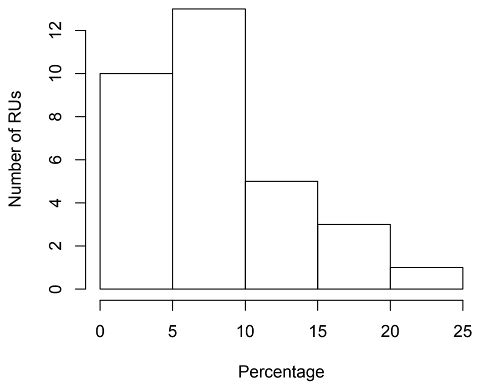

Figure 8.1: Diversification effect.

Comparing the resulting risk measures per SST entity with the ones where the copula is applied on group level (that is applied for the marginal distributions of the SST entity which encapsulates all the other companies – the mother company) will give an idea of the diversification effect between the entities. Included in Figure 8.1 is the diversification effect in percentages for all of the SST entities, i.e. $\frac{\text{Risk measure on SE level}}{\text{Risk measure on group level}}$, presented. We may note that the average diversification effect is 8.14% and the maximal 22.18%.

9 Impact of CRTI Model

What we really consider here is the difference between assuming no transactions between companies within the group and taking transactions in to account. Interestingly we shall get a lower value for Target Capital (Expected Shortfall minus mean) in the latter case. Table 9.1 presents changes between the two different approaches for the three largest SST Entities.

These numbers come from a run of 2163 simulations, however for the final report to FINMA the run will include 200000 simulations. Due to time limitations we can not wait for the final numbers, but those presented here will still reveal some insights.

SST entity	Change in TC
1	-0.27%
2	-0.24%
3	-2.52%

Table 9.1: BUS and LIL parameters.

There are two outliers among the 30 SST Entities: one change of 2800% and one of 650%. Not taking these two into account the average change is -11.89%. One reason for the very large outliers might be the small number of simulations. We believe that the changes are negative because we apply a LLPO (Limited Liability Put Option) - that is, we allow subsidiaries that default just be dropped. Even though their core capital is below zero, we cut the loss at zero.

While we may not disclose the actual numbers these changes are definitely substantial and in the hundreds of millions.

10 Conclusion

We have looked at several aspects of the SST and the Solvency II frameworks and examined some differences. While we do conclude that these can be quite big in terms of risk measures, it is hard to see which approach is the better one merely based on the findings in this paper. As we have seen choosing a copula can be hard and affect the numbers considerably. We also note that modeling the CRTIs between subsidiaries may change the risk measures quite a bit, however further research (and further runs with more simulations) will need to be done in this area.

A S Code

A.1 Numerically Computing Quantiles for the Copula

```
CopulaOrder.m.rt <- SimulateGaussCopulaOrder(CORR.rt.rt, m)

CopulaIndex.m.rt[, "BUS"] <-
        order(RT.BUS.LA.mbus)[ceiling(m * CopulaOrder.m.rt[, "BUS"])]
CopulaIndex.m.rt[, "LIL"] <-
        order(RT.LIL.LA.mlil)[ceiling(m * CopulaOrder.m.rt[, "LIL"])]

Loss.m.ru.rt[,, "BUS"] = RT.BUS.LA.mbus.ru[CopulaIndex.m.rt[, "BUS"],]
Loss.m.ru.rt[,, "LIL"] = RT.LIL.LA.mlil.ru[CopulaIndex.m.rt[, "LIL"],]
```

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