Master thesis

CPPI Structures on Funds Derivatives

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Abstract

With the ever-increasing complexity of financial markets and financial products, many investors now choose to benefit from a manager's expertise by investing in a fund. This fueled a rapid growth of the fund industry over the past decades, and the recent emergence of complex derivatives products written on underlying funds. The diversity (hedge funds, mutual funds, funds of funds, managed accounts…) and the particularities (liquidity, specific risks) of funds call for adapted models and suited risk management.

This thesis aims at understanding the issues and difficulties met when dealing with such products. In particular, we will deal in a great extent with CPPI (Constant Proportion Portfolio Insurance) structures written on funds, which combine the specificities of funds with particularities of such structures. Correctly assessing the corresponding market risks is a challenging issue, and is the subject of many investigations.

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Introduction

I have carried out my 6-month internship in the Market Activity Monitoring (MAM) department of Crédit Agricole CIB, one of the largest French investment banks. The MAM lies at the crossing paths of Front Office, Risk Management and Quantitative Analysis. It is in charge of the production and control of risk indicators on the trading books of the bank: risk exposures, value-at-risk, stress tests, explanation of daily P&L…

As a part of the Global Equity Derivatives sector, I was assigned to the Fund Derivatives perimeter which controls a variety of derivative products with funds underlyings (hedge funds, mutual funds, managed accounts…).

My main daily task was to perform the P&L (profits and losses) analysis: it consists in explaining the day to day changes in the trading book value with the variation of the market parameters which the book has sensitivity to. Although it does not require daily use of quantitative methods, this demands a good understanding of the framework: the products and their functioning (especially behavior in regards to market moves), which models are used and why, as well as an insight on the handling of the contracts by the FO (hedging schemes, risk provisions).

Consequently, my first task has been to acquire the necessary practical knowledge on the products I had to control. This proved to be a very instructive insight on the reality of financial products: after a few years spent studying mathematical models relying on idealized hypothesis, I have been confronted with products and market conditions that do not fall under these assumptions.

Therefore I chose to orientate my thesis towards the different issues that prove to be important for risk management. In particular I will focus on one structure that was among the most important in the book and appears to be also quite challenging: CPPI contracts.

In the first part I will present the Funds Derivatives framework: the different types of funds, the derivatives I have been working on. We will see that although funds are generally put in the equity basket, their specificities (liquidity, accessibility…) make them harder to handle than standard equity like shares or indexes. A special stress will be put on hedge funds and their strategies because of their wide diversity.

In part 2, I will talk about CPPI structures. In its simplest form, CPPI is just a structured product constituted of a risky asset and cash, where the allocation is made according to a single formula. However in practice these products are generally embedded with several features that affect the strategy. We will have a look at the performance profile of this structure and oversee the principal features. We will also see how introducing discrete hedging and/or jumps in equity give birth to an additional risk for the strategy.

Which brings us to part 3: we will investigate the particular gap risk of CPPI structures. I will show why basic models like Black-Scholes for the equity are very unsuited, and how to modify it to get more appropriate results. I will also quickly present some recently designed models on the subject.

The approach I will use is mainly descriptive. As we are to see the vast majority of performed computations need to be made via intensive Monte-Carlo simulations; so I decided to focus more on the different issues that are important when modeling and measuring risk with the described products, than on its quantification itself.

Part 1: The fund derivatives background

As its name indicates, a fund derivative is a financial derivative product whose underlying lies in the range of the fund family: hedge funds, mutual funds, funds of funds, private equity funds…

This particular asset class is attractive to investors because of its historic high returns and/or its specific diversification and risk profile. However, funds and particularly hedge funds carry an important systemic risk that can lead to huge losses in an adverse market context; this risk played a crucial role in the LTCM failure in 1998 (more details on this in the next section). The raise of awareness of those specific risks among investors triggered an explosive growth of fund derivatives products in the last decade.

These derivatives allow investors to benefit from the advantages of the underlying funds while having a better grip on the encountered risks. In particular, option-like products or guaranteed capital structures (among which CPPIs, which will be studied in a large extent in this paper) proved to be quite popular in the recent years.

To understand properly the risks borne by the issuer of such products, one first has to go through all the particularities of the fund-like assets. Consequently, I will present in this part the specificities of different types of funds (in particular hedge funds), and then give an overview of standard products in the fund derivatives business.

1.1 Hedge funds

It is rather hard to characterize hedge funds as there is no existing official definition of a hedge fund, but there is a set of features that is common to most of them.

1.1.1 Characteristics

Hedge funds are usually described as aggressive, lowly regulated investment companies. Indeed most funds limit their accessibility to a few wealthy investors. For instance funds in the US typically have less than one hundred investors, in order not to be constrained by the SEC (Security Exchange Commission) regulations.

One big difference between hedge funds and other investment funds is their principal objective: whereas investment companies such as mutual funds aim at outperforming a given benchmark, the purpose of hedge funds is to generate absolute returns independently of the behavior of markets. In economical terms, this means maximizing the fund's *alpha* (the absolute performance) while maintaining a very low *beta* (the fund's correlation to the market). In those conditions, hedge funds should be able to make benefits both in bear and bull markets; even if the last financial meltdown proved that hedge funds can be more exposed to adverse market conditions than announced.

Hedge funds usually have a separation between their legal structure and their management board: the managers and administrators of the fund are typically based in major financial places, with a legal structure registered in more fiscally favorable environments. It is estimated that about two thirds of hedge funds have an offshore legal structure (Cayman Islands, Bermuda…), and that most of the onshore legal structure are situated in countries with fiscal advantages such as Luxembourg or Ireland.

The fact that hedge funds are limited to a few investors allows them not to fall under the usual legislations for investment companies, e.g. with regards to liquidity, transparency, strategies… Indeed they tend to be highly secretive about their positions, and frequently use investment strategies that are not available to traditional investors. This implied an increased freedom in the investment tools, and allows for instance the use of high leverage and short selling. These funds also are notoriously illiquid instrument, with many clauses aimed at protecting the fund against investor panic. Since this point is of crucial importance in the management of fund derivatives, it will be the object of the next section.

One last characteristic of hedge funds is the high level of fees perceived by the managers. Those take the form of fixed management fees (percentage of the total asset value under management) and performance fees (percentage on the annual benefits made by the fund). It is not unusual to as high as 2% management fees and 20% fees. Those are accepted by investors because the rate of return after fees of hedge funds is still generally higher than the rest of the financial instruments. As a consequence, the managers of well-performing funds can reach enormous earnings, a fact that has often been criticized.

1.1.2 The liquidity issue

As mentioned earlier, liquidity is one of the key issues when dealing with fund derivatives. Indeed, hedge funds are far less liquid than classic instruments like stocks, bonds and even the other types of funds.

Indeed, liquidity restrictions are an almost vital feature for hedge funds. To see why this is the case, one has to remember that in their quest for opportunities, hedge funds can be invested in quite illiquid positions, and often use a high level of leverage. As a consequence, in order to implement his strategy in good conditions a hedge fund manager is looking for long-term investors, with an insight on their intentions of redemption. This materializes as liquidity constraints on redemption; but there exists also up-front liquidity constraints.

Here is a summary of classical constraints used by hedge funds:

- **Up-front illiquidity:** this is usually not a contractual constraint, but is a consequence of the investment policy of the fund. Indeed hedge funds are limited to a few important investors. For a first investment, one has to bring more than a certain threshold specified by the fund. This amount can vary greatly from one fund to another, but usually lies between 250 and 500k$. Moreover, once the manager decides that the fund has reached its maximum capacity in terms of capital under management (or in term of number of investors), the fund can be closed to new investments, either temporarily (it is then said to be *soft closed*) or permanently (*hard closed*).
- **Lock-up period:** this one prevents the investor to take his capital back for a fixed period after the initial investment (up to 3 years in some cases). This means either that the investor is in the contractual impossibility to demand redemption for this period (*hard lock-up*), or has to pay penalty fees to do so (*soft lock-up*).
- **Notice period:** when an investor asks for redemption of the invested capital, the operation does not take place instantaneously. There is a contractual delay (usually 1 to 6 months) before the redemption is met by the manager, which gives him time to reduce his positions without threatening his strategy.
- **Side-pocket:** this is used when a fund's investments become very illiquid. A new account containing the problematic assets is created. The investors at the time of the creation of this account become its owners, but capital cannot be redeemed from it until the fund manager manages to liquidate the assets. This feature has been massively used in the 2008 financial crisis, when managers had to face accrued illiquidity on the markets.

All these features are supposed to act as a protection for the fund. Indeed, many hedge funds strategies imply taking positions in quite illiquid assets, generally hedged with more liquid assets (although this is not always the case). Without any liquidity constraint, whenever a redemption event occurs, the manager of the fund would have to liquidate his most liquid assets to generate some cash, thus leaving him uncovered. This can result in major hits on the strategy, especially when there is high leverage. Notice periods are also a good way to protect the fund against investor panic, which is advantageous for the stable investment seeking manager.

However, those protections are not always enough when market conditions deteriorate. This is well illustrated by the 2008 financial crisis: with markets crashing, liquidity in markets did deteriorate and most investors were willing to exit their most volatile investments. When announcing their first severe losses, many funds had to face massive redemption orders, which only made things worse. Even with the use of side-pockets, many funds have been forced into liquidation.

1.1.3 Hedge funds typology

Despite the fact that hedge funds are very secretive about their positions and proceedings, and that it is not always obvious to assign a precise strategy to a given fund, there exists a commonly accepted typology depicting the hedge fund strategies' landscape.

These strategies vary mainly by the risk profile they provide to the investor: e.g. some are more or less correlated to the market, leveraged, or cover different classes of assets…

We will now oversee those strategies, following the classification given in (1).

Long/short equity

This is the strategy that was used by the first hedge funds. It mainly consists in investing in stocks with long and short positions, according to the manager's beliefs concerning their future evolution. The different amounts invested in short and long positions define the **bias** of the manager (**long bias**, **short bias**). There are usually more funds with long biases, because long positions are more convenient to manage and carry less risk (limited potential losses, unlike short positions). However this bias can evolve according to the market moves: typically biases will tend to be on the short side on bear markets. These strategies rely on a great extent on the skills of the manager on correctly selecting stocks, as well as on his judgment on market timing.

Some funds, known as **short sellers**, structurally maintain a large short bias, betting on the underperformance of the selected stocks. As mentioned before, these funds are riskier due to the potentially infinite losses on short positions. They are consequently very exposed in short squeeze situations: periods when the sudden growth of an instrument is explosively amplified by short sellers trying to get rid of their positions. This is why short seller funds are usually invested in as part of a larger portfolio of funds (or by funds of funds) to increase diversification.

Directional

Like L/S equity, directional strategies are among the first strategies developed by hedge funds. Actually L/S equity strategies are often also considered as directional, especially when they have an adaptive bias.

- **Global macro** funds specialize into forecasting macroeconomics trends, and place their bets accordingly. Those funds generally use a large range of assets (equities, bonds, FX, commodities…) to profit from the realization of the forecasted scenarios. They usually favor liquid assets, and can use leverage. Global macro funds are well-known for the variety of their strategies and their flexibility.

- **CTA** (commodity trading advisors) funds are similar to global macro funds, but focus on the futures market. Traditionally those funds used to take positions on the commodity futures market, and have now spread to the other liquid futures contracts (on indexes, foreign exchange…). The two main families of CTA funds are fundamental traders and systematic traders. Fundamental traders will rely on a fundamental analysis of the market to forecast future trends (therefore being close to global macro funds), whereas systematic traders will rather use quantitative methods to spot these trends.

Relative value

These strategies are based on the relative behavior of several financial instruments, looking for arbitrage or quasi-arbitrage opportunities. They mainly differ on the asset class they are dealing with.

- The most spread strategy in this category is the **equity market neutral**, which takes opposite positions on similar stocks (same sector, industry…) to benefit from their differences of spot price. Those opposite positions imply a low exposition to market variation, hence the name of equity neutral. The positions are usually taken with a high level of leverage.

- **Fixed income arbitrage**: those funds invest in fixed income securities, looking for pricing inefficiencies. The expansion of fixed income derivatives market allows the managers to use leverage through such instruments, thus benefiting from tiny credit spreads. Fixed income arbitrage requires a high technical level, and has been in the core of several crash periods: failure of LCTM in 1998, subprime crisis in 2008…

- **Convertible arbitrage**: uses prices inefficiencies between the different instruments involved in a convertible bond (the convertible bond and the associated stock and company debt instruments). This strategy lies at the crossroad of equity and fixed income arbitrage.

- The managers of **asset-backed securities** funds invest in the various class of ABS, and particularly in MBS (mortgage-based securities). Indeed, in addition of the traditional default risk and rate risk, MBS carry an early redemption risk that affects the instrument price. ABS-specialized funds generally use quantitative methods to detect inefficiencies in the valorization of this risk.

There exist many other relative value strategies (volatility arbitrage, statistical arbitrage which can be seen as intensive quantitative version of equity market neutral…), and many funds use more than one of the techniques described above. However two of their almost constant characteristics are their high use of leverage and their small correlation to the market.

Event driven

These strategies aim at taking advantage of special events, or situations that can generate large variations on the corresponding financial instrument. Among those we can find two main types of strategies:

- **Merger arbitrage** that specializes in M&A (mergers and acquisitions) events. Investors bet on the convergence of the two companies' market values, typically by taking a long position on the acquired company and short position on the acquirer. If however the merger is aborted, this will result in substantial losses as the gap will typically widen. This strategy is lowly correlated to the market, even though there are generally more opportunities in a bull market.
- **Distressed securities** whose investors focus on distressed companies that can be highly indebted, or experiencing a sudden loss in profitability. Those companies' stocks are valued very low, and are subject to big variations relative to predictions about the company's future. A distressed securities investor will then either bet on a recovery by buying the current cheap stocks, or bet on a depreciation by shorting the company's stocks or bonds.

1.1.4 Specific risks of hedge funds

Due to their specificities, hedge funds will carry additional risk for the investor than other equity products.

- **Liquidity risk:** as we have seen before, liquidity is limited when investing in hedge funds. Many funds have a weekly or monthly liquidity, and often orders are made according to an estimated net asset value of the fund. This means that at the time he posts an order, the investor has some uncertainty on the exact amount of the transaction.
- **Valorization risk:** most of hedge funds' positions are taken over the counter, and often imply complex structured products for which there is no marked-to-market valorization available. Consequently, the NAV (net asset value) published by the funds are dependant of the management's views and models.
- **Fraud risk:** although this remains rare, there are some cases of fraud in the hedge fund industry (embezzlement, Ponzi schemes…). If most of those swindling target non-expert investors (personal fortunes), some manage to fool even the most careful ones (e.g. Madoff's scandal that harmed many banks worldwide, see next paragraph)

1.1.5 Examples

From an outsider point of view, the hedge fund industry is very obscure: indeed hedge funds are not dedicated to mainstream investors, are not allowed to advertise and tend to be as secretive as possible concerning their operations and processes. However, several funds have made it to public notoriety because of their spectacular failures, frauds or noticeable operations they have carried out. Here are some examples of such funds:

- **Long Term Capital Management** (LTCM): this fund was launched in 1994 by John Meriwether and other former trading stars from Salomon Brothers fixed income trading group. Its management included high-end academics such as future Nobel Prize winners Myron Scholes and Robert Merton. This collection of talents helped raise a considerable amount of money, and the fund started its activity with more than $1 billion to invest. The fund specialized in fixed income arbitrage, using elaborate quantitative models to select opportunities.
  It immediately started with outstanding results: about 40% annual returns for its first years of existence, leading to an exponential growth of assets under management. Following this success, the fund started to take increasingly aggressive positions; at the beginning of 1998 its leverage ratio was as high as 25 to 1. It was mainly invested on quasi arbitrage deals (betting on the convergence of several similar assets, like long-maturity bonds with different liquidity). This level of leverage proved to be dramatic: in 1998 Russia defaulted on its bonds, generating a wave of panic on the fixed income market. Liquidity froze, and the supposedly converging gaps LCTM was invested in widened abruptly, resulting on huge losses for the fund. As it approached bankruptcy, an emergency bailout had to be set in Wall Street to avoid contagion to the markets. The full story of the fund, from its creation to its failure, can be found in (2).
  LTCM is now considered as a case study, and it is commonly accepted that hedge funds use way less leverage than prior to its bankruptcy.
- A famous illustration of the weight hedge funds can have in worldwide economy is given by George Soros' fund **Soros Fund Management**. In 1992 this fund shorted $10 billion of British pound in anticipation of its devaluation. This huge market move caused the government to exit the European Exchange Rate Mechanism and devaluate its currency, eventually generating $1 billion profit for the fund. Since then Soros has been referred to as "*the man who broke the Bank of England*". This shows the risk of hedge funds for market stability, when strategies are deployed in such a large quantity that they become self-fulfilling. This is why hedge funds are sometimes accused of manipulating markets, or triggering crisis for profit.
- Another well-known example of hedge fund is **Bernard L. Madoff Investment Securities LLC**, famous for settling the largest Ponzi scheme in history. The fraud has been going on for almost 30 years, for a total amount of nearly $65 billion (including $18 billion of actual losses to investors). Bernard Madoff, the manager of the fund, was sentenced in 2009 to 150 years in prison, at the age of 71.

These are of course only famous examples of hedge funds (and of their malpractices).

The interested reader will find other examples in (3), which reviews the evolution of several quantitative-oriented hedge funds and their managers.

1.2 Mutual funds

A mutual fund is a collective investment company that invests money collected from its different investors. Compared to hedge funds, these funds are more accessible to the public, and more closely regulated. In the US, mutual funds are governed by the Investment Company Act, and have to register with the SEC (Securities Exchange Commission). Therefore, they have some common characteristics that are appealing to the investor, e.g. a daily liquidity (in most cases). This comes however with a set of limitations (limited access to derivatives, fees…) restricting the manager's scope, which explains why hedge funds systematically avoid falling under this category.

In France, the equivalent of US mutual funds are OPCVM (Organismes de Placement Collectif en Valeurs Mobilières), and regroups SICAV (Société d'Investissement à Capital Variable) and FCP (Fonds Communs de Placement). They have to present a prospectus with a clearly defined strategy and a benchmark, and are legally bonded to stick within the disclaimed characteristics. This limits greatly the liberty of the manager to adapt his strategy to market conditions, for instance. Moreover, they shall use financial derivatives only for hedging purposes.

Those points explain why mutual funds usually perform lower returns than hedge funds (as well in the profits than in the losses). But their transparency and accessibility make them more likely to raise funds from "normal" investors. They also present a less aggressive profile than hedge funds, and therefore are more suitable as a sustainable investment (whereas an investment in HF usually corresponds to a small risky part of a portfolio). This accounts for the fact that the total value of assets under management of mutual funds is way larger than those of hedge funds: in early 2006 those totals amounted to $17770 billion and $1350 billion respectively (1).

1.3 Funds of funds

In response to the low accessibility of hedge funds, funds of funds (FoF) have quickly grown very popular. A FoF works the same way as other funds, except that they invest exclusively in other funds. An investment in a FoF thus provides a diversified position among several funds, which is usually impossible to obtain by direct investment for small investors. This is why funds of hedge funds are very popular (way more than funds of mutual funds for instance): currently between one third and one half of the investments in hedge funds are made via FoFs.

Investing in a FoF has several advantages. First of all, as mentioned above, it brings natural diversification, with the standard FoF containing between 20 and 50 underlying funds. Such a level of diversification requires a huge investment size, knowing the minimum level of investment to enter a hedge fund. Selecting funds is also a lot of work and requires a lot of upfront analysis, which can also be hard to bear for a single investor. They may also be the only way to gain participation in a hedge fund that is closed to new investors. Finally, they usually provide more liquidity than their underlying funds.

Funds of funds are by now a serious alternative to multistrategy hedge funds. While those two types of fund present a similar diversified profile, they both present some pros and cons: multistrategy HF will charge less (no intermediary fee perceived by the FoF) and be more reactive to the market because it has direct hold of its assets. On the other hand funds of funds provide another dimension of diversification (diversification among managers, which is important to lower correlation on the different underlyings and exposition to fraud-like risks), and will be faster to integrate new strategies because it will simply have to invest in a corresponding fund.

In order to give information to their customers with regard to their strategies, funds of funds generally guarantee a certain amount of constraints on the FoF's composition. Those can be:

- Maximum weight assigned to a single fund (e.g. no fund with more than 10% portfolio weight)
- Maximum weight assigned to a single manager (e.g. no manager above 20% portfolio weight)
- Minimum and maximum amount to be invested among the different strategies (CTA, Equity L/S, Event driven…)
- Constraint on total liquidity of the portfolio (e.g. at any time 50% of the portfolio should have a monthly liquidity, 80% a semi-annually liquidity…)

If one or more of those investment guidelines are not met by the FoF, it can trigger a termination event: the customer can withdraw his investment within the FoF at special conditions. It is then of crucial importance for a trading team on fund derivatives to monitor those guidelines closely. It allows both to check that the risk exposition is the one needed, and to take measures if a critical point is reached.

1.4 Managed accounts

To limit the inherent risks of hedge funds (fraud, valorization), a special structure has grown popular: managed accounts. Instead of investing directly in a fund, one goes through an independent platform or special purpose vehicle. This account management is delegated to a hedge fund manager, but all the operations of valorization and guideline checking are performed by the platform.

This provides more transparency for the investor on the fund's composition, and a guarantee against fraud risk (the fund manager has no authority on the account, and all the orders are to be made by the platform). It is also a good way to gain control over the nature of transactions (e.g. no OTC transaction without prior investor authorization).

Of course, this advantage for the investor comes at the price of extra fees, perceived by the platform. Moreover, many hedge fund managers are reluctant to use those structures, on the one hand because of the restrictive constraints that can be embedded, and on the other hand because the transparency over their positions could cause their strategies to be massively replicated, thus limiting profit on their side. Another constraint over the use of managed accounts is the implied infrastructural cost: such structures are once again not profitable under a quite important initial investment level (estimated at about $5 million in (1)).

1.5 Funds derivatives products

In this part I will present quickly the various fund derivatives products dealt with by Crédit Agricole CIB, and that I have overseen during my internship. I will only give a brief description of those in this part, and I will come back on the case of CPPI-type derivatives on the next chapter for a much more detailed study.

Note that from the bank's point of view, issued contracts must result on a negative delta in order to be hedged with long positions in fund shares.

1.5.1 Funds options

Funds options are in all points similar to standard equity options, except that the option's underlying is one or several fund share. We can find all classic features of options: call, put, asian, rainbow, bestOf, worstOf, caps and floors…

Compared to equity options, the pricing and hedging of funds options must take into account the particularities of the underlying. Indeed, as seen before fund products carry specific risks (default, fraud, non-respect of strategy) and have particular liquidity features. Remember that many funds allow investors' orders only at predetermined times, which makes "continuous" hedging impossible.

This constraint to hedge discretely is another risk for the provider of the product. Indeed, if we take for instance the Black-Scholes model, we know that the price of a derivative is derived from the replicating continuous strategy. This hypothesis is clearly not realistic, but does not induce much error for liquid enough assets; this is not longer the case when the gap between rebalancing dates widens.

> Example: in (4), the author investigates the effects of introducing discrete hedging for the Black-Scholes replication of a call option, by studying the hedging error:
>
> $$Error = Hedge_T - Payoff$$
>
> where $Hedge_T$ is the (stochastic) value of the hedging strategy at maturity. It appears that under the Black-Scholes framework, this induced error follows approximately a normal distribution, with zero mean and with estimated standard deviation:
>
> $$\sigma_{error} = \nu\sigma\sqrt{\frac{\pi}{4N}}$$
>
> Where ν and σ are the option's vega and volatility, and N is the number of rebalancing dates. Note that when we make N go to infinity, the error becomes zero almost surely, as we are back to the standard Black-Scholes framework.

The previous example shows how to evaluate the extra risk brought by discrete hedging. This can be computed on the other types of funds options (often with the use of Monte-Carlo simulations), and the provider of a product will have to take provisions accordingly. In the same way, provisions have to be booked to take into account the other funds related risks: illiquidity due to lock-ups, fraud risk, wrong information about the fund's expositions…

1.5.2 Options on vol-capped indexes

Vol-capped indexes are a simple form of structured products designed to profit from the underlying's performance, while having a control on the total investment's volatility. In general, these products contain:

- An investment in a risky asset (here, a fund),
- An investment in a non risky asset: usually cash bearing interest, but it can also be another fund with very low volatility,

- If necessary, a funding instrument to borrow cash. This is only used if the product allows leverage.

The goal of this product is to invest in the risky underlying with a targeted level of volatility σ_{target} (or if there is no leverage, a cap on volatility). To achieve this, one first has to compute the current volatility σ_t of the risky asset at each rebalancing date. The target allocation in the risky asset (over the whole index value) is then given by the formula:

$$TA_t = \max\left[w_{min}; min\left(w_{max}; \frac{\sigma_{target}}{\sigma_t}\right)\right]$$

If we consider that the non risky instruments have zero volatility, this amounts to set the volatility of the index equal to σ_{target}, provided that the weight of the risky asset lies in the interval $[w_{min}; w_{max}]$. If w_{max} is greater than 100%, leverage is used when the risky asset's volatility is lower than the target.

Here an important feature is the choice of the range over which the historical volatility σ_t is calculated. Since the purpose of the product is to reduce the exposition to the risky asset in turbulent times, the range should be relatively small, so that changes in volatility are more quickly detected. However it should remain large enough to be significant, and to avoid excessive rebalancing due to one extreme move for instance. Usually, this range lies between the last 20 and 60 available returns of the risky asset, according to the needs of the client.

1.5.3 Total Return Swaps (TRS)

A TRS is a contract in which the total performance of an asset is exchanged for a fixed or floating interest rate payment. The performance includes positive and negative returns and eventual coupons, dividends… there is therefore a transfer of both credit and market risk of the instrument.

This structure allows the investor to benefit from an asset's performance without including it in his balance sheet. This can thus allow access to a hard-to-reach asset class, or by adding contractual terms to use synthetic assets which are non-tradable.

1.5.4 CPPI

This is another type of structured product between a risky and a non-risky asset. CPPI stands for Constant Proportion Portfolio Insurance; it is a product with a guaranteed capital.

The underlying principle is rather simple: the guaranteed capital is materialized by a bond-like instrument. At any time, the investment in the risky asset should be a multiple of the difference between the current value of the contract and the present value of the guarantee. This allows to benefit from good performance of the risky asset over the non-risky one by increasing the risky exposition as the gap widens (leverage is generally allowed), while reducing this exposition as the cushion decreases.

The formalization, features and behavior with regards to chosen modeling of CPPI will be the subject of the next section.

1.5.5 Options on CPPI

Following the growing success of CPPI structures, many derivatives written on a CPPI have appeared. Most of them take the form of equity derivatives like options, with the underlying share or index replaced by a CPPI strategy. These can be basic vanilla options, or more exotic. Of course, combining an exotic option with a CPPI structure can result in a complex risk structure.

Part 2: Presentation of CPPI structures

In this part we will study in more details the characteristics of CPPI contracts, particularly with funds underlyings. CPPI and options on CPPI were some of the main products I was dealing with during my internship, and they require particular care when it comes to market monitoring and the production of risk indicators.

I will then present the formulas regulating a CPPI structure, as well as the different features present in usual CPPI contracts.

2.1 Definition and formulas

A CPPI (**Constant Proportion Portfolio Insurance**) contract is a structured product from Portfolio Insurance family. Portfolio insurance has been designed as various dynamic strategies providing participation to the performance of an asset while protecting the portfolio value. It boomed in the 80s as research provided more theory for those strategies.

The two most popular strategies are OBPI (Option Based Portfolio Insurance) and CPPI; in this paper we will focus on the latter.

As mentioned, CPPI is a security with guaranteed capital, based on two instruments: a risky and a non-risky asset (plus a loan instrument in the case of leverage). The allocation in the risky asset is made through a rather simple rule: at every rebalancing date, the allocation in the risky asset must be the product of a constant (the multiplier) and the difference between the portfolio value and the discounted guarantee (the buffer).

Therefore, if the asset performs poorly, the buffer will decrease, causing the allocation in risky asset to decrease as well. Conversely, when the risky asset performs well, the buffer will grow resulting in more and more allocation in the risky asset. Extra leverage can be used when necessary by borrowing money to invest in the risky asset.

The non-risky asset is usually a zero-coupon bond, and the underlying risky asset can be a stock, an index, a fund… As seen before, using funds as underlying raises issues with regards to liquidity.

In the following, we will use the following notations:

- NAV_t for the CPPI's Net Asset Value at time t
- RA_t is the value of the investment in the risky asset at t
- $Prot_t$ is the value of the investment in the non risky protection asset at time t
- $Loan_t$ is the amount borrowed in order to leverage the risky position
- FIC_t is the value at t of the fixed income instrument that constitutes the guarantee. At maturity, this is thus equal to the guarantee (e.g. if the guarantee is predetermined, this is the present value at t of a zero-coupon bond delivering the guarantee at maturity)

For convenience, we will have $Prot_t > 0$ and $Loan_t < 0$, and at any time t at least one of those two amounts is equal to zero (we don't borrow money if we can sell some non-risky asset). We will note

$$NRA_t = Prot_t + Loan_t$$

We will also define the buffer, which is the excess level of the CPPI value over the guarantee:

$$Buffer_t = NAV_t - FIC_t$$

Finally, the CPPI product is characterized by a parameter called **elasticity**, denoted λ here.

With the previous notations, the rebalancing formula that drives a CPPI product is the following: at each rebalancing date, where the CPPI value is NAV_t, the allocation in the risky asset is

$$\boxed{RA_t = \lambda . Buffer_t \qquad (1)}$$

In absence of transaction cost, we then deduce the non-risky asset allocation:

$$NRA_t = NAV_t - RA_t$$

The formula (1) justifies the name of **constant proportion** portfolio insurance. We can see that if the buffer is large enough, the structure will be leveraged. A quick calculation shows that it is the case when:

$$NAV_t \geq \frac{\lambda}{\lambda - 1} FIC_t$$

Note that with this formula, if the rebalancing is done continuously, the allocation in risky asset will decrease as the buffer reduces; if the latter goes all the way to zero then the entire portfolio is invested in the non-risky asset. If the fixed-income curve and the non-risky asset use the same underlying (we will assume it is the case), the product will stay equal to the fixed income curve until maturity in order not to jeopardize the guarantee.

This raises an important point for the discrete rebalancing case: what if the buffer has moved down enough between two rebalancing dates to become negative? Formula (1) suggests that we now take a short position in the risky asset. But this would be contradictory with the principle of guaranteeing a fixed amount at maturity (indeed if the risky asset was to rise back in value, the NAV would keep falling below the needed cash amount to reach the guarantee at maturity). So from a practical point of view, all the position in risky asset is liquidated whenever the buffer becomes non-positive:

$$Buffer_t \leq 0 \ \Rightarrow RA_t = 0$$

Clearly if the buffer becomes strictly negative, the final payoff will fall slightly below the guarantee at maturity. We will discuss the consequences of this point in section 2.3.

Let's now illustrate the mechanism of CPPI with a simple example.

Consider a simple 5 year CPPI, with initial value $NAV_0 = 100€$*, elasticity* $\lambda = 4$*, and monthly rebalancing. Suppose that the non-risky asset is a zero coupon bond, with fixed interest rate of 5% per annum. The reference level for the guarantee is this same zero coupon bond.*

The first allocation is made as follows:

At t=0, the FIC is equal to 77.88€ (it is the price at t=0 of the 5y-ZCB), so the buffer is equal to $NAV_0 - FIC_0 = 22.12€$*. Therefore, the first allocation is:*

$$RA_0 = \lambda . Buffer_0 = 88.48€$$
$$NRA_0 = NAV_0 - RA_0 = 11.52€$$

Now we consider two scenarios for the risky asset:

> *<u>Case1:</u> the risky asset raised by 20% in the first month: the value of the strategy prior to rebalancing is now* $NAV_1 = 88.48 \times 1.2 + 11.52.\frac{78.2}{77.88} = 117.74$*. The new allocation is:*
>
> $$RA_1 = \lambda . Buffer_1 = 158.17€$$
> $$NRA_1 = NAV_1 - RA_1 = -40{,}43€$$
>
> *The structured is now leveraged, as some cash has to be borrowed to invest more in the risky asset.*

Case 2: *the risky asset dropped by 20% in the first month: we then have the net asset value NAV_1=82.35€, leading to the new allocation:*

$$RA_1 = elasticity\ .Buffer_1 = 16.61€$$
$$NRA_1 = NAV_1 - RA_1 = 65.75€$$

We can see that in this second case, the total value of the strategy comes close to the present value of the guarantee. Therefore most of the strategy's value is reallocated to the non-risky asset.

The figure below displays one simulated path for the CPPI structure described in the example. The risky asset has been modeled here as a geometrical Brownian motion with annualized volatility of 20%. As expected, we can see that the allocation in the risky asset increases quickly when the difference between the NAV and the fixed income curve widens, leading the structure to be leveraged on several occasions.

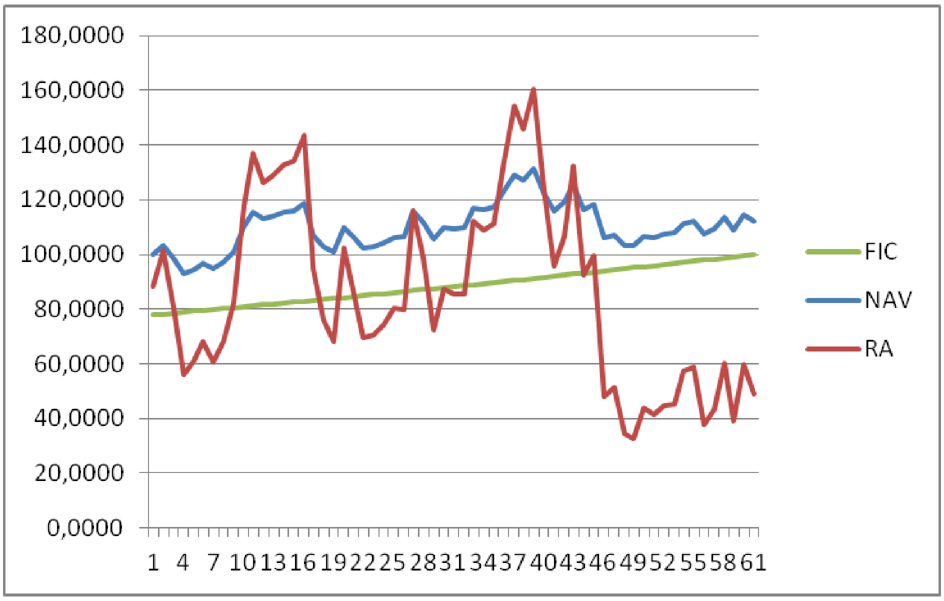


Figure 1: Simulated path for the CPPI structure

The next figure displays the final value of the structure (vertical axis) as a function of the final value of the underlying (horizontal axis), over 10.000 simulations similar to the one used for the figure above. Although the path-dependency of CPPI structures prevents those two values to be linked with an exact relation, a clear profile appears: we have a call-like payoff with an exponential in-the-money area.[1]

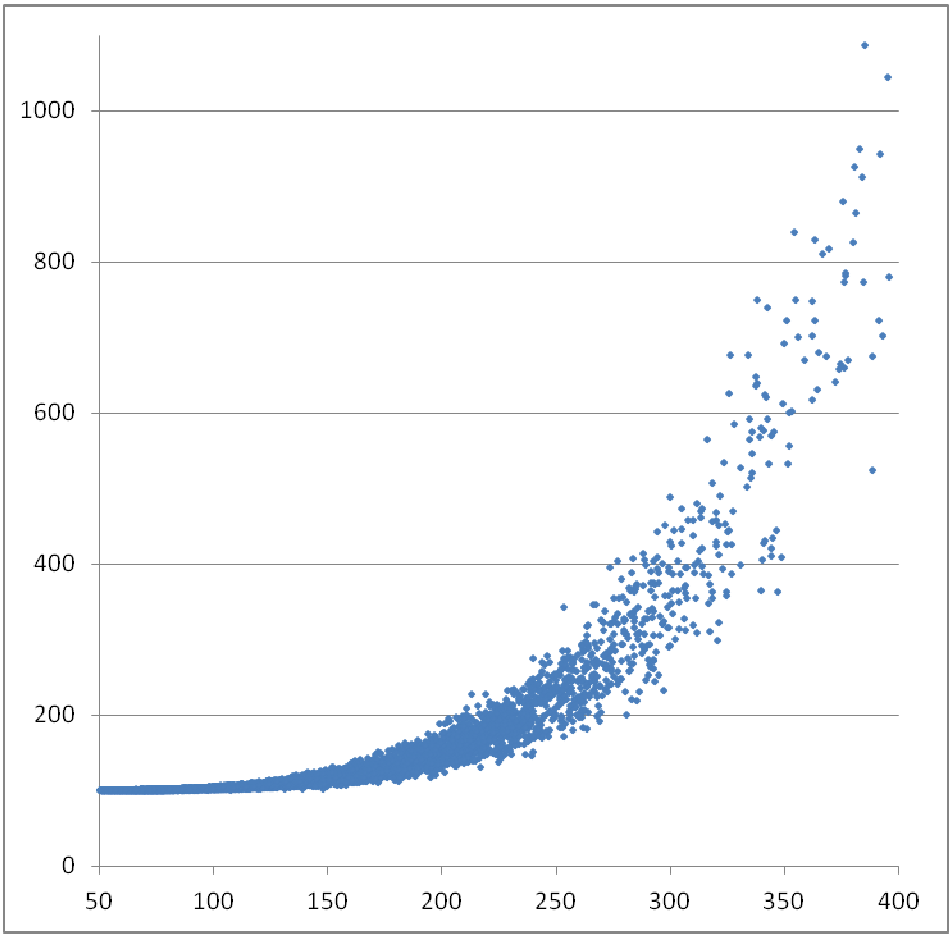


Figure 2: Final value of the CPPI as a function of the final value of the risky asset

The following graphic shows the distribution of the payoff according to these simulations.

[1] Note that this graph depends on the chosen model: the apparent correlation here can be less obvious with a different modelling, or a more volatile underlying.

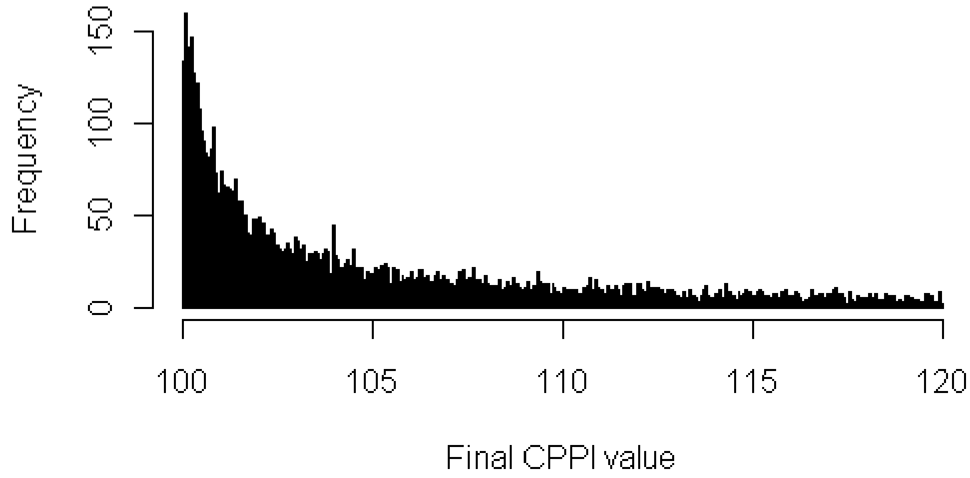


Figure 3: Histogram of the final values for the simulations

This gives a good overview of the CPPI performance profile: many paths lie near the final guarantee amount, but in the tail part the decay is quite slow, providing a few very high returns.

2.2 Additional features

The formulas in the previous section describe theoretical CPPI products. Nevertheless, practical CPPI usually come along with additional features to limit gap risk, control leverage or take into account transaction costs. There follows a set of features I've been seeing on the CPPIs I've been working on.

2.2.1 Liquidation trigger

The effect of a liquidation trigger is to liquidate the position in risky asset when the buffer size relatively to the total value is too small. Typically, this condition will be of the form:

$$\frac{Buffer_t}{NAV_t} \leq LT_{\%} \Rightarrow RA_t = 0$$

As seen previously, setting the risky allocation to zero makes the CPPI entirely driven by the fixed income curve. When a trigger event occurs, the CPPI is then worth the guarantee plus the eventual remaining buffer converted into non-risky asset.

This feature is more useful for the seller of the product who actually replicates the strategy. When the buffer is getting critically small, rebalancing imply buying or selling very small quantities of equity, which can be costly and inconvenient. Moreover, holding such a small quantity of risky asset is almost nonsense: if the target allocation is 1% of the total value, returns on the risky asset will have to be as high as several hundred percent on a given period for it to gain back some weight in the product. Therefore, the CPPI value at maturity will very probably stay close to the guarantee. It is then understandable that it is usually not worth loosing time and money on monitoring and issuing orders for those products: it is more convenient to have a "trigger" to rebalance them entirely into a bond or so.

Let's go back to figure 3, and see what the effects of a liquidation trigger on the payoff distribution are.

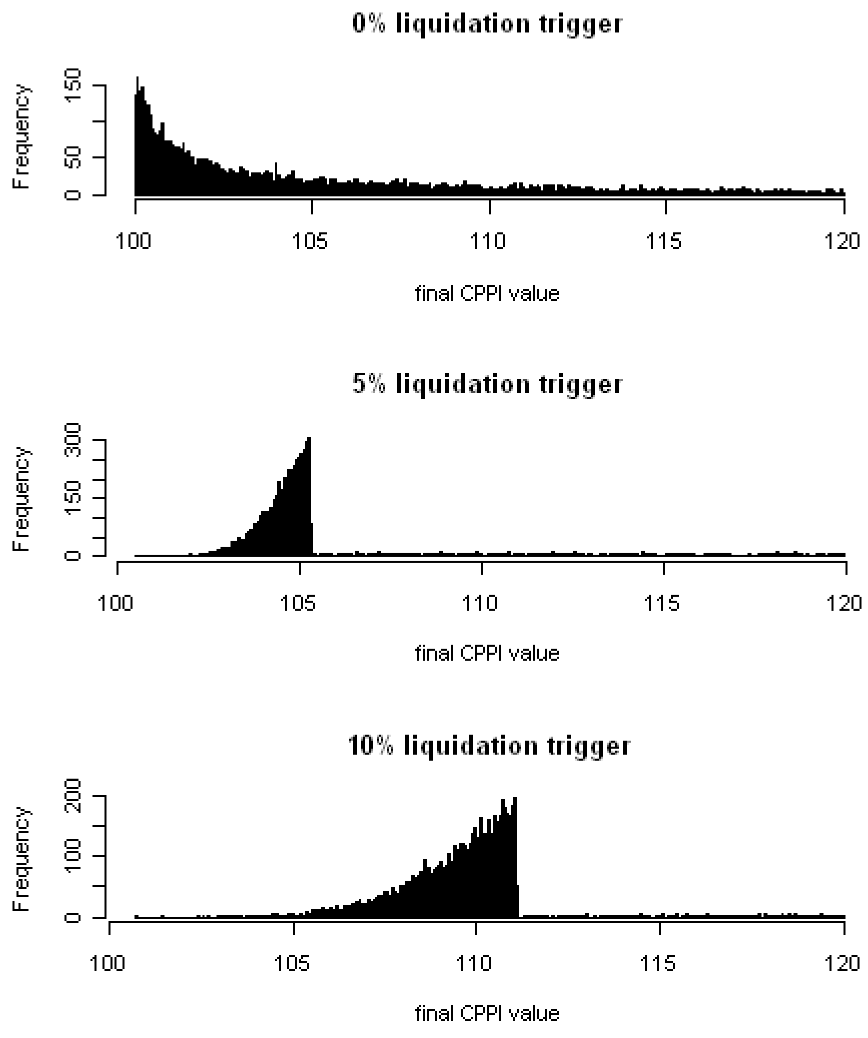


Figure 4 : Effect of the liquidation trigger on the payoff distribution

We can see that the liquidation trigger brings the final value of close-to-guarantee paths up, but it also kills a lot of paths: there are therefore much less possibility of high returns when the trigger is high.

2.2.2 Leverage cap

Remember the allocation formula for a CPPI on a rebalancing date:

$$RA_t = \lambda . Buffer_t$$

As such, we see that this formula does not constrain leverage: the higher the buffer, the larger the position in risky asset has to be. This principle is the core of the CPPI strategy, but it should only be valid up to a certain point.

Indeed, excessive leverage brings additional risks:

- More exposition to market risk
- More exposition to specific funds risks
- Also, as we will see, from the seller's point of view a CPPI carries gap risk. The losses occurring on a gap event depend on two factors for a CPPI: the jump size and the level of leverage when the jump occurs.

Those points justify the use of limitations on leverage for CPPI. Those come usually under two classes:

- A limit on the weight of the risky asset in the total value of the product, of the form:

  $$RA_t \leq w_{max} . NAV_t$$

  This constraint can either eliminate (set w_{max} to 100% or lower) or bound leverage. Alternatively, this constraint can be put on the weight of the loan, which is equivalent.
- A limit on the total loan (hard cap): it is a limit on the product's credit line. The limit is usually set as a multiple of the product's notional:

  $$Loan_t \leq C . notional$$

  As for the previous limit, we can constrain the total amount invested in the risky asset, but this would make less sense.

Note that whenever one of those constraints is filled, the real allocation in risky asset is reduced compared to the one given by the CPPI formula. This confirms that those constraints are useful to reduce exposure to equity risk. It is not uncommon to see a mixture of those two constraints in a CPPI product: the relative constraint is the first to be hit, providing a linear leveraging with the growth of the risky asset (instead of exponential before); then the hard cap kicks in, stabilizing the structure below a certain level of debt and leverage.

2.2.3 Transaction costs and minimum order amount

As for most derivatives products, transaction costs have to be taken into account for an accurate pricing and monitoring of CPPI. Indeed, many funds charge subscription and/or redemption fees, and the charge on a loan is usually higher than the return on held cash.

In particular, the fix part of transaction costs (the amount to be paid to issue an order, as opposite to the proportional costs on this order) put a large restriction on quasi-continuous hedging. Issuing an order isn't worthy in general if the order amount is too small.

Consequently, CPPI strategies are often embedded with a minimal order amount clause at each rebalancing date. That is too say, the rebalancing is performed only if the amount to be ordered is higher than a given limit. This limit is usually set as a percentage of the total allocation in risky asset.

Example: the order is performed only if there needs to be purchased/sold more than 3% of the current total allocation. This condition writes

$$\left|\frac{RA_{target}}{RA_t} - 1\right| \geq 3\%$$

Where RA_t is the total value of the investment in the risky asset right before the eventual rebalancing.

As seen above, this feature avoids posting orders when judged unnecessary. There are however some drawbacks: imagine that the risky asset grows steadily, so as the minimum order amount is not reached for a few rebalancing dates. When eventually the asset price reaches the trigger point, the order will be posted at the current market price, which results in a total order that can be much higher than if the rebalancing had been made all along the asset growth.

This shows that the use of minimum order amounts should be set with care: it has to be large enough to avoid losses due to transaction cost, but it should not be too high because it would amplify the "buy high, sell low" effect that is already present in CPPIs (see paragraph 2.5).

2.3 Risks associated with CPPI

To manage risks properly on CPPI products, one has to figure to which parameters this structure is exposed. We will focus here on the sell-side of the product, as I was doing my internship in an investment bank providing them. Independently of the underlying, a CPPI structure is exposed to two main market risks.

The first and most important of those risks is gap risk. We know by now that CPPI are products with guaranteed capital… in theory. Indeed with the CPPI formula, continuous rebalancing and perfectly liquid assets, we see that if the risky asset value decreases, the target allocation for the risky asset continuously goes to zero to preserve the guarantee.

In reality, rebalancing is done at discrete times. It is then possible that a brutal down move of the risky asset causes the buffer to fall below zero. In that case all the investment remaining in the risky asset has to be liquidated, and the total cash remaining is not sufficient to meet the theoretical guarantee (this can be worsened by liquidity issues, and those are unfortunately frequent when a fund experiences sudden large losses). Despite this, customers buying CPPI are looking for real portfolio insurance; nobody is interested in an insurance product that works only when the underlying behaves nicely. Therefore, CPPI contracts usually present a payoff off the type:

$$\text{payoff} = \max[\text{guarantee}, NAV_T]$$

This means that the customer will get at least the guarantee back, and that the losses in case of a large downside move will have to be borne by the issuer who is replicating the strategy. This is called gap risk. It has to be provisioned by the issuer, and thus to be incorporated in the price of the product. Consequently, the main challenge for the issuer of a CPPI product is to put a price on that risk. This is not an easy task, because it focuses exclusively on rare events, which are notoriously hard to handle. The next paragraph will define more precisely the gap risk encountered by the issuer, and the modeling issues will be exposed in part 3.

We have seen that the gap event can be caused by a sudden drop of price of the equity. This isn't however the only source of gap risk: since the buffer is calculated relatively to a credit instrument, a sudden move of interest rates can as well make the buffer fall under zero. So CPPI also carry an exposition to rate risk, but from the issuer's point of view, a rate-triggered gap is possible only when the NAV of the index is close to the present level of the guarantee. At this point the product is almost entirely made of cash (totally is there is a liquidation trigger), so it is easily hedgeable. That is why we will focus in the following on gap risk induced by a drop in equity.

Note that in a fund context, this gap event can be triggered or amplified by the specific funds risks we have overseen in part 1.

As an illustration of this phenomenon, the following table counts the number of gap events, with different liquidation trigger levels and an artificially increased volatility for the risky asset (the 20% volatility did not generate any gap event, see part 3).

Volatility	Liquidation trigger	Gap events	Trigger events
30%	0%	273	270
	5%	135	8786
	10%	91	9396
40%	0%	3757	3714
	5%	1108	9684
	10%	762	9807
50%	0%	8251	8188
	5%	2740	9928
	10%	2097	9946

The difference between the values for a 0% liquidation trigger are due to the gap events occurring during the last rebalancing period (since the position is liquidated at maturity anyway, liquidation trigger does not apply to it).

2.4 Adapting the elasticity parameter

In this part we will formalize gap events and see how it is related to the elasticity coefficient chosen for the CPPI structure.

2.4.1 Tolerable drop in equity

Let's first investigate what we call gap event here. Remember that for the issuer, a large down move is no problem as long as the buffer stays positive until the next rebalancing date: the allocation is revised accordingly, and the guarantee is not hurt. So what we are interested here is to see how large the move has to be to get over the guarantee cap.

Let's consider two successive rebalancing times, $t_1 < t_2$. Right after rebalancing at t_1, we have:

$$Buffer_{t_1} = NAV_{t_1} - FIC_{t_1} = RA_{t_1} + NRA_{t_1} - FIC_{t_1}$$
$$RA_{t_1} = \lambda . Buffer_{t_1}$$

Let's now place ourselves just before rebalancing at t_2. If we note J the excess return of the underlying over the period, so that $S_{t_2} = (1+J)S_{t_1}$, we have then:

$$NAV_{t_2} = RA_{t_2} + NRA_{t_2} = (1+J)RA_{t_1} + NRA_{t_2}$$

$$\begin{aligned} Buffer_{t_2} &= (1+J)RA_{t_1} + NRA_{t_2} - FIC_{t_2} \\ &= Buffer_{t_1} + J.RA_{t_1} + \left(NRA_{t_2} - NRA_{t_1}\right) - \left(FIC_{t_2} - FIC_{t_1}\right) \end{aligned}$$

If we take t_1 and t_2 close enough and assume no unusual behavior of interest rates over the period, we have the following approximations:

$$NRA_{t_2} \approx NRA_{t_1}$$
$$FIC_{t_2} \approx FIC_{t_1}$$

Consequently:

$$Buffer_{t_2} \approx Buffer_{t_1} + J.RA_{t_1}$$

Re-injecting the target allocation at time t_1:

$$Buffer_{t_2} \approx \left(\frac{1}{\lambda} + J\right) RA_{t_1}$$

So the return on equity that makes the buffer fall below zero is given by (note that RA_{t_1} is necessarily nonnegative):

$$Buffer_{t_2} \leq 0 \ \Rightarrow J \leq -\frac{1}{\lambda}$$

The maximum size of negative jump bearable for the investor is then approximately equal to $1/\lambda$; it is call the **tolerable drop in equity**. Past this limit, the issuer will have to put extra cash to pay back the full guarantee.

Remark: in the continuous rebalancing scheme where the underlying dynamics contain jumps, the maximum tolerable jump at any time is exactly $1/\lambda$.

2.4.2 Consequences on the choice of λ

We just saw that the tolerable drop in equity is directly related to the chosen elasticity coefficient. Therefore the choice of this parameter will be crucial for the stability of the CPPI structure. Since the tolerable drop in equity represents the largest equity jump for which the issuer doesn't suffer losses, it should be calibrated so that this kind of events almost never happens (according to the issuer's views).

This puts a bound on the elasticity used for the structure: in order not to be exposed to losses, the issuer first evaluates J_{max}, the largest possible jump size for the risky asset over a rebalancing period. This will set a range for acceptable elasticity values:

$$\lambda \in \left[0; {}^{1}\!/_{J_{max}}\right]$$

The final elasticity is then chosen within this range according to the customer's preferences. An investor looking to benefit highly from the equity's good returns will be looking for an elasticity coefficient around the limit, whereas risk-adverse investors will prefer it close to 1.

2.5 Remarks

This part was dedicated on the presentation of CPPI structures and of their classical features. In particular, we went over the profile of such products: it is path-dependant but is of optional kind, with high returns in the case of good underlying performance, due to the use of leverage.

Several points can however be raised about the pattern used by the strategy. CPPI structures are one form of "buy high/sell low" (or convex) strategies: more risky asset is bought when its value rises and vice versa.

- Clearly better returns will be obtained in times when the market is directional (constantly rising or dropping): since you buy more equity when its raises, you hope that it will keep that way. Therefore returns will be lower in volatile markets with numerous up-and-downs.
- Convex strategies can be dangerous when they are implemented largely enough to influence significantly market prices. Indeed if a purchase order is large enough to drive the equity market value up, you will have to buy more of it and so forth. On the way up, this phenomenon will lead to unrealistic high market values; on the way down it can trigger a dramatic drop of the equity price.
  This phenomenon has been identified as one of the causes of 1987's Black Monday (5). Needless to say that illiquidity in the market only increases the risk of this happening.

Let's also note a particular property of CPPI due to its rebalancing scheme. We have seen that in absence of leverage caps, the structure value will rise quickly with positive returns of the equity; but it takes only one large negative return to kick the strategy out of equity. After such an event, the contract value will have little or no chance to rise above the guarantee (depending on if all equity has to be sold or just a large part).

This illustrates the importance of a good calibration of the elasticity parameter in order not to be endangered by negative jumps; the use of leverage caps being an additional way to "secure" a good performance.

Part 3: Modeling issues

The previous part was all about formalizing CPPI structures and their features, as well as illustrating the payoff profile they provide. These points are useful from a portfolio construction perspective.

In this part we will focus on the risk management view of the problem. We have seen that the issuer of a CPPI will often be the one bearing the gap risk to offer a real guarantee to the client. This risk will be charged to the client (together with fees for implementing the strategy), therefore its evaluation is of crucial importance for a CPPI issuer: it has to be sufficiently taken into account to provision issuer's losses, but should not be overvalued (which would generate uncompetitive prices).

Consequently, models and evaluation methods for gap risk are being an active field of research in financial mathematics. In the following, we will build progressively from a basic model to a more adapted one, to stress what issues have to be dealt with for CPPI modeling.

3.1 Starting point: the Black-Scholes framework

Almost every issue in modeling equity derivatives is first handled with the Black-Scholes framework, before switching to a more appropriate model if necessary.

3.1.1 Theoretical CPPI with continuous rebalancing

In this section we will consider the simplest case for a CPPI: no special feature and hypothetical continuous rebalancing. It is shown in (6) that in those conditions, there is a nice closed form for the dynamics of the CPPI value at maturity.

Let's use the usual notations for modeling:

- ✓ $(S_t)_{t\geq 0}$ represents the dynamics of the underlying asset price,
- ✓ $(B_t)_{t\geq 0}$ is the dynamic of a zero-coupon bond paying the guarantee K at maturity. It is also taken as a reference for the non-risky asset
- ✓ V_t is the value at of the strategy at time t

✓ C_t is the buffer size at time t

In the Black-Scholes framework, we have the following dynamics:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$
$$dB_t = rB_t dt$$

At any moment the portfolio is constituted of the risky and the non-risky asset, therefore its dynamics are of the form:

$$dV_t = n_S dS_t + n_B dB_t$$

Where $n_S = \lambda C_t / S_t$ because of the continuous rebalancing scheme, and by definition we have $n_B = (V_t - \lambda C_t)/B_t$. Since the buffer is defined as $C_t = V_t - B_t$,

$$dV_t = \lambda C_t \frac{dS_t}{S_t} + \left(\frac{C_t - \lambda C_t}{B_t} + 1\right) dB_t$$

Therefore:

$$dC_t = dV_t - dB_t = \lambda C_t \frac{dS_t}{S_t} + (1-\lambda) C_t \frac{dB_t}{B_t}$$
$$= (\lambda\mu + (1-\lambda)r) C_t\, dt + \lambda\sigma C_t dW_t$$

We recognize the SDE for the exponential geometric Brownian motion, which solves:

$$C_T = C_0 \exp\left(\left(\lambda\mu + (1-\lambda)r - \frac{\lambda^2\sigma^2}{2}\right)T + \lambda\sigma W_t\right)$$

So the final value of the CPPI writes:

$$V_t = K + (V_0 - Ke^{-rT}) \exp\left(\left(\lambda\mu + (1-\lambda)r - \frac{\lambda^2\sigma^2}{2}\right)T + \lambda\sigma W_t\right)$$

We can see that, as implied by the theoretical rebalancing scheme, the final value of the CPPI will always be more than K, the guarantee.

The formula above is very nice and could be used to price derivatives easily, but it proves terrible in practice. Indeed it doesn't allow considering the gap risk induced by discrete rebalancing and discontinuous price processes.

3.2.2 Introduction of discrete rebalancing

In a first attempt of improving the previous model, let's put away the hypothesis of continuous rebalancing and introduce discrete rebalancing dates. This will make gap risk possible, if the risky asset drops enough between two rebalancing dates.

We have seen in the previous section that if $s < t$ are two rebalancing dates, a gap event will occur in the interval $[s, t]$ if:

$$\frac{S_t}{S_s} \leq 1 - \frac{1}{\lambda}$$

Given the price dynamics between two rebalancing dates, we can compute the gap event probability:

$$S_t = S_s \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)(t-s) + \sigma\sqrt{t-s}.Z\right), \qquad Z \sim N(0,1)$$

Therefore:

$$\begin{aligned} P\left[\frac{S_t}{S_s} \leq 1 - \frac{1}{\lambda}\right] &= P\left[\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)(t-s) + \sigma\sqrt{t-s}.Z\right) \leq 1 - \frac{1}{\lambda}\right] \\ &= P\left[Z \leq \frac{\ln\left(1-\frac{1}{\lambda}\right) - \left(\mu - \frac{\sigma^2}{2}\right)(t-s)}{\sigma\sqrt{t-s}}\right] \\ &= \Phi\left[\frac{\ln\left(1-\frac{1}{\lambda}\right) - \left(\mu - \frac{\sigma^2}{2}\right)(t-s)}{\sigma\sqrt{t-s}}\right] \end{aligned}$$

Therefore, if there is N rebalancing dates, the probability of a gap event occurring over the CPPI's lifetime is $1-(1-p)^N$, where p is the probability of a gap event over one rebalancing period. So the gap event probability is:

$$P_{gap\ event} = 1 - \left[1 - \Phi\left[\frac{\ln\left(1-\frac{1}{\lambda}\right) - \left(\mu - \frac{\sigma^2}{2}\right)(t-s)}{\sigma\sqrt{t-s}}\right]\right]^N$$

We can now compute the theoretical values for the gap probability with this model, and compare it to the results obtained in section 2.3:

Volatility	Theoretical gap probability	Number of gaps	Estimated probability
20%	$1{,}5.10^{-5}$	0	0
30%	$2{,}6.10^{-2}$	273	$2{,}7.10^{-2}$
40%	0,33	3757	0,38
50%	0,79	8251	0,83

Table 1: Theoretical and estimated gap probabilities

We can see that the previous results are consistent with the theory. However, we observe that the theoretical gap probability is very low for "normal" volatilities: with this model, it is almost non-existent for 20% volatility (where 20% is a common volatility for the equity asset class, and the parameters used are quite high: usually elasticity is around 2-3 instead of 4). This type of event is in reality more frequent, so this model is still not suitable for gap risk.

This should not be a surprise: the Black-Scholes model is notoriously inappropriate to model extreme events, due to its thin tail density.

3.2 Improvements and other models

To handle CPPI and gap risk correctly, one has to use a model that is efficient for describing extreme probabilities. Indeed those are the ones important for CPPI pricing: the lower tail provides crucial information about gap risk, while the upper tail provides the highest returns on the equity. Knowing that the CPPI profile is made of many outcomes close to the guarantee and a few very outstanding outcomes with influence on the contract value, high returns have to be correctly assessed in order to generate correct product price.

3.2.1 Adding a jump component

One way of making our model more realistic is to add a jump component to the previous diffusion model. If calibrated correctly, the continuous process will represent the "normal" behavior of the asset, and the jump process will handle exceptional variations. This will allow getting back to a continuous framework with a control over the modeling of extreme events.

Structure

Let's first define the jump diffusion structure. The following is proposed in (7):

- The underlying asset is still driven by a Black-Scholes-like process, with an adjusted trend:
$$dS_t = (r - A)S_t dt + \sigma S_t dW_t$$
where A is a constant,
- Jump events can occur, the elapsed time between two jump events has an exponential distribution with parameter a.
- At every jump date t_i, we have the following jump in the process:
$$S_{t_i^+} = (1 + J_i)S_{t_i^-}$$
- The $(J_i)_{i\geq 1}$ are iid, and are independent of the random occurrence times and of the driving Wiener process; the occurrence times and the Wiener process are independent as well.

Under those assumptions, the number of gap events over a given period follows a Poisson process of intensity a. This is a common distribution to model the number of occurrences of an event; it is widely used in credit risk for instance. We do not specify a probability distribution for the jumps yet, this will be discussed later.

Since the jump part is independent of the continuous process, we have an explicit form for the asset price process:

$$S_t = S_0 \exp\left(\left(r - A - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)\prod_{i=1}^{N_t}(1 + J_i)$$

So the expected value of the process is

$$E[S_t] = S_0 \exp\left(\left(r - A - \frac{\sigma^2}{2}\right)t\right).E[\exp(\sigma W_t)].E\left[\prod_{i=1}^{N_t}(1 + J_i)\right]$$

With $E[\exp(\sigma W_t)] = \exp\left({}^{\sigma^2 t}/_2\right)$ and

$$\begin{aligned}E\left[\prod_{i=1}^{N_t}(1 + J_i)\right] &= \sum_{n=0}^{\infty} E\left[\prod_{i=1}^{N_t}(1 + J_i)\middle| N_t = n\right] P(N_t = n)\\
&= \sum_{n=0}^{\infty} E\left[\prod_{i=1}^{n}(1 + J_i)\right] P(N_t = n)\\
&= \sum_{n=0}^{\infty}\left[e^{-at}\frac{(at)^n}{n!}\prod_{\mathrm{i}=1}^{\mathrm{n}}(1 + \mathrm{E}\,[\mathrm{J_i}])\right] = \sum_{n=0}^{\infty}\left[e^{-at}\frac{(at)^n}{n!}(1 + \mathrm{E}[\mathrm{J_1}])^{\mathrm{n}}\right]\\
&= e^{-at}\exp(at(1 + E[J_1])\,)\end{aligned}$$

So all put together:

$$E[S_t] = S_0 \exp\big((r - A - aE[J_1])t\big)$$

So, in order to be consistent with the risk-free valuation we will set $A = aE[J_1]$.

The nice thing about this model is that we are free to pick any iid random variables for the jump part as long as we adjust the drift with the product of the intensity and the expected value of jumps.

Jump distribution

So we are left with a certain freedom concerning the choice of the distribution for the asset jumps; this can be used at our advantage to reflect efficiently the extreme behavior of the asset. Several points are to be considered here:

- **What data do we use**: there are two approaches. If we already have a predetermined intensity a for the Poisson process, we can focus on the aN highest returns in absolute value over the N data points (this is the expected number of jumps occurring in this time interval according to the model). We can also choose on which part of the data we want to work (e.g. absolute returns higher than 5%, or more than 2 standard deviations with the dataset…) and set the intensity parameter accordingly. Note that in any case, we have to pick relevant data: for our purpose there is no point in selecting a too large intensity or too low returns to calibrate the distribution.
- According to the observed data, we can use different models or parameters for **positive and negative jumps** (or choose to model only negative ones if relevant). It is often observed for instance that jumps in equity value happen mostly in sharp market drops, whereas growth periods tend to be steadier. Instead of finding a corresponding all-in-one distribution, one can then imagine to separate those two types of jumps and measure the probability of one or the other occurring.

Next comes the question of the model to be chosen. Since we introduced the jump part to balance the low participation of the lognormal model in extreme events, this should be a model able to thicken the tails of the global distribution.

When it comes to tail modeling, many studies reveal a fractal behavior of extreme returns and favor a power law model (since Mandelbrot's study of cotton prices, (8)).

Therefore we can use the density of the power law distribution found in (9) for the negative jump returns (similar techniques can be used for positive returns, but in the context of gap risk we will focus on negative ones here):

$$\phi_J(x) = \frac{C}{x^{\alpha+1}} 1_{x_{min} \leq x \leq 1}$$

The decay parameter α can be obtained from data by computing the empirical estimate of mean excess beyond a threshold u (with $u > x_{min}$), $E[J|J > u]$. Indeed this is theoretically equal to:

$$E[J|J > u] = \frac{E[J.1_{J>u}]}{E[1_{J>u}]} = \frac{\int_u^1 x\phi_J(x)dx}{\int_u^1 \phi_J(x)dx}$$
$$= \frac{\int_u^1 \frac{C}{x^\alpha}dx}{\int_u^1 \frac{C}{x^{\alpha+1}}dx} = \frac{\frac{C}{\alpha-1}\left(\frac{1}{u^{\alpha-1}} - 1\right)}{\frac{C}{\alpha}\left(\frac{1}{u^\alpha} - 1\right)}$$
$$= \frac{u\alpha(1 - u^{\alpha-1})}{(\alpha-1)(1-u^\alpha)}$$

So the empirical estimation of $E[J|J > u]$ gives an estimate of α. The two other parameters C and x_{min} are then given by the condition $\int \phi(x)dx = 1$ and by the mean value given to the jump distribution. This mean can thus be given any desired value: it can be estimated empirically, or can be related to a stress test of different scenarios for the equity for instance.

The latter approach can be particularly useful if we want indeed to include a stress test value into the model to reflect the investor's views, or to introduce conservative parameters.

However, if our goal is to match some data, this approach can seem approximate: the threshold is typically chosen as the mean of negative returns, and the parameters are also chosen according to a desired mean. In order to have a more accurate fitting of the tail part (which is the real issue here), one could instead opt for the Peaks Over Threshold methodology, as described in the Risk Management module (10). This is an iterative method consisting in:

- Selecting a threshold for the tail distribution (corresponding here to the threshold over which we consider that a jump has occurred in the data)
- Fitting a generalized Pareto distribution to the excess distribution: the parameters are obtained by using a least squares estimation on the quantile values or by using a log-likelihood function.

These steps are repeated with different thresholds until obtaining a satisfying fit. Note that the distribution one would obtain would of course still be easily integrated in the jump-diffusion structure described in this section.

Whether one should use one approach or the other is not an easy question. The latter provides a nice fit on extreme values, while the former can still be used when there is few data to work on, and allows a certain liberty in the choice of the jump intensity. My opinion is that if the data allows it, the two methods should be carried out and the results should be compared: too important differences could be a warning sign that further investigation is needed.

3.2.2 Other models

Evaluating gap risk is a difficult issue, and has been the subject of many recent researches. The goal of this part in not to enter into details about those models (some require mathematical tools that are beyond the scope of this study); I will only give a few examples.

We have already seen in the previous sections the modeling of CPPI in the Black-Scholes framework, as well as the clear limitations of this modeling. Then we have added a jump component to take into account the extreme values taken by the common underlyings. This approach has the advantage to be easily understandable, and does not present major difficulty in calibration and simulation. In some simple cases, closed-form formulas can even be exhibited.

Other techniques have been developed to estimate gap risk. In the early 00s, extreme value theory was used in (11). A generalization of jump-diffusion models with Levy processes and stochastic volatility can be found in (12). The use of Levy processes is increasingly popular in mathematical finance for their properties; they also have fat tails suited for our purpose.

Nevertheless these models present some drawbacks: they usually fit simple CPPI structures but do not include several frequent features of such contracts. Furthermore they usually require the use of Monte-Carlo methods to produce numerical results. This can be problematic: indeed the particular structure of CPPI (path-dependency, sensitivity to extreme events) makes the convergence of Monte-Carlo simulations very slow. It can take hours of computations on a standard computer to come up with an accurate value for a CPPI or an option on CPPI.

To tackle the computational issue, an approach using Markov process has been developed in (13) under the assumption of independent price increments for the risky asset. This allows fast computations, even in presence of special features.

3.3 Underlying-related issues

To calibrate a model, one usually uses historical data of the underlying asset(s) for the derivative. But in the funds derivatives business, it is not uncommon for a contract to be written on a **new fund** for which there is no previous data to exploit. In this case we need to use a proxy with supposed similar behavior instead.

We have seen that a typology is performed to classify the funds according to their strategy. For this purpose this is very useful: to build a model on a newly created fund, one will look at historical returns for funds with similar strategy, liquidity features, etc. Of course, this has to be done with care: the information collected in this way is obviously less relevant than real returns for the new fund. Some provisions have to be made to account for the increase in model risk.

Another underlying issue is when writing a contract on a **fund of funds**. Indeed as mentioned in the first part, one may know punctually the composition of the FoF, but this is subject to changes since the FoF is regularly rebalancing its positions. Consequently historical returns of the FoF may as well not be very relevant. To calibrate a model, one will then rather use a mix of actual returns and an aggregation of returns from the current composition to be more accurate. As for new funds, extra attention will be paid due to accrued model risk.

This also influences stress testing: instead of just testing different scenarios to come up with worst-case results, one will have to combine those with potential evolutions of the composition of the FoF. The scenarios will therefore be tested on the different allocations that are allowed by the investment guidelines of the FoF.

Conclusion

Throughout this thesis, I have presented the different issues met when dealing with CPPI on Funds Derivatives. Part 1 was dedicated to the description of the different types of funds: we have seen that they come in great variety, and that their particularities can make them much trickier to handle than standard equity assets. In particular they usually have reduced accessibility and liquidity, there can be uncertainty on published NAVs, and they are subject to additional risks. Those points by themselves make the management and modeling of fund derivatives more difficult.

Complexity is further added when those products are included in structured products. I have chosen to study CPPI because this strategy raises an additional problem: it relies on a great extent on one's ability to estimate extreme values distributions, which is always quite complicated. Therefore combining a CPPI structure with specific products like fund shares generates overall quite a complex product, especially when contractual features are added to the structure (this is not uncommon). I tried to give starting points here for the management of such products and to put a stress on how precautious one should be at each step of the process. Finally additional issues can appear when dealing with newly created funds and FoFs, which again is rather frequent.

Coming up with a complete and effective methodology for the management of such products would be a long and tricky piece of work. For what I have seen, the methodology used by CA CIB is quite massive, uses many mathematical and statistical techniques together with a lot of data.

And this does not end at the issue of the product: at CA CIB those are carefully monitored through their entire validity period, using sensibilities to market parameters, risk indicators (various VaR methodologies), stress tests…

I have been truly surprised by this gap between the usual hypothesis and assumptions made when building a theoretical model, and the particularities and actual market conditions in which some products are traded. If standard models can be efficient to describe some simple products, for others they are clearly insufficient (as we have seen for funds CPPI). For this last category, the math is not enough, and has to be combined with accurate knowledge on the products: it is indeed impossible to build a correct model without a deep understanding of the workings and issues related to the corresponding product. This, of course, includes the eventual adjustments of the theoretical products: for instance many models were designed for CPPIs that does not support the rather usual contractual details presented in part 2.

In the same way, one cannot manage properly this kind of products without understanding the models used and their limits. This is especially the case when there is some uncertainty concerning the appropriateness of the chosen modeling techniques or the data used to do the calibration. This calls for a critical view on the risk exposure to the corresponding products when risk is quantified in model-dependant terms such as sensitivities to a parameter or value-at-risk; some more importance can be accorded to measures which are more robust (but often more subjective) like stress tests and worst-case scenarios.

To conclude this thesis, I would like to say how rewarding I think this internship has been. I learned a lot on the practical side of finance: functioning of the market and its agents, the products, the strategies… I reckon this is a valuable experience to my studies in financial mathematics, and it confirmed my interest in risk modeling of complex financial products.

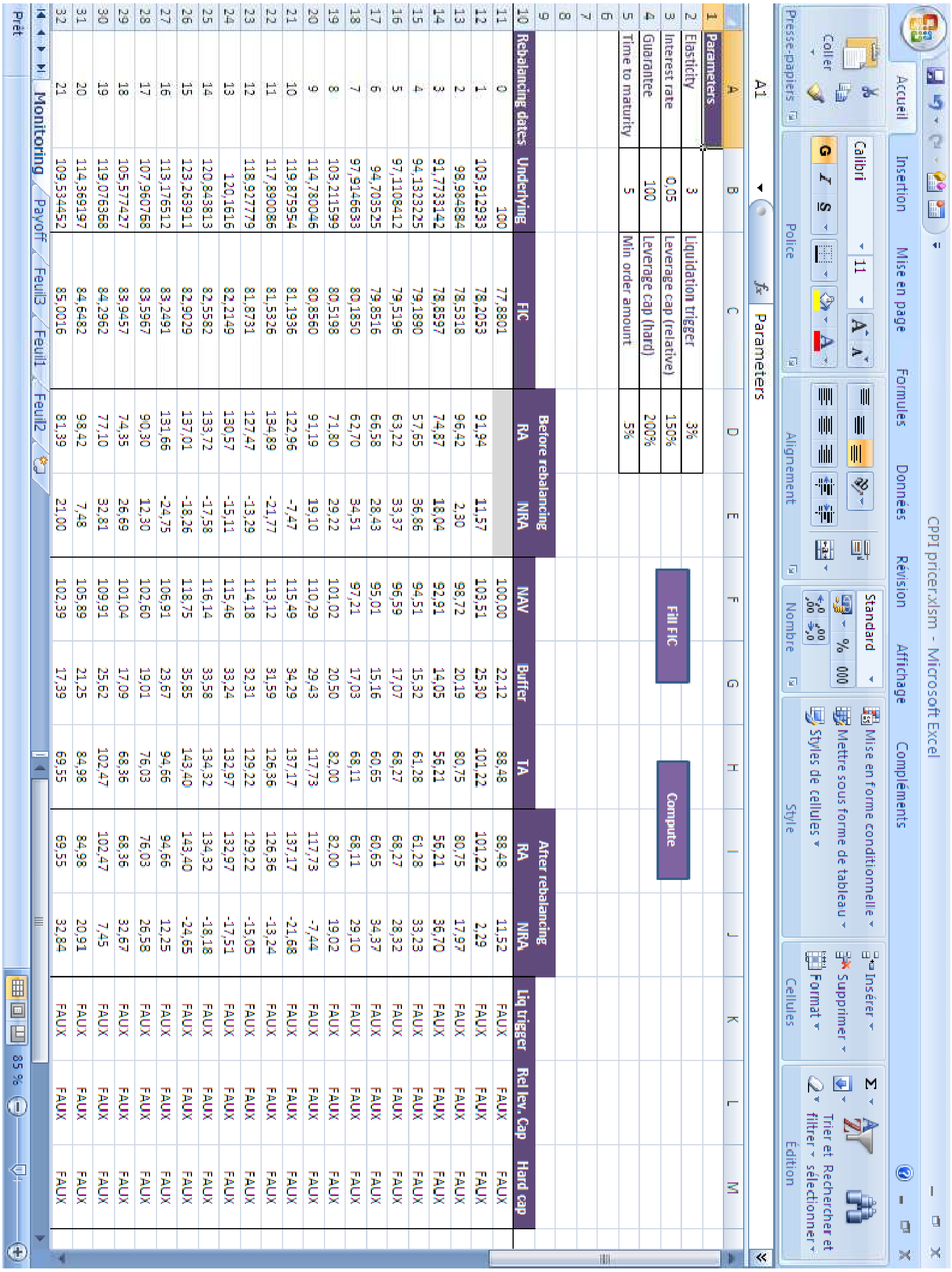
Annex: computational tools

The various computations performed in this paper needed to implement CPPI strategies. To that end, I have used two (similar) tools:

- One "monitoring" tool that details the evolution of the CPPI strategy. It takes as inputs the simulated (or observed) paths for the risky and the non-risky asset, the parameters of the CPPI structure (time to maturity, elasticity, guarantee level), and allows all the features described in part 2: liquidation trigger, hard and relative leverage caps, minimum order amounts.
  The full strategy is then calculated in Visual Basic, and the details of the allocation are displayed in an Excel worksheet (see figure 5 thereafter). This is particularly useful to observe the effects of the different parameters on a single path for the underlyings.
  This tool has been created for the purpose of this thesis, but is largely inspired of the monitoring spreadsheets for vol-capped funds I've had to develop during my internship.

- The full process of simulating inputs for the monitoring tool, computing the strategy and displaying it is relatively time-consuming for a standard computer. For more intensive computation (like estimating the density of CPPI distributions), the computations were made directly with VBA code, including only the necessary features for the current computation for computational gain. The results were then tapped in R to generate the graphical outputs.

The second approach generated results much faster, but even then getting accurate values was still very time-consuming on a standard computer. This is one point that comforted me in focusing on issues that didn't need intensive computations.



Parameters			
Elasticity	3	Liquidation trigger	3%
Interest rate	0,05	Leverage cap (relative)	150%
Guarantee	100	Leverage cap (hard)	200%
Time to maturity	5	Min order amount	5%

			Before rebalancing					After rebalancing				
Rebalancing dates	Underlying	FIC	RA	NRA	NAV	Buffer	TA	RA	NRA	Liq trigger	Rel lev. Cap	Hard cap
0	100	77,8801			100,00	22,12	88,48	88,48	11,52	FAUX	FAUX	FAUX
1	103,912933	78,2053	91,94	11,57	103,51	25,30	101,22	101,22	2,29	FAUX	FAUX	FAUX
2	98,984884	78,5318	96,42	2,30	98,72	20,19	80,75	80,75	17,97	FAUX	FAUX	FAUX
3	91,7733142	78,8597	74,87	18,04	92,91	14,05	56,21	56,21	36,70	FAUX	FAUX	FAUX
4	94,1323225	79,1890	57,65	36,86	94,51	15,32	61,28	61,28	33,23	FAUX	FAUX	FAUX
5	97,1108412	79,5196	63,22	33,37	96,59	17,07	68,27	68,27	28,32	FAUX	FAUX	FAUX
6	94,703525	79,8516	66,58	28,43	95,01	15,16	60,65	60,65	34,37	FAUX	FAUX	FAUX
7	97,9146633	80,1850	62,70	34,51	97,21	17,03	68,11	68,11	29,10	FAUX	FAUX	FAUX
8	103,211599	80,5198	71,80	29,22	101,02	20,50	82,00	82,00	19,02	FAUX	FAUX	FAUX
9	114,780046	80,8560	91,19	19,10	110,29	29,43	117,73	117,73	-7,44	FAUX	FAUX	FAUX
10	119,875954	81,1936	122,96	-7,47	115,49	34,29	137,17	137,17	-21,68	FAUX	FAUX	FAUX
11	117,890086	81,5326	134,89	-21,77	113,12	31,59	126,36	126,36	-13,24	FAUX	FAUX	FAUX
12	118,927779	81,8731	127,47	-13,29	114,18	32,31	129,22	129,22	-15,05	FAUX	FAUX	FAUX
13	120,1616	82,2149	130,57	-15,11	115,46	33,24	132,97	132,97	-17,51	FAUX	FAUX	FAUX
14	120,843813	82,5582	133,72	-17,58	116,14	33,58	134,32	134,32	-18,18	FAUX	FAUX	FAUX
15	123,263911	82,9029	137,01	-18,26	118,75	35,85	143,40	143,40	-24,65	FAUX	FAUX	FAUX
16	113,176512	83,2491	131,66	-24,75	106,91	23,67	94,66	94,66	12,25	FAUX	FAUX	FAUX
17	107,960768	83,5967	90,30	12,30	102,60	19,01	76,03	76,03	26,58	FAUX	FAUX	FAUX
18	105,577427	83,9457	74,35	26,69	101,04	17,09	68,36	68,36	32,67	FAUX	FAUX	FAUX
19	119,076368	84,2962	77,10	32,81	109,91	25,62	102,47	102,47	7,45	FAUX	FAUX	FAUX
20	114,369197	84,6482	98,42	7,48	105,89	21,25	84,98	84,98	20,91	FAUX	FAUX	FAUX
21	109,534452	85,0016	81,39	21,00	102,39	17,39	69,55	69,55	32,84	FAUX	FAUX	FAUX

Figure 5: CPPI monitoring spreadsheet

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