Asian Option Pricing and Volatility

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Abstract

An Asian option is a path-depending exotic option, which means that either the settlement price or the strike of the option is formed by some aggregation of underlying asset prices during the option lifetime. This thesis will focus on European style Arithmetic Asian options where the settlement price at maturity is formed by the arithmetic average price of the last seven days of the underlying asset.

For this type of option it does not exist any closed form analytical formula for calculating the theoretical option value. There exist closed form approximation formulas for valuing this kind of option. One such, used in this thesis, approximate the value of an Arithmetic Asian option by conditioning the valuation on the geometric mean price. To evaluate the accuracy in this approximation and to see if it is possible to use the well known Black-Scholes formula for valuing Asian options, this thesis examines the bias between Monte-Carlo simulation pricing and these closed form approximate pricings. The bias examination is done for several different volatility schemes.

In general the Asian approximation formula works very well for valuing Asian options. For volatility scenarios where there is a drastic volatility shift and the period with higher volatility is before the average period of the option, the Asian approximation formula will underestimate the option value. These underestimates are very significant for OTM options, decreases for ATM options and are small, although significant, for ITM options.

The Black-Scholes formula will in general overestimate the Asian option value. This is expected since the Black-Scholes formula applies to standard European options which only, implicitly, considers the underlying asset price at maturity of the option as settlement price. This price is in average higher than the Asian option settlement price when the underlying asset price has a positive drift. However, for some volatility scenarios where there is a drastic volatility shift and the period with higher volatility is before the average period of the option, even the Black-Scholes formula will underestimate the option value. As for the Asian approximation formula, these over-and underestimates are very large for OTM options and decreases for ATM and ITM options.

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Sammanfattning

En Asiatisk option är en vägberoende exotisk option, vilket betyder att antingen settlement-priset eller strike-priset beräknas utifrån någon form av aggregering av underliggande tillgångens priser under optionens livstid. Denna uppsats fokuserar på Aritmetiska Asiatiska optioner av Europeisk karaktär där settlement-priset vid lösen bestäms av det aritmetiska medelvärdet av underliggande tillgångens priser de sista sju dagarna.

För denna typ av option finns det inga slutna analytiska formler för att beräkna optionens teoretiska värde. Det finns dock slutna approximativa formler för värdering av denna typ av optioner. En sådan, som används i denna uppsats, approximerar värdet av en Aritmetisk Asiatisk option genom att betinga värderingen på det geometriska medelpriset. För att utvärdera noggrannheten i denna approximation och för att se om det är möjligt att använda den väl kända Black-Scholes-formeln för att värdera Asiatiska optioner, så analyseras differenserna mellan Monte-Carlo-simulering och dessa slutna formlers värderingar i denna uppsats. Differenserna analyseras utifrån ett flertal olika scenarion för volatiliteten.

I allmänhet så fungerar Asiatapproximationsformeln bra för värdering av Asiatiska optioner. För volatilitetsscenarion som innebär en drastisk volatilitetsförändring och där den perioden med högre volatilitet ligger innan optionens medelvärdesperiod, så undervärderar Asiatapproximationen optionens värde. Dessa undervärderingar är mycket påtagliga för OTM-optioner, avtar för ATM-optioner och är små, om än signifikanta, för ITM-optioner.

Black-Scholes formel övervärderar i allmänhet Asiatiska optioners värde. Detta är väntat då Black-Scholes formel är ämnad för standard Europeiska optioner, vilka endast beaktar underliggande tillgångens pris vid optionens slutdatum som settlement-pris. Detta pris är i snitt högre än Asiatisk optioners settlement-pris när underliggande tillgångens pris har en positiv drift. Men, för vissa volatilitetsscenarion som innebär en drastisk volatilitetsförändring och där den perioden med högra volatilitet ligger innan optionens medelvärdesperiod, så undervärderar även Black-Scholes formel optionens värde. Som för Asiatapproximationen så är dessa över- och undervärderingar mycket påtagliga för OTM-optioner och avtar för ATM och ITM-optioner.

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Options

Standard options

A standard option (also called Plain vanilla) is a financial contract which gives the owner of the contract the right, but not the obligation, to buy or sell a specified asset to a prespecified price (strike price) at a prespecified time (maturity). The specified asset (underlying asset) can be for example stocks, indexes, currencies, bonds or commodities. The option can be either a call option, which gives the owner the right to buy the underlying asset, or it can be a put option, which gives the owner the right to sell the underlying asset. Moreover the option can either only be exercised at maturity, i.e. European option, or it can be exercised at any time before maturity, i.e. American option.

To buy an option the buyer must pay an option premium to the one who writes (sell) the option. The writer of the option is thus obligated to sell or buy the underlying asset to the prespecified price if the owner (buyer) of the option decides to exercise. This premium is the option price.

Standard options all share the characteristics: one underlying asset, the effective starting time is present, only the price of the underlying asset at the option´s maturity affects the payoff of the option, whether an option is a call or a put is known when sold, the payoff is always the difference between the underlying asset price and the strike price.

Exotic options and the Asian option

Exotic options are options that do not share one or more of the characteristics of the plain vanilla options. There are two main types of exotic options, Correlation options and Path dependent options. Correlation options are options whose payoffs are affected by more than one underlying asset. Path dependent options are options whose payoffs are affected by how the price of the underlying asset at maturity was reached, the price path of the underlying asset. One particular path dependent option, called Asian option, will be of main focus throughout this thesis.

Asian options

Asian options are one of the most popular path dependent options and are also called average-price options. The characteristic of an Asian option is that the payoff is dependent of the average price of the underlying asset, over some prespecified period and prespecified frequency, during the lifetime of the option.

The average price of the underlying asset can either determine the underlying settlement price (average-price Asian options) or the option strike price (average-strike Asian options). Furthermore the average prices can be calculated using either the arithmetic mean or the geometric mean. The type of Asian options that will be examined throughout this thesis is *arithmetic-price Asian options*. As will be explained later, the difference between arithmetic-and geometric-price Asian options is very important when it comes to pricing the option, [2]

Volatility

The price (or premium) of Plain vanilla options is determined by five components: price of the underlying asset, Strike price, lifetime of the option, risk-free interest rate and volatility of the underlying asset price. If the underlying is a stock, expected dividends during the life of the option is also a component of pricing the option.

Usually, volatility is the most interesting parameter in option pricing due to its impact on the option price combined with the great difficulty in estimating it. Volatility can be described as the speed and magnitude of the price movement of the underlying asset. In the case of option pricing, it can also be described as a measure of the uncertainty about the future price movements of the underlying asset.

The volatility of a stock price can be defined as the standard deviation of the return provided by the stock in one year when the return is expressed using continuous compounding. To illustrate, an example of how to estimate the volatility from historical data is

Define:

$n+1$: Number of observations

S_i: Stock price at end of interval i, with $i = 0, 1, \dots, n$

τ: Length of time interval in years

and let

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$$

for $i = 1, 2, \dots, n$.

An estimate of the standard deviation of u_i is given by

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_i - \bar{u})^2}$$

where $\bar{u}$ is the arithmetic mean of the u_i. Since this exampled calculated the standard deviation of the stock return over intervals of τ years, the volatility, σ, therefore is,

$$\sigma = \frac{s}{\sqrt{\tau}}$$

[1]. Estimating volatility is a wide and well written subject and will not be discussed further in this thesis. In option pricing volatility is generally expressed in percentage of underlying asset price, and for stocks it is typically between 15% and 60%.

Pricing Arithmetic Asian option

Properties of the probability distribution of the stock price stochastic process make it possible to obtain an analytical closed form formula to the price of a standard European option, the Black-Scholes formula. These properties also make it possible to obtain an analytical formula for the price of a Geometric Asian option due to the properties of a geometric mean, [2]. Arithmetic means does not share these vital properties with geometric means and Arithmetic Asian option prices are thus, plausibly, impossible to express in a closed form formula. As will be described, it is possible to approximate Arithmetic Asian option prices using the geometric mean prices, [4]. In order to price Arithmetic Asian option accurately numerical methods has to be used, and one such is Monte Carlo Simulation.

Monte Carlo simulation

Using Monte Carlo simulation to calculate the price of an option is a useful technique when the option price is dependent of the path of the underlying asset price. The simulation is carried out by simulating a large number of samples of the underlying asset price path, between some starting time and the maturity of the option. Then these samples are used to calculate the statistics of the option price. Since each sample includes all prices of the underlying asset, with some updating frequency, it is easy to calculate the arithmetic mean over any averaging period.

The concern when using Monte Carlo simulation to price option is that accurate estimates are very time consuming to obtain. As will be described, since the accuracy of the estimates is proportional to the number of simulations, there are variance reduction techniques to improve the efficiency of Monte Carlo simulation.

Derivation of the path constructing formula – Stocks with constant volatility

The stock price in a risk neutral world, [1], is assumed to follow the stochastic process of a Geometric Brownian motion,

$$dS = rSdt + \sigma Sdz$$

where S is the stock price, r is the risk-free interest rate, σ is the volatility and dz is a Wiener process. A Wiener process has the following properties:

1) *The change Δz in a short period of time Δt is*

$$\Delta z = \varepsilon\sqrt{\Delta t}$$

where ε has a standardized normal distribution $N(0,1)$.

2) *The values of Δz for any two different short intervals of time, Δt, are independent*

$N(x, y)$ denotes a normal distribution with mean x and variance y.

In discrete time notation the process of the stock price becomes,

$$\Delta S = rS\Delta t + \sigma S\Delta z = rS\Delta t + \sigma S\varepsilon\sqrt{\Delta t}$$

$$\Rightarrow \quad \frac{\Delta S}{S} = r\Delta t + \sigma\varepsilon\sqrt{\Delta t}$$

$$\Rightarrow \quad \frac{\Delta S}{S} \sim N(r\Delta t, \sigma^2\Delta t)$$

which means the percentage rate of return for a stock over a short period of time, Δt, has a normal distribution with mean $r\Delta t$ and standard deviation $\sigma\sqrt{\Delta t}$.

In order to find an explicit formula for the stock price at some future time, S_t, it is necessary to use the Itô Lemma:

If a stochastic variable X follows the Itô-process

$$dX = a(X,t)dt + b(X,t)dz$$

a function of X and t, $f(X,t)$, follows the process

$$df = \left(\frac{\partial f}{\partial X}a + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}b^2\right)dt + \frac{\partial f}{\partial X}bdz$$

[3]. Applying this to the stock price process, with $f = \ln(S)$, gives

$$dln(S) = \left(\frac{1}{S}rS + 0 + \frac{1}{2}\left(-\frac{1}{S^2}\right)\sigma^2 S^2\right)dt + \frac{1}{S}\sigma S dz = \left(r - \frac{1}{2}\sigma^2\right)dt + \sigma dz$$

In discrete time notation this becomes

$$\Delta \ln(S) = \left(r - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\varepsilon\sqrt{\Delta t}$$

This gives

$$\ln(S_{t+\Delta t}) - \ln(S_t) = \left(r - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\varepsilon\sqrt{\Delta t}$$

$$\Rightarrow \quad S_{t+\Delta t} = S_t e^{\left(r-\frac{1}{2}\sigma^2\right)\Delta t + \sigma\varepsilon\sqrt{\Delta t}} \qquad \text{(Eq. 1)}$$

Eq. 1 is the path constructing formula for the Monte Carlo simulation of the stock price.

Derivation of the Black-Scholes-Merton differential equation

As mentioned earlier the stock price process shares some important properties with the geometric mean. This property is the lognormal distribution. In order to understand this property and the Black-Scholes formula for European options, it is important to understand the derivation of the Black-Scholes-Merton differential equation.

A stochastic variable follow a lognormal distribution if the logarithm of the variable follow a normal distribution.

If $\ln(X) \sim N(a,b)$*, then*

$$X \sim lognormal(a,b)$$

$$E[X] = e^{(a+\frac{1}{2}b^2)}$$

$$Var(X) = e^{(2(a+b^2))} - e^{(2a+b^2)}$$

[6]. Taking the logarithm of Eq. 1 gives,

$$\ln(S_{t+\Delta t}) = \ln(S_t) + \left(r - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\varepsilon\sqrt{\Delta t} \sim N(\ln(S_t) + \left(r - \frac{1}{2}\sigma^2\right)\Delta t, \sigma^2\Delta t)$$

which means $S_{t+\Delta t}$ is lognormally distributed with

$$E[S_{t+\Delta t}] = S_t e^{r(t+\Delta t)}$$

$$Var(S_{t+\Delta t}) = S_t^2 e^{2r(t+\Delta t)}(e^{\sigma^2(t+\Delta t)} - 1)$$

The trick to derive the Black-Scholes-Merton differential equation is to form a risk-free portfolio consisting of the derivative, $f(S,t)$, and the underlying asset, S. If the portfolio is set up as,

$$\Pi \begin{cases} -1: f \\ \dfrac{\partial f}{\partial S}: S \end{cases}$$

this gives

$$\Pi = -f + \frac{\partial f}{\partial S}S \;\Rightarrow \Delta\Pi = -\Delta f + \frac{\partial f}{\partial S}\Delta S$$

and with

$$\Delta S = rS\Delta t + \sigma S\Delta z$$

$$\Delta f = \left(\frac{\partial f}{\partial S}rS + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t + \frac{\partial f}{\partial S}\sigma S\Delta z$$

this gives

$$\Delta\Pi = (-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2)\Delta t$$

Another necessary condition is the assumption of an arbitrage free market. This means the change of the portfolio value cannot differ from the return of an investment of an equal amount in a risk-free asset,

$$\varDelta\Pi = r\Pi\varDelta t$$

$$\Rightarrow \quad -\left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\varDelta t = -r(f - \frac{\partial f}{\partial S}S)\varDelta t$$

$$\Rightarrow \quad \frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}rS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 = rf \qquad \text{(Eq. 2)}$$

Eq. 2 is called the Black-Scholes-Merton differential equation.

Variance reduction

Monte Carlo simulation is a way of generating independent random samples from a stochastic variable X and then calculating the average of these samples,

$$\widehat{M} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

where x_i are the independent random samples and N is the number of samples. The estimator M is unbiased since,

$$E[\widehat{M}] = E\left[\frac{1}{N}\sum_{i=1}^{N} x_i\right] = \frac{1}{N}\sum_{i=1}^{N} E[x_i] = \frac{1}{N}NM = M$$

The interesting measure in this case is the variance,

$$Var(\widehat{M}) = Var\left(\frac{1}{N}\sum_{i=1}^{N} x_i\right) = \frac{1}{N^2}(\sum_{i=1}^{N} Var(x_i) + 2\sum_{i\neq j} Cov(x_i, x_j)) = \frac{Var(X)}{N} \qquad \text{(Eq. 3)}$$

[6]. Since the x_i are independent. Since the variance is reduced proportionally to the number of samples, the number of simulations in the case of Monte Carlo simulation, a large number of simulations is required to estimate the option price accurately. In order to reduce the total simulation time, it is important to find a method to effectively reduce the variance.

Antithetic sampling

Looking at Eq. 3 it is clear that if the covariance term is negative this would reduce the variance. One way to do this is by antithetic sampling.

Antithetic sampling is a technique where two paths for the asset price are simulated at once. First a price path S_1 is calculated using the random sample from the normal distribution ε, then another price path S_2 is calculated instantaneously by just changing the sign on ε to $-\varepsilon$. Then the final estimate of the option price is calculated as,

$$\bar{f} = \frac{f(S_1) + f(S_2)}{2}$$

[1]. This works because each pair $(\varepsilon_i, -\varepsilon_i)$ generates negatively correlated price paths and thus reduce the variance of the estimator.

Black-Scholes pricing formulas

With Eq. 2, the Black-Scholes-Merton differential equation, as foundation it is possible to derive a closed form formula for pricing a standard European option on a non dividend-paying stock as,

$$c = S_0 N(d_1) - K e^{-rT} N(d_2) \tag{Eq. 4a}$$

$$p = K e^{-rT} N(-d_2) - S_0 N(-d_1) \tag{Eq. 4b}$$

where

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}$$

c is the price of a call option, p the price of a put option and $N(x)$ is the cumulative probability distribution function for a standardized normal distribution. S_0 is the stock price at time 0, K is the strike price, r is the risk-free interest rate, σ is the stock price volatility and T is the time to maturity of the option, [1].

This formula is called the Black-Scholes (or Black-Scholes-Merton) pricing formula.

Asian approximation formula

Geometric Asian options

As mentioned earlier it is possible to derive a closed form formula for pricing Geometric Asian option. This is possible because the geometric average share a vital property with stock price process. This property is the lognormal distribution, which was showed for the stock price earlier. The lognormal distribution of the geometric mean will not be proved here, but the pricing formula for a non dividend-paying stock can be expressed as,

$$c = S_0 A_j N\left(d_{n-j} + \sigma\sqrt{T_{2,n-j}}\right) - K e^{-rT} N(d_{n-j}) \tag{Eq. 5a}$$

$$p = K e^{-rT} N(-d_{n-j}) - S_0 A_j N\left(-d_{n-j} - \sigma\sqrt{T_{2,n-j}}\right) \tag{Eq. 5b}$$

where

$$d_{n-j} = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right) T_{1,n-j} + \ln(B_j)}{\sigma\sqrt{T_{2,n-j}}}$$

$$A_j = e^{-r(T - T_{1,n-j}) - \sigma^2 (T_{1,n-j} - T_{2,n-j})/2} B_j$$

$$T_{1,n-j} = \frac{n-j}{n}\left(T - \frac{(n-j-1)h}{2}\right)$$

$$T_{2,n-j} = \left(\frac{n-j}{n}\right)^2 T - \frac{(n-j)(n-j-1)(4n-4j+1)}{6n^2} h$$

$$B_j = \left(\prod_{j=1}^{n} \frac{S(T-(n-j)h}{S}\right)^{1/n}, B_0 = 1$$

n is the number of observations to form the average, h is the observation frequency, j is the number of observations past in the averaging period, [2]. The other parameters are the same as in Black-Scholes formula.
For a standard European option, with $n = 1$ and $j = 0$, it is easy to see that this formula becomes the Black-Scholes pricing formula.

Arithmetic Asian options

The arithmetic mean does not follow a lognormal distribution and because of that it is not possible to obtain a closed form formula to price Arithmetic Asian option. However, since it is possible to approximate the arithmetic mean using the geometric mean, it is possible to derive an approximation of the price of an Arithmetic Asian option. One such way is to value the Arithmetic Asian option by conditioning on the geometric mean price of the underlying asset,

$$c = e^{(-rT)}E\big[E[\max(A-K,0) \mid G]\big]$$

and

$$p = e^{(-rT)}E\big[E[\max(K-A,0) \mid G]\big]$$

where A is the arithmetic and G is the geometric mean of the underlying asset price, [4]. In the case when the averaging period has not yet started, the price for an non-dividend paying Arithmetic Asian option can be approximated by,

$$c \approx e^{-rT}\left[\left(\frac{1}{n}\sum_{i=1}^{n} e^{\mu_i+\sigma_i^2/2}N\left(\frac{\mu-\ln(\widehat{K})}{\sigma_x}+\frac{\sigma_{xi}}{\sigma_x}\right)\right)-KN\left(\frac{\mu-\ln(\widehat{K})}{\sigma_x}\right)\right]$$

and

$$p \approx e^{-rT}\left[KN\left(-\frac{\mu-\ln(\widehat{K})}{\sigma_x}\right)-\left(\frac{1}{n}\sum_{i=1}^{n} e^{\mu_i+\sigma_i^2/2}N\left(\frac{\mu-\ln(\widehat{K})}{\sigma_x}+\frac{\sigma_{xi}}{\sigma_x}\right)\right)\right]$$

where

$$\mu_i = \ln(S) + (r-\sigma^2/2)(t_1+(i-1)\Delta t)$$

$$\sigma_i = \sigma\sqrt{(t_1+(i-1)\Delta t)}$$

$$\sigma_{xi} = \sigma^2(t_1+\Delta t((i-1)-i(i-1)/2n)$$

$$\mu = \ln(\mathrm{S}) + (r-\sigma^2/2)(t_1+(n-1)\Delta t/2)$$

$$\sigma_x = \sigma\sqrt{t_1+\Delta t(n-1)(2n-1)/6n}$$

$$\widehat{K} = 2K - \frac{1}{n}\sum_{i=1}^{n} e^{\mu_i+\frac{\sigma_{xi}(\ln(K)-\mu)}{\sigma_x^2}+\frac{\sigma_i^2-\sigma_{xi}^2/\sigma_x^2}{2}}$$

t_1 is the time to the first average point, Δt is the time between averaging points and all other parameters are the same as in previous sections. The derivation of this approximation will not be done here but the results will be used in the simulations.

Non constant volatility

The simulations performed in this thesis are focusing on the effects that different volatility scenarios have on the price of Arithmetic European-style Asian options. Four main volatility scenarios will be examined: *Constant volatility, Half-time changing volatility, Volatility peaks and Stochastic volatility.*

Stochastic volatility

If the volatility of the stock price itself is supposed to be stochastic, the stock price process described earlier would follow the process,

$$dS = rSdt + \sigma Sdz$$

where σ itself follow a stochastic process. One such process could be the Hull-White model, [5], defined as,

$$d\sigma^2 = \mu\sigma^2 dt + \xi\sigma^2 dw$$

In the general case the two Wiener processes, dz and dw, have correlation ρ and both μ and ξ may depend on σ and t.

An analytical closed form solution to this equation has not been derived and might be impossible to achieve. It is, however, possible to derive a series solution to the Hull-White model. Another way to calculate the option price based on stochastic volatility is to simulate the volatility in the same way as earlier described for the stock price. If the two Wiener processes is assumed to be uncorrelated, $\rho = 0$, and ξ is assumed constant, a rather simple path constructing formula for the volatility can be formed as,

$$\sigma_{t+\Delta t}^2 = \sigma_t^2 e^{\left(\mu - \frac{1}{2}\xi^2\right)\Delta t + \xi\varepsilon\sqrt{\Delta t}} \qquad \text{(Eq. 7)}$$

in the same way as for the stock price earlier, with ε being a random sample from a standardized normal distribution $N(0, 1)$. The variable μ in this formula is important to investigate a little closer. If μ would be constant and non-zero, the volatility itself would have a drift instead of being mean-reverting. Since this is never observed empirically μ should be formed so that σ^2 follow a mean-reverting process. A simple way to do this is by defining μ as,

$$\mu = a(\sigma^m - \sigma) \qquad \text{(Eq. 8)}$$

where a and σ^m are constants, [5].

Part II – Simulations, Results and Conclusions

Description of simulations

The purpose of this thesis is to examine the difference, or bias, between three models for pricing an Asian (European style) Arithmetic call option. The three models are: the Black-Scholes model, the Arithmetic Asian Approximation model and Monte Carlo simulation. Furthermore, the bias examination will be done for four volatility scenarios: constant volatility, half-time changing volatility, volatility peaks and stochastic volatility.

Formulas used for each model and volatility scenario are:

	Volatility scenario			
Model	**Constant volatility**	**Half-time changing volatility**	**Volatility peaks**	**Stochastic volatility**
Monte Carlo simulation	Eq. 1	Eq. 1	Eq. 1	Eq. 1 & Eq. 7
Arithmetic Asian Approximation model	Eq. 6a	Eq. 6a*	Eq. 6a*	Eq. 6a**
Black-Scholes model	Eq. 4a	Eq. 4a*	Eq. 4a*	Eq. 4a**

* The volatility used in these cases is the arithmetic mean of the volatility scheme

** The volatility used in these cases is the arithmetic mean of the simulated volatility in Eq. 7

For each volatility scenario the bias between the models is examined for options *At the money (ATM), Out of the money (OTM)* and *In the money (ITM)*.

ATM: A call option is said to be "At the money" when the forward price of the underlying asset equals the strike price of the option, i.e. $S_t e^{r(T-t)} = K$.

OTM: A call option is said to be "Out of the money" when the forward price of the underlying asset is lower than strike price of the option, i.e. $S_t e^{r(T-t)} < K$.

ITM: A call option is said to be "In the money" when the forward price of the underlying asset is higher than strike price of the option, i.e. $S_t e^{r(T-t)} > K$.

Accuracy in estimations

Before simulating the Asian Option prices it is important to analyze the accuracy (or error) in the prices estimated by simulation. Since Monte-Carlo simulation, in this case, is carried out by simulating a large number of paths for the stock price and the calculating the option price by taking the mean of each simulated option value, i.e.

$$f_{mean} = \frac{1}{n}\sum_{i=1}^{n} f_i$$

it follows that (since each simulation is statistically independent),

$$Var(f_{mean}) = \frac{1}{n^2}\sum_{i=1}^{n} Var(f_i) = \frac{Var(f)}{n}$$

$$SE(f_{mean}) = \sqrt{\frac{Var(f)}{n}}$$

where f is the option price, $Var(f)$ is the variance of the estimated price and $SE(f)$ the standard error.

Since the standard error of the estimate decreases proportionally to the square root of the number of simulations, the accuracy improves with larger number of simulations. Since a very large number of simulations are very time consuming, it is desirable to find the smallest number of simulations that still gives a satisfying accuracy.

To analyze this, the results of Monte-Carlo simulated prices of at the money standard European call options was compared to the corresponding Black-Scholes option price.

European option, ATM, K=100, r=4%, *T=30 days*					
No. of simulations	Volatility	Simulated price	Black-Scholes price	Relative deviation (Simulated/Black-Scholes)	Standard error of simulated price
100	0,1	1,02	1,14	-11 %	8,4 %
	0,3	3,82	3,42	12 %	6,7 %
	0,6	6,33	6,83	-7 %	9,0 %
1 000	0,1	1,14	1,14	0 %	2,4 %
	0,3	3,43	3,42	0 %	2,5 %
	0,6	6,71	6,83	-2 %	2,8 %
10 000	0,1	1,14	1,14	0 %	0,8 %
	0,3	3,40	3,42	0 %	0,8 %
	0,6	6,84	6,83	0 %	0,9 %
100 000	0,1	1,15	1,14	0 %	0,2 %
	0,3	3,42	3,42	0 %	0,3 %
	0,6	6,84	6,83	0 %	0,3 %

Table 1a

European option, ATM, K=100, r=4%, *T=60 days*					
No. of simulations	Volatility	Simulated price	Black-Scholes price	Relative deviation (Simulated/Black-Scholes)	Standard error of simulated price
100	0,1	1,71	1,61	6 %	7,1 %
	0,3	5,03	4,82	4 %	9,1 %
	0,6	9,35	9,62	-3 %	8,7 %
1 000	0,1	1,63	1,61	1 %	2,4 %
	0,3	5,06	4,82	5 %	2,7 %
	0,6	10,14	9,62	5 %	2,9 %
10 000	0,1	1,63	1,61	1 %	0,8 %
	0,3	4,85	4,82	1 %	0,9 %
	0,6	9,66	9,62	0 %	1,0 %
100 000	0,1	1,61	1,61	0 %	0,3 %
	0,3	4,84	4,82	0 %	0,3 %
	0,6	9,61	9,62	0 %	0,3 %

Table 1b

European option, ATM, K=100, r=4%, *T=90 days*					
No. of simulations	Volatility	Simulated price	Black-Scholes price	Relative deviation (Simulated/Black-Scholes)	Standard error of simulated price
100	0,1	1,87	1,96	-4 %	7,7 %
	0,3	6,66	5,88	13 %	9,2 %
	0,6	11,19	11,73	-5 %	8,8 %
1 000	0,1	1,94	1,96	-1 %	2,5 %
	0,3	6,18	5,88	5 %	2,7 %
	0,6	11,43	11,73	-3 %	3,3 %
10 000	0,1	1,96	1,96	0 %	0,8 %
	0,3	5,88	5,88	0 %	0,9 %
	0,6	11,59	11,73	-1 %	1,0 %
100 000	0,1	1,96	1,96	0 %	0,3 %
	0,3	5,85	5,88	0 %	0,3 %
	0,6	11,74	11,73	0 %	0,3 %

Table 1c

Table 1a-c shows the standard error of the simulated option prices and the deviation between simulated and analytically calculated option prices. The values are shown for different number of simulations and volatilities. Tables display results for options with maturities in 30, 60 and 90 days respectively.

As can be seen in above tables it is necessary to carry out at least 100 000 simulations in order to reach a deviation between the "true" and simulated value of less than 0,5 %. Also, with 100 000 simulations, the standard error of the simulated option prices is around 0,3 %, which means the "true" value is with ~68 % certainty within $\pm$0,3 % of the estimated value. This is considered accurate enough for this thesis, but it will be important to analyze the simulation results with respect to the standard error. Throughout this thesis the standard error of the simulated option prices will be used to form a 95% confidence interval for the price, using the fact that

$$f_{mean} \pm 1{,}96 * SE(f_{mean})$$

forms the boundary for the 95% confidence interval, [6]. Even though f_{mean} actually has a Student´s t – distribution, since the standard error is estimated, since the degrees of freedom are very large ($\sim$ number of simulations) it can be approximated with a standard normal distribution.

Results of bias analysis

Results of simulations and bias analysis for the different volatility schemes are shown below. Analysis and conclusions are done only for Asian call options with maturity in 30 days and with an average period of the last 7 days of the option lifetime.

Simulations and bias analysis has been done in the context of this thesis for options with lifetimes of more than 30 days. This will not be presented in this report, but can be handed out upon request.

Constant volatility

In order to get an idea of how volatility affects the option value, simulations are done with a constant volatility parameter. Also, a bias examination is done between simulated values and the values of the Asian approximation formula and Black-Scholes formula respectively. This gives benchmark biases since a well suited closed form formula should calculate the option value reasonably accurate when volatility, and other parameters, is assumed constant.

OTM:

Spot S=90, Strike K=100, r=4%, T=30 days, Avg. period=last 7 days

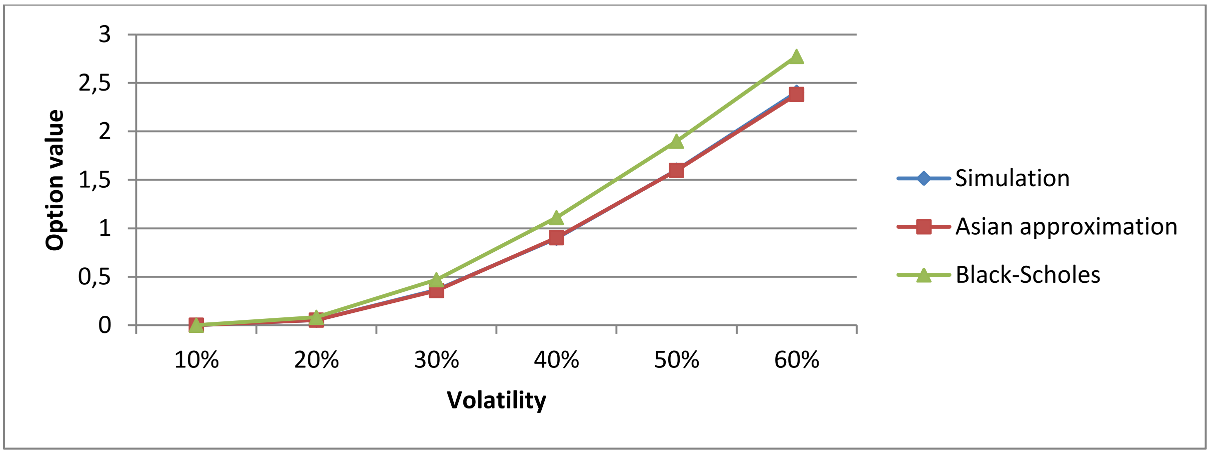

Fig. 1a

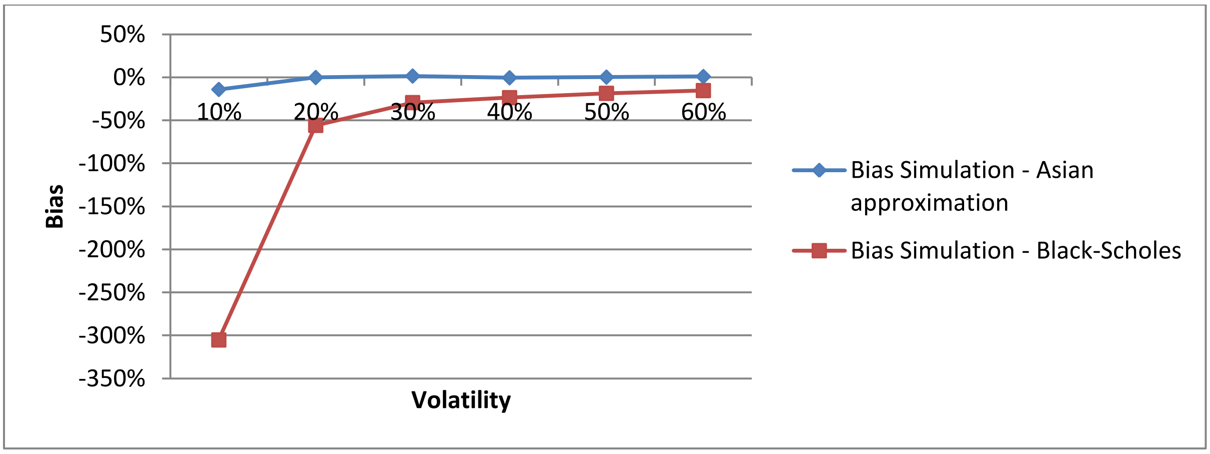

Fig. 1b

		95% Confidence Interval of simulated value			
Volatility	**Simulation price**	**Lower bound**	**Upper bound**	**Asian Approximation price**	**Black-Scholes price**
10 %	0,00003	0,00000	0,00006	0,00004	0,00012
20 %	0,052	0,050	0,054	0,052	0,081
30 %	0,363	0,356	0,369	0,357	0,469
40 %	0,897	0,885	0,909	0,902	1,110
50 %	1,600	1,582	1,618	1,596	1,898
60 %	2,402	2,378	2,427	2,380	2,772

Table 2

Key observations:

- The option value increases exponentially with increasing volatility
- There is no significant difference between the simulated option value and the value of the Asian approximation formula, independent of volatility size. The Asian approximation value is covered by the 95% confidence interval of simulated value.
- Black-Scholes formula overestimates the option value for any volatility size. This is concurrent with theory since Black-Scholes formula only considers the end price of the underlying stock, and the average price of a stock with a positive drift tends to be lower than the end price.
- The overestimate of Black-Scholes formula decreases with increasing volatility from about 300% to 15% for volatilities of 10% to 60%

ATM:

Spot S=K*exp(-rT), Strike K=100, r=4%, T=30 days, Avg. period=last 7 days

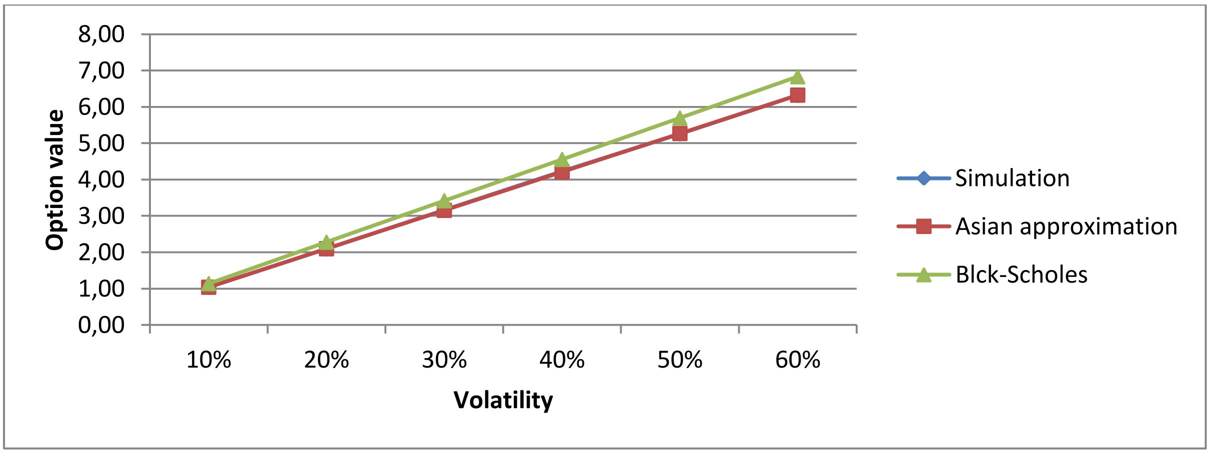

Fig. 2a

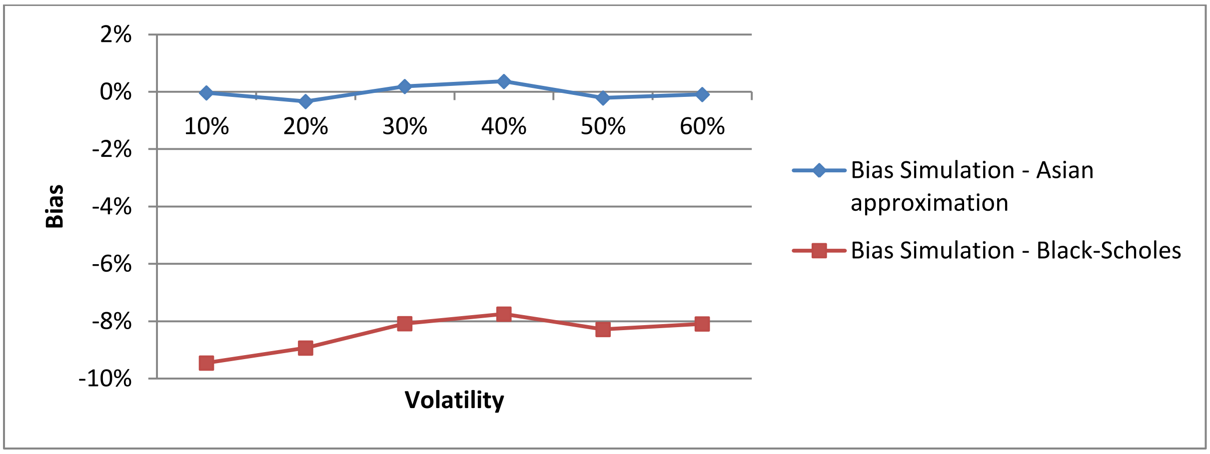

Fig. 2b

Volatility	Simulation price	95% Confidence Interval of simulated value: Lower bound	Upper bound	Asian Approximation price	Black-Scholes price
10 %	1,041	1,036	1,047	1,042	1,140
20 %	2,093	2,082	2,103	2,100	2,280
30 %	3,163	3,147	3,179	3,157	3,419
40 %	4,230	4,207	4,252	4,214	4,557
50 %	5,259	5,231	5,288	5,271	5,695
60 %	6,320	6,285	6,355	6,326	6,831

Table 3

Key observations:

- The option value increases linearly with increasing volatility
- There is no significant difference between the simulated option value and the value of the Asian approximation formula, independent of volatility size. The Asian approximation value is covered by the 95% confidence interval of simulated value.
- Black-Scholes formula overestimates the option value for any volatility size, but not by as much as for OTM options. This is logical since for the very low option values of OTM options a small absolute difference in option value can give rise to large relative differences. But since ATM options have higher value, the relative differences are smaller.
- There is no apparent correlation between the size of the overestimate of Black-Scholes formula and volatility size. The overestimate is about 8%-9% of the simulated value.

ITM:

Spot S=110, Strike K=100, r=4%, T=30 days, Avg. period=last 7 days

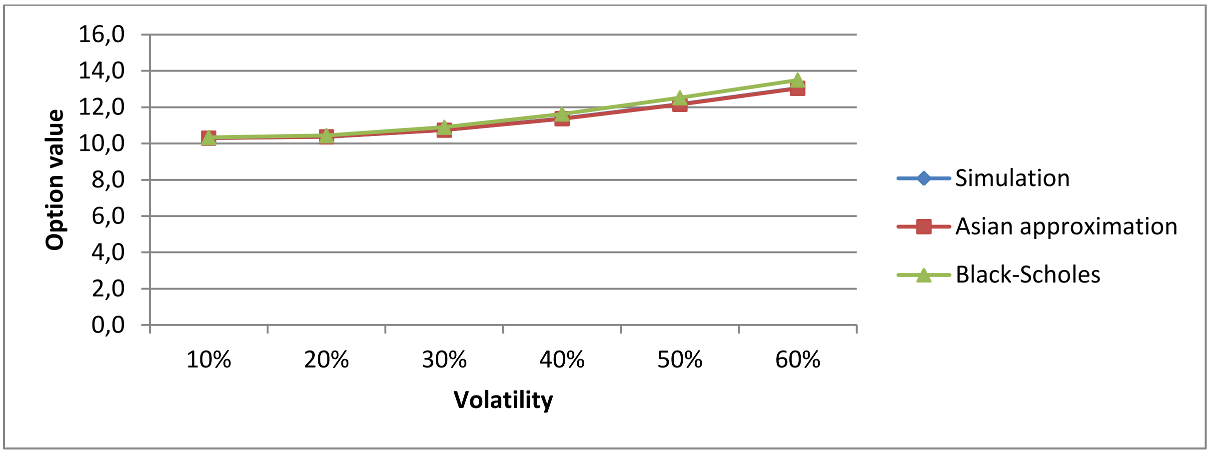

Fig. 3a

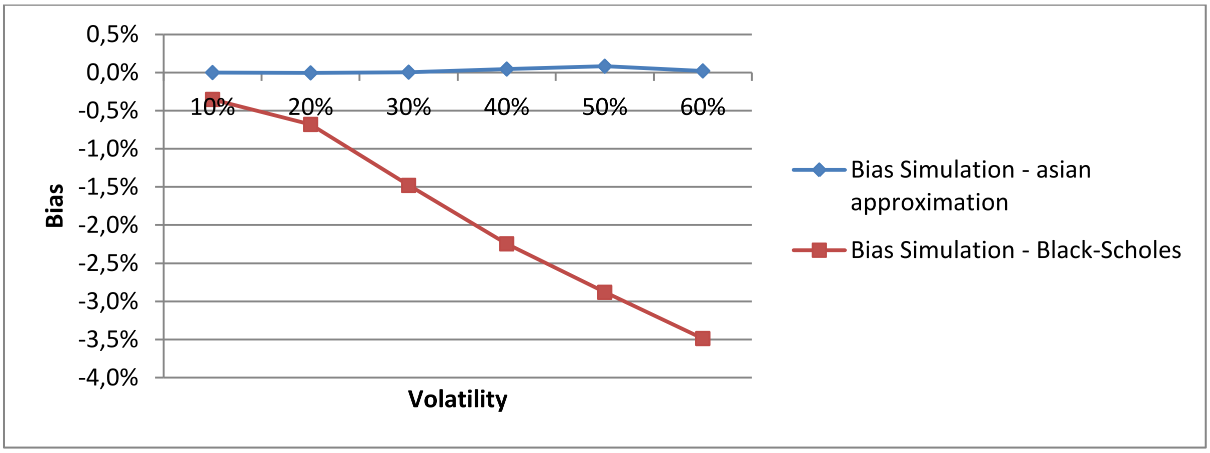

Fig. 3b

		95% Confidence Interval of simulated value			
Volatility	**Simulation price**	**Lower bound**	**Upper bound**	**Asian Approximation price**	**Black-Scholes price**
10 %	10,2922	10,2918	10,2925	10,2921	10,3285
20 %	10,3624	10,3590	10,3658	10,3628	10,4329
30 %	10,7325	10,7226	10,7423	10,7319	10,8912
40 %	11,3691	11,3516	11,3866	11,3639	11,6244
50 %	12,1638	12,1382	12,1893	12,1537	12,5142
60 %	13,0400	13,0060	13,0739	13,0372	13,4946

Table 4

Key observations:

- The option value increase exponentially with increasing volatility
- There is no significant difference between the simulated option value and the value of the Asian approximation formula, independent of volatility size. The Asian approximation value is covered by the 95% confidence interval of simulated value.

- Black-Scholes formula overestimates the option value for any volatility size, but by even less than for ATM options, which is logical according to the same reasoning as for ATM options.
- The overestimate of Black-Scholes formula increases with increasing volatility, contrary to the behavior for OTM options. The overestimate increases from about 0,5% to 3,5% for volatilities of 10% to 60%

Half-time changing volatility

OTM:

Spot S=90, Strike K=100, r=4%, T=30 days, Avg. period=last 7 days

Volatility change from low to high

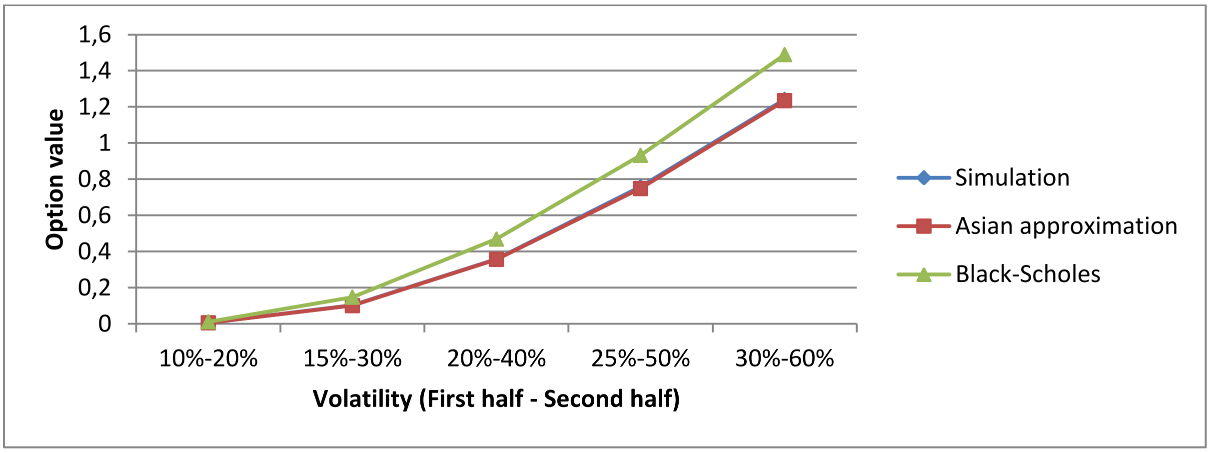

Fig. 4a

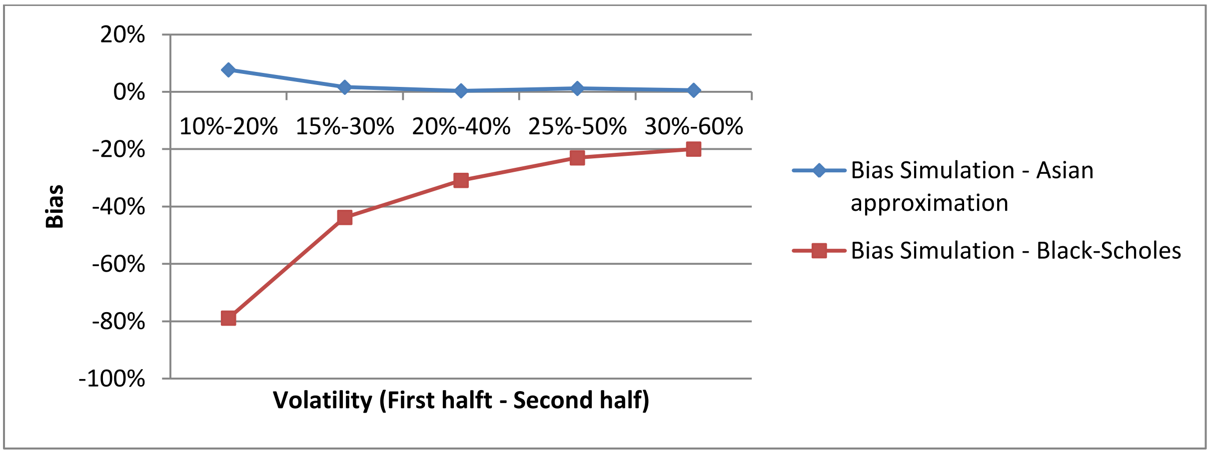

Fig. 4b

Volatility	Simulation price	95% Confidence Interval of simulated value: Lower bound	Upper bound	Asian Approximation price	Black-Scholes price
10 %-20 %	0,00670	0,00615	0,00726	0,00619	0,01199
15 %-30 %	0,10290	0,09998	0,10581	0,10124	0,14794
20 %-40 %	0,35857	0,35202	0,36512	0,35722	0,46924
25 %-50 %	0,75801	0,74720	0,76882	0,74890	0,93241
30 %-60 %	1,24215	1,22691	1,25739	1,23513	1,48991

Table 5

Key observations:

- There is no significant difference between the simulated option value and the value of the Asian approximation formula, independent of volatility size. The Asian approximation value is covered by the 95% confidence interval of simulated value.

- Black-Scholes formula overestimates the option value for any volatility size.
- The overestimate of Black-Scholes formula decreases with increasing volatility, from about 80% to 20% for mean volatilities of 15% to 45%. With respect to the mean volatility size, this is in line with the analysis of constant volatility for OTM Asian options.

Volatility change from high to low

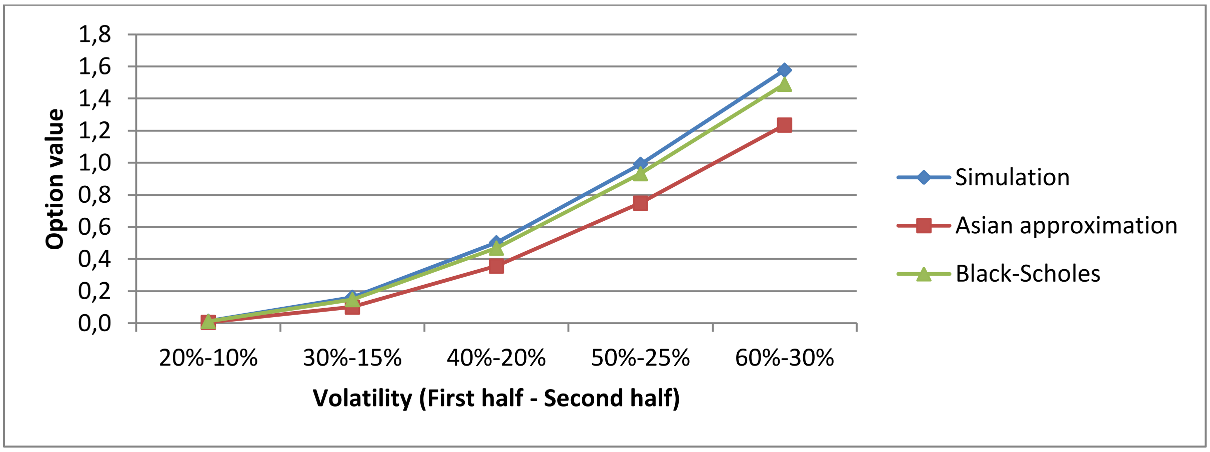

Fig. 5a

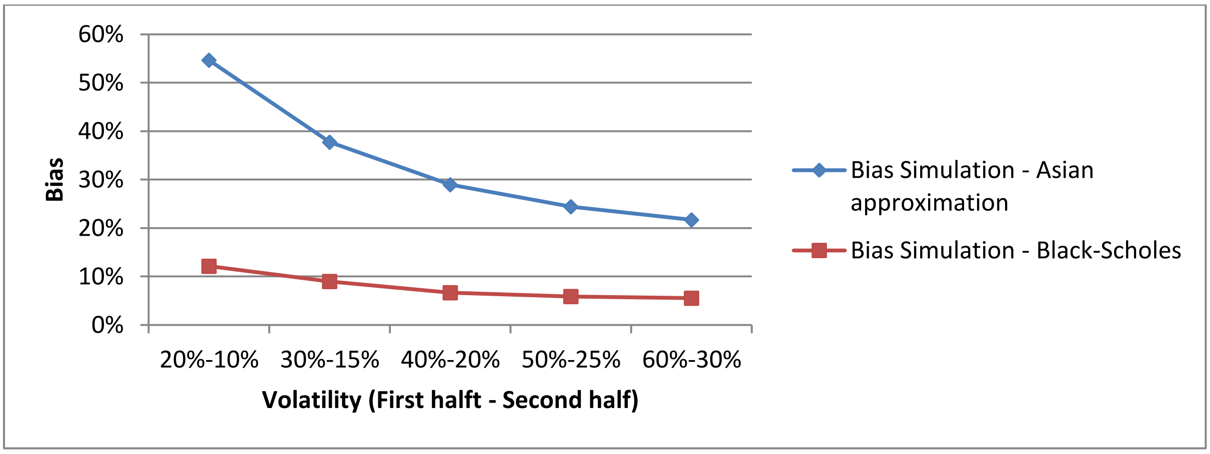

Fig. 5b

		95% Confidence Interval of simulated value			
Volatility	**Simulation price**	**Lower bound**	**Upper bound**	**Asian Approximation price**	**Black-Scholes price**
20 %-10 %	0,01365	0,01281	0,01449	0,00619	0,01199
30 %-15 %	0,16256	0,15865	0,16647	0,10124	0,14794
40 %-20 %	0,50277	0,49457	0,51097	0,35722	0,46924
50 %-25 %	0,99070	0,97769	1,00370	0,74890	0,93241
60 %-30 %	1,57731	1,55925	1,59538	1,23513	1,48991

Table 6

Key observations:

- Neither the Asian approximation formula nor Black-Scholes formula manage to capture the effect of volatility change with the higher volatility prior to the averaging period of the option. The effective mean volatility in this case is apparently higher than the arithmetic mean volatility.

- The Asian approximation formula underestimates the option value significantly when using the mean volatility
- The underestimate of the Asian approximation formula decreases with increasing mean volatility, from about 50% to 20% for mean volatilities of 15% to 45%
- Black-Scholes formula also underestimates the option value, but not by as much as the Asian approximation.
- The underestimate of Black-Scholes decreases with increasing mean volatility, from about 10% to 5% for mean volatilities of 15% to 45%.

ATM:

Spot S=K*exp(-rT), Strike K=100, r=4%, T=30 days, Avg. period=last 7 days

Volatility change from low to high

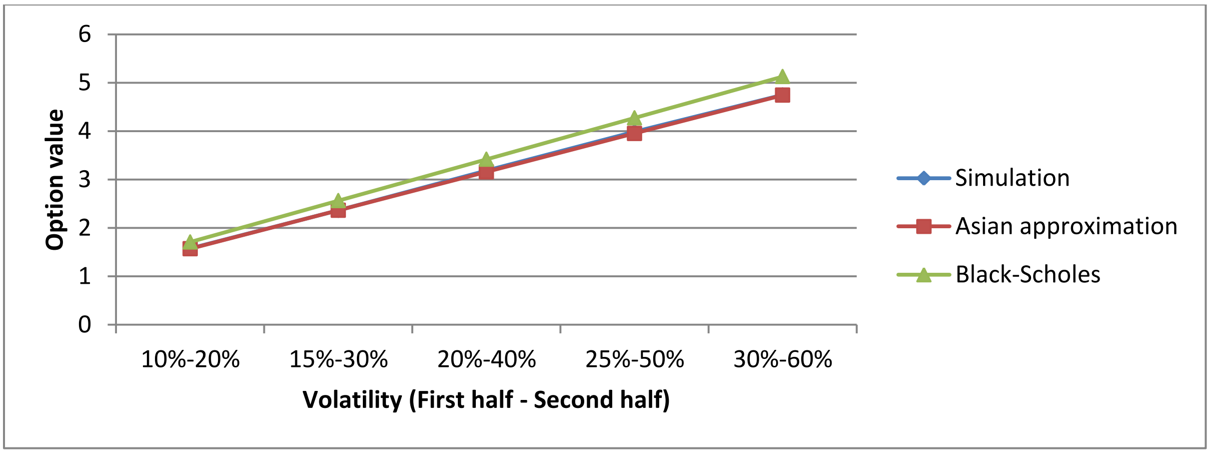

Fig. 6a

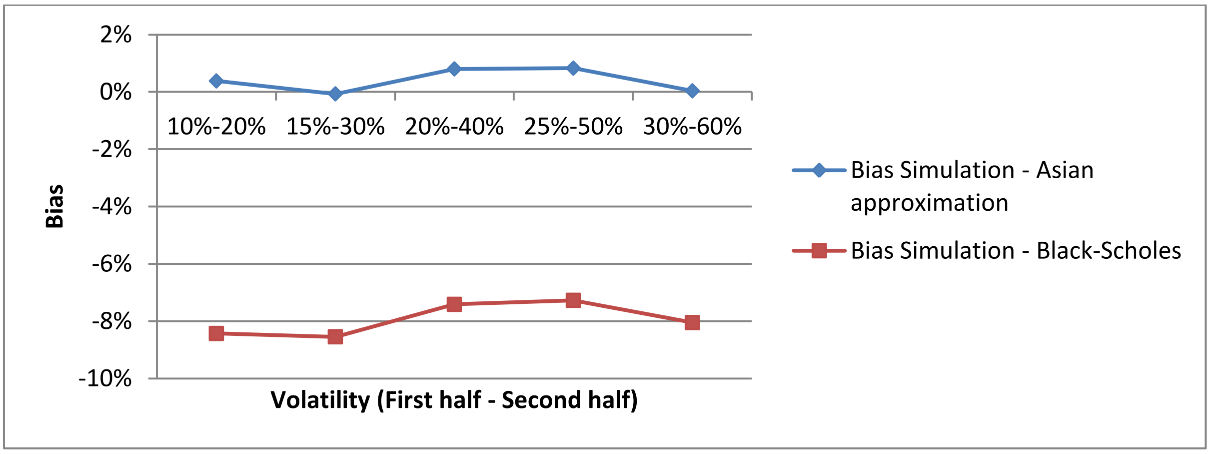

Fig. 6b

		95% Confidence Interval of simulated price			
Volatility	**Simulation price**	**Lower bound**	**Upper bound**	**Asian Approximation price**	**Black-Scholes price**
10 %-20 %	1,5769	1,5691	1,5846	1,5708	1,7098
15 %-30 %	2,3626	2,3506	2,3745	2,3642	2,5645
20 %-40 %	3,1828	3,1665	3,1992	3,1573	3,4189
25 %-50 %	3,9831	3,9623	4,0038	3,9501	4,2729
30 %-60 %	4,7443	4,7191	4,7696	4,7426	5,1263

Table 7

Key observations:

- There is no significant difference between the simulated option value and the value of the Asian approximation formula, independent of volatility size. The Asian approximation value is covered by the 95% confidence interval of simulated value.
- Black-Scholes formula overestimates the option value for any volatility size, but not by as much as for OTM options
- There is no apparent correlation between size of the overestimate of Black-Scholes formula and volatility size. The overestimate is about 8%-9% of the simulated value. With respect to the mean volatility size, this is in line with the analysis of constant volatility for ATM Asian options.

Volatility change from high to low

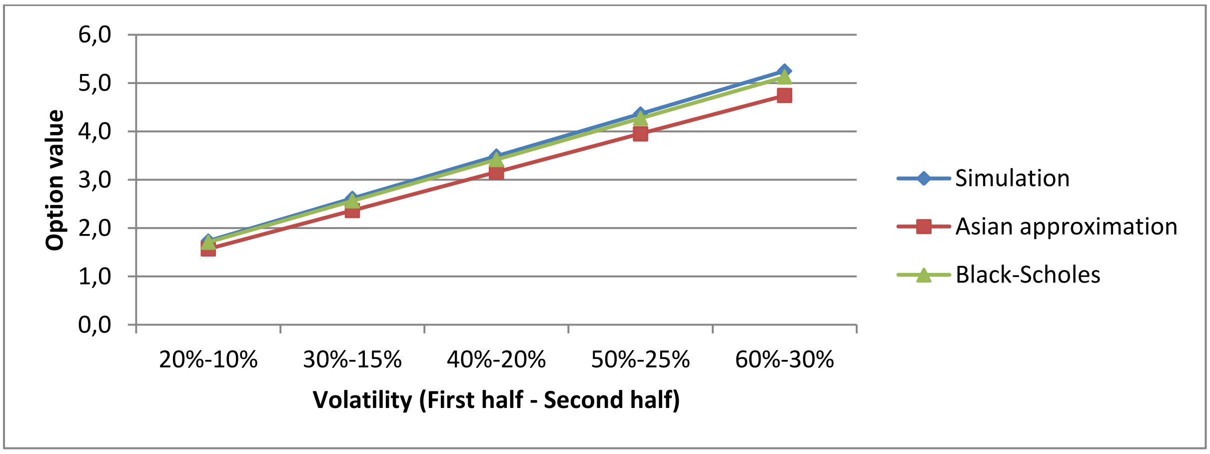

Fig. 7a

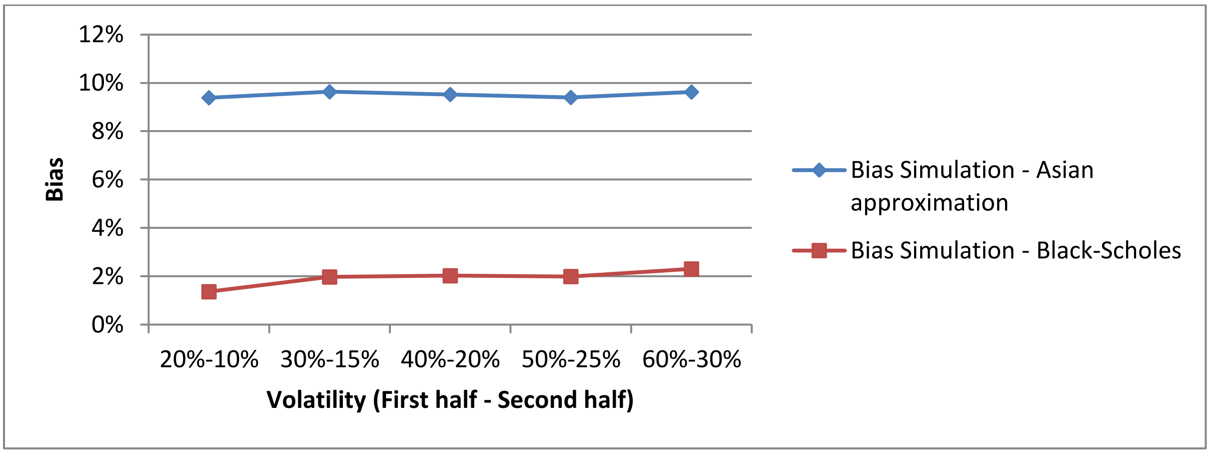

Fig. 7b

		95% Confidence Interval of simulated value			
Volatility	**Simulation price**	**Lower bound**	**Upper bound**	**Asian Approximation price**	**Black-Scholes price**
20 %-10 %	1,7335	1,7250	1,7421	1,5708	1,7098
30 %-15 %	2,6162	2,6030	2,6295	2,3642	2,5645
40 %-20 %	3,4897	3,4716	3,5077	3,1573	3,4189
50 %-25 %	4,3599	4,3369	4,3829	3,9501	4,2729
60 %-30 %	5,2475	5,2193	5,2758	4,7426	5,1263

Table 8

Key observations:

- Neither the Asian approximation formula nor Black-Scholes formula manage to capture the effect of volatility change with the higher volatility prior to the averaging period of the option. The effective mean volatility in this case is apparently higher than the arithmetic mean volatility.
- The Asian approximation formula underestimates the option value significantly when using the mean volatility, but not by as much as for OTM options.
- There is no apparent correlation between the size of the underestimate of the Asian approximation formula and mean volatility size. The underestimate is about 10% of the simulated value.
- Black-Scholes formula also underestimates the option value, but not by as much as the Asian approximation.
- There is no apparent correlation between the size of the underestimate of Black-Scholes formula and mean volatility size. The underestimate is about 2% of the simulated value.

ITM:

Spot S=110, Strike K=100, r=4%, T=30 days, Avg. period=last 7 days

Volatility change from low to high

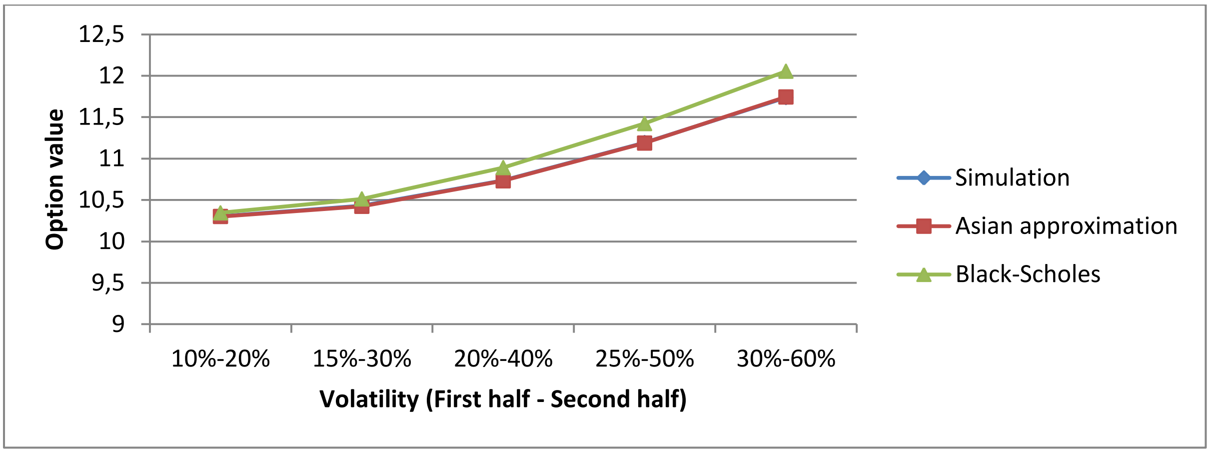

Fig. 8a

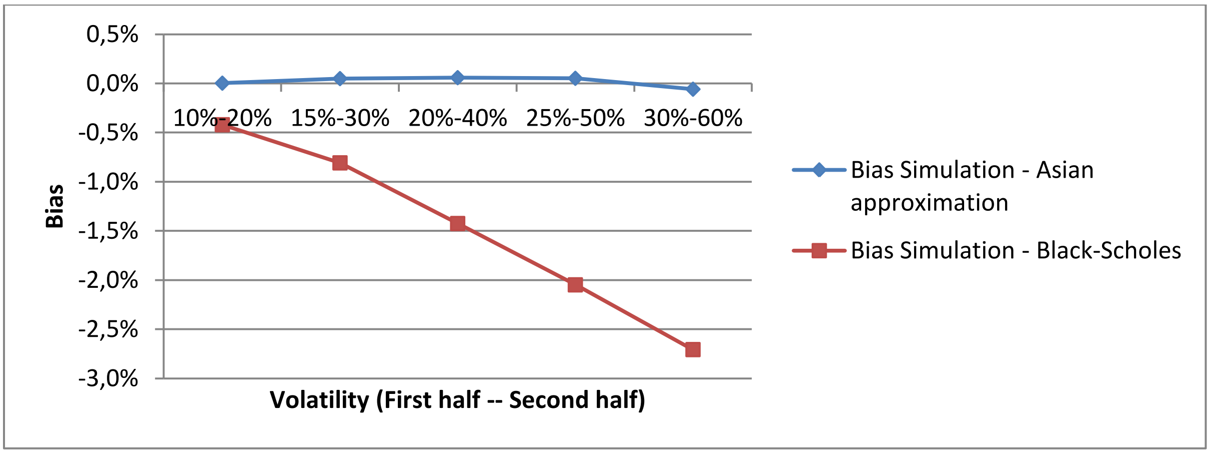

Fig. 8b

		95% Confidence Interval of simulated price			
Volatility	**Simulation price**	**Lower bound**	**Upper bound**	**Asian Approximation price**	**Black-Scholes price**
10 %-20 %	10,3018	10,3006	10,3031	10,3015	10,3451
15 %-30 %	10,4294	10,4245	10,4342	10,4242	10,5136
20 %-40 %	10,7383	10,7284	10,7481	10,7319	10,8912
25 %-50 %	11,1935	11,1780	11,2091	11,1877	11,4228
30 %-60 %	11,7371	11,7156	11,7586	11,7439	12,0548

Table 9

Key observations:

- There is no significant difference between the simulated option value and the value of the Asian approximation formula, independent of volatility size. The Asian approximation value is covered by the 95% confidence interval of simulated value.

- Black-Scholes formula overestimates the option value for any volatility size, but by even less than for ATM options
- The overestimate of Black-Scholes formula increases with increasing volatility, contrary to the behavior for OTM options. The overestimate increases from about 0,5% to 3% for mean volatilities of 15% to 45%. With respect to the mean volatility size, this is in line with the analysis of constant volatility for ITM Asian options.

Volatility change from high to low

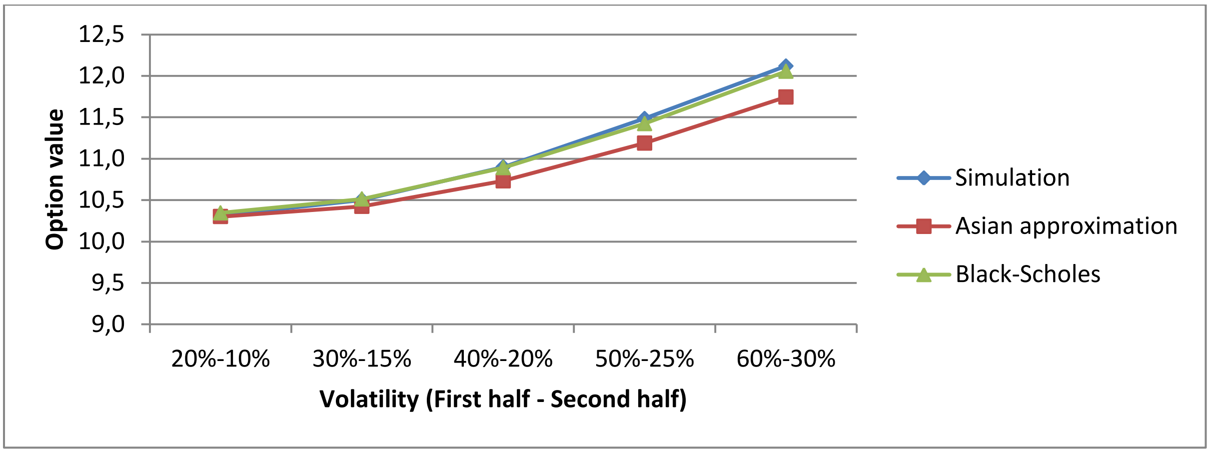

Fig. 9a

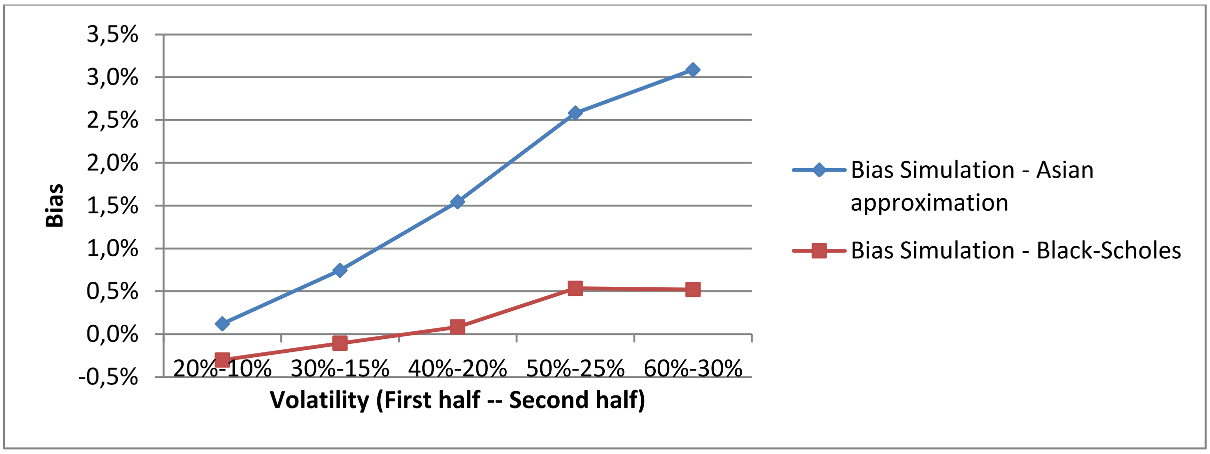

Fig. 9b

		95% Confidence Interval of simulated value			
Volatility	**Simulation price**	**Lower bound**	**Upper bound**	**Asian Approximation price**	**Black-Scholes price**
20 %-10 %	10,3137	10,3119	10,3155	10,3015	10,3451
30 %-15 %	10,5023	10,4960	10,5086	10,4242	10,5136
40 %-20 %	10,9003	10,8881	10,9124	10,7319	10,8912
50 %-25 %	11,4840	11,4654	11,5027	11,1877	11,4228
60 %-30 %	12,1178	12,0927	12,1429	11,7439	12,0548

Table 10

Key observations:

- Neither the Asian approximation formula nor Black-Scholes formula manage to capture the effect of volatility change with the higher volatility prior to the averaging period of the option. The effective mean volatility in this case is apparently higher than the arithmetic mean volatility.
- The Asian approximation formula underestimates the option value significantly when using the mean volatility, but not by as much as for OTM or ATM options when volatility changes from high to low.
- The underestimate of the Asian approximation formula increases with increasing mean volatility, from about 0% to 3% for mean volatilities of 15% to 45%
- Black-Scholes formula overestimates the option value for mean volatilities of 15% to 25%, and underestimates the option value for mean volatilities of 35% to 45%. The over- and underestimates are small and between 0% to 0,5%.

Volatility peaks

OTM:

Spot S=90, Strike K=100, r=4%, T=30 days, Avg. period=last 7 days, Length of volatility peak=2 days

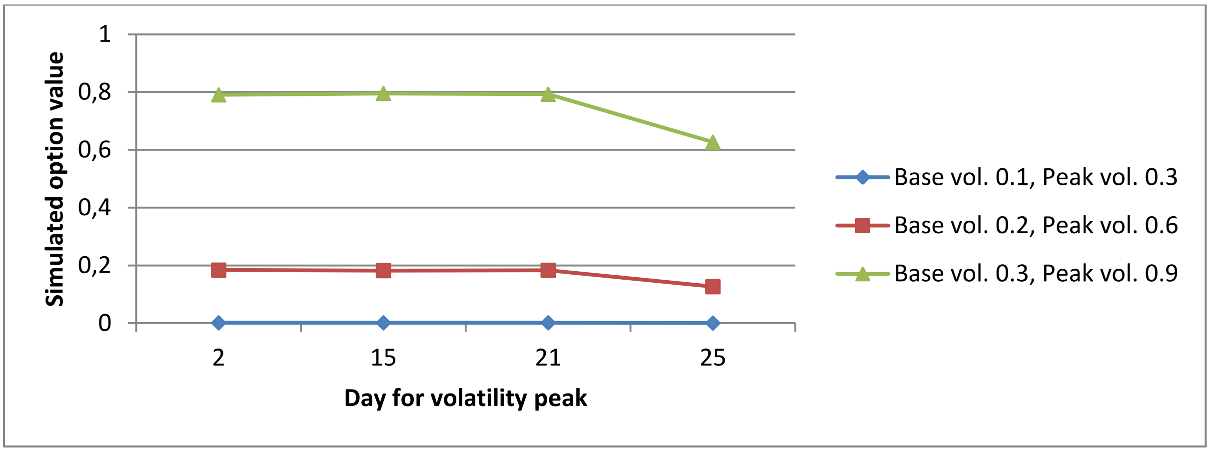

Fig. 10

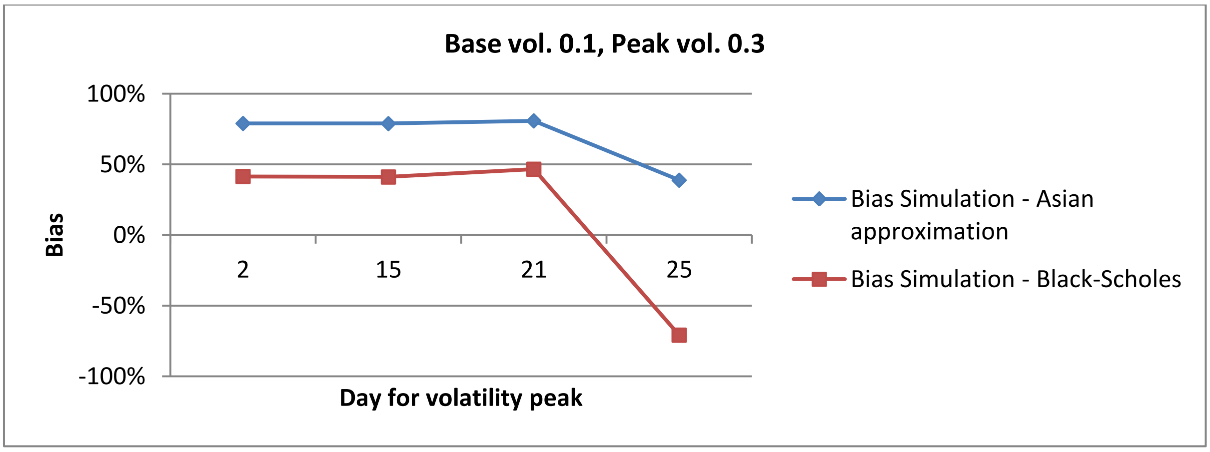

Fig. 11a

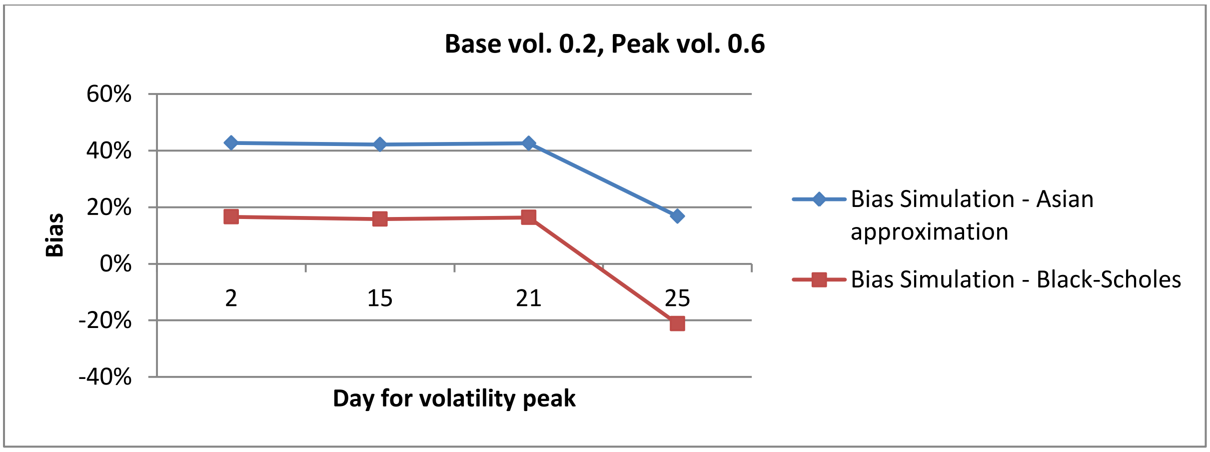

Fig. 11b

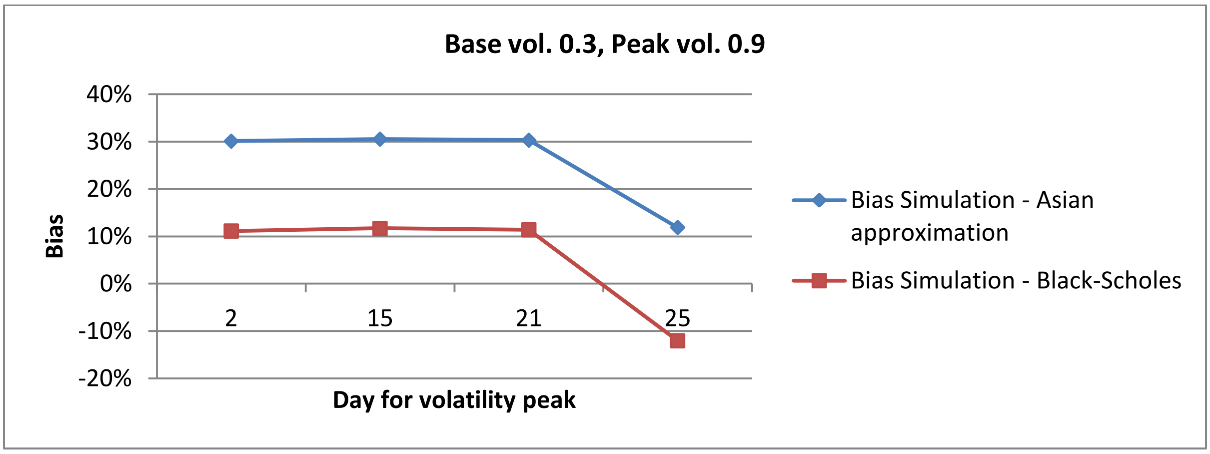

Fig. 11c

Base vol.	Peak vol.	Day of peak	Simulation price	95% Confidence Interval of simulated price: Lower bound	95% Confidence Interval of simulated price: Upper bound	Asian approximation price	Black-Scholes price
10%	**30%**	2	0,00121	0,00100	0,00142	0,00025	0,00071
		15	0,00120	0,00098	0,00142		
		21	0,00133	0,00110	0,00156		
		25	0,00041	0,00030	0,00053		
15%	**45%**	2	0,04018	0,03853	0,04182	0,01748	0,03032
		15	0,03806	0,03649	0,03962		
		21	0,03937	0,03776	0,04097		
		25	0,02258	0,02145	0,02371		
20%	**60%**	2	0,18370	0,17943	0,18797	0,10516	0,15315
		15	0,18192	0,17772	0,18611		
		21	0,18322	0,17899	0,18744		
		25	0,12639	0,12305	0,12973		
25%	**75%**	2	0,43438	0,42695	0,44182	0,28707	0,38363
		15	0,44400	0,43646	0,45154		
		21	0,43654	0,42909	0,44400		
		25	0,32756	0,32139	0,33373		
30%	**90%**	2	0,79016	0,77904	0,80128	0,55203	0,70199
		15	0,79504	0,78392	0,80617		
		21	0,79243	0,78123	0,80363		
		25	0,62663	0,61707	0,63618		

Table 11

Key observations:

- For simulated options values the time of the volatility peak is highly significant to the option value.
- For volatility peaks occurring prior to the averaging period, it does not seem to matter when the peak occurs as long as it is before the averaging period. Even though there exists significant differences between some of the option values with peaks at different times but

prior to the averaging period, the values are very close to the boundaries of the confidence intervals.

- If the peak occurs during the averaging period, the option value drops drastically. This is logical since the stock price is averaged and thus reduces the effect of the peak.
- The relative drop in option value if the peak occurs during the average period is depended of the mean volatility size and decreases with increasing volatility. The option value drops between 60% to 20% for base volatilities of 10% to 30%
- The Asian approximation formula significantly underestimates the option value when there is a volatility peak, even when the peak occurs during the averaging period.
- The underestimate of the Asian approximation value decreases with increasing mean volatility, from about 80% to 30% if the peak occurs prior to the averaging period, and about 40% to 10% if the peak occurs during the averaging period, for base volatilities of 10% to 30%.
- The Black-Scholes formula significantly underestimates the option value when there is a volatility peak if the peak occurs prior to the averaging period. If the peak occurs during the averaging period the Black-Scholes formula significantly overestimates the option value.
- The underestimate of the Black-Scholes value for peaks occurring prior to the averaging period decreases with increasing mean volatility, from about 40% to 10% for base volatilities of 10% to 30%.
- The overestimate of the Black-Scholes formula for peaks occurring during the averaging period decreases with increasing mean volatility, from about 70% to 10% for base volatilities of 10% to 30%.

ATM:

Spot S=K*exp(-rT), Strike K=100, r=4%, T=30 days, Avg. period=last 7 days, Length of volatility peak=2 days

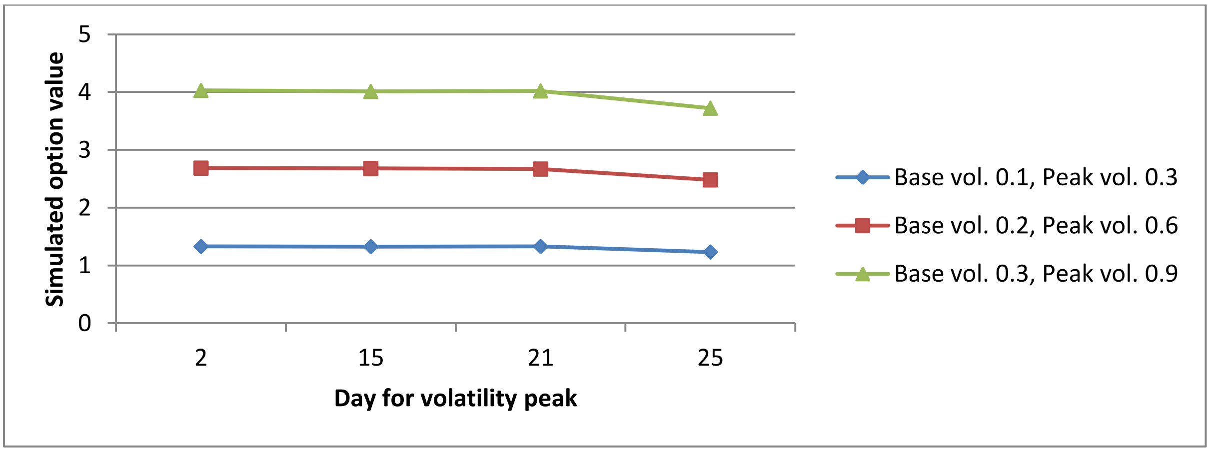

Fig. 12

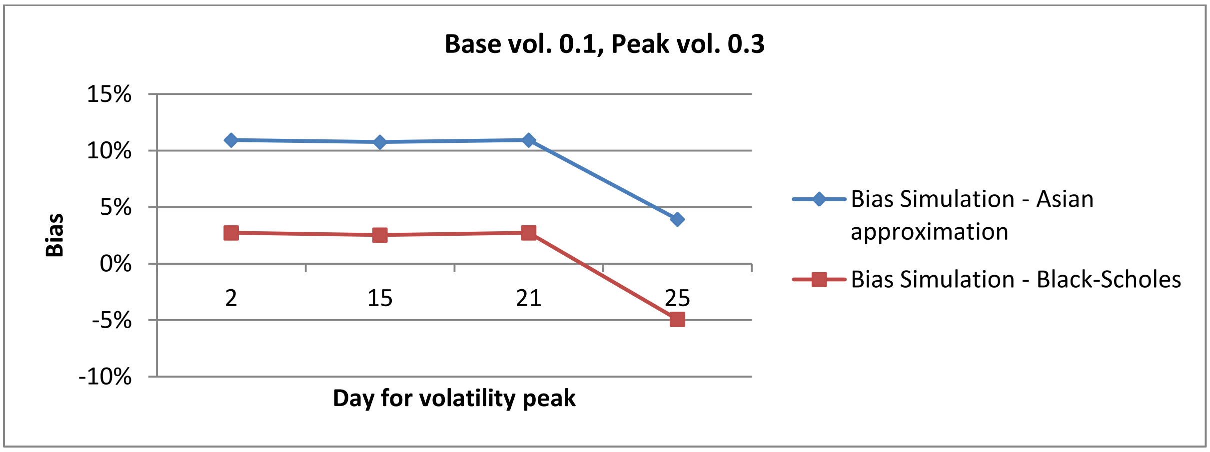

Fig. 13a

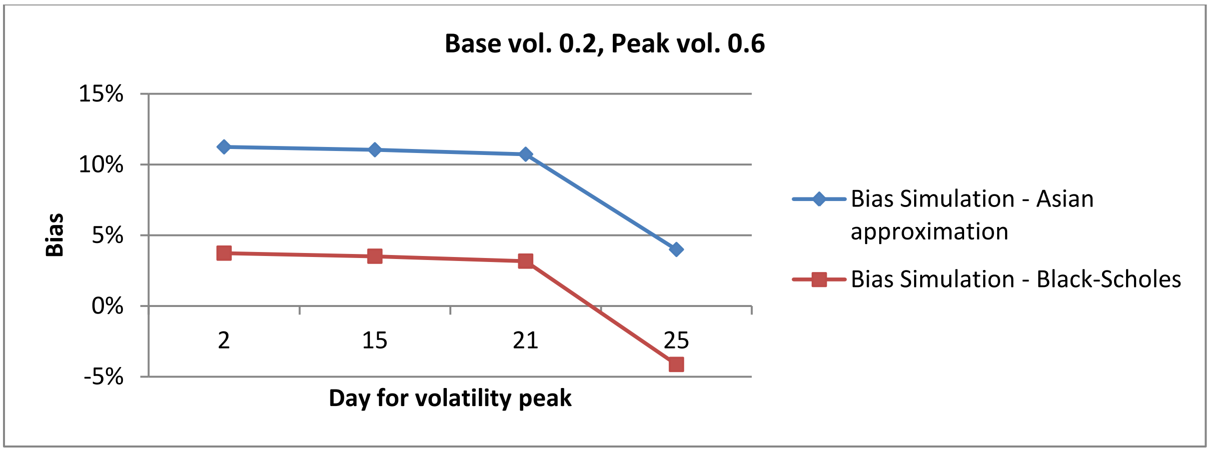

Fig.13b

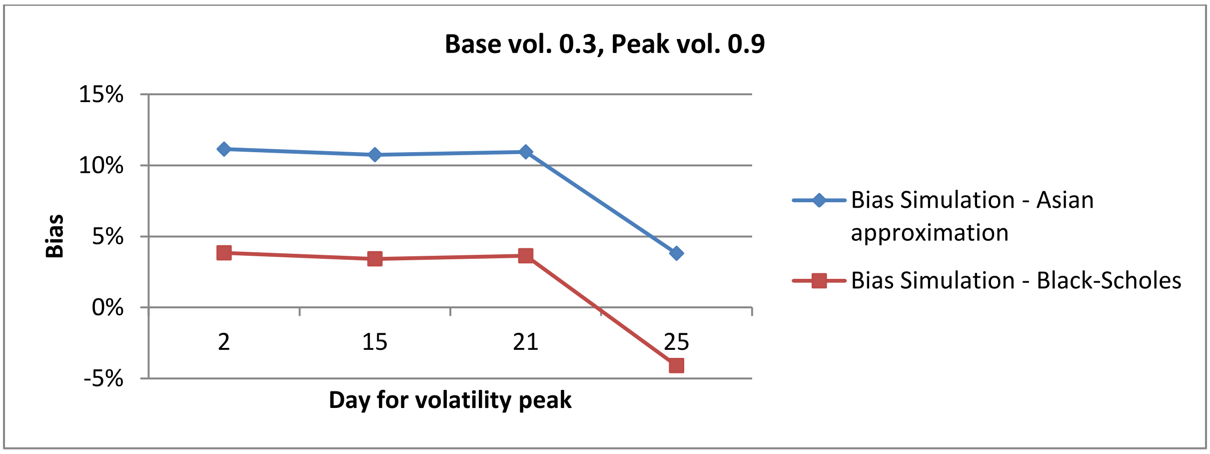

Fig. 13c

				95% Confidence Interval of simulated price			
Base vol.	**Peak vol.**	**Day of peak**	**Simulation price**	**Lower bound**	**Upper bound**	**Asian approximation price**	**Black-Scholes price**
10 %	**30 %**	2	1,3282	1,3216	1,3347	1,1829	1,2919
		15	1,3255	1,3190	1,3320		
		21	1,3282	1,3216	1,3348		
		25	1,2313	1,2252	1,2373		
15 %	**45 %**	2	1,9933	1,9833	2,0032	1,7824	1,9378
		15	1,9995	1,9896	2,0095		
		21	2,0014	1,9915	2,0113		
		25	1,8485	1,8393	1,8577		
20 %	**60 %**	2	2,6837	2,6701	2,6973	2,3818	2,5835
		15	2,6776	2,6640	2,6912		
		21	2,6680	2,6545	2,6816		
		25	2,4809	2,4684	2,4933		
25 %	**75 %**	2	3,3424	3,3251	3,3596	2,9811	3,2290
		15	3,3532	3,3359	3,3705		
		21	3,3514	3,3341	3,3688		
		25	3,0979	3,0820	3,1138		
30 %	**90 %**	2	4,0292	4,0081	4,0502	3,5802	3,8744
		15	4,0116	3,9906	4,0326		
		21	4,0205	3,9995	4,0415		
		25	3,7218	3,7025	3,7411		

Table 12

Key observations:

- For simulated options values the time of the volatility peak is highly significant to the option value.
- For volatility peaks occurring prior to the averaging period, it does not seem to matter when the peak occurs as long as it is before the averaging period. Even though there exists significant differences between some of the option values with peaks prior to the averaging period, the values are very close to the boundaries of the confidence intervals.
- If the peak occurs during the averaging period, the option value drops, but not as drastically as for OTM options. This is logical since the stock price is averaged and thus reduces the effect of the peak.
- The relative drop in option value if the peak occurs during the average period is independent of the mean volatility size. The option value drops about 7,5% for any base volatility
- The Asian approximation formula significantly underestimates the option value when there is a volatility peak, even when the peak occurs during the averaging period, but not by as much as for OTM options.
- The underestimate of the Asian approximation value is independent of mean volatility size, about 11% if the peak occurs prior to the averaging period, and about 4% if the peak occurs during the averaging period, for any base volatility.

- The Black-Scholes formula significantly underestimates the option value when there is a volatility peak occurring prior to the averaging period. If the peak occurs during the averaging period the Black-Scholes formula significantly overestimates the option value. These under- and overestimates are not by as much as for OTM options
- The underestimate of the Black-Scholes value for peaks occurring prior to the averaging period is independent of the mean volatility size, about 4% for any base volatility.
- The overestimate of the Black-Scholes value for peaks occurring during the averaging period is also independent of the mean volatility size, about 4% for any base volatility.

ITM:

Spot S=110, Strike K=100, r=4%, T=30 days, Avg. period=last 7 days, Length of volatility peak=2 days

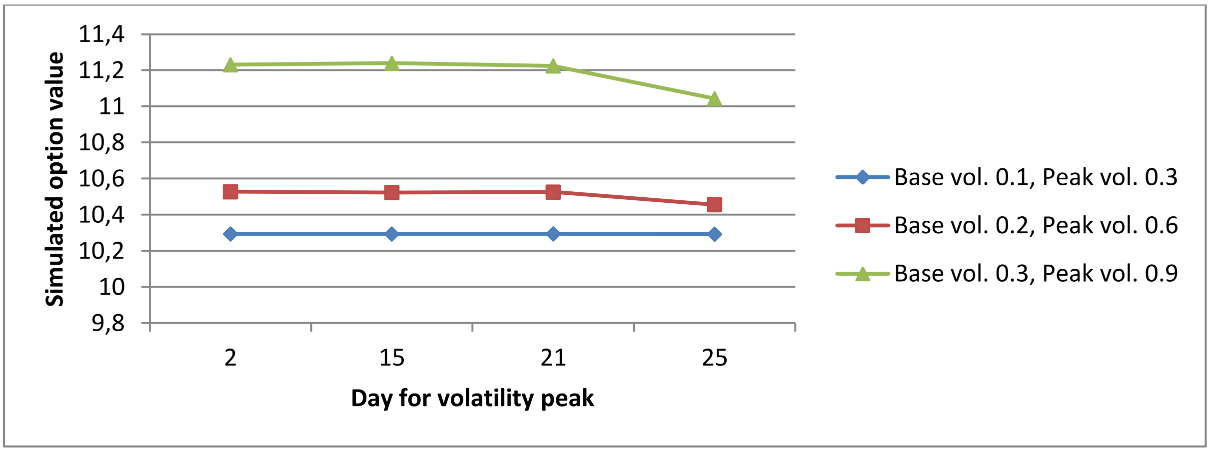

Fig. 14

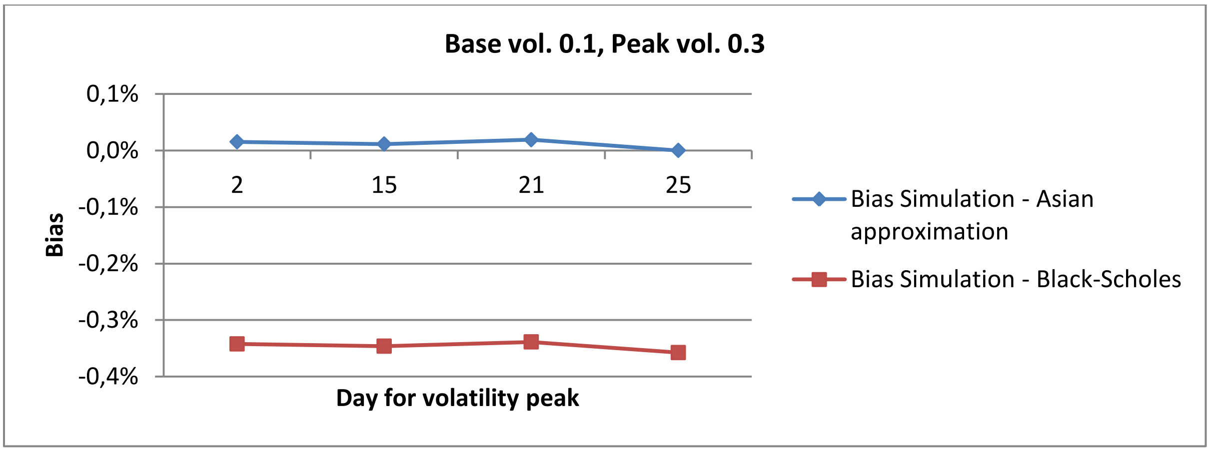

Fig. 15a

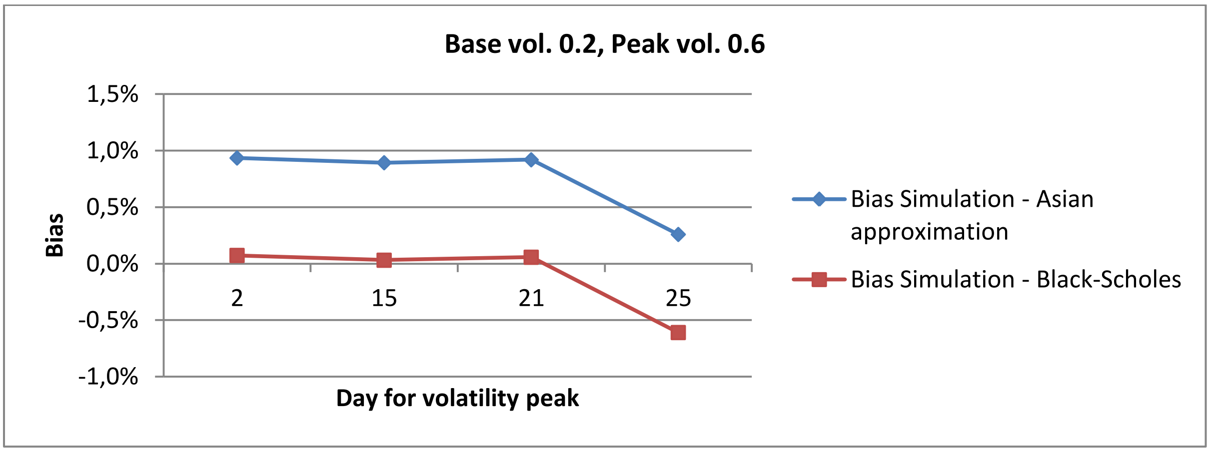

Fig.15b

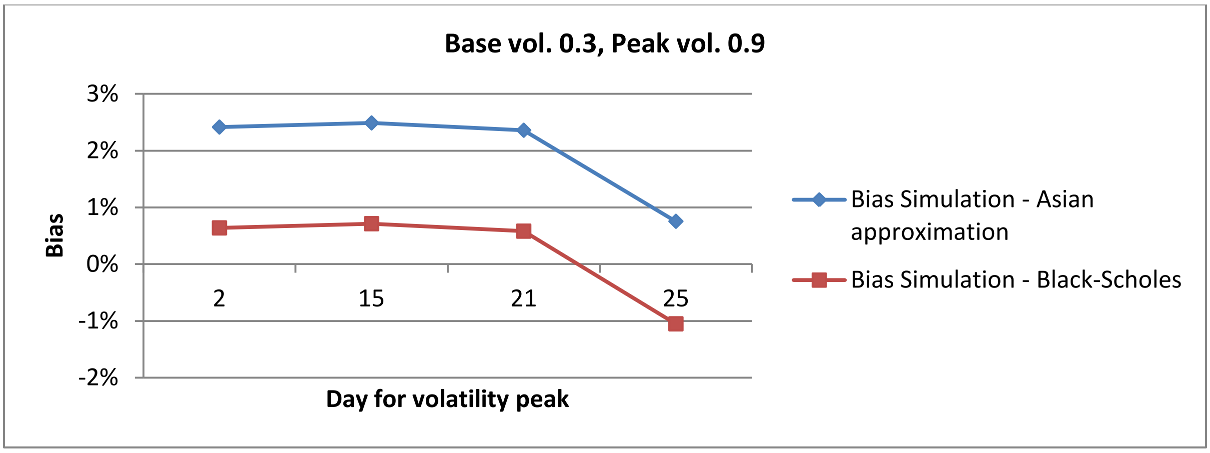

Fig. 15c

Base vol.	Peak vol.	Day of peak	Simulation price	95% Confidence Interval of simulated price: Lower bound	95% Confidence Interval of simulated price: Upper bound	Asian approximation price	Black-Scholes price
10 %	**30 %**	2	10,2941	10,2934	10,2948	10,2925	10,3294
		15	10,2937	10,2931	10,2944		
		21	10,2945	10,2938	10,2952		
		25	10,2926	10,2920	10,2931		
15 %	**45 %**	2	10,3472	10,3443	10,3502	10,3171	10,3689
		15	10,3461	10,3432	10,3490		
		21	10,3443	10,3414	10,3472		
		25	10,3247	10,3225	10,3270		
20 %	**60 %**	2	10,5276	10,5209	10,5343	10,4291	10,5198
		15	10,5231	10,5165	10,5298		
		21	10,5259	10,5192	10,5326		
		25	10,4562	10,4507	10,4617		
25 %	**75 %**	2	10,8237	10,8127	10,8348	10,6489	10,7918
		15	10,8314	10,8202	10,8425		
		21	10,8322	10,8211	10,8434		
		25	10,7001	10,6907	10,7096		
30 %	**90 %**	2	11,2310	11,2151	11,2470	10,9599	11,1595
		15	11,2394	11,2233	11,2554		
		21	11,2246	11,2087	11,2405		
		25	11,0431	11,0292	11,0570		

Table 13

Key observations:

- For simulated options values the time of the volatility peak is significant to the option value, but much less significant as for OTM and ATM options.
- For volatility peaks occurring prior to the averaging period, it does not seem to matter when the peak occurs as long as it is before the averaging period. Even though there exists significant differences between some of the option values with peaks prior to the averaging period, the values are very close to the boundaries of the confidence intervals.
- If the peak occurs during the averaging period, the option value drops, but not at all drastic and by much less than for ATM options.
- The relative drop in option value if the peak occurs during the average period is depended of the mean volatility size and increases with increasing volatility. The option value drops between 0% to 1,5% for base volatilities of 10% to 30%.
- The Asian approximation formula underestimates the option value when there is a volatility peak, even when the peak occurs during the averaging period, but not by as much as for ATM options and by much less than for OTM options.

- The underestimate of the Asian approximation value increases with increasing mean volatility, from about 0,2% to 2,5% if the peak occurs prior to the averaging period, and from about 0% to 0,8% if the peak occurs during the averaging period, for base volatilities from 10% to 30%.
- The Black-Scholes formula overestimates the option value for small mean volatilities and underestimates the value for larger mean volatilities when the volatility peak occurs prior to the averaging period. Although significant, the over- and underestimates are very small and ranges between 0% to 0,5%
- When the peak occurs during the averaging period, the Black-Scholes formula overestimates the value for all base volatilities from 10% to 30%.
- The overestimate of the Black-Scholes formula increases with increasing mean volatility from about 0,5% to 1% for base volatilities from 10% to 30%.

Stochastic volatility

OTM:

Spot S=90, Strike K=100, r=4%, T=30 days, Avg. period=last 7 days, a=10

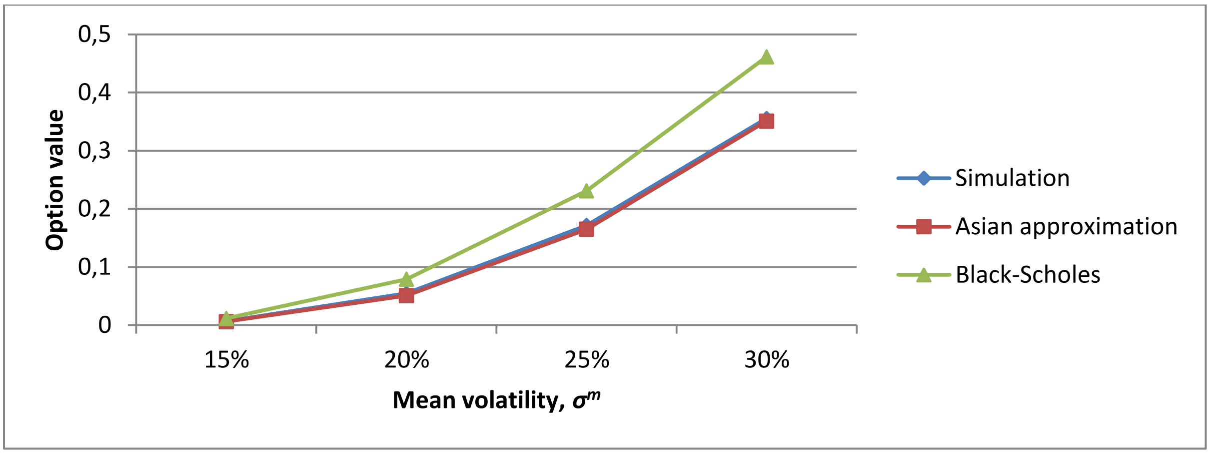

Fig. 16a

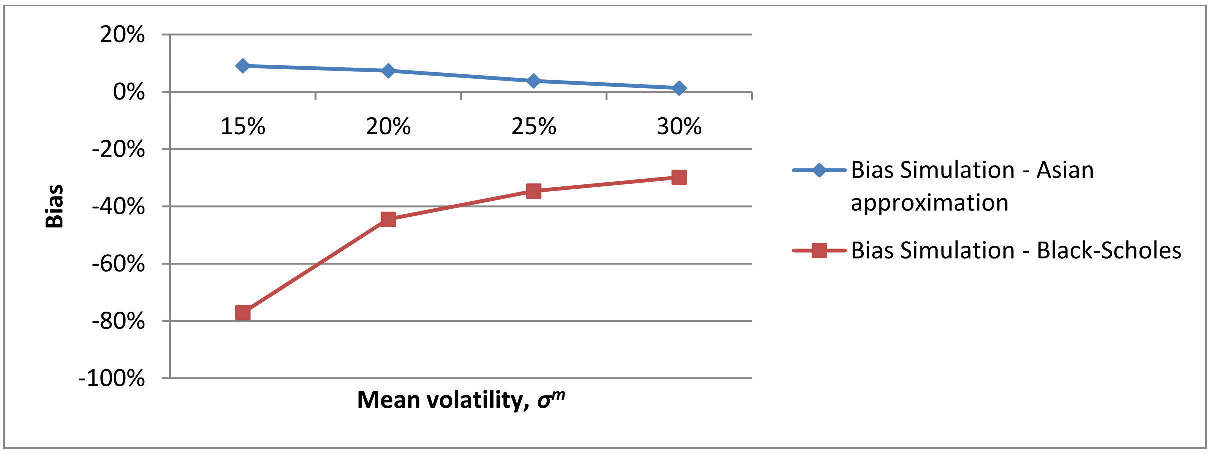

Fig. 16b

			95% Confidence Interval of simulated price			
d* in Hull-White process	**Mean volatility**	**Simulation**	**Lower bound**	**Upper bound**	**Asian approximation price**	**Black-Scholes price**
0,15	**14,93 %**	0,0065	0,0060	0,0071	0,0059	0,0115
0,2	**19,89 %**	0,0547	0,0527	0,0567	0,0507	0,0791
0,25	**24,87 %**	0,1715	0,1674	0,1755	0,1650	0,2309
0,3	**29,85 %**	0,3552	0,3487	0,3618	0,3507	0,4613

Table 14

Key observations:

- The Asian approximation formula underestimates the option value significantly for low mean volatilities. The Asian approximation values are outside the 95% confidence interval of the simulated value for mean volatilities from 10% to 25%, but inside the confidence interval for a mean volatility of 30%.

- The underestimate of the Asian approximation formula decreases with increasing volatility, from about 9% to 1% for mean volatilities of 15% to 30%.
- The Black-Scholes formula overestimates the option value for any mean volatility size.
- The overestimate of Black-Scholes decreases with increasing mean volatility, from about 80% to 30% for mean volatilities from 15% to 30%.

ATM:

Spot S=K*exp(-rT), Strike K=100, r=4%, T=30 days, Avg. period=last 7 days

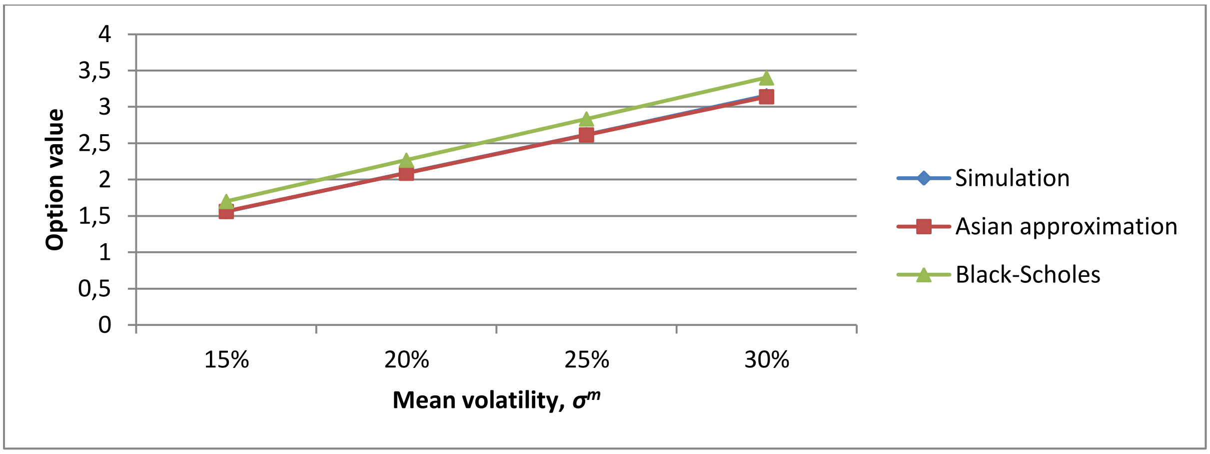

Fig. 17a

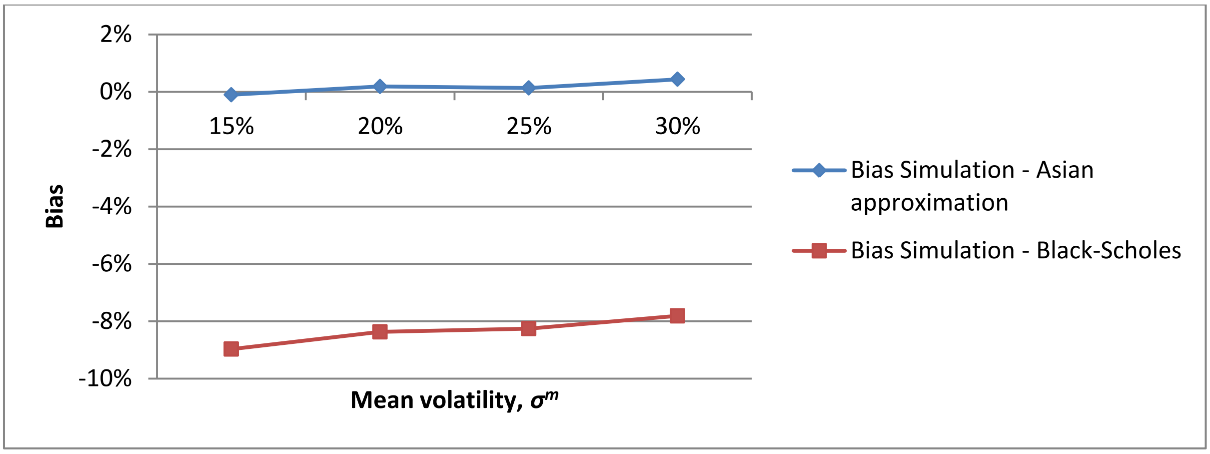

Fig. 17b

d* in Hull-White process	Mean volatility	Simulation	95% Confidence Interval of simulated price Lower bound	 Upper bound	Asian approximation price	Black-Scholes price
0,15	**14,92 %**	1,5611	1,5533	1,5688	1,5628	1,7012
0,2	**19,90 %**	2,0929	2,0824	2,1035	2,0891	2,2682
0,25	**24,88 %**	2,6193	2,6060	2,6326	2,6159	2,8356
0,3	**29,84 %**	3,1541	3,1379	3,1703	3,1404	3,4007

Table 15

Key observations:

- There is no significant difference between the simulated option value and the value of the Asian approximation formula, independent of volatility size. The Asian approximation value is covered by the 95% confidence interval of simulated value.
- The Black-Scholes formula overestimates the option value for any mean volatility size.
- There is no apparent correlation between the size of the overestimate of Black-Scholes formula and mean volatility size, about 8% for mean volatilities between 15% to 30%

ITM:

Spot S=110, Strike K=100, r=4%, T=30 days, Avg. period=last 7 days

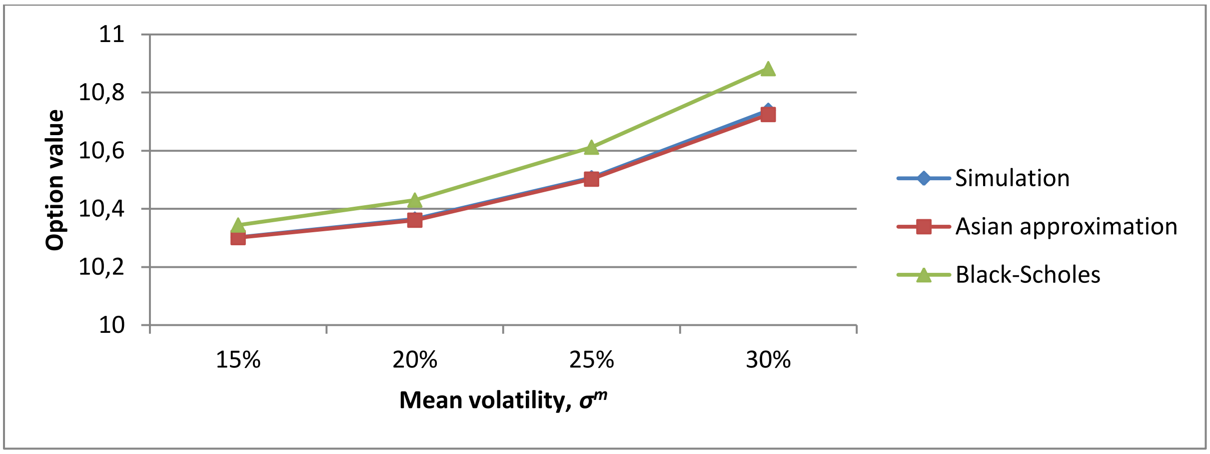

Fig. 18a

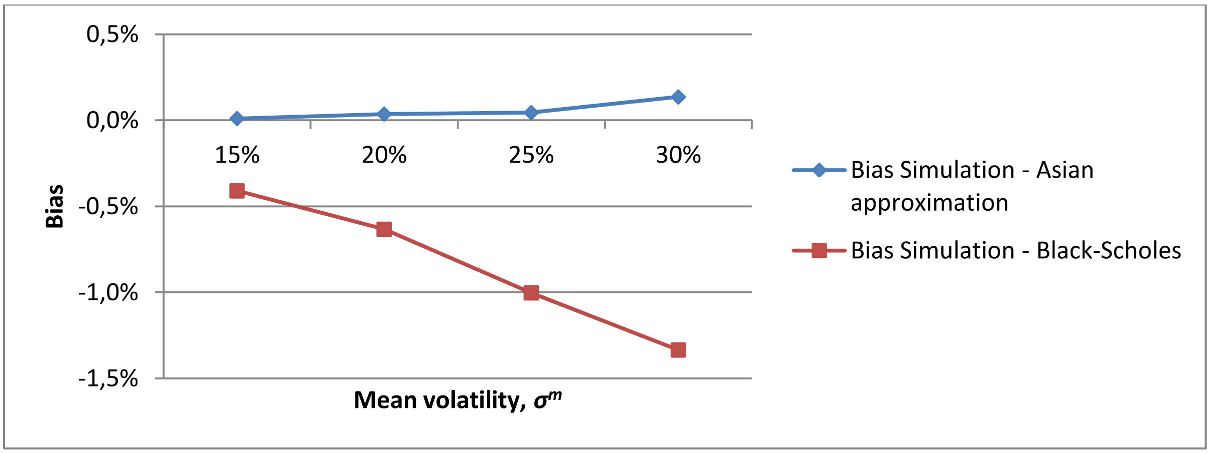

Fig. 18b

			95% Confidence Interval of simulated price			
d* in Hull-White process	**Mean volatility**	**Simulation**	**Lower bound**	**Upper bound**	**Asian approximation price**	**Black-Scholes price**
0,15	**14,92 %**	10,3021	10,3008	10,3034	10,3011	10,3444
0,2	**19,90 %**	10,3646	10,3611	10,3680	10,3608	10,4302
0,25	**24,86 %**	10,5067	10,5002	10,5131	10,5020	10,6120
0,3	**29,85 %**	10,7389	10,7289	10,7488	10,7243	10,8822

Table 16

Key observations:

- There is no significant difference between the simulated option value and the value of the Asian approximation formula for low mean volatilities. Asian approximation value covered by the 95% confidence interval of simulated value for volatilities from 15% to 25%.
- The Asian approximation formula underestimates the option value for higher mean volatility. The Asian approximation value is below the 95% confidence interval of simulated value for mean volatility of 30%.
- The Black-Scholes formula overestimates the option value for any mean volatility, but by much less than for OTM and ATM options.
- The overestimate of the Black-Scholes formula increases with increasing mean volatility, from about 0,5% to 1,5% for mean volatilities from 15% to 30%.

Conclusions

”Out of The Money” Asian options

The Asian approximation formula presented in this thesis works well for OTM Asian options only if volatility is assumed constant or changing from lower to higher volatility during a fairly long period covered by the average period of the option.

If volatility is stochastic, changing from higher to lower volatility during a period covered by the average period of the option, the Asian approximation formula will underestimate the option value. If a volatility peak occurs at any time during the option lifetime the Asian approximation formula will also underestimate the option value.

The reason for these results is because just using the average volatility does not seem to capture the full effect a period of high volatility has on the option value, if this effect is not smoothed out by the average period. Even if a short time volatility peak is smoothed out by the average period of the option, the effect is not captured by the average volatility since a short period of time does not have enough impact on the average volatility calculation.
Another interesting result is that the underestimate of OTM options by the Asian approximation formula decreases with increasing mean volatility. This means that for OTM options, higher mean volatility makes the value of the Asian approximation less sensitive to volatility changes.

As expected, the Black-Scholes formula overestimates the option value in most volatility scenarios. This is expected since the Black-Scholes formula implicitly only considers the stock price at the day of maturity. Since the stochastic process followed by the stock price has a positive drift, the stock price at maturity is higher, in average, than the average price of the, in this case, last week.

In one of the examined volatility scenarios the Black-Scholes formula underestimates the option value. This occurs when the volatility changes from higher to lower during a period covered by the average period of the option. The reason for this is the same as for the underestimate by the Asian approximation formula; the average volatility does not capture the full effect a period of high volatility has on the option value. Since even the Black-Scholes formula underestimates the option value in this case, it seems like this volatility scenario has large impact on the option value.

As for the Asian approximation formula, both the over- and underestimate of OTM option by the Black-Scholes formula decreases with increasing mean volatility. The same conclusion holds; for OTM options, higher mean volatility makes the value of the Black-Scholes formula less sensitive to volatility changes.

“At The Money” Asian options

For ATM Asian options the Asian approximation formula works better than for OTM Asian options. The Asian approximation works well when volatility is constant, changing from lower to higher volatility during a fairly long period covered by the average period of the option, as for OTM options. But for ATM options the Asian approximation formula also works well when volatility is stochastic.

If volatility is changing from higher to lower volatility during a period covered by the average period of the option, the Asian approximation formula underestimates ATM options. Also if a volatility peak occurs at any time during the option lifetime the Asian approximation formula will underestimate the ATM option value. The underestimates of ATM options by the Asian approximation are by much less than for OTM options.

This means that also for ATM options just taking the average volatility does not capture the full effect of the volatility changes, as for OTM options. The reason for this is the same as for OTM options, but the underestimates are much smaller.
Unlike OTM options, the underestimate by the Asian approximation formula is indifferent to the size of the mean volatility. This means that for ATM Asian options the Asian approximation is equally sensitive to volatility changes regardless of the size of the mean volatility.

The Black-Scholes formula overestimates ATM Asian options if volatility is constant, stochastic, changing from lower to higher volatility during a period covered by the average period of the option or if a volatility peak occurs during the average period of the option. The reason for this is the same as for OTM options when the Black-Scholes formula overestimates the options, but the overestimates of ATM options are much smaller than for OTM options.

If volatility is changing from higher to lower volatility during a period covered by the average period of the option or if a volatility peak occurs prior to the average period of the option, the Black-Scholes formula underestimates the ATM option value. This is explained in the same way as for the underestimates of OTM options, but the underestimates are much smaller than for OTM options.

Like the underestimates of ATM options by the Asian approximation formula, the over- and underestimates by the Black-Scholes formula are indifferent to the size of the mean volatility. This means that for ATM Asian options the Black-Scholes formula is equally sensitive to volatility changes regardless of the size of the mean volatility.

A possible reason why the under- and overestimates of ATM options by both the Asian approximation and Black-Scholes formulas are smaller than for OTM options, is because the relative, percentage, change of the option value when volatility increases is much less than for OTM options. That is, the relative slope of the volatility-option value curve is much less steep. This makes the ATM options less sensitive to volatility changes than OTM options. The cumulative average growth of the OTM option value when volatility increases in 10%-steps from 10% to 60%, is about 850%. Corresponding growth for ATM options is 43%.
A possible reason why the under-and overestimates of ATM options by both the Asian approximation and Black-Scholes formulas are indifferent to the mean volatility size, is because for ATM options the value increases linearly with increasing volatility. For OTM options the value increases exponentially with increasing volatility, making the option value more differently sensitive to volatility changes depending on volatility size.

"In The Money" Asian options

For ITM Asian options the Asian approximation formula works even better than for ATM Asian options. The Asian approximation works well when volatility is constant, changing from lower to higher volatility during a fairly long period covered by the average period of the option, as for ATM options. If volatility is stochastic, the Asian approximation formula works well when the mean volatility is low.

If volatility is stochastic with high mean volatility or changing from higher to lower volatility during a period covered by the average period of the option, the Asian approximation formula underestimates ITM options. Also if a volatility peak occurs at any time during the option lifetime the Asian approximation formula will underestimate the ITM option value. The underestimates of ITM options by the Asian approximation are significant but very small, much smaller than for ATM options.

In contrary to OTM options, the underestimate of ITM options by the Asian approximation formula increases with increasing mean volatility. This means that for ITM options, higher mean volatility makes the value of the Asian approximation more sensitive to volatility changes.

The Black-Scholes formula overestimates ATM Asian options if volatility is constant, stochastic, changing from lower to higher volatility during a period covered by the average period of the option or if a volatility peak occurs during the average period of the option. The reason for this is the same as for OTM and ATM options when the Black-Scholes formula overestimates the options, but the overestimates of ITM options are much smaller than for ATM options.

If volatility is changing from higher to lower volatility during a period covered by the average period of the option or if a volatility peak occurs prior to the average period of the option, the Black-Scholes formula both over- and underestimates the ITM option value. When the mean volatility increases in these volatility scenarios, the Black-Scholes formula goes from overestimating to underestimating. The correlation between mean volatility and the bias of Black-Scholes formula is the same as for the Asian approximation of ITM options.

A possible reason why the under- and overestimates of ITM options by both the Asian approximation and Black-Scholes formulas are smaller than for OTM and ATM options, is the same as for ATM options relative OTM options. That is, the relative slope of the volatility-option value curve is much less steep for ITM options than even for ATM. This makes the ITM options less sensitive to volatility changes than OTM and ATM options. The cumulative average growths of the OTM and ATM option values when volatility increases in 10%-steps from 10% to 60%, are about 850% and 43% respectively. Corresponding growth for ITM options is 5%.

References

[1] Hull John C. *Options, futures and other derivatives, 6th ed.* Pearson Prentice Hall, New Jersey, 2005

[2] Zhang Peter G. *Exotic options: a guide to second generation options, 2th ed.* World Scientific Publishing, Singapore, 1998

[3] Djehiche Boualem. *Stochastic Calculus. An introduction with Applications.* Royal Institute of Technology, Stockholm.

[4] Curran Michael. *Valuing Asian and Portfolio Options by Conditioning on the Geometric Mean Price.* The Institute of Management Sciences, Management Science/Vol. 40, No. 12, 1994

[5] Hull John C and White Alan. *The Pricing of Options on Assets with Stochastic Volatilities.* The Journal of Finance, Vol. 42, Issue 2, 281-300, 1987

[6] Gut Alan. *An Intermediate Course in Probability.* Springer-Verlag, New York, 1995