Risk assessment of portfolios of exotic derivatives

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April 10, 2012

Abstract

Trading with derivatives is getting increasingly popular. A consequence is that the risk of the portfolio becomes less transparent and more difficult to evaluate. Simple derivatives like European options are easily priced using the Black-Scholes formula. However, when the derivatives are path-dependent, finding closed-form expressions gets a lot trickier and you have to rely on approximations or even simulations to price them. As the portfolios are getting larger the computational cost becomes an issue. In this thesis we attempt to find a model that is accurate while still maintaining a low computational cost. We estimate Value-at-Risk and Expected Shortfall using Monte-Carlo Simulation. Estimating the risk factors is always a challenge. We test three different methods and evaluate their performance using a simple backtest.

Acknowledgements

I would like to thank my supervisor at KTH Henrik Hult for valuable feedback and guidance. I would also like to thank my supervisors at Carnegie, Kristoffer von Freymann and Magnus Ekdahl for very interesting ideas and discussions.

Stockholm, April 2012
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1 Contents

2 Introduction

2.1 Background

The purpose of this thesis is to risk evaluate a portfolio with a high concentration of exotic derivatives. Due to the complexity in valuating these derivatives and the large number of products in the portfolio a major challenge is to find appropriate tools to price the components as effectively as possible. In this study we consider three types of computer heavy pricing derivatives. These are American Options, Asian options and Barrier Options. Our portfolios also include more simple derivatives like futures and forwards but the pricing of these are standard and will not be emphasized in the thesis.

2.2 Outline

In Section 3, 5-7 the theoretical background will be explained in detail. Section 3 will introduce the reader to the options treated in this study and explain the pricing methods used. Section 4 gives an overview of the structure of the portfolios that are risk-evaluated. Section 5 describes how we simulate scenarios used to risk evaluate. Section 6 explains how we classify the risk. Section 7 describes the simple backtesting method used to compare the estimations to actual results. In Section 8 the results are given. In Section 9 we draw conclusions from the results obtained in Section 8.

3 Pricing the derivatives

To be able to risk-evaluate our portfolios we need to be able to price the components. Since we have quite large portfolios with rather complex derivatives, pricing them exactly would be very time-consuming. Therefore our pricing methods are chosen to allow us to risk-evaluate as exactly as possible while still having good performance which will allow us to run sufficient number of simulations in a reasonable amount of time.

For all closed form solutions we assume the underlying to follow a geometric Brownian motion:

$$dS = \mu S dt + \sigma S dz$$

Where μ is the expected rate of return of the asset, σ is the variance of the rate of return and dz is a Wiener process. We also assume that the risk-free rate and volatility is constant through the life of the option.

3.1 Pricing American Options

American options, unlike European options, can be exercised any day. This makes them path-dependent which makes exact closed-form solutions impossible to find. Therefore the pricing of American options is usually done numerically. The most common method is to use the Binomial Option Pricing Method ($BOPM$). $BOPM$ is fairly fast and accurate so we will use this method to price our American options.

The theory behind $BOPM$ is the following. We assume that the underlying moves either up or down at each step with some probabilities $P(up)$ and $P(down)$. At each step we evaluate the optimal action, which is to exercise the option or just keep it. Starting at maturity and stepping back n steps, while discounting at every step, we determine the value of the option today.

To determine the amplitude of the movements at each time step we use the Cox, Ross, & Rubinstein (CRR) method.

$$up = e^{\sigma\sqrt{t}}$$

$$down = e^{-\sigma\sqrt{t}} = \frac{1}{up}$$

This method makes the tree recombinant, i.e. an "up and then down movement" will end up at the same value as a "down and then up movement", which is a nice feature since it speeds up computations.

The probabilities of up and down movements are chosen in accordance with the no arbitrage assumption, giving us the following probabilities:

$$p_{up} = \frac{e^{(r-q)\Delta t} - down}{up - down}$$

$$p_{down} = 1 - p_{up}$$

The number of binomial steps (n) is chosen depending how much accuracy and performance preferred. Larger n will give better accuracy at the expense of computing time. In this study we are using $n = 10$ because we have a large portfolio and we need to limit the computational cost.

3.2 Pricing Barrier Options

Barrier options come in many different forms. The barrier options in our portfolio are options that are either activated or de-activated when hitting a specified barrier level. After being activated, the option will behave as a standard European option with corresponding specifications. Usually there is a rebate that is paid out if the option never get activated (knocked in) during its life. We are treating four different types of barrier options, each type can be either a put or a call, in this study:

Up & In (uic and uip)
Up & Out (uoc and uop)
Down & In (dic and dip)
Down & Out (doc and dop)

Denote the price of the Up & In call option as $\boldsymbol{c_{up-in}}$ **(**the price of an Up & In put option will then be denoted $\boldsymbol{p_{up-in}}$) and denote the other option types in a similar way. Then we will get, in the Black-Scholes framework, the following formulas (Reiner & Rubinstein, 1991) when the barrier level is lower or equal to the strike:

$$(\boldsymbol{c_{up-in}} \mid \boldsymbol{H} \leq \boldsymbol{X}) = c_{BS}$$

$$(\boldsymbol{c_{up-out}} \mid \boldsymbol{H \le X}) = c_{BS} = \left(c_{up-in} \,\middle|\, H > X\right)$$

$$(\boldsymbol{c_{down-in}} \mid \boldsymbol{H \le X}) = Se^{-DT}\left(\frac{H}{S}\right)^{2m} N(y) - Xe^{-rT}\left(\frac{H}{S}\right)^{2m-2} N(y - \sigma\sqrt{T})$$

$$(\boldsymbol{c_{down-out}} \mid \boldsymbol{H \le X}) = c_{BS} - (c_{down-in} \mid H \le X)$$

$$\begin{aligned}(\boldsymbol{p_{up-out}} \mid \boldsymbol{H \le X}) \\ &= Se^{-DT}N(-x_1) + Xe^{-rT}N\left(x_1 + \sigma\sqrt{T}\right) + Se^{-DT}\left(\frac{H}{S}\right)^{2m} N(-y_1) \\ &- Xe^{-rT}\left(\frac{H}{S}\right)^{2m-2} N(-y_1 - \sigma\sqrt{T})\end{aligned}$$

$$(\boldsymbol{p_{down-out}} \mid \boldsymbol{H \le X}) = p_{BS} - (p_{down-in} \mid H \le X)$$

$$\begin{aligned}(\boldsymbol{p_{down-in}} \mid \boldsymbol{H \le X}) \\ &= -Se^{-DT}N(-x_1) + Xe^{-rT}N\left(-x_1 + \sigma\sqrt{T}\right) + Se^{-DT}\left(\frac{H}{S}\right)^{2m} [N(y) \\ &- N(y_1)] - Xe^{-rT}\left(\frac{H}{S}\right)^{2m-2} [N\left(y - \sigma\sqrt{T}\right) - N\left(y_1 - \sigma\sqrt{T}\right)]\end{aligned}$$

$$(\boldsymbol{p_{up-in}} \mid \boldsymbol{H \le X}) = p_{BS} - \left(p_{up-out} \,\middle|\, H \le X\right)$$

Where X is the strike price, H is the barrier value, S is the asset price, r the risk-free rate of return, σ is the volatility, T is the time to maturity, D the dividend yield (which we set to zero since we use discrete dividends), c_{BS} and p_{BS} are the prices of a European call and put option respectively under the Black-Scholes framework.

When the barrier is greater than the strike price:

$$(\boldsymbol{c_{up-out}} \mid \boldsymbol{H > X}) = c_{BS}$$

$$(\boldsymbol{c_{down-out}} \mid \boldsymbol{H} > \boldsymbol{X}) = Se^{-DT}N(x_1) - Xe^{-rT}N\left(x_1 - \sigma\sqrt{T}\right) - Se^{-DT}\left(\frac{H}{S}\right)^{2m}N(y_1) + Xe^{-rT}\left(\frac{H}{S}\right)^{2m-2}N(y_1 - \sigma\sqrt{T})$$

$$(\boldsymbol{c_{down-in}} \mid \boldsymbol{H} > \boldsymbol{X}) = c_{BS} - (c_{down-out} \mid H > X)$$

$$(\boldsymbol{c_{up-in}} \mid \boldsymbol{H} > \boldsymbol{X}) = Se^{-DT}N(x_1) - Xe^{-rT}N\left(x_1 - \sigma\sqrt{T}\right) - Se^{-DT}\left(\frac{H}{S}\right)^{2m}[N(-y) - N(-y_1)] + Xe^{-rT}\left(\frac{H}{S}\right)^{2m-2}\left[N\left(-y + \sigma\sqrt{T}\right) - N\left(-y_1 + \sigma\sqrt{T}\right)\right]$$

$$(\boldsymbol{p_{down-in}} \mid \boldsymbol{H} > \boldsymbol{X}) = p_{BS}$$

$$(\boldsymbol{p_{down-out}} \mid \boldsymbol{H} > \boldsymbol{X}) = 0$$

$$(\boldsymbol{p_{up-in}} \mid \boldsymbol{H} > \boldsymbol{X}) = -Se^{-DT}\left(\frac{H}{S}\right)^{2m}N(-y) + Xe^{-rT}\left(\frac{H}{S}\right)^{2m-2}N(-y + \sigma\sqrt{T})$$

$$(\boldsymbol{p_{up-out}} \mid \boldsymbol{H} > \boldsymbol{X}) = p_{BS} - \left(p_{up-in} \mid H > X\right)$$

Where:

$$y = \frac{\ln\left(\frac{H^2}{SX}\right)}{\sigma\sqrt{T}} + m\sigma\sqrt{T}, \qquad m = \frac{r - D + 0.5\sigma^2}{\sigma^2}$$

$$x_1 = \frac{\ln\left(\frac{S}{H}\right)}{\sigma\sqrt{T}} + m\sigma\sqrt{T}, \qquad y_1 = \frac{\ln\left(\frac{H}{S}\right)}{\sigma\sqrt{T}} + m\sigma\sqrt{T}$$

For coding versatility I will use a generalization of these formulas provided by Haug (1998). Reiner & Rubinstein and Haug formulas are equivalent and merely presented in a different way.

$$A = \phi S e^{-DT} N(\phi x_1) - \phi X e^{-rT} N(\phi x_1 - \phi\sigma\sqrt{T})$$

$$B = \phi S e^{-DT} N(\phi x_2) - \phi X e^{-rT} N(\phi x_2 - \phi\sigma\sqrt{T})$$

$$C = \phi S e^{-DT} \left(\frac{H}{S}\right)^{2(m+1)} N(\eta y_1) - \phi X e^{-rT} \left(\frac{H}{S}\right)^{2m} N(\eta y_1 - \eta\sigma\sqrt{T})$$

$$D = \phi S e^{-DT} \left(\frac{H}{S}\right)^{2(m+1)} N(\eta y_2) - \phi X e^{-rT} \left(\frac{H}{S}\right)^{2m} N(\eta y_2 - \eta\sigma\sqrt{T})$$

$$E = K e^{-rT} \left[N\left(\eta x_2 - \eta\sigma\sqrt{T}\right) - \left(\frac{H}{S}\right)^{2m} N(\eta y_2 - \eta\sigma\sqrt{T})\right]$$

$$F = K \left[\left(\frac{H}{S}\right)^{m+\lambda} N(\eta z) + \left(\frac{H}{S}\right)^{m-\lambda} N(\eta z - 2\eta\lambda\sigma\sqrt{T})\right]$$

Where:

$$\lambda = \sqrt{m^2 + \frac{2(r-D)}{\sigma^2}}, \qquad m = \frac{r - D + 0.5\sigma^2}{\sigma^2}, \qquad z = \frac{\ln\left(\frac{H}{S}\right)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}$$

$$x_1 = \frac{\ln\left(\frac{S}{X}\right)}{\sigma\sqrt{T}} + (1+m)\sigma\sqrt{T}, \qquad x_2 = \frac{\ln\left(\frac{S}{H}\right)}{\sigma\sqrt{T}} + (1+m)\sigma\sqrt{T}$$

$$y_1 = \frac{\ln\left(\frac{H^2}{SX}\right)}{\sigma\sqrt{T}} + (1+m)\sigma\sqrt{T}, \qquad y_2 = \frac{\ln\left(\frac{H}{S}\right)}{\sigma\sqrt{T}} + (1+m)\sigma\sqrt{T}$$

K is the rebate being paid out if the option has not been activated (knocked in) during its lifetime.

With these formulas we can obtain the price of the different option types according to Table 1.

TABLE 1

X<H	Value	η	ϕ
Down-In Call	A-B+D+E	1	1
Down-Out Call	B-D+F	1	1
Up-In Call	B-C+D+E	-1	1
Up-Out Call	A-B+C-D+F	-1	1
Down-In Put	A+E	1	-1
Down-Out Put	F	1	-1
Up-In Put	C+E	-1	-1
Up-Out Put	A-C+F	-1	-1
X>H	**Value**	η	ϕ
Down-In Call	C+E	1	1
Down-Out Call	A-C+F	1	1
Up-In Call	A+E	-1	1
Up-Out Call	F	-1	1
Down-In Put	B-C+D+E	1	-1
Down-Out Put	A-B+C-D+F	1	-1
Up-In Put	A-B+D+E	-1	-1
Up-Out Put	B-D+F	-1	-1

Here we have assumed we check for a barrier breach continuously. In reality this checking interval is discrete but since we check for barrier breaches every single "market tick" it is approximately equivalent as a continuous checking.

3.3 Pricing Asian Options

Asian options are similar to European options, but with the difference that the payoff is determined not by the underlying asset, but instead on an average value of the underlying asset.

Payoff for Asian call option: $\max(0, S_{average} - K)$

Payoff for Asian put option: $\max(0, K - S_{average})$

$S_{average}$ is calculated by checking the value of the underlying at a number of specified dates(reset dates) and calculating the arithmetic average of these values.

Asian options are clearly path-dependent and are a bit tricky to value since it's hard to find a good approximation and BOPM is not suitable for pricing method. Therefore we use Monte-Carlo-Simulation to price them. The simulation approach will yield an accurate price at the cost of computational time. The simulations are carried out similar to the method described in section 4.

4 The Portfolios

In this study we are going to evaluate two portfolios which have different characteristics. We will call them Portfolio 1 and Portfolio 2. Both portfolios consist of derivatives with underlying assets traded on the Swedish stock market. The underlying assets are usually stocks included in OMXS30 (the index itself is also an underlying asset for some derivatives), but there are also some other stocks which makes a total of roughly 50 underlying assets.

4.1 Portfolio 1

This portfolio consists of more than 500 derivatives. The derivatives in this portfolio are a mixture of American options, forward contracts and equities.

4.2 Portfolio 2

This portfolio is smaller than portfolio 1 since it contains less than 500 derivatives. On the other hand this portfolio consists of more complex path-dependent derivatives such as Asian options and barrier options.

5 Monte Carlo Simulations

To evaluate the risk of these portfolios we use Monte-Carlo-Simulation. The underlying assets are simulated 10 000 times based on historical returns and volatilities. A decay model for historic volatilities is also implemented. For simplicity and for computation effectiveness we assume the underlying assets to be independent, though not accurate it is widely assumed in computations. This will yield 10 000 different scenarios which we will use to determinate the risk measures of the portfolio. Note that when estimating historical returns and volatilities we use data up to one year old, which means we use $n = 250$.

5.1 Standard volatility model

The most simple volatility estimation method, sometimes referred to as the Equally Weighted Moving Average (MA) estimator, where you treat all the data equally regardless of how old they are. The estimator is

$$MA_t = \frac{1}{n}\sum_{i=0}^{n-1}(R_{t-i} - \bar{R})^2$$

Where $\bar{R}$ is the expected return and R_t is the logreturn at time t.

5.2 Decay volatility model

The decay model, also known as the Exponential Weighted Moving Average (EWMA), is a refined estimator where recent observations are given more weight than older observations.

$$EWMA_t = (1 - \lambda)\sum_{i=1}^{n}\lambda^i R_{t-i}^2$$

Where λ is the decay factor which we set to the standard value 0.94 since we have daily data.

5.3 Historical simulation

Historical simulation is using past returns to simulate future returns. This is done by storing past underlying returns, we have access to data a year back so we have 250 scenarios, and randomly drawing a day from history and use those returns in the simulation. This procedure is repeated n number of times. The advantage of this method is that we can simulate the unknown parameters, including correlation, in a simple way.

A drawback of this method is that only past returns can occur in the future, which of course is unrealistic especially with the relative small sample size of historical returns we use.

6 Risk Measures

To determine the risk evaluate a portfolio we need to define the risk measures we are using. In this study we will use the standard risk measures Value at Risk (VaR) and Expected Shortfall (ES, $CVaR$). To get a sense of the correctness of our risk values we will also evaluate the uncertainty of the risk measures.

6.1 Value at Risk (VaR)

Value at Risk is defined as the threshold value where a loss will exceed that threshold with a certain probability. For instance, imagine you own a portfolio, define the one-day VaR and choose the confidence level 95%. Then the probability that your portfolio loss over one day will exceed that value is 5%. The time horizon and the confidence level can be chosen arbitrarily but the most common parameters are for confidence levels 95% and 99% and for time horizons one day and ten days.

Furthermore we get the mathematical definition using the following arguments. Define X as the net worth of the portfolio, and then we can define Value at Risk as

$VaR_\alpha(X) = F_L^{-1}(1-\alpha)$, where

$L = -X/(1+r_f)$ is the discounted loss of the portfolio.

Given the fact that we use simulation to obtain the empirical Value at Risk, we conclude that the empirical estimate of $VaR_\alpha(X)$ is given by

$$\widehat{VaR}_\alpha(X) = L_{[n\alpha]+1,n}$$

Where $L_{1,n} \geq \cdots \geq L_{n,n}$ is the ordered sample.

In this study we will calculate VaR with confidence level 95% and 99% using a one day time horizon. We will not calculate ten-day VaR and the reason behind that is that ten day VaR is much more complicated to compute, since the estimators are likely to change during this ten day period, without making a lot of approximations.

Value at Risk quantifies the risk in a single number quite nicely but it is not flawless. The biggest drawback being that it does not capture the risk of extreme scenarios in a satisfying way. This is of course essential in risk management and therefore we will discuss Expected Shortfall in the next section.

6.2 Expected Shortfall (ES, $CVaR$ etc)

Expected Shortfall, also known as Conditional Value at Risk, is a measure that eliminates some of the shortcomings of VaR. Expected shortfall is the average losses of the losses greater than the VaR threshold and is defined as follows. Using the definitions in the previous section Expected Shortfall at level α is defined as:

$$ES_\alpha(X) = \frac{1}{\alpha}\int_{1-\alpha}^{1} {F_L}^{-1}(p)dp$$

Introducing the empirical distribution function F_n we can define the empirical expected shortfall estimator

$$\widehat{ES}_\alpha(X) = \frac{1}{\alpha}\int_{1-\alpha}^{1} {F_n}^{-1}(p)dp$$

Since ${F_n}^{-1}$ is piecewise constant the estimator can be written as

$$\widehat{ES}_\alpha(X) = \frac{1}{\alpha}\left(\sum_{i=1}^{[n\alpha]}\frac{L_{i,n}}{n} + \left(\alpha - \frac{[n\alpha]}{n}\right)L_{[n\alpha]+1,n}\right)$$

Which, if $\alpha = k/n$ for some integer k>1, is reduced to

$$\widehat{ES}_{k/n}(X) = \frac{1}{k}\sum_{i=1}^{k} L_{i,n}$$

The advantage of Expected Shortfall is that it captures heavy tails and is therefore desirable when evaluating portfolios with potentially extreme outcomes.

6.3 Uncertainty of the risk measures

Suppose we have $x_1, x_{2,\ldots,} x_n$ observations from iid random variables $X_1, X_{2,\ldots,} X_n$with the same unknown continuous distribution function F. Suppose we want to construct a confidence interval (a, b) for the quantile $F^{-1}(\alpha)$, where $a = f_a(x_{1,\ldots,} x_n)$ and $b = f_b(x_{1,\ldots,} x_n)$ such that

$$P(A < F^{-1}(\alpha) < B) = p, \;\; P\big(A \geq F^{-1}(\alpha)\big) = P\big(B \leq F^{-1}(\alpha)\big) = (1-p)/2$$

Where p is the confidence level. F is unknown so we can't find a and b. However we can find $i > j$ and the smallest $p' \geq p$ such that

$$P\big(X_{i,n} < F^{-1}(\alpha) < X_{j,n}\big) = p'$$

$$P\left(X_{i,n} \geq F^{-1}(\alpha)\right) \leq (1-p)/2\,, \quad P\left(X_{j,n} \leq F^{-1}(\alpha)\right) \leq (1-p)/2 \qquad (6.1)$$

Where $X_{1,n} \geq \cdots \geq X_{n,n}$ is the ordered sample.

Let $Y_\alpha = \#\{X_k > F^{-1}(\alpha)\}$ i.e. the number of sample points exceeding $F^{-1}(\alpha)$

Y_α is $Binomial(n, 1-\ \alpha)) - distributed$. Note that

$$P\left(X_{1,n} \leq F^{-1}(\alpha)\right) = P(Y_\alpha = 0),$$
$$P\left(X_{2,n} \leq F^{-1}(\alpha)\right) = P(Y_\alpha \leq 1),$$
$$\dots$$
$$P\left(X_{j,n} \leq F^{-1}(\alpha)\right) = P(Y_\alpha \leq j-1),$$

Similarly $P\left(X_{i,n} \geq F^{-1}(\alpha)\right) = 1 - P(Y_\alpha \leq i-1)$. Hence we can compute $P\left(X_{j,n} \leq F^{-1}(\alpha)\right)$ and $P\left(X_{i,n} \geq F^{-1}(\alpha)\right)$ for different i and j until we find indices that satisfy (6.1). This results in the confidence interval $(X_{i,n}\,, X_{j,n})$.

7 Backtesting method

While the risk measures and statistical uncertainty quantifies the risk in a very nice way, it could be interesting to compare the model used with real market outcomes. This is done by using some kind of backtesting method. In this study we will use a pretty straightforward and simple backtest.

The observed returns for the following 100 daily returns are calculated and used when pricing the portfolio. Basically we use the same procedure as when performing the historical simulation with the difference that every scenario is only calculated once. Based on these values the number of Value at Risk breaches and Expected Shortfall are observed and compared with the estimated values. This method will be quite inaccurate since we have to assume that the risk factors are constant during this period and the sample size is small, but it will give a good sense of how accurate the model is with reality.

8 Results

In this section the results of all the simulations will be presented. The results for each portfolio will be presented separately. Also after each return distribution histogram there will be a backtest which compares the estimated risk measures with the future returns calculated as described in section 7.

8.1 Portfolio 1

FIGURE 1

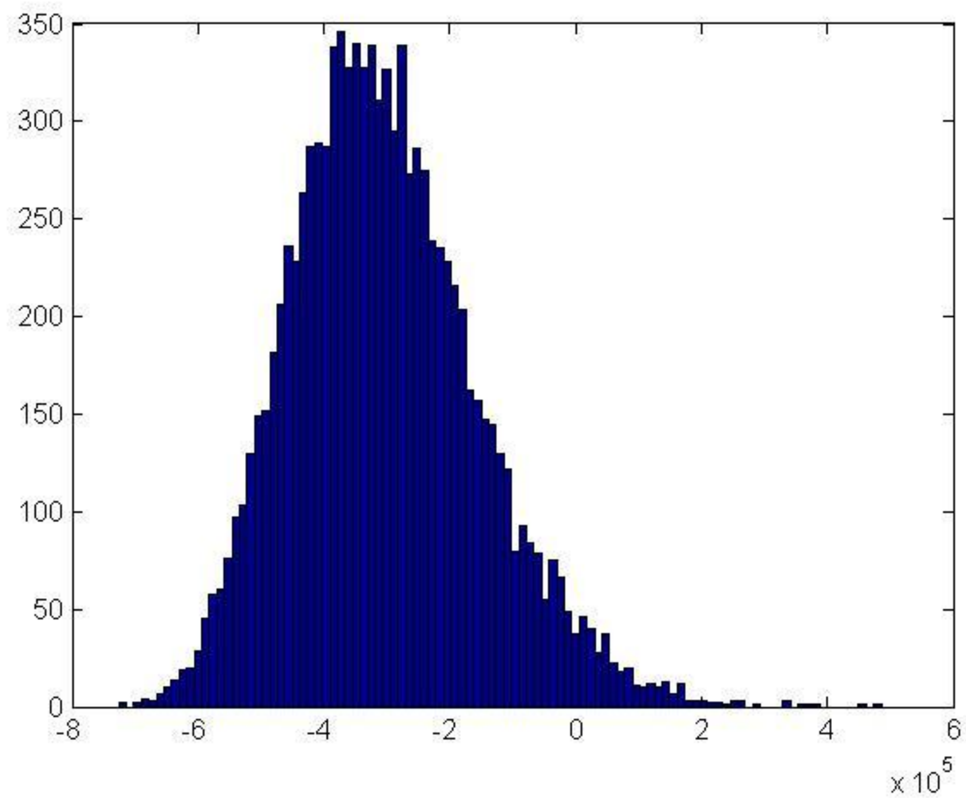


Figure 1 shows the distribution of absolute portfolio returns generated by our Monte Carlo Simulations (N = 10000) when using the standard volatility estimator, where the x-axis shows the absolute returns and the y-axis is the number of outcomes. From this distribution we calculate the risk measures. For the comfort of the reader these measures are displayed in the table below.

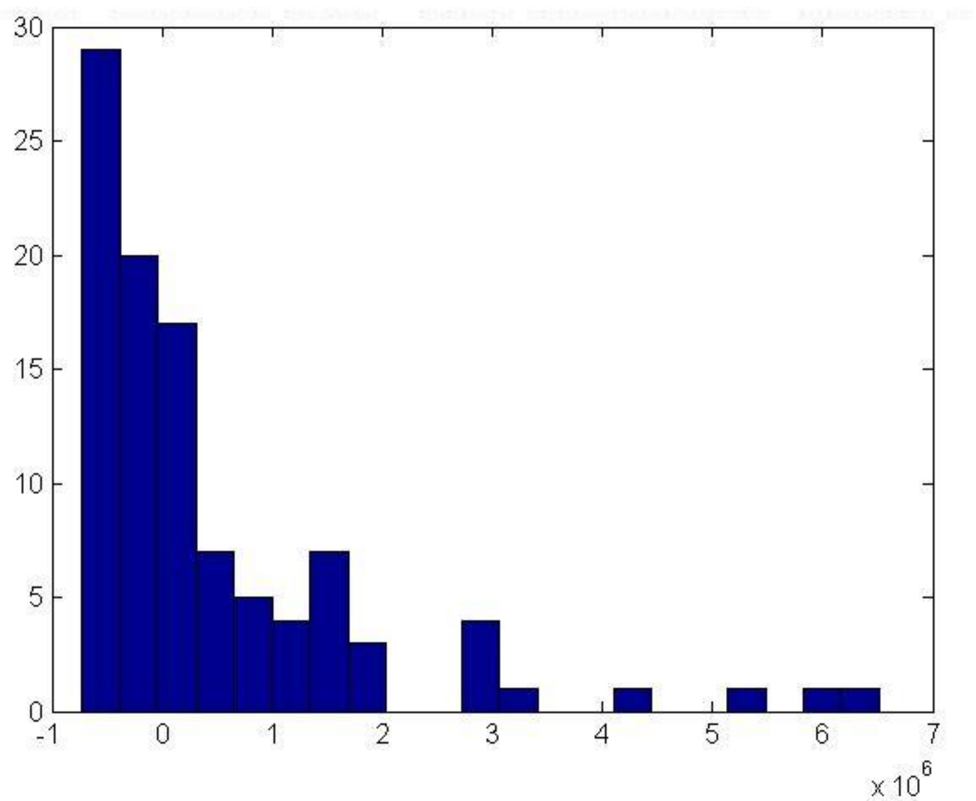
VaR95:	527 621
VaR99:	596 429
ES95:	571 070

ES99:	630 903
Mean return:	-304 633
Standard deviation:	149 146

Using theory in section 6.3 we calculate the estimated confidence intervals for the VaR estimators. For the 95% VaR we get the interval (522 640, 533 600) and doing the same for the 99% VaR we get the interval (589 450, 609 390).

The actual change in portfolio value, based on the market movements the following 100 (which gives us N = 100) market days are presented in figure 2.

FIGURE 2

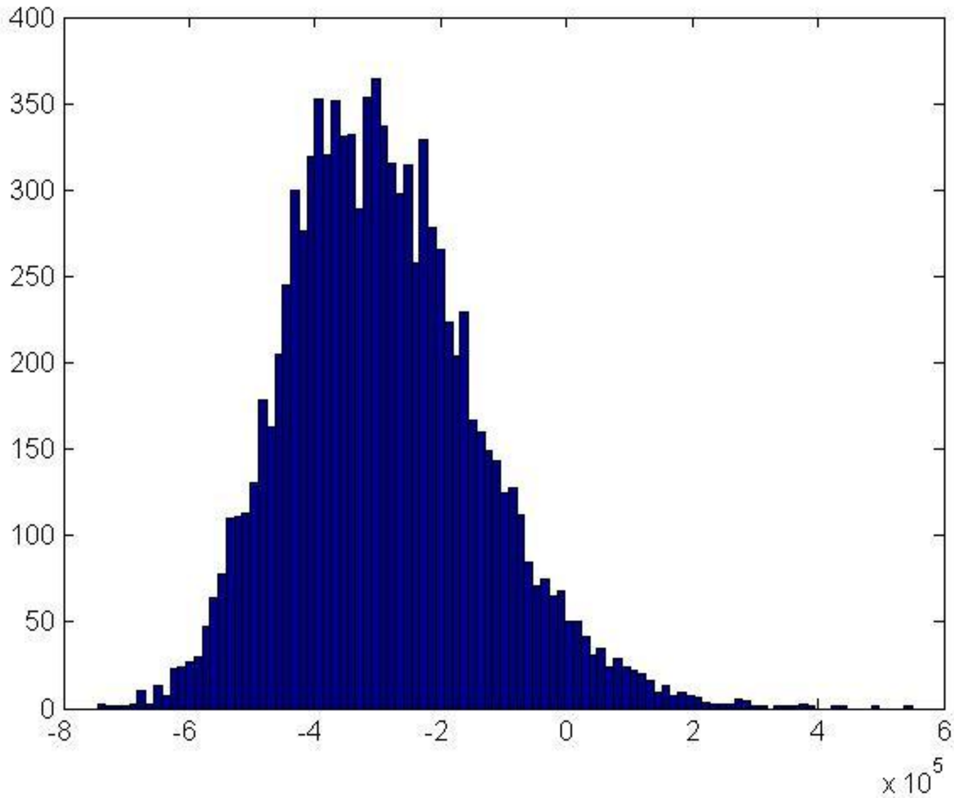


VaR95 breaches:	22
VaR99 breaches:	0
ES95:	552 597
ES99:	
Mean return:	460 595
Standard deviation:	1 370 357

We observe that our model, with this calibration, greatly underestimated the variance of the portfolio. Because the extreme returns were positive our risk measures wasn’t as inaccurate as the variance. With a sample size of 100 we would have expected the VaR95 threshold to be exceeded approximately 5 times compared to the 22 times we observed, while the VaR99 threshold were exceeded 0 times compared to the expected 1 time. The average losses of the VaR95 breaches was 552 597 compared to the 571 070 we expected. Though it should be noted that a sample size of 124 is very small and could yield inaccurate values. The main issue with this configuration is probably the assumption of zero correlation between the individual assets in combination with extreme market conditions during the test period which will make our estimated returns and volatilities quite inaccurate. Obviously there exists some market correlation, which probably is significant when all the underlying assets are traded on the same market as is the case in this portfolio.

Figure 3 shows the outcomes when using the more sophisticated volatility estimator, the decay model introduced in section 5.

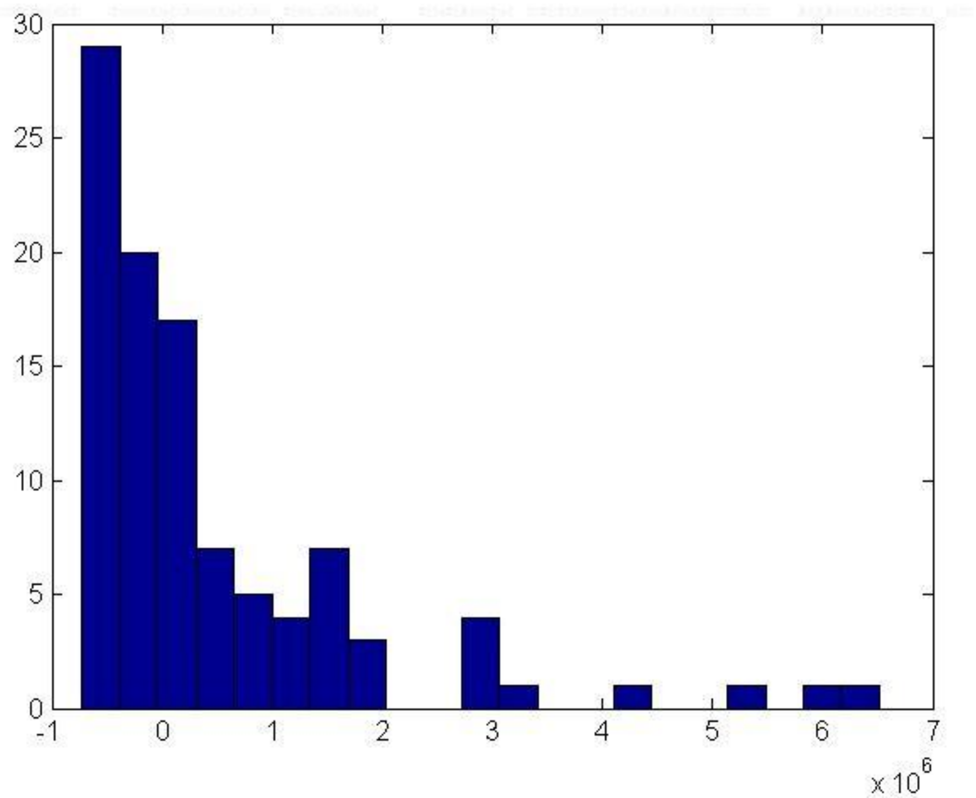
FIGURE 3



VaR95:	520 293
VaR99:	597 078
ES95:	566 484
ES99:	633 516
Mean return:	-290 935
Standard deviation:	153 556

Using this calibration we get a pretty similar return distribution as when using the standard volatility estimator. What we can observe is a slight increase in the variance and a slight increase in the mean return. The confidence intervals calculated as usual are for VaR^{95} (516 300, 526 280) and for VaR^{99} (586 110, 608 050).

FIGURE 4

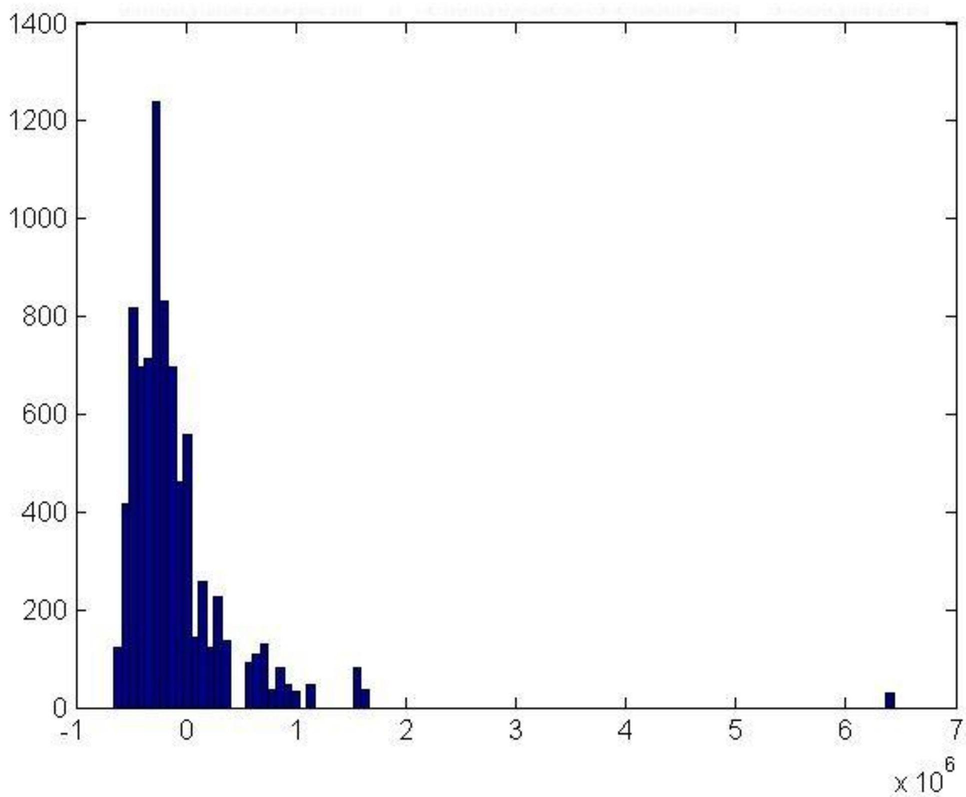


VaR95 breaches:	22
VaR99 breaches:	0
ES95:	552 597
ES99:	
Mean return:	460 595
Standard deviation:	1 370 357

Our comparison with the market returns actually yields the same results as with the standard volatility estimator. This suggests that the two models are quite equal and that something else is responsible for the difference between expected returns and market returns. While the main issues probably is correlation and extreme market conditions we will test the first one when using historical simulation.

Figure 5 shows the outcomes when simulating with the Historical Simulation method rather than the Monte Carlo Method.

FIGURE 5

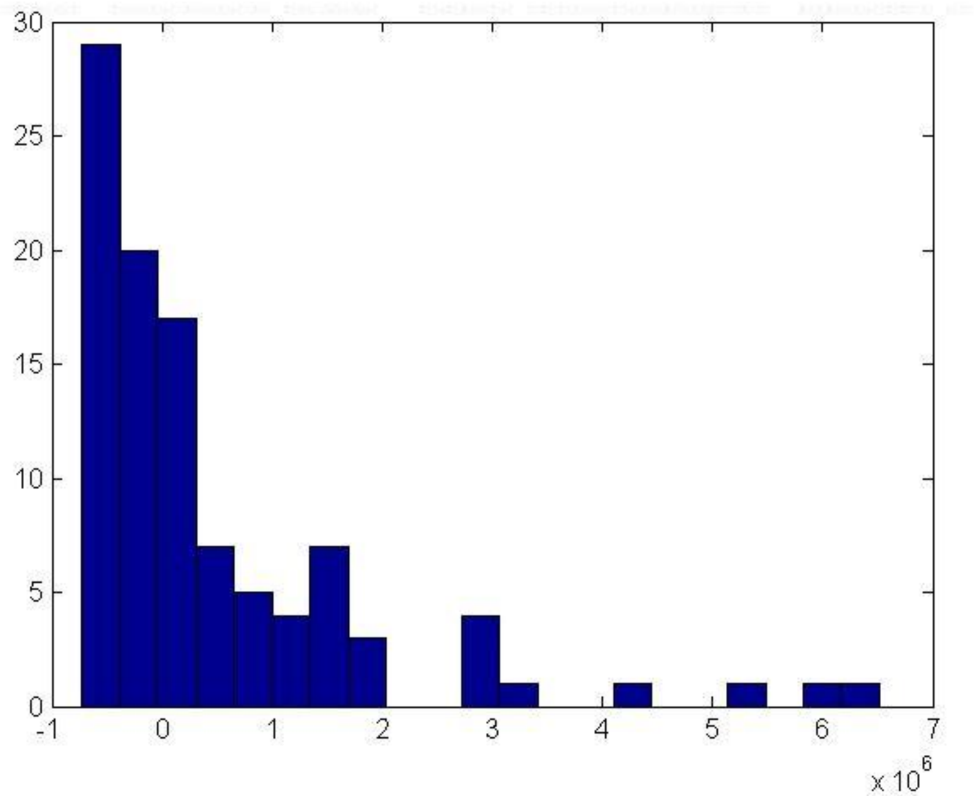


VaR95:	516 652
VaR99:	606 401
ES95:	561 644
ES99:	613 492
Mean return:	-96 658
Standard deviation:	562 040

The confidence intervals calculated as usual are for VaR^{95} (516 650, 517 650) and for VaR^{99} (606 400, 608 400).

Using historical simulation, which includes correlation, we get a substantially larger variance in our portfolio. The mean return is less negative than with the previous configurations, while the other risk measures are quite similar.

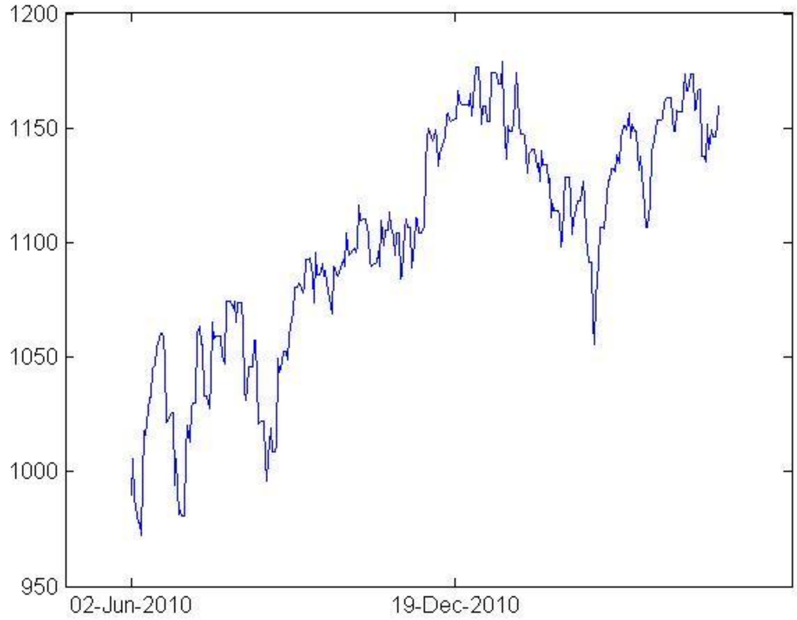
FIGURE 6



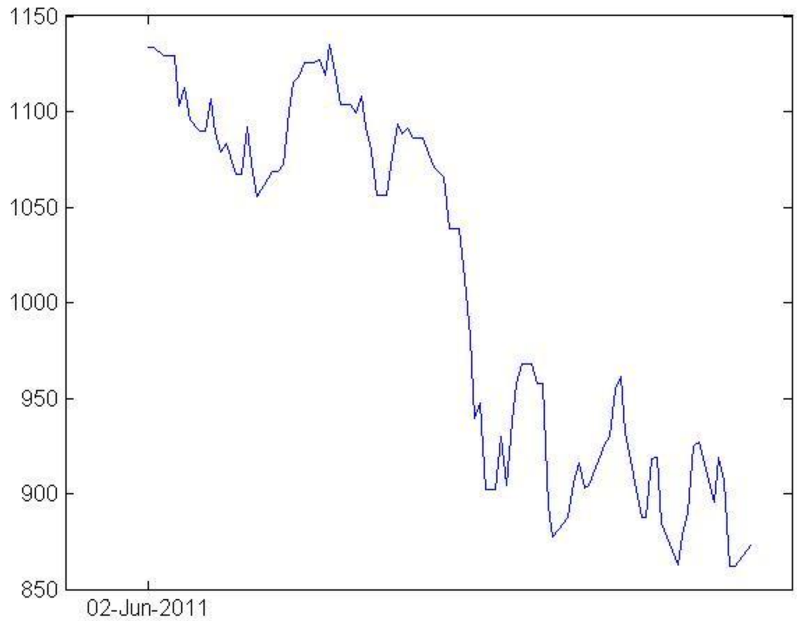
VaR95 breaches:	22
VaR99 breaches:	0
ES95:	552 597
ES99:	
Mean return:	460 595
Standard deviation:	1 370 357

Once again we get the same values when comparing with the following market returns. This is of course because the three different configurations yielded similar risk measures, in combination with the positive returns that were observed during the test period.

To understand the inaccuracy of our model we need to take a look at the market conditions. The graph below shows the movements of the index OMXS30 at the time we collected data for the estimators.



Compare this with the next graph, which shows the OMXS30 movements during the time interval used to determine the "real" returns.

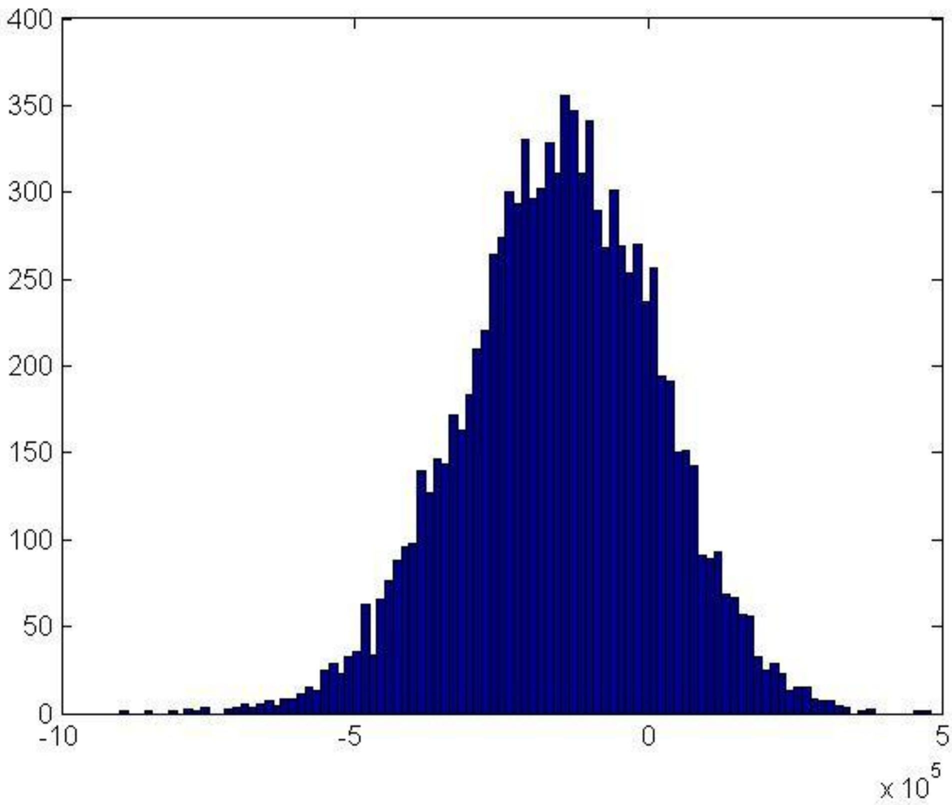


One can observe completely different returns and a much larger variance than in the data used to estimate the expected returns and variance. This is very likely the main reason that the models are quite inaccurate.

8.2 Portfolio 2

The first figure shows the distribution when using the standard setup with equally weighted volatilities. The risk measures are presented in the same way as the other portfolio.

FIGURE 8-1

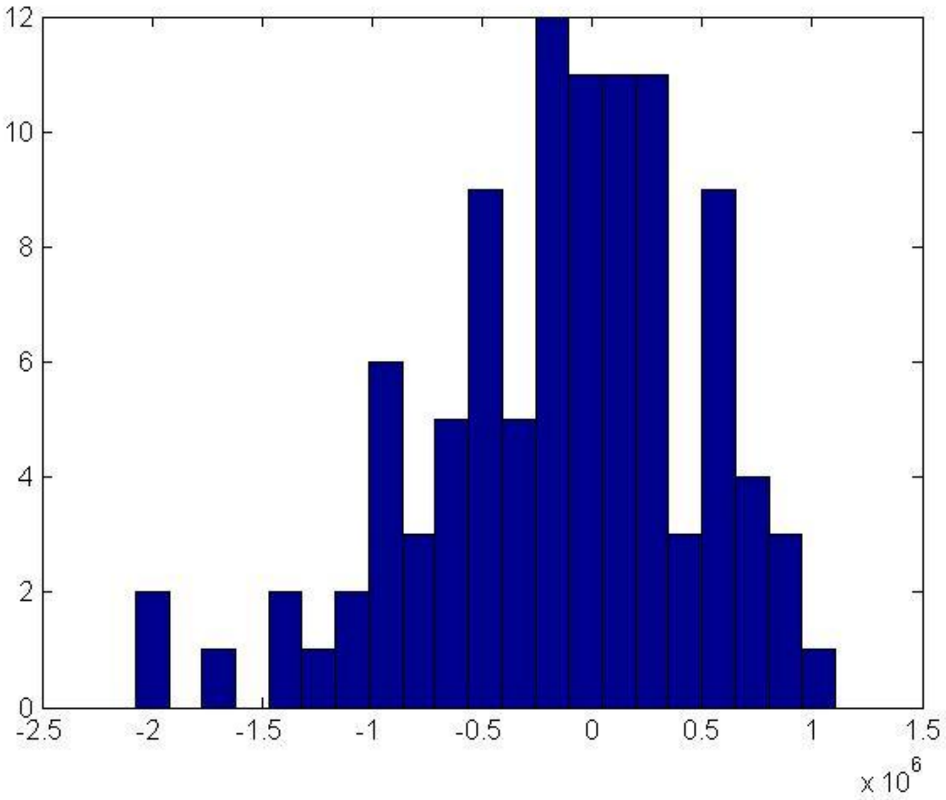


VaR95:	432 024
VaR99:	554 381
ES95:	508 619
ES99:	626 900
Mean return:	-151 138
Standard deviation:	166 464

The confidence intervals are for VaR^{95} (424 540, 437 310) and for VaR^{99} (538 130, 575 320).

The figure below shows the "actual distribution" using returns observed in the following 100 market days. These results are then compared with our estimated risk measures.

FIGURE 8-2

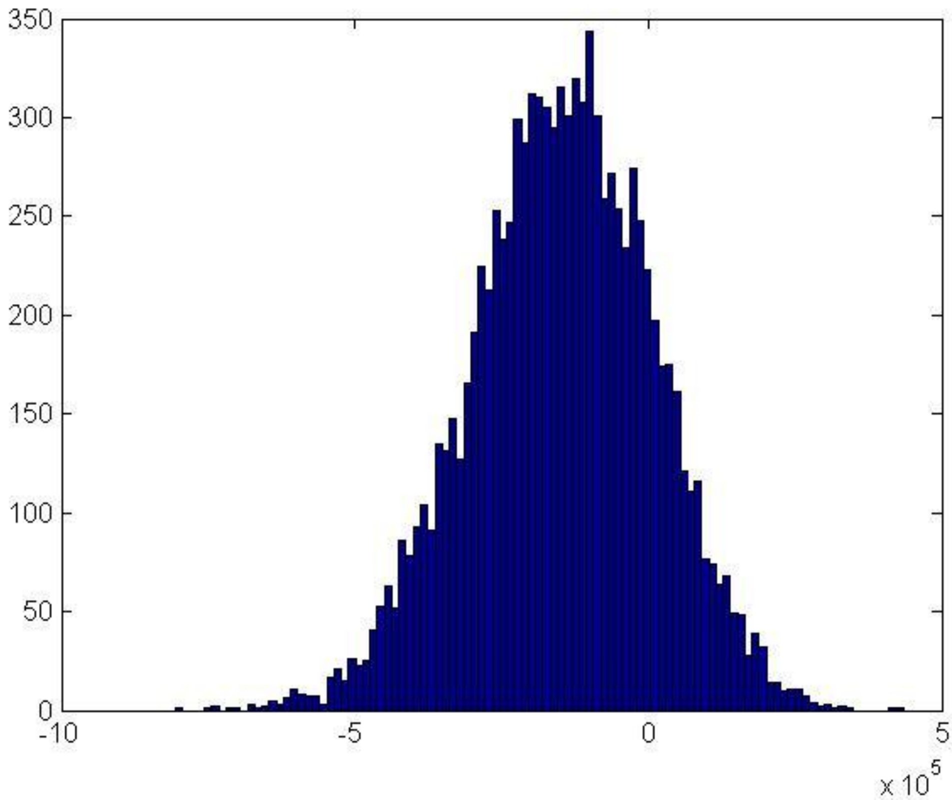


VaR95 breaches:	30
VaR99 breaches:	22
ES95:	914 903
ES99:	1 064 691
Mean return:	-153 759
Standard deviation:	624 835

As with Portfolio 1 we observe a different behavior between our estimated distribution and the observed distribution. A difference is that this portfolio would not perform as well as Portfolio 1 and therefore the large variance of the observed returns would have an large impact on the VaR and ES observations. Since the observed returns are more balanced for this portfolio, our model will underestimate both VaR and ES. Observing the number of breaches confirms this (30 and 22 compared to the expected 5 and 1).

The following figure shows the estimated distribution using the decay volatility model.

FIGURE 8-3

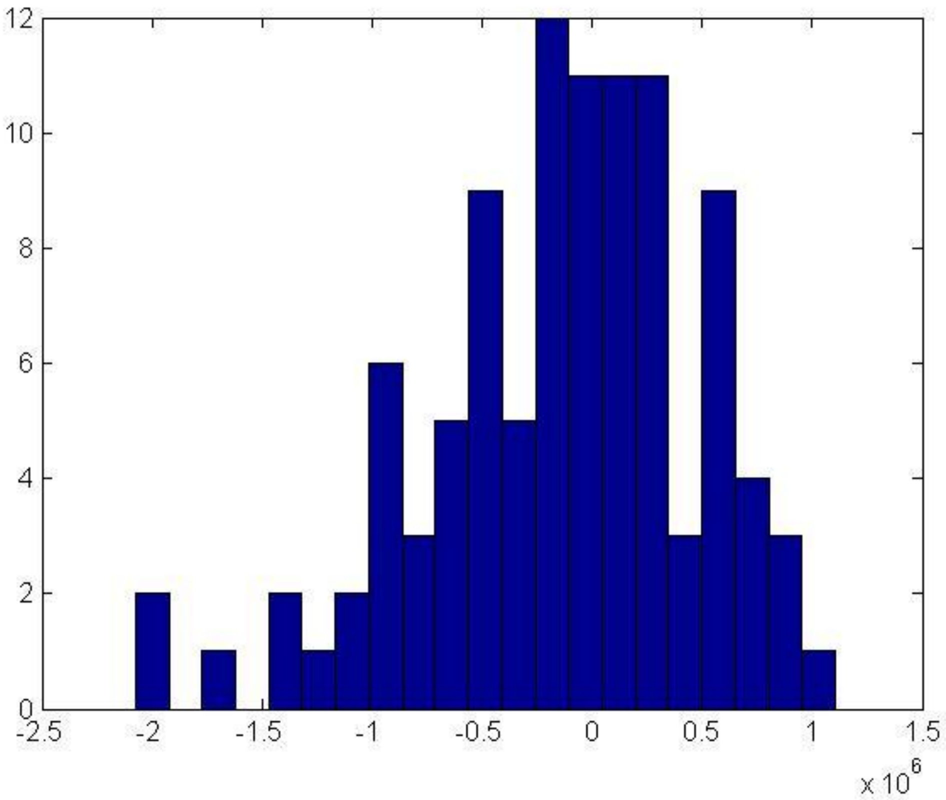


VaR95:	410 137
VaR99:	523 918
ES95:	478 090
ES99:	585 297
Mean return:	-147 693
Standard deviation:	154 758

The decay model gives a pretty similar distribution as the equally weighted volatility model. This model estimates the portfolio to be slightly less risky but the difference is very small. The confidence intervals are for VaR^{95} (403 750, 416 520) and for VaR^{99} (503 970, 538 180).

This is confirmed in the comparison with observed returns presented in the figure below.

FIGURE 8-4

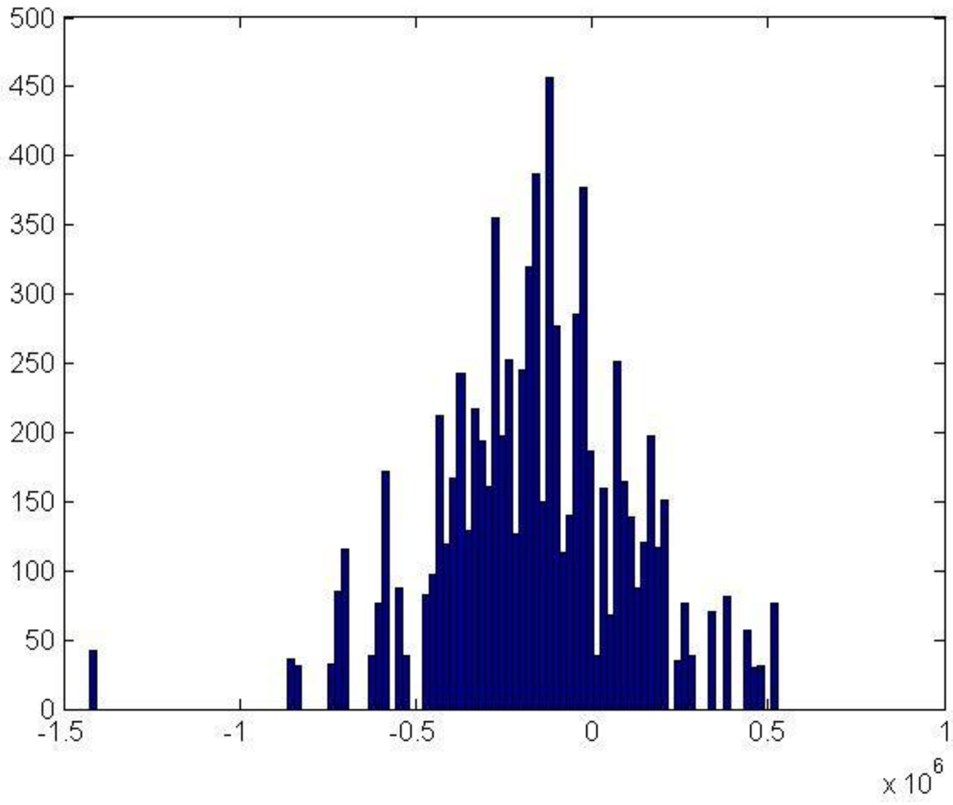


VaR95 breaches:	31
VaR99 breaches:	24
ES95:	899 053
ES99:	1 020 611
Mean return:	-153 759
Standard deviation:	624 835

Because this model was quite similar to the original model the comparison yields similar results. The volatility and risk is underestimated and we once again observe more VaR breaches than expected.

Finally we are testing historical simulation. The results are presented below.

FIGURE 8-5

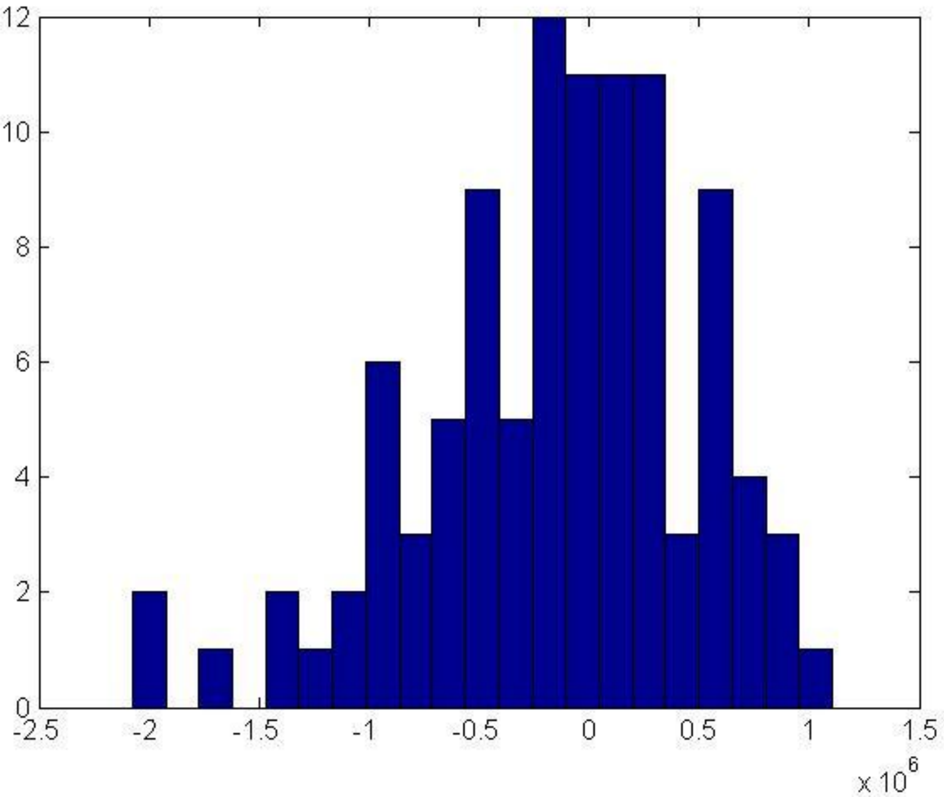


VaR95:	613 712
VaR99:	833 398
ES95:	788 870
ES99:	1 142 412
Mean return:	-159 773
Standard deviation:	273 000

With this model, where correlation is incorporated, we get a greater variance of the returns. This results in a more risky portfolio with larger VaR and ES. This is more in line with the observed returns presented in the figure below.

The confidence intervals are for VaR^{95} (604 940, 620 990) and for VaR^{99} (833 400, 857 830).

FIGURE 8-6



VaR95 breaches:	19
VaR99 breaches:	16
ES95:	1 139 388
ES99:	1 226 103
Mean return:	-153 759
Standard deviation:	624 835

The results yielded by the historical simulation are the most similar to the observed returns. The risk is still underestimated, since the observed variance is again greater and yields more extreme outcomes. We observe 19 Var95 breaches and 16 VaR99 breaches which is more than expected. The standard deviation observed is 624 835 compared to 273 000 we estimated in our model.

9 Conclusions

This study highlights the difficulties of estimating risk measures in extreme market conditions when there is a sudden shift in market behavior. There are two ways to solve this problem. First option is to improve the estimation method in some way to make the estimators represent future movements better. This was done in this study by introducing the decay method for variance estimation. This gives more accurate results if recent results are representative for future returns. Unfortunately in this scenario it was more of an issue of a sudden shift in market behavior right when we started the test. This is of course a very difficult situation to deal with if using historical events to predict the future. What you can do if you know that the market conditions are extreme is that you could try classifying the past time periods and only use data from those periods you consider similar to the current market conditions. Although this is very difficult and unpractical to implement since it requires subjective evaluation of market states. The other method for improving model-accuracy is to use implied volatilities. This is quite easy to implement since all you need is market data. The drawback is of course that you assume that the market is correctly valuated and if that is not the case your estimations will become inaccurate.

Another reason for inaccurate estimations is the correlation. Mostly we assumed zero correlation for computation effectiveness, only when using historical simulation the correlations were incorporated in the simulations, which sometimes is quite inaccurate. The results also suggested¸ since the configuration with historical simulation was the most accurate, that there was indeed some correlation between the underlying assets.

The inclusion of a proper correlation model and the introduction of implied volatility would probably the most interesting features to be implemented to the model, considering the market conditions and the portfolio structure with underlying assets in the same market.

Also the backtesting method used can be quite inaccurate since we use static return and volatility estimations in our models, while when we observe future returns we assume these estimates to be constant. In reality this is unlikely the case, you would expect the market state to change a bit in 100 market days. The objective of the backtest was more to give a simple example of how well it predicted the risk in a point estimate. If one were to evaluate a model more thoroughly it is advised to use a more sophisticated backtesting method.

As a final comment I would say that this study is a great example how VaR and ES, estimated by historical estimation, are only accurate in a stable market state. In very volatile market states these risk measures will be inaccurate. This is of course a big issue since those are the times when risk management is essential.

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