Classification of Probability of Default and Rating Philosophies

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Abstract

Basel II consists of international recommendations on banking regulations, mainly concerning how much capital banks and other financial institutions should be made to set aside in order to protect themselves from various types of risks. Implementing Basel II involves estimating risks; one of the main measurements is Probability of Default. Firm specific and macroeconomic risks cause obligors to default. Separating the two risk factors in order to define which of them affect the Probability of Default through the years. The aim of this thesis is to enable a separation of the risk variables in the structure of Probability of Default in order to classify the rating philosophy.

Acknowledgements

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Chapter 1

Introduction

Enabling banks to give credit, each obligor has to be assigned a credit worthiness. Banks develop models which they use to estimate credit risk. Probability of default (PD) is one of the major measurements in credit risk modelling used to estimates losses which measures how likely obligors are to default during the upcoming year. The great importance of estimating the PD is in gaining a good comprehension of a specific obligor's credit quality. By making a comparison between the real and the estimated defaults, it is possible to see different properties over one business cycle or more. A higher number of defaults during recession is natural, while expansion will result in a reduction of defaults. The impact of macroeconomic conditions will affect the PD. The opportunity to develop PD estimation models of one's own resulted in an increase in existing ones, but the difference between them should be mentioned. Some PD models are affected by macroeconomic conditions while others are almost completely unaffected, depending on the input when developing them. It is reasonable to say that the frequency of defaulted obligors will almost be the same as PD if all macroeconomic conditions are taken into account at any given moment.

At this point, we will introduce the terms **point-in-time** (PIT) and **through-the-cycle** (TTC). Those are two different philosophies that describe the behaviour of the PD. There are no fixed definitions for PIT or TTC, but there are some common ways of describing them. In general, PIT PD is described as a rating system that follows the business cycle and changes over time, while the TTC PD philosophy is almost unaffected by economical conditions. Then there is also a range of hybrid rating systems between those two pure ones.

This area is not well studied in practice and there are not many established ways that describe the characteristics of PIT and TTC. One recurrent description of the philosophies is that PIT PD takes all available external information into account and that it changes as the economy fluctuates, while TTC PD is constant throughout the business cycle.

Because of the poor extent of research on this subject, the selections of methods are few. We have focused on two methods that estimate a factor showing whether the PD tends to PIT or TTC.

Chapter 2

Probability of Default

The Basel II framework issued by the Basel Committee on Banking Supervision contains recommendations on banking laws and regulations. The aim of Basel II is to set international standards on how much capital a bank must hold in order to protect itself against financial and other types of risks. This is the cost price a bank has to pay for doing business and it is divided into two parts; **Expected Loss** (EL) and **Unexpected Loss** (UL). Banks implement Basel II framework to estimate EL, which contains the parameters Probability of Default (PD), Loss Given Default (LGD) and Exposure At Default (EAD). As the name says, EL is the loss that can be estimated. EAD is the estimated outstanding amount in the event of an obligor's default. LGD is the credit loss if an obligor defaults, i.e., the percentage of exposure that the bank may lose if an obligor defaults.

$$EL = PD \cdot EAD \cdot LGD$$

Also included is UL, but there are difficulties forecasting the occurrence of UL because it is the unknown part although the importance of UL is significant and banks have to put aside capital for this type of risk. Thus, EL can be calculated while UL as the name implies is unexpected. The graphs on the next page give an example how the losses could be spread. Figure 2.1 (courtesy The Basel Committee on Banking Supervision (2005)) shows us that UL can vary considerably if something unexpected happens. The probable manner in which those two different types of losses are distributed is shown in Figure 2.2. (courtesy The Basel Committee on Banking Supervision (2005)).

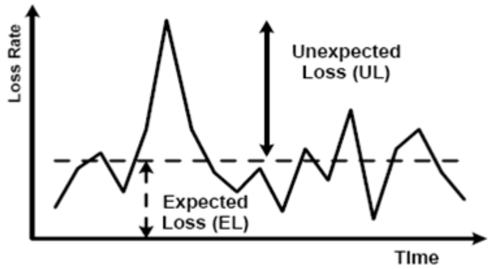

Figure 2.1: The loss rates over time.

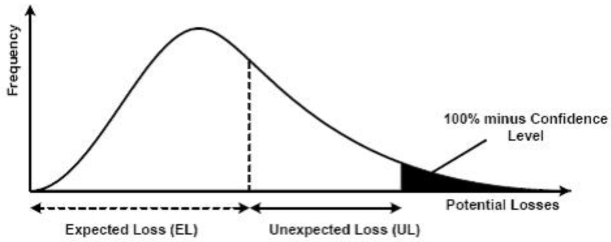

Figure 2.2: Here is probability graph of losses.

PD is the probability that an obligor will default during the upcoming year. There are alternative ways to determine PD. One is to study historical data on defaults and use regression analysis; another could involve observations made of asset prices. The definition of a default according to The Basel Committee on Banking Supervision's (2006) is as follows:[1]

> *"A default is considered to have occurred with regard to a particular obligor when either or both of the two following events have taken place.*
> • *The bank considers that the obligor is unlikely to pay its credit obligations to the banking group in full, without recourse by the bank to actions such as realising security (if held).*
> • *The obligor is past due more than 90 days on any material credit obligation to the banking group. Overdrafts will be considered as being past due once the customer has breached an advised limit or been advised of a limit smaller than current outstandings."*

With Basel II, banks were permitted to use their own models for credit risk calculations; this is known as the Internal Ratings Based approach. However, there are still credit rating agencies that have external ratings. Banks with internal ratings and external rating agencies both rate the obligors according to a scale of credit worthiness called rating grades, e.g., from Aaa to Ca-C, where Aaa is the best and Ca-C is the lowest, such as for the external rating agency Moody's. Those rating grades can be seen as "buckets". Obligors with the same rating grade are put in the same bucket and share the same PD.

[1] Paragraph 452

The characterisation of PD is described in terms of PIT, TTC or a hybrid of those two. Thus, what is the difference between them and what do they mean? There is no one clear definition of PIT and TTC, but a few common ones recur. As shown in Figure 2.3, PIT PD changes over time, while TTC PD is stable and the hybrid stays somewhere between. The figure shows PIT, TTC and a hybrid of 50 %. A hybrid of 50 % means that the PD is exactly between PIT and TTC. This is of course an idealised case resulting in a perfect sine wave.

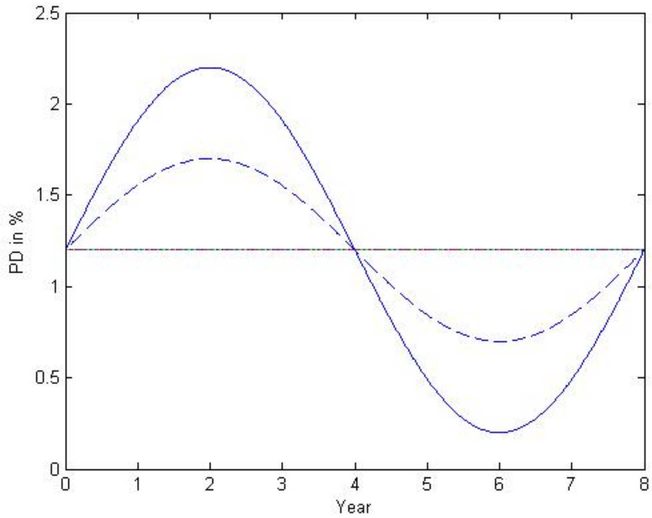

Figure 2.3: The solid curve is the PIT PD which has the highest amplitude, TTC PD is the flat line and of the dashed curve is a hybrid.

One way to describe the PIT and TTC is to say that the two approaches consist of different risk factors. When talking about risk factors, in this case, they can be separated into two categories: risk due to macroeconomic variables and idiosyncratic risk. The macroeconomic risk is common to all obligors and has in one way or another something to do with the fluctuations in the business cycle. When the economy fluctuates, it also naturally has an impact on single obligors, e.g., during recessions, obligors in total have a higher PD. Then the idiosyncratic risk, as the name implies is the risk that is obligor specific and what affects one obligor does not necessarily affect another. PIT PD will just include macroeconomic risk and vice versa for the TTC PD. Thus, those two rating systems are estimated in regard to different factors.

PD is the expected defaults, but what about the real defaults that will occur? **Default Frequency** (DF) is the frequency of actual defaults that have been measured. All obligors have to be rated in order to assess how likely they are to default. Another way to describe the terms PIT and TTC is in terms of how they are related to DF, the defaults that have actually occurred. This way to describe the two philosophies is to compare the estimate with the real occurrence of default. The DF is the same in total for a given year, but the DF within the buckets differs with regard to which philosophy is used.

$$DF = \frac{defauled\ obligors}{total\ number\ of\ obligors}.$$

If the PD follows the DF perfectly, it is a PIT PD approach, whereas TTC PD is a mean of the DF. This corresponds to the sitution shown shown in Figure 2.3. It is the PD for the whole portfolio we are talking about. N is the total number of obligors and we sum up for all years and ratings.

$$Portfolio\ DF = \sum_y \sum_r \frac{N_y^r DF_y^r}{N_y},$$

$$Portfolio\ PD = \sum_y \sum_r \frac{N_y^r PD^r}{N_y}.$$

Where r is rating and y is year. Continuing with the buckets, every bucket has an associated pooled PD, i.e., every rating grade has a given PD. The pooled PDs belong to an exponential PD scale. (See Figure 2.4).

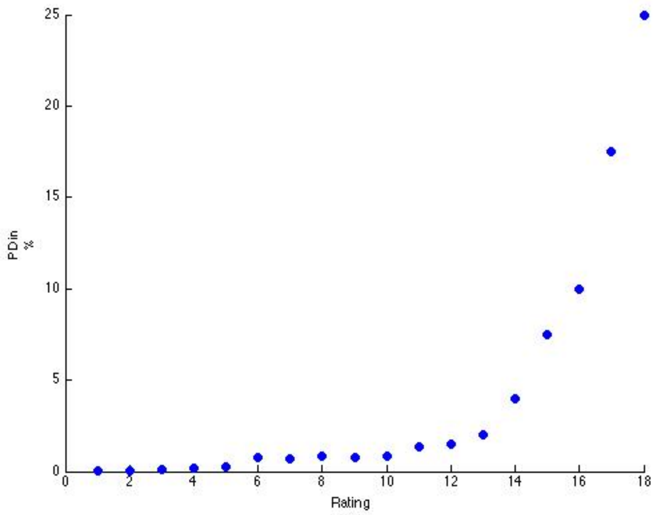

Figure 2.4: PD scale, where rating 1 is the best with the lowest PD.

It is observed while using PIT philosophy that the DF in the buckets stays quite the same over the years, while the obligors migrated among them. In contrast, having a TTC PD the DF in each bucket changes with the changes for the total yearly DF. So, the obligors do not have to migrate as their respective PD vary. With poor migrations, almost all obligors stay within the same rating bucket, which indicates a TTC approach. There is more migration to higher ratings during economic boom times for a PIT.

	rating 1	rating 2	rating 3	rating 4	rating 5	rating 6	rating 7	rating 8	rating 9	rating 10	rating 11	rating 12	rating 13	rating 14	rating 15	rating 16	rating 17	rating 18	Default
rating 1	92,778	5,129	2,083	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,010
rating 2	3,489	81,135	9,973	5,373	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,030
rating 3	3,262	10,576	75,619	7,035	2,329	1,078	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,100
rating 4	0,347	2,919	7,906	77,179	7,833	2,237	1,378	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,200
rating 5	0,030	0,158	2,279	6,390	81,491	6,797	1,432	0,965	0,209	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,250
rating 6	0,261	0,102	0,260	3,298	6,029	78,211	8,536	1,944	0,278	0,331	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,750
rating 7	0,000	0,000	0,688	0,336	1,789	8,244	78,013	6,609	2,432	0,969	0,252	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,667
rating 8	0,000	0,000	0,000	0,166	0,358	1,658	8,117	79,913	5,889	2,067	0,852	0,147	0,000	0,000	0,000	0,000	0,000	0,000	0,833
rating 9	0,000	0,000	0,000	0,000	0,119	0,688	1,446	7,626	78,451	7,586	1,292	0,290	1,551	0,197	0,000	0,000	0,000	0,000	0,750
rating 10	0,000	0,000	0,000	0,000	0,000	0,041	0,387	2,292	9,886	73,946	6,840	2,338	2,496	0,865	0,000	0,000	0,000	0,000	0,875
rating 11	0,000	0,000	0,000	0,000	0,000	0,341	0,110	0,376	2,053	12,087	72,980	7,278	2,497	0,621	0,316	0,000	0,000	0,000	1,333
rating 12	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,238	1,018	4,366	11,134	68,787	9,043	2,380	1,271	0,255	0,000	0,000	1,500
rating 13	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,305	1,153	3,432	10,744	67,808	9,795	3,706	0,655	0,385	0,000	2,000
rating 14	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,196	2,727	3,591	10,529	69,215	5,761	2,611	1,122	0,324	4,000
rating 15	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,420	0,670	3,380	10,911	63,188	8,956	3,466	1,465	7,500
rating 16	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	3,879	0,928	2,225	9,346	62,721	8,116	3,690	10,000
rating 17	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,906	0,357	2,877	7,412	66,492	5,827	17,500
rating 18	0,000	0,000	0,067	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	4,128	8,649	62,155	25,000

Table 2.1: DF within each rating grade for a TTC rating system.

The table above more or less shows the TTC migration matrix [2] expressed as a percentage.

[2]Details regarding matrix found in Appendix.

2.1 Properties of Rating Philosophies

The data used here are fictive values over a 4-year period and 18 rating grades, where a rating of 1 is the best with lowest pooled PD. A pure PIT will be the case when the PD matches the DF perfectly on the portfolio level. Here, the PD-scale is an average of the percentage defaults for a particular rating grade. In saying that PD follows the DF, it is the portfolio PD and DF, which is the weighted average of PD or DF for the whole year within each bucket, as previously described. Using that information the DF can be calculated, but so can the total PD and DF for the portfolio. As for the pure PIT, the DF within each rating bucket will stay unchanged through the years and coincide with the PD-scale. In contrast, the pure TTC is clearly an average of the DF, the PD will form an average of the portfolio DF and opposite here will the DF changes in the buckets. The PIT PD has more migration downturns when PD increases and vice versa if PD decreases. The obligors migrate as the DF changes and in that way so does the PD for the obligors. As opposed to the previous case, there is no migration among the buckets; the position in the current bucket is not affected by the DF. For the constructed PIT data set, the DF is consistent for a given bucket and the obligors migrate between the buckets. TTC PD has a variable DF in the buckets through the years, while the obligors stay in the respective bucket.

	year 1	year 2	year 3	year 4
rating 1	0,01%	0,01%	0,01%	0,01%
rating 2	0,03%	0,03%	0,03%	0,03%
rating 3	0,10%	0,10%	0,10%	0,10%
⋮	⋮	⋮	⋮	⋮
rating 18	25,00%	25,00%	25,00%	25,00%

Table 2.2: DF within each rating grade for a PIT rating system.

	year 1	year 2	year 3	year 4
rating 1	0,02%	0,00%	0,02%	0,00%
rating 2	0,04%	0,02%	0,04%	0,02%
rating 3	0,13%	0,07%	0,13%	0,07%
⋮	⋮	⋮	⋮	⋮
rating 18	30,00%	20,00%	30,00%	20,00%

Table 2.3: DF within each rating grade for a TTC rating system.

	year 1	year 2	year 3	year 4
Portfolio DF	4,14%	3,71%	4,14%	3,71%
Portfolio PD	4,14%	3,71%	4,14%	3,71%

Table 2.4: Portfolio DF and PD for a PIT rating system.

	year 1	year 2	year 3	year 4
Portfolio DF	4,14%	3,71%	4,14%	3,71%
Portfolio PD	3,92%	3,92%	3,92%	3,92%

Table 2.5: Portfolio DF and PD for a TTC rating system.

Chapter 3

Analysis

3.1 Credit Cycle Indices and Transformation into PIT and TTC

With the knowledge that a bank's PD is somewhere between PIT and TTC, it is possible first to present the current "PIT-ness" with a factor, and with that information convert to a pure PIT or TTC PD. The interesting part is the factor that tells how much PIT or TTC the PD is, which takes us toward the aim of this thesis. Thus, we can establish the fact that PD for an obligor obtains both systematic and idiosyncratic risks. The TTC PD does not stay completely unchanged for one specific obligor, but it will be quite similar if looking at the portfolio PD. If the PDs over the course of years show a pattern in their changes that reflect the business cycle or anything else in common, then it is said that the behaviour is caused by a systematic risk.

Primary to the following steps is the significance of the credit cycle index c_t, derived from Kang (2012), with almost the same notation. The PD could be everywhere between pure PIT and TTC, the credit cycle index is in a way a measure of the systematic risk. $\gamma \cdot c_t$ is the gap between the hybrid PD and the pure TTC PD (see Figure 2.3). γ is the degree of "PIT-ness", a $\gamma = 1$ is a pure PIT while $\gamma = 0$ indicates a pure TTC. Following we have

$$c_t = \Phi^{-1}(PD^g_{t,PIT}) - \Phi^{-1}(PD^g_{t,TTC}).$$

However, since $PD^g_{t,PIT}$ and $PD^g_{t,TTC}$ are unknown this formula cannot be used. One way of solving the equation is to use an approximation of the cyclical indices that are calculated from the default observations.

$$c_t = -[\Phi^{-1}(\bar{d}_t) - \Phi^{-1}(\bar{d})],$$

where $\bar{d}_t = \frac{1}{N}\sum_{i=1}^{N} d_{i,t}$ and $\bar{d} = \frac{1}{N(T-1)}\sum_{i=1}^{T-1}\sum_{i=1}^{N} d_{i,t}$. $d_{i,t}$ is 1 if the obligor changes his status from non-default to default at time t and 0 otherwise. It is possible to make smoother and more stabile cyclical indices with a Kalman filter, e.g., if having quarterly data or noisy data for some other reason.[1] Below are the estimates for pure PIT and TTC PD, for a known γ.

$$\widehat{PD}^{g}_{t,PIT} = \Phi(\Phi^{-1}(PD^g) - (1-\gamma)c_t),$$

$$\widehat{PD}^{g}_{t,TTC} = \Phi(\Phi^{-1}(PD^g) - \gamma c_t).$$

Then γ also has to be estimated. Continuing with the Maximum Likelihood in order to estimate the factor γ that is the degree of "PIT-ness".[2]

$$\hat{\gamma} = \arg\max_{\gamma}\left(\sum_{g,t} D^g_t \cdot ln(\widehat{PD}^{g}_{t,PIT}) + (N^g_t - D^g_t)\cdot ln(1-\widehat{PD}^{g}_{t,PIT})\right),$$

where N^g_t is the total number of obligors within rating grade g at time t and D^g_t is the defaulted obligors.

Except for the data previously used, here it is reasonable to present data over a longer time period. Thus it could be easier to assign the estimated pure PIT and TTC. Here are the yearly data over a 15-year period using fictive values. Figure 4.1 shows the PD and DF. Just by looking at the graphs, one can make the assumption that this is hybrid PD with a quite high degree of "PIT-ness".

As described previously, the credit cyclical indices are the first to be determined. Only the 15-year period example will be described in detail. The credit cyclical indices are filtered and predicted for 3 years into future (see Figure 3.2). It may be beneficial to add a filter in case of noisy data, but since having rather smooth data may be insignificant here, it would make more sense to use quarterly data, which produces a jagged graph.

On the other plot (see Figure 3.3) the estimates; $\widehat{PD}^{g}_{t,PIT}$ and $\widehat{PD}^{g}_{t,TTC}$ are shown. The estimated $\widehat{PD}^{g}_{t,PIT}$ follows the DF nearly. Which shows that the estimated pure PIT

[1] Details on how to use a Kalman filter is shown in the Appendix.

[2] The equation which yields the maximum likelihood estimation is solved using the quasi-Newton method, in this case in the language SAS/IML).

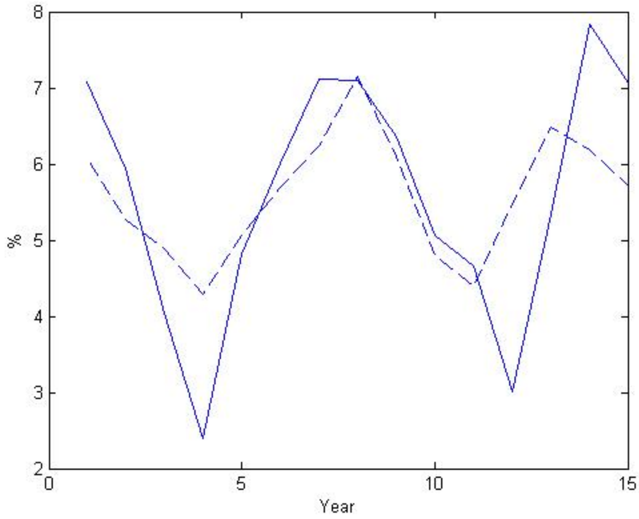

Figure 3.1: DF is the solied curve and the dashed is the PD.

is good. The $\widehat{PD}^{g}_{t,TTC}$ is not perfectly flat and does not give an average over the years but it gives a reduction of the initial PD. And the estimated degree of "PIT-ness", $\hat{\gamma}$, is 0.72.

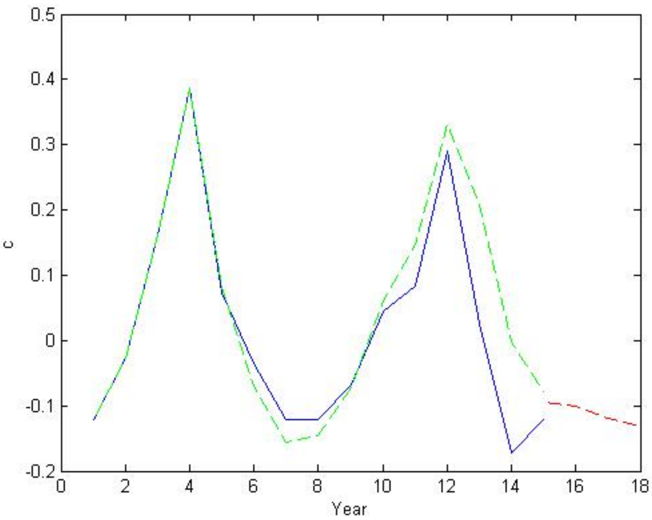

Figure 3.2: Credit cycle indices, filtered and predicted.

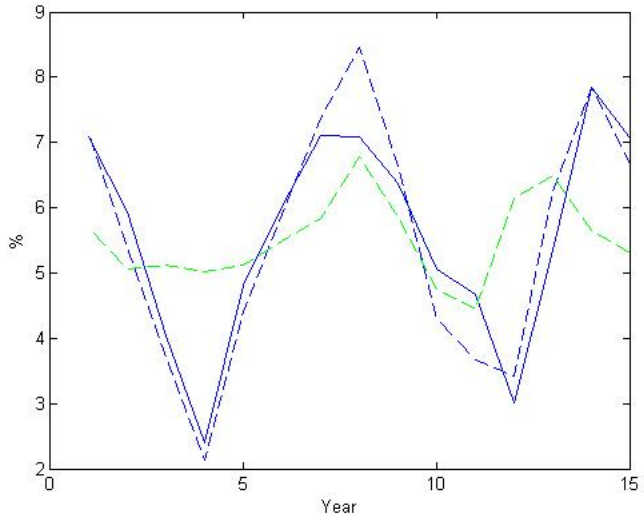

Figure 3.3: The dashed curves represent the estimates, the blue represents the PIT and the green represents the TTC.

The results below are from the pure PIT and TTC.

	$\hat{\gamma}$
PIT	1.01
TTC	-0.05

3.2 One Factor Model and Optimisation

The aim of this model is to calculate a factor which shows the degree of PIT and TTC, with separating the influence of macroeconomic variables due to the default and the total default probability. According to Basel Committee on Banking Supervision (2005) there is assumed that the rating of each obligor consists of an obligor specific and a macroeconomic risk factor, the first-mentioned is an idiosyncratic risk which specific to one single obligor and last one is the risk that affects the whole market. The process is divided into two steps. Firstly, the rating migrations are explored where one could show how likely the obligors are to migrate between the rating buckets.

Every obligor's rating consists of two factors, one is the obligor specific and the other is the macroeconomic risk factor. This is all the information that is known about the obligor i at time t derived from Cornaglia and Morone (2009), with almost the same notation here and the following parts in this section).

$$R_t^i = \beta_W W_t^i + \beta_Y Y_t.$$

R_t^i stands for the rating of obligor i at time t and consists of the two random variables; W_t^i is the obligor specific risk factor and Y_t is the macroeconomic risk factor and those are uncorrelated standard normal variables. Here β_W and β_Y describe how large an impact the factors have. β_W describes the cyclicality of the ratings and $\beta_W = \sqrt{1-\beta_Y^2}$. The one-factor model is used to exhibit the one year asset value A_{t+1}^i, given the info

$$A_{t+1}^i = \alpha_i + R_t^i + \sqrt{\rho} X_{t+1} + \sqrt{1-\rho}\varepsilon_{t+1}^i,$$

where $-\alpha_i$ is the debt threshold. the case $A_{t+1}^i \leq 0$ will give a defaulted obligor. X_{t+1} is a systematic risk variable which is the same for all obligors and ε_{t+1}^i is the obligor specific risk variable. The risk factors are standard normal distributed, orthogonal and time independent.

ρ is a conditional correlation between obligor A^i and A^j. A ρ equal to zero, i.e., no correlations would indicate a pure TTC PD.

$$\rho = \frac{E[A_{t+1}^i \cdot A_{t+1}^i] - E[A_{t+1}^i]E[A_{t+1}^i]}{\sigma(A_{t+1}^i)\sigma(A_{t+1}^i)}.$$

The factor showing the degree of "PIT-ness" is represented by γ.

$$\gamma = \frac{\beta_Y^2}{\beta_Y^2 + \rho_c}.$$

Where this quotient is the contribution from the macroeconomic factor and the variance of the asset value. If γ is 1 then the model is a pure PIT, if 0 then pure TTC and everything in between is a hybrid. Continuing with the optimisation of β_Y and ρ, we start with the first part where β_Y is supposed to be estimated. The only thing that is needed here is historical migration matrices for the given years, which in this case is the number of obligors. The rating R_t^i is theoretically in the interval between $R^{g,k}$ and $R^{g,k-1}$, where the change is from rating grade g to k, for all rating grades up to G. $R^{g,k}$ and $R^{g,k-1}$ are the thresholds for the space within which g lies. The macroeconomic factor Y_t changes due to the common changes in obligors ratings.

$$\hat{P}_t^{g,k} = P(R^{g,k} < R_t^i < R^{g,k-1}) = P(R^{g,k} < \beta_Y Y_t + \sqrt{1-\beta_Y^2 W_t^i} < R^{g,k-1}) =$$
$$= P\left(\frac{R^{g,k} - \beta_Y Y_t}{\sqrt{1-\beta_Y^2}} < W_t^i < \frac{R^{g,k-1} - \beta_Y Y_t}{\sqrt{1-\beta_Y^2}}\right) = \Phi\left(\frac{R^{g,k-1} - \beta_Y Y_t}{\sqrt{1-\beta_Y^2}}\right) - \Phi\left(\frac{R^{g,k} - \beta_Y Y_t}{\sqrt{1-\beta_Y^2}}\right).$$

$n_t^{g,G}$ is the number of obligors that have migrated from rating grade g to k. $\hat{P}_t^{g,k}$ is multinomially distributed.

$$\hat{\beta} = \arg\max_{\beta}\left(\sum_{t=1}^{T}\int\prod_{g=1}^{G}\frac{n_t^g!}{n_t^{g,1}!\dots n_t^{g,G}!}(\hat{P}_t^{g,1})^{n_t^{g,1}}\dots(\hat{P}_t^{g,G})^{n_t^{g,G}}dF(Y_t)\right).$$

β_Y shows the impact of Y_t for the current portfolio. A β_Y close to zero indicates a pure PIT PD.

Continuing with the second part, where the square root of ρ_c stands for the influence of X_{t+1}, realised systemic risk variable is common to all obligors. The PD for a given rating grade is conditional on the systematic risk. The systematic factor is normally distributed.

$$PD^g|X_{t+1} = \Phi\left(\frac{\alpha_g - \sqrt{\rho_c^g}\cdot X_{t+1}}{\sqrt{1-\rho_c^g}}\right),$$

here, $\alpha_g = \Phi(PD_g)^{-1}$. The result comes out from the Bernoulli trial with success, where the PD given the systematic factor is the success.

$$\hat{\rho_c} = \arg\max_{\rho_c} \left(\sum_{t=1}^{T} \int \prod_{g=1}^{G} \binom{N_t^g}{D_{t+1}^g} PD^g|X_{t+1}{}^{D_{t+1}^g}(1 - PD^g|X_{t+1})^{N_t^g - D_{t+1}^g} dF(X_{t+1}) \right),$$

where N_t^g is number of defaulted obligors, while D_{t+1}^g is the number of total obligors. Here are the results after implementing the model, for the constructed pure PIT PD, TTC PD and the hybrid.

	$\hat{\beta_Y}$	$\hat{\rho_c}$	$\hat{\gamma}$
PIT	0.15	0	1
TTC	0	0.10	0

The correlation ρ_c is 0 in the PIT case which is consistent with the theory that there is no asset volatility and that it is opposite for the other cases. For the TTC, $\beta_Y = 0$ which is expected since no impact of the macroeconomic factor.

3.2.1 Simulation

In this section we present simuleted data from the model. All the assumptions below are assumptions that we have made, in order to make it possible to do a simulation. Since all variables are standard normal, Y_t may depend on Y_{t+1} and X_t, the remaining variables are independent (according to Basel Committee on Banking Supervision (2005)). For the PIT we choose $\rho = 0$ and an arbitrary β_Y. The obligor specific risk factor W^i can remain the same for a given obligor, while making a change to the obligor specific risk ε_{t+1}^i. All those are assumptions for enabling a statistical test. We sample the standard normal variables and do the calculations in SAS. For both Y_t and X_{t+1} we take variables between -0.5 to 0.5 so that they follow a randomly picked curve. Here,

$$R_t^i = \beta_W W^i + \beta_Y Y_t,$$

$$A_{t+1}^i | t - \alpha_i = R_t^i + \varepsilon_{t+1}^i.$$

For a TTC PD $\beta_Y = 0$ and an arbitrary ρ. The assumptions give,

$$R_t^i = W^i,$$

$$A^i_{t+1}|t - \alpha_i = R^i_t + \sqrt{\rho}X_{t+1} + \sqrt{1-\rho}\varepsilon^i_{t+1},$$

where R^i_t are random standard normal distributed and have to be bucket appropriately into ratings. Assuming that the obligors are normally distributed, we have highest number of obligors in the middle and fewer in the higher respective lower rating classes. In the interval between -1.96 and 1.96 (this is just an interval that we are choosing, it is possible to select an other). $A^i_{t+1}|t - \alpha_i < -1.96$ is a default and everything above that value is the distance to default. Once we have all the obligors bucketed and separated into defaulted and non defaulted we can calculate the DF and it is easy to continue with calculating of the portfolio PD and DF, in the same way as before. The graphs in Figure 3.5 and 3.6 show the simulated data.

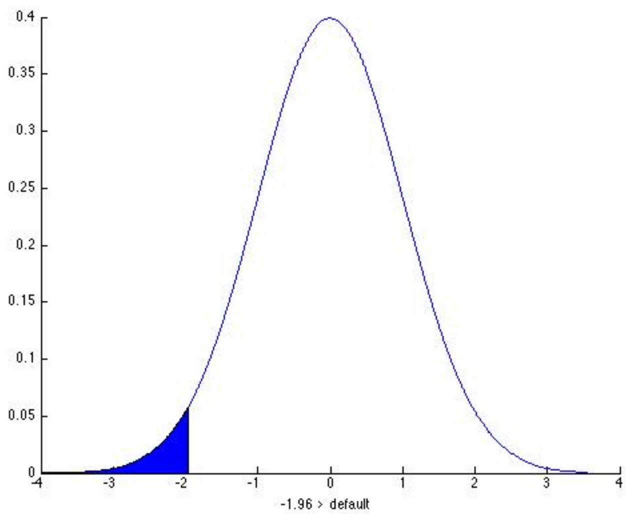

Figure 3.4: The blue selection represents defaulted obligors.

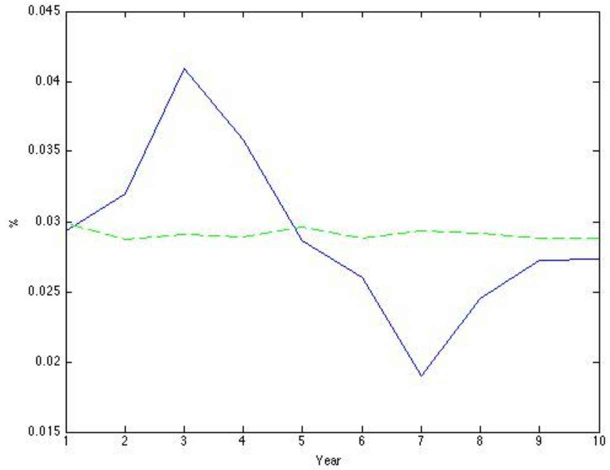

Figure 3.5: This is a simulated TTC PD and its respective DF

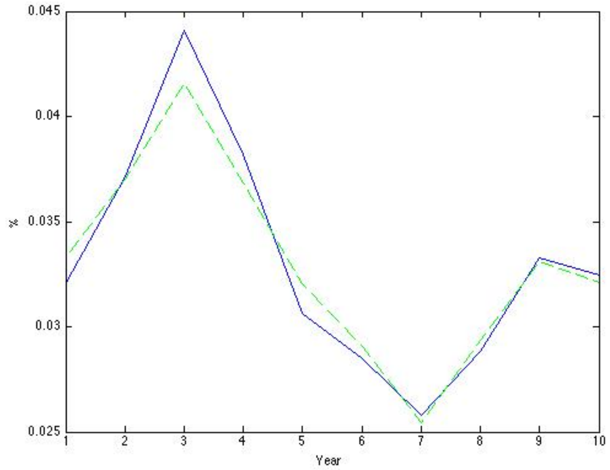

Figure 3.6: This is a simulated PIT PD and its respective DF

Chapter 4

Conclusions

This report presents two different methods estimating the grade of PIT and TTC for a certain portfolio. None of the two methods have any limitations concerning either the number of obligors or the number of years, which allows the feasibility of data size, small as well as large. Both exhibit good results; our assessment is that both methods work well on all data, but with different advantages.

It is not always easy to draw conclusions from financial data. Since we know that real data does not appear as perfect as sine curves, due to recession and expansion not occur within exact intervals. Therefore, mathematical methods for calculations of grade of PIT and TTC it is good to use, as it can be difficult to conclude which grade of PIT and TTC a portfolio has by just analyse graphs of DF and PD.

There is a slight difference between the outcome of γ, which is the grade of PIT and TTC, from the two methods.

When using the method in section "Credit Cycle Indices and Transformation into PIT and TTC" we did not get the exact result for PIT and TTC portfolios. A reason could be that the assumptions used in this method probably is more adapted to data not pure PIT or TTC. Instead, it is fitted to hybrid data, that can be converted into pure PIT or TTC.

To enable estimation of the cyclicality of the method in section "One Factor Model and Optimisation" we need to possess the migration matrix regarding the portfolio in term. While in the "Credit Cycle Indices and Transformation into PIT and TTC" the cyclicality are calculated with the Credit Cycle Indices.

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Appendix A

Time Series Analysis in Brief

The following is a mathematical description and notation according to P. J. Brockwell and R. A. Davis *Introdution to Time Series and Forecasting.*

A time series x_t is a sequence of observations measured at time points t.

$$X_t = m_t + s_t + c_t + Z_t$$

The trend component m_t is a function that changes slowly, seasonal component s_t has a given period and this function is known and cyclical component c_t is also a known function. Z_t is the noise, here it is a white noise i.e if X_t is a sequence of uncorrelated random variables, with zero mean and variance σ^2, the notation is $X_t \sim \mathrm{WN}(0, \sigma^2)$

A.1 AR(1) process

This process shows how future values can depend on current values, which often come up when dealing with financial data. Hence, what occurs today is connected with the event tomorrow. If X_t is a stationary series and the following is satisfied

$$X_t = \phi X_{t-1} + Z_t, \qquad t = 0, \pm 1, ...,$$

where $Z_t \sim \mathrm{WN}(0, \sigma^2)$ and $|\phi| < 1$, then it is an AR(1) process.

A.2 State Space Form

The state space model is on form

$$\boldsymbol{Y_t} = G\boldsymbol{X_t} + \boldsymbol{W_t}, \qquad t = 0, \pm 1, ...,$$

$$\boldsymbol{X_{t+1}} = F\boldsymbol{X_t} + \boldsymbol{V_t}, \qquad t = 0, \pm 1, ...,$$

where $\boldsymbol{V_t} \sim \mathrm{WN}(\mathbf{0}, Q)$ and $\boldsymbol{W_t} \sim \mathrm{WN}(\mathbf{0}, R)$. Here $\boldsymbol{Y_t}$ is the observation equation and $\boldsymbol{X_{t+1}}$ stands for the state equation.

A.3 State Space Form and Business Cycle

The cyclical behaviour on state space form

$$\begin{pmatrix} \psi_t \\ \psi_t^* \end{pmatrix} = \begin{pmatrix} \rho\ cos\lambda & \rho\ sin\lambda \\ -\rho\ sin\lambda & \rho\ cos\lambda \end{pmatrix} \begin{pmatrix} \psi_{t-1} \\ \psi_{t-1}^* \end{pmatrix} + \begin{pmatrix} \varepsilon_{t-1} \\ \varepsilon_{t-1}^* \end{pmatrix}$$

where $0 < \rho < 1$ and $0 < \lambda < \pi$. At time t, the amplitude is $\rho^t \sqrt{c_0^2 + c_0^{*2}}$ and the phase is $\tan^{-1}(\frac{\alpha}{\beta})$.

If the period is $\frac{2\pi}{\lambda}$ then the cycle is deterministic.

$\binom{\varepsilon_{t-1}}{\varepsilon_{t-1}^*}$ is the white noise, if the variance is zero then the model can be reduced to a deterministic model.

$$\begin{pmatrix} \mu_t \\ \beta_t \\ \psi_t \\ \psi_t^* \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \rho\ cos\lambda & \rho\ sin\lambda \\ 0 & 0 & -\rho\ sin\lambda & \rho\ cos\lambda \end{pmatrix} \begin{pmatrix} \mu_{t+1} \\ \beta_{t+1} \\ \psi_{t+1} \\ \psi_{t+1}^* \end{pmatrix} + \begin{pmatrix} \eta_{t+1} \\ \zeta_{t+1} \\ \kappa_{t+1} \\ \kappa_{t+1}^* \end{pmatrix}$$

A.4 Kalman Filter

In time series analysis filters are used to separate the time series properties into trend, seasonal, cyclical and error components. The use of filters allows for the splitting up the different properties, but perhaps the major function of a filter is removing the noise. The Kalman filter is a standard algorithm for time series, which is often used solving problems in financial mathematics. It is a recursive algorithm that works on input data with noise. The aim of the process is to give an estimate of the state system, so that the error will be minimised. In order to make it possible to apply the Kalman filter the model has to be written in state space form. $\boldsymbol{X_t}$ is the state-vector expressed in terms of the observations $\boldsymbol{Y_0}, \boldsymbol{Y_1}, ..., \boldsymbol{Y_t}$. Having a random vector $\boldsymbol{X} = (X_1, ..., X_n)'$ and $P_t(\boldsymbol{X}) := (P_t(X_1), ..., P_t(X_n))'$ the best linear predictor of X_i is $P_t(X_i) := P(X_i|\boldsymbol{Y_0}, \boldsymbol{Y_1}, ..., \boldsymbol{Y_t})$. Kalman Prediction: The prediction for one step $\hat{\boldsymbol{X_t}} := P_{t-1}(\boldsymbol{X_t})$ and the error covariance matrix $\Omega_t = E[(\boldsymbol{X_t} - \hat{\boldsymbol{X_t}})(\boldsymbol{X_t} - \hat{\boldsymbol{X_t}})']$, so

$$\hat{\boldsymbol{X_1}} = P(\boldsymbol{X_1}|\boldsymbol{Y_0}), \quad \Omega_t = E[(\boldsymbol{X_1} - \hat{\boldsymbol{X_1}})(\boldsymbol{X_1} - \hat{\boldsymbol{X_1}})']$$

the recursions, t=1, . . .,

$$\hat{\boldsymbol{X}}_{\boldsymbol{t+1}} = F_t \hat{\boldsymbol{X}}_{\boldsymbol{1}} + \Theta_t \Delta_t^{-1} (\boldsymbol{Y_t} - G_t \hat{\boldsymbol{X}}_{\boldsymbol{t}})$$

$$\Omega_{t+1} = F_t \Omega_t F_t' + Q_t - \Theta_t \Delta_t^{-1} \Theta_t'$$

where $\Delta_t = G_t \Omega_t G_t' + R_t$ and $\Theta_t = F_t \Omega_t G_t'$. Kalman Filtering: The filtered estimates $\boldsymbol{X_{t|t}} = P_t(\boldsymbol{X_t})$ and the error covariance matrices $\Omega_{t|t} = E[(\boldsymbol{X_t} - \boldsymbol{X_{t|t}})(\boldsymbol{X_t} - \boldsymbol{X_{t|t}})']$

$$P_t \boldsymbol{X_t} = P_{t-1} \boldsymbol{X_t} + \Omega_t G_t' \Delta_t^{-1} (\boldsymbol{Y_t} - G_t \hat{\boldsymbol{X}}_{\boldsymbol{t}})$$

and

$$\Omega_{t|t} = \Omega_t - \Omega_t G_t' \Delta_t^{-1} G_t \Omega_t'$$

Appendix B

Migration matrix

	rating 1	rating 2	rating 3	rating 4	rating 5	rating 6	rating 7	rating 8	rating 9
rating 1	92,778	5,129	2,083	0,000	0,000	0,000	0,000	0,000	0,000
rating 2	3,489	81,135	9,973	5,373	0,000	0,000	0,000	0,000	0,000
rating 3	3,262	10,576	75,619	7,035	2,329	1,078	0,000	0,000	0,000
rating 4	0,347	2,919	7,906	77,179	7,833	2,237	1,378	0,000	0,000
rating 5	0,030	0,158	2,279	6,390	81,491	6,797	1,432	0,965	0,209
rating 6	0,261	0,102	0,260	3,298	6,029	78,211	8,536	1,944	0,278
rating 7	0,000	0,000	0,688	0,336	1,789	8,244	78,013	6,609	2,432
rating 8	0,000	0,000	0,000	0,166	0,358	1,658	8,117	79,913	5,889
rating 9	0,000	0,000	0,000	0,000	0,119	0,688	1,446	7,626	78,451
rating 10	0,000	0,000	0,000	0,000	0,000	0,041	0,387	2,292	9,886
rating 11	0,000	0,000	0,000	0,000	0,000	0,341	0,110	0,376	2,053
rating 12	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,238	1,018
rating 13	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,305
rating 14	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
rating 15	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
rating 16	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
rating 17	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
rating 18	0,000	0,000	0,067	0,000	0,000	0,000	0,000	0,000	0,000

Table B.1: First part of DF within each rating grade for a TTC rating system.

	rating 10	rating 11	rating 12	rating 13	rating 14	rating 15	rating 16	rating 17	rating 18	Default
rating 1	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,010
rating 2	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,030
rating 3	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,100
rating 4	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,200
rating 5	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,250
rating 6	0,331	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,750
rating 7	0,969	0,252	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,667
rating 8	2,067	0,852	0,147	0,000	0,000	0,000	0,000	0,000	0,000	0,833
rating 9	7,586	1,292	0,290	1,551	0,197	0,000	0,000	0,000	0,000	0,750
rating 10	73,946	6,840	2,338	2,496	0,865	0,000	0,000	0,000	0,000	0,875
rating 11	12,087	72,980	7,278	2,497	0,621	0,316	0,000	0,000	0,000	1,333
rating 12	4,366	11,134	68,787	9,043	2,380	1,271	0,255	0,000	0,000	1,500
rating 13	1,153	3,432	10,744	67,808	9,795	3,706	0,655	0,385	0,000	2,000
rating 14	0,196	2,727	3,591	10,529	69,215	5,761	2,611	1,122	0,324	4,000
rating 15	0,000	0,420	0,670	3,380	10,911	63,188	8,956	3,466	1,465	7,500
rating 16	0,000	0,000	3,879	0,928	2,225	9,346	62,721	8,116	3,690	10,000
rating 17	0,000	0,000	0,000	0,906	0,357	2,877	7,412	66,492	5,827	17,500
rating 18	0,000	0,000	0,000	0,000	0,000	0,000	4,128	8,649	62,155	25,000

Table B.2: Second part of DF within each rating grade for a TTC rating system.