Pricing Inflation Derivatives

A survey of short rate- and market models

D A M R   T E W O L D E   B E R H A N

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Abstract

This thesis presents an overview of strategies for pricing inflation derivatives. The paper is structured as follows. Firstly, the basic definitions and concepts such as nominal-, real- and inflation rates are introduced. We introduce the benchmark contracts of the inflation derivatives market, and using standard results from no-arbitrage pricing theory, derive pricing formulas for linear contracts on inflation. In addition, the risk profile of inflation contracts is illustrated and we highlight how it's captured in the models to be studied studied in the paper.

We then move on to the main objective of the thesis and present three approaches for pricing inflation derivatives, where we focus in particular on two popular models. The first one, is a so called HJM approach, that models the nominal and real forward curves and relates the two by making an analogy to domestic and foreign fx rates. By the choice of volatility functions in the HJM framework, we produce nominal and real term structures similar to the popular interest-rate derivatives model of Hull-White. This approach was first suggested by Jarrow and Yildirim[1] and it's main attractiveness lies in that it results in analytic pricing formulas for both linear and non-linear benchmark inflation derivatives.

The second approach, is a so called market model, independently proposed by Mercurio[2] and Belgrade, Benhamou, and Koehler[4]. Just like the - famous - Libor Market Model, the modeled quantities are observable market entities, namely, the respective forward inflation indices. It is shown how this model as well - by the use of certain approximations - can produce analytic formulas for both linear and non-linear benchmark inflation derivatives.

The advantages and shortcomings of the respective models are eveluated. In particular, we focus on how well the models calibrate to market data. To this end, model parameters are calibrated to market prices of year-on-year inflation floors; and it is evaluated how well market prices can be recovered by theoretical pricing with the calibrated model parameters. The thesis is concluded with suggestions for possible extensions and improvements.

Contents

Chapter 1

Introduction

1.1 Inflation Basics

1.1.1 Inflation, nominal value and real value

An investor is concerned with the real return of an investment. That is, interested in the quantity of goods and services that can be bought with the nominal return. For instance, a 2% nominal return and no increase in prices of goods and services is preferred to a 10% nominal return and a 10% increase in prices of goods and services. Put differently, the real value of the nominal return is subjected to inflation risk, where inflation is defined as the relative increase of prices of goods and services.

Inflation derivatives are designed to transfer the inflation risk between two parties. The instruments are typically linked to the value of a basket, reflecting prices of goods and services used by an average consumer. The value of the basket is called an Inflation Index. Well known examples are the HICPxT(EUR), RPI(UK), CPI(FR) and CPI(US) indices.

The index is typically constructed such that the start value is normalized to 100 at a chosen base date. At regular intervals the price of the basket is updated and the value of the index ix recalculated. The real return of an investment can then be defined as the excess nominal return over the relative increase of the inflation index.

1.1.2 Inflation Index

Ideally, we would like to have access to index values on a daily basis, so that any cash flow can be linked to the corresponding inflation level at cash flow pay date. For practical reasons this is not possible. It takes time to compute and publish the index value. Due to this, inflation linked cash flows are subjected to a so-called indexation lag. That is, the cash flow is linked to an index value referencing a historical inflation level. The lag differs for different markets. For example the lag for the HICPxt daily reference number is three months. As a consequence, an investor who wishes to buy protection against inflation, has inflation exposure during the last three months before maturity of the inflation protection instrument. That is, the last three months it is effectively a nominal instrument.

1.2 Overview of Inflation-Linked Instruments

1.2.1 Inflation-Linked Bond

Definition

An inflation-linked zero coupon bond is a bond that pays out a single cash flow at maturity T_M, consisting of the increase in the reference index between issue date and maturity. We set the reference index to I_0 at issue date ($t = 0$) and a contract size of N units. The (nominal) value is denoted as $\mathbf{ZCILB}(t, T_M, I_0, N)$. The nominal payment consists of

$$\frac{N}{I_0} I(T_M) \tag{1.2.1}$$

nominal units at maturity. The corresponding real amount is obtained by normalizing with the time T_M index value. That is, we receive

$$\frac{N}{I_0} \tag{1.2.2}$$

real units at maturity. It's thus clear that an inflation-linked zero coupon bond pays out a *known real amount*, but an *unknown nominal amount*, which is fixed when we reach the maturity date.

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Let $P_r(t, T_M)$ denote the time t real value of 1 unit paid at time T_M. Then

$$\frac{N}{I_0} P_r(t, T_M)$$

expresses the time t real value of receiving N/I_0 units at T_M, which is the definition of the payout of the ZCILB. And since the time t real value of the ZCILB is obtained by normalizing the nominal value with the inflation index we have

$$\frac{\mathbf{ZCILB}(t, T_M, I_0, N)}{I(t)} = \frac{N P_r(t, T_M)}{I_0} \tag{1.2.3}$$

Defining the bonds unit value as $P_{IL}(t, T_M) := \mathbf{ZCILB}(t, T_M, 1, 1)$ we get

$$P_{IL}(t, T_M) = I(t) P_r(t, T_M) \tag{1.2.4}$$

Thus the price of the bond is dependent on inflation index levels and the real discount rate.

In practice, as in the nominal bond market, the inflation bonds issued are typically coupon bearing. The coupon inflation bond can be replicated as the sum of a series of zero coupon inflation bonds. With C denoting the annual coupon rate , T_M the maturity date, N the contract size , I_0 the index value at issue date and assuming annual coupon frequency, the nominal time t value of a coupon bearing inflation-linked bond is given as

$$\begin{aligned} \mathbf{ILB}(t, T_M, I_0, N) &= \frac{N}{I_0}\left[C\sum_{i=1}^{M} P_{IL}(t, T_i) + P_{IL}(t, T_M)\right] \\ &= \frac{I(t)}{I_0} N \left[C\sum_{i=1}^{M} P_r(t, T_i) + P_r(t, T_M)\right] \end{aligned} \tag{1.2.5}$$

1.2.2 Zero Coupon Inflation Swap

Definition

A zero coupon inflation swap is a contract where the inflation seller pays the inflation index rate between today and T_M, and the inflation buyer pays a fixed rate. The payout on the inflation leg is given by

$$N\left[\frac{I(T_M)}{I_0} - 1\right] \tag{1.2.6}$$

Thus, the inflation leg pays the net increase in reference index. The payout on the fixed side of the swap is agreed upon at inception and is given as

$$N\left[(1 + b(0, T_M))^{T_M} - 1\right] \tag{1.2.7}$$

where b is the so called breakeven inflation rate. In the market, b is quoted such that the induced T_M maturity zero coupon inflation swap has zero value today. It's analogous to the par rates quoted in the nominal swap market.

From the payout of the inflation leg, it's clear that it can be valued in terms of an inflation linked and a nominal zero coupon bond. However we shall proceed with a bit more formal derivation as it will be useful when proceeding to more complicated instrument types.

Foreign markets and numeraire change

Consider a foreign market where an asset with price X_f is traded. Denote by Q_f the associated (foreign) martingale measure. Assume that the foreign money market account evolves according to the process B_f. Analogously, consider a domestic market with domestic money market account evolving according to the process B_d. Let the exchange rate between the two currencies be modeled by the process H, so that 1 unit of the foreign currency is worth $H(t)$ units of domestic currency at time t. Let $\mathtt{F} = \{\mathcal{F}_t : 0 \le t \le T_M\}$ be the filtration generated by the above processes.

If we think of X_f as a derivative that pays out $X_f(T_M)$ at time T_M, by standard no-arbitrage pricing theory, the arbitrage free price in the foreign market at time t

is

$$V_f(t) = B_f(t)E^{Q_f}\left[\left.\frac{X_f(T_M)}{B_f(T_M)}\right|\mathcal{F}_t\right] \tag{1.2.8}$$

Or expressed as a price in the domestic currency

$$V_d(t) = H(t)B_f(t)E^{Q_f}\left[\left.\frac{X_f(T_M)}{B_f(T_M)}\right|F_t\right] \tag{1.2.9}$$

Note, that for a *domestic investor* who buys the (foreign) asset X_f, the payout at time T_M is $X_f(T_M)H(T_M)$. Now consider a *domestic* derivative which at time T_M pays out $X_f(T_M)H(T_M)$. To avoid arbitrage, the price of this instrument must be equal to price of the foreign asset multiplied with the spot exchange rate. So we get the relation

$$H(t)B_f(t)E^{Q_f}\left[\left.\frac{X_f(T_M)}{B_f(T_M)}\right|\mathcal{F}_t\right] = B_d(t)E^{Q_d}\left[\left.\frac{H(T_M)X_f(T_M)}{B_d(T_M)}\right|\mathcal{F}_t\right] \tag{1.2.10}$$

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Using well known results from standard no-arbitrage pricing theory, with obvious choice of notations, we get the time t value of the inflation leg as

$$\mathbf{ZCILS}(t, T_M, I_0, N) = NE^{Q_n}\left[\left.e^{-\int_t^{T_M} r_n(u)du}\left(\frac{I(T_M)}{I_0} - 1\right)\right|F_t\right] \tag{1.2.11}$$

We draw a foreign currency analogy, namely that real prices correspond to foreign prices and nominal prices correspond to domestic prices. The inflation index value then corresponds to the domestic currency/foreign currency spot exchange rate. Applying the result from (1.2.10) we then obtain

$$I(t)P_r(t, T_M) = I(t)E^{Q_r}\left[\left.e^{-\int_t^{T_M} r_r(u)du}\right|\mathcal{F}_t\right] = E^{Q_n}\left[\left.I(T_M)\, e^{-\int_t^{T_M} r_n(u)du}\right|\mathcal{F}_t\right] \tag{1.2.12}$$

Putting this into (1.2.11) yields

$$\mathbf{ZCILS}(t, T_M, I_0, N) = N\left[\frac{I(t)}{I(0)}P_r(t, T_M) - P_n(t, T_M)\right] \tag{1.2.13}$$

which at time $t = 0$ simplifies to

$$\mathbf{ZCILS}(0, T_M, I_0, N) = N\left[P_r(0, T_M) - P_n(0, T_M)\right] \tag{1.2.14}$$

Important to note is that these prices do not depend on any assumptions of the dynamics of the interest rate market, but rather follow from the absence of arbitrage. This is an imortant result, as it will enable us to calibrate the real rate discount

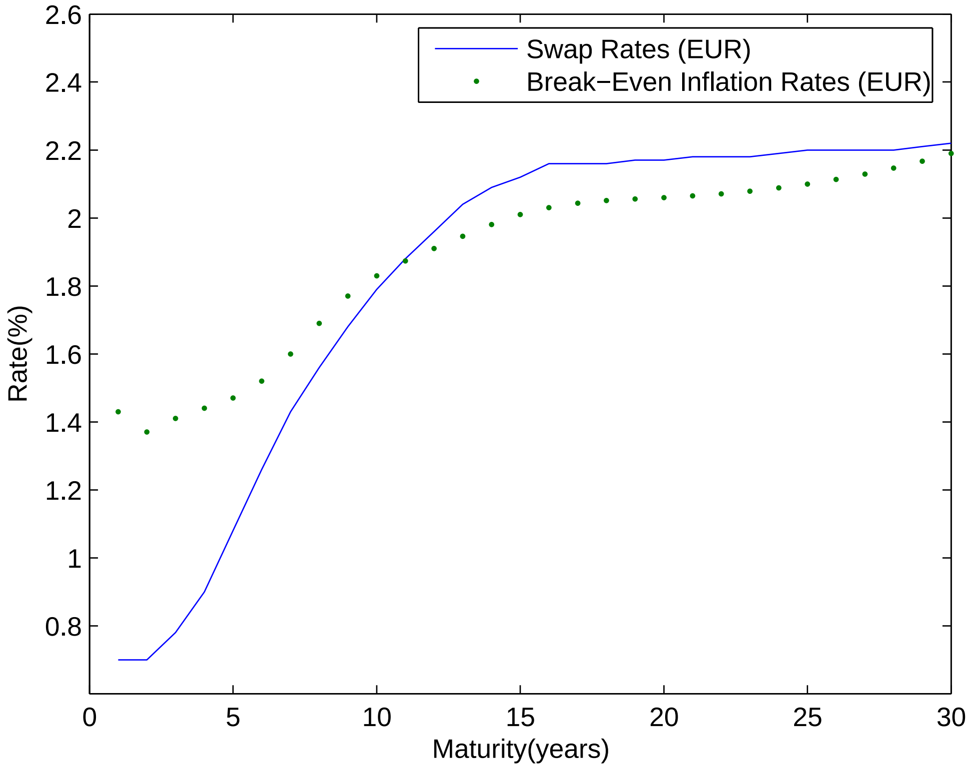

Figure 1.1: Quotes for European nominal- and Zero-Coupon Inflation Swaps, 13 jul-2012

curve from prices of index linked zero-coupon swaps by, for each swap, solving the par value relation. That is , when entering the contract, the value of the pay leg should equal the receive leg.

$$NP_n(0,T_M)[(1+b(0,T_M))^{T_M}-1] = N\left[P_r(0,T_M) - P_n(0,T_M)\right] \qquad (1.2.15)$$

which gives us the real discount rate as

$$P_r(0,T_M) = P_n(0,T_M)(1+b(0,T_M))^{T_M} \qquad (1.2.16)$$

where b is the (market quoted) break-even inflation rate and $P_n(0,T_M)$ can be recovered from bootstrapping the nominal discount curve.

Figure 1.1 shows quotes for break-even inflation rates and nominal swap rates. The maturities up to the 10-year mark reveal something interesting. The nominal swap quote is the fixed rate one would demand in exchange for paying floating cash flows of EURIBOR until maturity. Whereas the break-even inflation quote is the fixed rate one would would demand in exchange for paying realized inflation rate until maturity. The gap between the two, i.e. that the swap quote is lower, indicates that expected inflation is higher than expected EURIBOR. This in turn implies negative real rates. We discuss break-even inflation visavis expected inflation in more detail in section (1.3.1).

1.2.3 Year On Year Inflation Swap

Definition

The inflation leg on a Year On Year Inflation Swap pays out a series of net increases in index reference

$$N \sum_{i=1}^{M} \left[\frac{I(T_i)}{I(T_{i-1})} - 1 \right] \psi_i \tag{1.2.17}$$

where ψ_i is the time in years on the interval $[T_{i-1}, T_i]; T_0 := 0$ according to the contracts day-count convention.

The fixed leg pays a series of fixed coupons

$$N \sum_{i=1}^{M} \psi_i C \tag{1.2.18}$$

Just as for Zero Coupon Inflation swaps , Year On Year Inflation Swaps are quoted in the market in terms of their fixed coupon. However out of the two, the former is more liquid , and is considered to be the primary benchmark instrument in the inflation derivatives market.

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We can view (1.2.17) as a series of forward starting Zero Coupon Swap Inflation legs. Then the price of each leg is

$$\mathbf{YYILS}(t, T_{i-1}, T_i, \psi_i, N) = N\psi_i E^{Q_n} \left[e^{-\int_t^{T_i} r_n(u)du} \left(\frac{I(T_i)}{I(T_{i-1})} - 1 \right) \middle| \mathcal{F}_t \right] \tag{1.2.19}$$

If $t > T_{i-1}$ so that $I(T_{i-1})$ is known then it reduces to the pricing of a regular Zero Coupon Swap Inflation leg. If $t < T_{i-1}$ then we use repeated expectation to get

$$\begin{aligned} &\mathbf{YYILS}(t, T_{i-1}, T_i, \psi_i, N) \\ &= N\psi_i E^{Q_n} \left[E^{Q_n} \left[e^{-\int_t^{T_i} r_n(u)du} \left(\frac{I(T_i)}{I(T_{i-1})} - 1 \right) \middle| \mathcal{F}_{T-1} \right] \middle| \mathcal{F}_t \right] \\ &= N\psi_i E^{Q_n} \left[e^{-\int_t^{T_{i-1}} r_n(u)du} E^{Q_n} \left[e^{-\int_{T-1}^{T_i} r_n(u)du} \left(\frac{I(T_i)}{I(T_{i-1})} - 1 \right) \middle| \mathcal{F}_{T-1} \right] \middle| \mathcal{F}_t \right] \end{aligned} \tag{1.2.20}$$

The inner expection is recognized as $\mathbf{ZCILS}(T_{i-1}, T_i, I(T_{i-1}), 1)$ so that we finally obtain

$$\mathbf{YYIIS}(t, T_{i-1}, T_i, \psi_i, N) = N\psi_i E^{Q_n}\left[e^{-\int_t^{T_{i-1}} r_n(u)du}\left[P_r(T_{i-1}, T_i) - P_n(T_{i-1}, T_i)\right]\middle| \mathcal{F}_t\right]$$
$$= N\psi_i E^{Q_n}\left[e^{-\int_t^{T_{i-1}} r_n(u)du} P_r(T_{i-1}, T_i)\middle| \mathcal{F}_t\right] - N\psi_i P_n(t, T_i) \tag{1.2.21}$$

The last expectation can be interpreted as the nominal price of a derivative paying out at time T_{i-1} (in nominal units) the T_i maturity real zero coupon bond price. If real rates were deterministic then we would get

$$\begin{aligned} E^{Q_n}\left[e^{-\int_t^{T_{i-1}} r_n(u)du} P_r(T_{i-1}, T_i)\middle| \mathcal{F}_t\right] &= P_n(t, T_{i-1}) P_r(T_{i-1}, T_i) \\ &= P_n(t, T_{i-1}) \frac{P_r(t, T_i)}{P_r(t, T_{i-1})} \end{aligned}$$

which is simply the nominal present value of the T_{i-1} forward price of the T_i maturity real bond. In practice however it's not realistic to assume that real rates are deterministic . Real rates are stochastic so that the expectation in (1.2.21) is model dependent.

1.2.4 Inflation Linked Cap/Floor

Definition

An Inflation-Linked Caplet (ILCLT) is a call option on the net increase in forward inflation index. Whereas an Inflation-Linked Floorlet (ILFLT) is a put option on the same quantity. At time T_i the ILCFLT pays out

$$N\psi_i\left[\omega\left(\frac{I(T_i)}{I(T_{i-1})} - 1 - \kappa\right)\right]^+ \tag{1.2.22}$$

where κ is the IICFLT strike, ψ_i is the contract year fraction for the interval $[T_{i-1}, T_i]$, N is the contract nominal, and $\omega = 1$ for a caplet and $\omega = -1$ for a floorlet.

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Setting $K := 1 + \kappa$ we get the time t value of the payoff (1.2.22) as

$$\begin{aligned} &\mathbf{ILCFLT}(t, T_{i-1}, T_i, \psi_i, K, N, \omega) \\ &= N\psi_i E^{Q_n}\left[e^{-\int_t^{T_i} r_n(u)du}\left[\omega\left(\frac{I(T_i)}{I(T_{i-1})} - K\right)\right]^+\middle| \mathcal{F}_t\right] \\ &= N\psi_i P_n(t, T_i) E^{T_i}\left[\left[\omega\left(\frac{I(T_i)}{I(T_{i-1})} - K\right)\right]^+\middle| \mathcal{F}_t\right] \end{aligned} \tag{1.2.23}$$

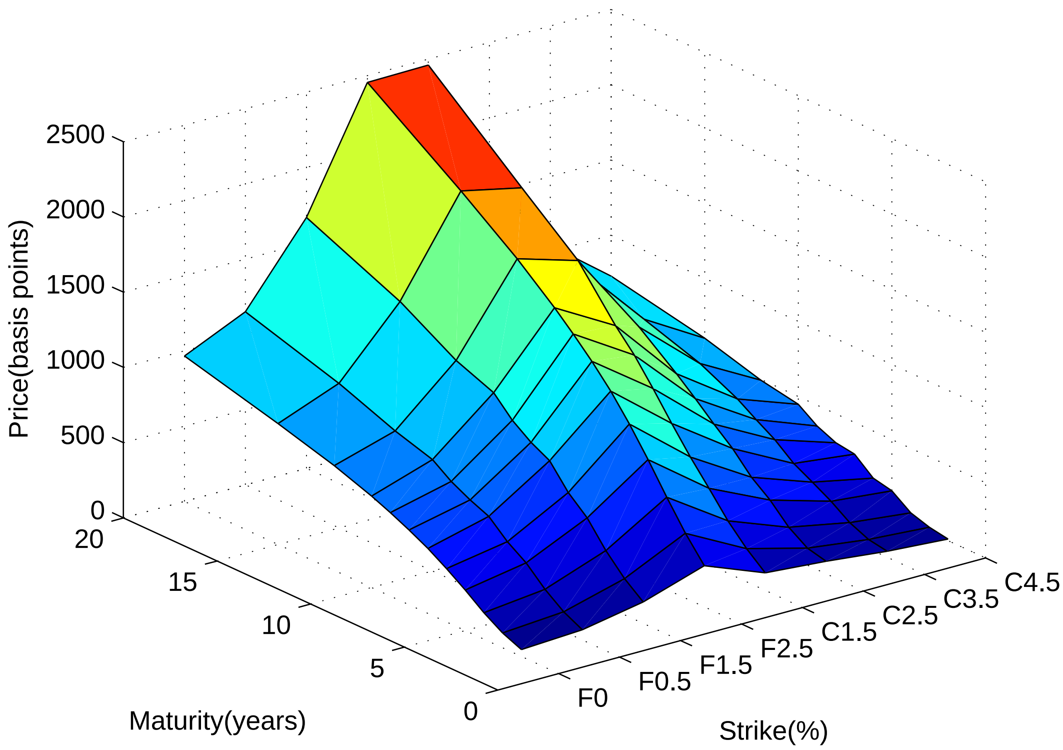

Figure 1.2: Quotes for Year-On-Year Inflation Caps and Floors, 13 jul-2012.

Where E^{T_i} denotes expectation under the (nominal) T_i forward measure. The price of the Inflation Linked Cap/Floor(ILCF) is obtained by summing up over the individual Caplets/Florlets. Clearly this price is model dependent as well.

1.3 Inflation and interest rate risk

1.3.1 Breakeven inflation vs expected inflation

Compounding effect

It seems intuitive to think of break-even inflation rate as a measure of *expected* future inflation. There are however, a number of problems with that assumption. The first is a simple *compounding effect*. Denote the annualized inflation rate between $t = 0$ and T as

$$i(0,T) = \left[\frac{I(T)}{I_0}\right]^{1/T} - 1 \tag{1.3.1}$$

By the definition of the breakeven-inflation rate and Jensens Inequality we have

$$[1 + b(0,T)]^T = E\left[[1 + i(0,T)]^T\right] \geq [1 + E\left[i(0,T)\right]]^T \tag{1.3.2}$$

Hence, the break-even rate is an overestimation of future inflation.

Inflation risk premium

The second argument to why break-even rates can not be directly translated into expected inflation, is that nominal rates are thought to carry a certain *inflation risk premium*. A risk averse bond investor would demand a premium(a higher yield) to compensate for the scenario where *realized inflation* turns out to be higher than expected inflation.

Consider a risk-averse investor who wishes to obtain a real return. The investor can either buy a T-maturity inflation linked bond, receiving a real rate of return $y_r(0,T)$ or a T-maturity nominal bond, receiving a nominal rate of return $y_n(0,T)$. Assuming that both bonds are issued today, the real return on the nominal bond is

$$\frac{I_0}{I(T)}\left[1 + y_n(0,T)\right]^T$$

whereas the real return for the index linked bond is

$$[1 + y_r(0,T)]^T$$

To compensate for the inflation risk, i.e. the scenario where realized inflation over $[0,T]$ turns out be greater than the expected inflation, the risk averse investor would demand an additional return on y_n, in effect demanding a higher yield than motivated by inflation expectations

$$[1 + y_n(0,T)]^T \geq [1 + y_r(0,T)]^T\, E\left[\frac{I(T)}{I_0}\right] = [1 + y_r(0,T)]^T\, E\left[[1 + i(0,T)]^T\right]$$

Denoting the inflation risk premium over $[0,T]$ as $p(0,T)$, we can express the nominal return as

$$[1 + y_n(0,T)]^T = [1 + p(0,T)]^T\,[1 + y_r(0,T)]^T\, E\left[[1 + i(0,T)]^T\right]$$

Consequently , break-even inflation rates will include the risk premium , i.e overestimate future inflation rates.

Assuming a correction factor $c(0,T) \geq 0$ such that we can rewrite (1.3.2) as

$$E\left[[1 + i(0,T)]^T\right] = [1 + c(0,T)]^T\,[1 + E\left[i(0,T)\right]]^T \tag{1.3.3}$$

, then we can express the nominal return in the style of the famous *Fisher Equation*

$$1 + y_n(0,T) = [1 + p(0,T)]\,[1 + y_r(0,T)]\,[1 + c(0,T)]\,[1 + E\left[i(0,T)\right]] \tag{1.3.4}$$

1.3.2 Inflation risk

Since the annually compounded nominal yield y_n is defined as

$$y_n(t,T) = P_n(t,T)^{-1/T} - 1 \tag{1.3.5}$$

by (1.2.5) and (1.2.16) we can write the price of an ILB in terms of the break-even inflation curve and the nominal yield curve

$$\mathbf{ILB}(t,T_M,I_0,N,s_b,s_n) = \frac{I(t)}{I_0}N\left[C\sum_{i=1}^{M}\frac{(1+b(t,T_i)+s_b)^{T_i}}{(1+y_n(t,T_i)+s_n)^{T_i}} + \frac{(1+b(t,T_M)+s_b)^{T_M}}{(1+y_n(t,T_M)+s_n)^{T_M}}\right] \tag{1.3.6}$$

where s_b and s_n should equal zero in order for the price to be fair. Thus it's clear that the price is sensitive to shifts in the inflation curve as well as to shifts in the nominal interest curve.

The effect of a parallel shift in the nominal interest curve is then obtained as

$$\frac{\partial\mathbf{ILB}(t,T_M,I_0,N,0,0)}{\partial s_n} = -\frac{I(t)}{I_0}N\left[C\sum_{i=1}^{M}T_i\frac{P_r(t,T_i)}{1+y_n(t,T_i)} + T_M\frac{P_r(t,T_M)}{1+y_n(t,T_M)}\right] \tag{1.3.7}$$

And the effect of a parallel shift in the inflation curve as

$$\frac{\partial\mathbf{ILB}(t,T_M,I_0,N,0,0)}{\partial s_b} = \frac{I(t)}{I_0}N\left[C\sum_{i=1}^{M}T_i\frac{P_r(t,T_i)}{1+b(t,T_i)} + T_M\frac{P_r(t,T_M)}{1+b(t,T_M)}\right] \tag{1.3.8}$$

Since the inflation delta and the nominal yield delta have opposite signs, the *net effect* will be small if the inflation and nominal curves are equally shifted. Typically, a rise in inflation expectation pushes up the nominal interest rates, so it's natural to impose some correlation $\rho_{n,I}$ between the two, by - for instance - setting $s_b = s_n \times \rho_{n,I}$. Indeed when modeling the evolution of interest rates and inflation in the short rate model of Jarrow and Yildirim[1] and the market model of Mercurio[2] and Belgrade, Benhamou, and Koehler[4], a correlation structure is proposed in the model dynamics.

Chapter 2

The HJM framework of Jarrow and Yildirim

2.1 Definitions

Using a foreign currency analogy, Jarrow and Yildirim reasoned that real prices correspond to foreign prices, nominal prices correspond to domestic prices and the inflation index corresponds to the spot exchange rate from foreign to domestic currency. We introduce the notation which will be used throughout this section.

- $P_n(t,T)$: time t price of a nominal zero coupon bond maturing at time T
- $I(t)$: time t value of the inflation index
- $P_r(t,T)$: time t price of a real zero-coupon bond maturing at time T
- $f_k(t,T)$: time t instantaneous forward rate for date T where $k \in \{r,n\}$ i.e.

$$P_k(t,T) = e^{-\int_t^T f_k(t,s)ds}$$

- $r_k(t) = f_k(t,t)$: the time t instantaneous spot rate for $k \in \{r,n\}$
- $B_k(t)$: time t money market account value for $k \in \{r,n\}$

2.2 Model specification

Under the real world probability space $(\Omega, \mathcal{F}, P)$, Jarrow and Yildirim introduce the filtration $\{\mathcal{F}_t : t \in [0,T]\}$ generated by the three brownian motions

$$dW_n^P(t), dW_r^P(t), dW_I^P(t) \tag{2.2.1}$$

The brownian motions are started at 0 and their correlations are given by

$$\begin{aligned} dW_n^P(t)\, dW_r^P(t) &= \rho_{nr} dt \\ dW_n^P(t)\, dW_I^P(t) &= \rho_{nI} dt \\ dW_r^P(t)\, dW_I^P(t) &= \rho_{rI} dt \end{aligned} \tag{2.2.2}$$

Thus, we will be working with a three-factor model.

Given the initial nominal forward rate curve, $f_n^*(0,T)$, it's assumed that the nominal T-maturity forward rate has a stochastic differential under the objective measure P given by

$$\begin{aligned} df_n(t,T) &= \alpha_n(t,T)dt + \sigma_n(t,T)dW_n^P(t) \\ f_n(0,T) &= f_n^*(0,T) \end{aligned} \tag{2.2.3}$$

where α_n is random and σ_n is deterministic.

Similarly, given the initial market real forward rate curve, $f_r^*(0,T)$, it's assumed that the real T-maturity forward rate has a stochastic differential under the objective measure P given by

$$\begin{aligned} df_r(t,T) &= \alpha_r(t,T)dt + \sigma_r(t,T)dW_r^P(t) \\ f_r(0,T) &= f_r^*(0,T) \end{aligned} \tag{2.2.4}$$

where α_n is random and σ_n is deterministic. The final entity to be modeled is the inflation index with dynamics

$$\frac{dI(t)}{I(t)} = \mu_I(t)dt + \sigma_I(t)dW_I^P(t) \tag{2.2.5}$$

where μ_I is random and σ_I is deterministic. The deterministic volatility in (2.2.5) implies that the inflation index follows a geometric brownian motion so that the logarithm of the index will be normally distributed.

Jarrow and Yildirim go on to show the evolutions introduced so far are arbitrage free and the market is complete if there exists a unique equivalent probability measure Q such that

$$\frac{P_n(t,T)}{B_n(t)}, \frac{I(t)P_r(t,T)}{B_n(t)} \text{ and } \frac{I(t)B_r(t,T)}{B_n(t)} \text{ are } Q \text{ martingales} \tag{2.2.6}$$

Furthermore, by Girsanov's theorem, given that $\{dW_n^P(t), dW_r^P(t), dW_I^P(t)\}$ is a P-Brownian motion, and that Q is a equivalent probability measure, then there exists market prices of risk $\{\lambda_n(t), \lambda_r(t), \lambda_I(t)\}$ such that

$$W_k^Q(t) = W_k^P(t) - \int_0^t \lambda_k(s)ds, \quad k \in \{n, r, I\} \tag{2.2.7}$$

are Q-brownian motions.

Finally, they provided the following proposition, which characterizes the necessary and sufficient conditions on the various price dynamics such that the economy is arbitrage free.

Proposition 2.2.1 (Arbitrage Free Term Structure)
$\frac{P_n(t,T)}{B_n(t)}$, $\frac{I(t)P_r(t,T)}{B_n(t)}$ and $\frac{I(t)B_r(t,T)}{B_n(t)}$ are Q martingales if and only if

$$\alpha_n(t,T) = \sigma_n(t,T)\left(\int_t^T \sigma_n(t,s)ds - \lambda_n(t)\right) \tag{2.2.8}$$

$$\alpha_r(t,T) = \sigma_r(t,T)\left(\int_t^T \sigma_r(t,s)ds - \sigma_I(t)\rho_{rI} - \lambda_r(t)\right) \tag{2.2.9}$$

$$\mu_I(t) = r_n(t) - r_r(t) - \sigma_I(t)\lambda_I(t) \tag{2.2.10}$$

(2.2.8) is recognized as the well known HJM drift condition for the nominal forward rates under the objective probability measure. Analogously, (2.2.9) is the drift condition for the real forward rates. It is to be noted that the inflation volatility and the inflation-real rate correlation appears in this expression. Finally (2.2.10) is a fisher equation(compare with the heuristics in (1.3.4)), relating nominal and real interest rates to expected inflation(μ_I) and inflationary risk premium.

2.3 Zero Coupon Bond term structure

2.3.1 General form

It can be shown(see [5]) that, for $k \in \{n, r\}$, the log-bond price process can be written as

$$\begin{aligned}\ln P_k(t,T) &= -\int_t^T f_k(t,u)du = \\ &= \ln P_k(0,T) - \int_0^t \left[\int_v^T \alpha_k(v,u)du\right] dv - \int_0^t \left[\int_v^T \sigma_k(v,u)du\right] dW_k^P(v) \\ &+ \int_0^t r_k(v)dv\end{aligned} \tag{2.3.1}$$

Let

$$a_k(t,T) = -\int_t^T \sigma_k(t,u)du \tag{2.3.2}$$

$$b_k(t,T) = -\int_t^T \alpha_k(t,u)du + \frac{1}{2}a_k^2(t,T) \tag{2.3.3}$$

Then we can write

$$\begin{aligned}\ln P_k(t,T) = \ln P_k(0,T) + \int_0^t \left[r_k(v) + b_k(v,T)\right] dv - \frac{1}{2}\int_0^t a_k^2(v,T)dv \\ + \int_0^t a_k(v,T)dW_k^P(v)\end{aligned} \tag{2.3.4}$$

Or

$$d\ln P_k(t,T) = \left[r_k(t) + b_k(t,T) - \frac{1}{2}a_k^2(t,T)\right] dt + a_k(t,T)dW_k^P(t) \tag{2.3.5}$$

Applying Itô's lemma yields the bond price process

$$\begin{aligned}\frac{dP_k(t,T)}{P_k(t,T} &= \left[r_k(t) + b_k(t,T)\right] dt + a_k(t,T)dW_k^P(t) \\ &= \left[r_k(t) - \int_t^T \alpha_k(t,u)du + \frac{1}{2}a_k^2(t,T)\right] dt + a_k(t,T)dW_k^P(t)\end{aligned} \tag{2.3.6}$$

2.3.2 Jarrow Yildirim drift conditions

Nominal bond price

We note that for the nominal drift condition

$$\alpha_n(t,u) = \sigma_n(t,u)\left(\int_t^u \sigma_n(t,s)ds - \lambda_n(t)\right) = \frac{1}{2}\frac{d\left(a_n^2(t,u)\right)}{du} + \frac{d\left(a_n(t,u)\right)}{du}\lambda_n(t) \tag{2.3.7}$$

So that with (2.3.6), under P, the dynamics of the nominal zero coupon bond is given as

$$\frac{dP_n(t,T)}{P_n(t,T} = \left[r_n(t) - a_n(t,T)\lambda_n(t)\right]dt + a_n(t,T)dW_n^P(t) \tag{2.3.8}$$

and under Q

$$\frac{dP_n(t,T)}{P_n(t,T} = r_n(t)dt + a_n(t,T)dW_n^Q(t) \tag{2.3.9}$$

Real bond price

Similarly, for the real drift condition

$$\begin{aligned} \alpha_r(t,u) &= \sigma_r(t,u)\left(\int_t^u \sigma_r(t,s)ds - \sigma_I(t)\rho_{rI} - \lambda_r(t)\right) \\ &= \frac{1}{2}\frac{d\left(a_r^2(t,u)\right)}{du} + \frac{d\left(a_r(t,u)\right)}{du}\left[\sigma_I(t)\rho_{rI} + \lambda_r(t)\right] \end{aligned} \tag{2.3.10}$$

So that under P the dynamics of the real zero coupon bond is given as

$$\frac{dP_r(t,T)}{P_r(t,T} = \left[r_r(t) - a_r(t,T)\left\{\sigma_I(t)\rho_{rI} + \lambda_r(t)\right\}\right]dt + a_r(t,T)dW_r^P(t) \tag{2.3.11}$$

and under Q

$$\frac{dP_r(t,T)}{P_r(t,T} = \left[r_r(t) - a_r(t,T)\sigma_I(t)\rho_{rI}\right]dt + a_r(t,T)dW_r^Q(t) \tag{2.3.12}$$

These results, and applying the drift conditions on the forward rates and the inflation index processes and integration by parts on the process $I(t)P_r(t,T)$, yields the following proposition

Proposition 2.3.1 (Price processes under the martingale measure) *The following price processes hold under the martingale measure*

$$df_n(t,T) = -\sigma_n(t,T)a_n(t,T)dt + \sigma_n(t,T)dW_n^Q(t) \tag{2.3.13}$$

$$df_r(t,T) = -\sigma_r(t,T)\left[a_r(t,T) + \rho_{rI}\sigma_I(t)\right]dt + \sigma_r(t,T)dW_r^Q(t) \tag{2.3.14}$$

$$\frac{dI(t)}{I(t)} = \left[r_n(t) - r_r(t)\right]dt + \sigma_I(t)dW_I^Q(t) \tag{2.3.15}$$

$$\frac{dP_n(t,T)}{P_n(t,T)} = r_n(t)dt + a_n(t,T)dW_n^Q(t) \tag{2.3.16}$$

$$\frac{dP_r(t,T)}{P_r(t,T} = \left[r_r(t) - a_r(t,T)\sigma_I(t)\rho_{rI}\right]dt + a_r(t,T)dW_r^Q(t) \tag{2.3.17}$$

$$\frac{dP_{IL}(t,T)}{P_{IL}(t,T)} := \frac{d(I(t)P_r(t,T))}{I(t)P_r(t,T)} = r_n(t)dt + \sigma_I(t)dW_I^Q(t) + a_r(t,T)dW_r^Q(t) \tag{2.3.18}$$

We note here that with these expressions, the nominal and real forward rates are normally distributed, whereas the inflation index is log-normally distributed.

2.4 Hull-White specification

2.4.1 Nominal term structure

For the nominal economy, Jarrow and Yildirim chose a one factor volatility function with an exponentially declining volatility of the form

$$\sigma_n(t,T) = \sigma_n e^{-\kappa_n(T-t)} \tag{2.4.1}$$

This yields the zero coupon bond volatility as

$$a_n(t,T) = -\int_t^T \sigma_n(t,u)du = -\sigma_n \int_t^T e^{-\kappa_n(u-t)}du = -\sigma_n\beta_n(t,T) \tag{2.4.2}$$

where

$$\beta_n(t,T) = \frac{1}{\kappa_n}\left[1 - e^{-\kappa_n(T-t)}\right] \tag{2.4.3}$$

and that the forward rate under Q evolves as

$$f_n(t,T) = f_n(0,T) + \sigma_n^2 \int_0^t \beta_n(s,T)e^{-\kappa_n(T-s)}ds + \sigma_n \int_0^t e^{-\kappa_n(T-s)}dW_n^Q(s) \tag{2.4.4}$$

, the spot rate as

$$\begin{aligned} r_n(t) = f_n(t,t) &= f_n(0,t) + \sigma_n^2 \int_0^t \beta_n(s,t)e^{-\kappa_n(t-s)}ds + \sigma_n \int_0^t e^{-\kappa_n(t-s)}dW_n^Q(s) \\ &= f_n(0,t) + \frac{\sigma_n^2}{2}\int_0^t \frac{\partial\beta_n^2(s,t)}{\partial t}ds + \sigma_n \int_0^t e^{-\kappa_n(t-s)}dW_n^Q(s) \\ &= f_n(0,t) + \frac{\sigma_n^2}{2}\frac{\partial}{\partial t}\left(\int_0^t \beta_n^2(s,t)ds\right) + \sigma_n \int_0^t e^{-\kappa_n(t-s)}dW_n^Q(s) \end{aligned} \tag{2.4.5}$$

and

$$\int_0^t r_n(u)du = -\ln P_n(0,t) + \frac{\sigma_n^2}{2}\int_0^t \beta_n^2(s,t)ds + \int_0^t \left[\sigma_n \int_0^u e^{-\kappa_n(u-s)}dW_n^Q(s)\right]du \tag{2.4.6}$$

We need to do some work in order to evaluate the double integral. Introducing the process $Y(t) = \int_0^t e^{as}dW_n^Q(s)$ we have

$$d(e^{-at}Y(t)) = e^{-at}dY(t) - ae^{-at}Y(t)dt = dW_n^Q(t) - ae^{-at}Y(t)dt \tag{2.4.7}$$

Integrating, we get

$$e^{-at}Y(t) = W_n^Q(t) - \int_0^t ae^{-au}Y(u)du \tag{2.4.8}$$

Inserting the definition of $Y(\cdot)$ in the expression above yields

$$\begin{aligned}
a\int_0^t \left[e^{-au}\int_0^u e^{as}dW_n^Q(s)\right]du &= W_n^Q(t) - e^{-at}\int_0^t e^{au}dW_n^Q(u) \\
&= \int_0^t \left(1-e^{-a(t-u)}\right)dW_n^Q(u) \\
&= a\int_0^t \beta(u,t)dW_n^Q(u)
\end{aligned} \tag{2.4.9}$$

Applying the result above in (2.4.6), we get

$$\int_0^t r_n(u)du = -\ln P_n(0,t) + \frac{\sigma_n^2}{2}\int_0^t \beta_n^2(s,t)ds + \sigma_n\int_0^t \beta_n(s,t)dW_n^Q(s) \tag{2.4.10}$$

Substituting this in the solution to the zero coupon bond price process yields

$$\begin{aligned}
P_n(t,T) &= P_n(0,T)\exp\left\{\int_0^t \left(r_n(s) - \frac{a_n^2(s,T)}{2}\right)ds + \int_0^t a_n(s,T)dW_n^Q(s)\right\} \\
&= P_n(0,T)\exp\left\{\int_0^t \left(r_n(s) - \frac{\sigma_n^2}{2}\beta_n^2(s,T)\right)ds - \sigma_n\int_0^t \beta_n(s,T)dW_n^Q(s)\right\} \\
&= \frac{P_n(0,T)}{P_n(0,t)}\exp\left\{\frac{\sigma_n^2}{2}\int_0^t \left[\beta_n^2(s,t)-\beta_n^2(s,T)\right]ds + \sigma_n\int_0^t \left[\beta_n(s,t)-\beta_n(s,T)\right]dW_n^Q(s)\right\}
\end{aligned} \tag{2.4.11}$$

Noting from (2.4.5) that

$$\begin{aligned}
-\beta_n(t,T)r_n(t) = -\beta_n(t,T)f_n(0,t) &+ \sigma_n^2\int_0^t \left[\beta_n^2(s,t) - \beta_n(s,T)\beta_n(s,t)\right]ds \\
+ \sigma_n\int_0^t &\left[\beta_n(s,t)-\beta_n(s,T)\right]dW_n^Q(s)
\end{aligned} \tag{2.4.12}$$

, then the term inside the exponential in (2.4.11) simplifies to

$$\begin{aligned}
&\beta_n(t,T)\left[f_n(0,t)-r_n(t)\right] \\
&-\frac{\sigma_n^2}{2}\int_0^t \left[\beta_n^2(s,t)+\beta_n^2(s,T)-2\beta_n(s,T)\beta_n(s,t)\right]ds \\
&= \beta_n(t,T)\left[f_n(0,t)-r_n(t)\right] - \frac{\sigma_n^2}{2}\int_0^t \left[\beta_n(s,t)-\beta_n(s,T)\right]^2 ds \\
&= \beta_n(t,T)\left[f_n(0,t)-r_n(t)\right] - \frac{\sigma_n^2}{4\kappa_n}\beta_n^2(t,T)\left[1-e^{-2\kappa_n t}\right]
\end{aligned} \tag{2.4.13}$$

So that we get the nominal term structure in terms of the short rate

$$P_n(t,T) = \frac{P_n(0,T)}{P_n(0,t)}\exp\left\{\beta_n(t,T)\left[f_n(0,t)-r_n(t)\right] - \frac{\sigma_n^2}{4\kappa_n}\beta_n^2(t,T)\left[1-e^{-2\kappa_n t}\right]\right\} \tag{2.4.14}$$

2.4.2 Real term structure

For the real economy, Jarrow and Yildirim chose again chose we a one factor volatility function with an exponentially declining volatility of the form

$$\sigma_r(t,T) = \sigma_r e^{-\kappa_r(T-t)} \tag{2.4.15}$$

This yields the real zero coupon bond volatility as

$$a_r(t,T) = -\int_t^T \sigma_r(t,u)du = -\int_t^T \sigma_r e^{-\kappa_r(u-t)}du = -\sigma_r\beta_r(t,T) \tag{2.4.16}$$

where

$$\beta_r(t,T) = \frac{1}{\kappa_r}\left[1 - e^{-\kappa_r(T-t)}\right] \tag{2.4.17}$$

For the inflation index process we assume a constant volatility, σ_I. Similar calculations as in the previous section then renders the real term structure as

$$\begin{aligned} P_r(t,T) = \frac{P_r(0,T)}{P_r(0,t)} \exp\left\{\frac{\sigma_r^2}{2}\int_0^t \left[\beta_r^2(s,t) - \beta_r^2(s,T)\right]ds + \sigma_r\int_0^t [\beta_r(s,t) - \beta_r(s,T)]dW_r^Q(s)s\right\} \\ \times \exp\left\{-\rho_{rI}\sigma_I\sigma_r\int_0^t [\beta_r(s,t) - \beta_r(s,T)]ds\right\} \end{aligned} \tag{2.4.18}$$

or in terms of the real short rate

$$P_r(t,T) = \frac{P_r(0,T)}{P_r(0,t)} \exp\left\{\beta_r(t,T)\left[f_r(0,t) - r_r(t)\right] - \frac{\sigma_r^2}{4\kappa_r}\beta_r^2(t,T)\left[1 - e^{-2\kappa_r t}\right]\right\} \tag{2.4.19}$$

2.5 Year-On-Year Inflation Swap

It turns out that it's convenient to derive the price of the inflation leg under the T-forward measure. By (1.2.21) and a change of measure we get

$$\mathbf{YYIIS}(t,T_{i-1},T_i,\psi_i,N) = N\psi_i\left(P_n(t,T_{i-1})E^{T_{i-1}}\left[P_r(T_{i-1},T_i)|\,\mathcal{F}_t\right] - P_n(t,T_i)\right) \tag{2.5.1}$$

So we need to work out the dynamics of $P_r(t,T_2)$ under the T_1-forward measure. Applying the toolkit specified in Proposition (A.1.1) in (2.3.17), we get the following dynamics for $P_r(t,T_2)$ under the T_1-forward measure

$$\frac{dP_r(t,T_2)}{P_r(t,T_2)} = \left[r_r(t) - a_r(t,T_2)\sigma_I(t)\rho_{rI} + a_r(t,T_2)a_n(t,T_1)\rho_{nr}\right]dt + a_r(t,T_1)dW_r^{T_1}(t) \tag{2.5.2}$$

with solution

$$P_r(t,T_2) = P_r(0,T_2)\exp\left\{\int_0^t (r_r(s) - a_r(s,T_2)\sigma_I(s)\rho_{rI} + a_r(s,T_2)a_n(s,T_1)\rho_{nr})\,ds\right\}$$
$$\times \exp\left\{-\int_0^t \frac{a_r^2(s,T_2)}{2}ds + \int_0^t a_r(s,T_2)dW_r^{T_1}(s)\right\} \tag{2.5.3}$$

And after some straightforward calculations we find that

$$\frac{P_r(t,T_2)}{P_r(t,T_1)} = \frac{P_r(0,T_2)}{P_r(0,T_1)}\mathcal{E}\left(\int_0^t [a_r(s,T_2) - a_r(s,T_1)]\,dW_r^{T_1}(s)\right)$$
$$\times \exp\left\{\int_0^t [a_r(s,T_2) - a_r(s,T_1)]\,[a_n(s,T_1)\rho_{nr} - \sigma_I(s)\rho_{rI} - a_r(s,T_1)]\,ds\right\} \tag{2.5.4}$$

where $\mathcal{E}$ denotes the Doléans-Dade exponential, defined as

$$\mathcal{E}(X(t)) = \exp\left\{X(t) - \frac{1}{2}\langle X, X\rangle(t)\right\} \tag{2.5.5}$$

So that, with $t = T_1$ we get

$$P_r(T_1,T_2) = \frac{P_r(0,T_2)}{P_r(0,T_1)}\mathcal{E}\left(\int_0^{T_1} [a_r(s,T_2) - a_r(s,T_1)]\,dW_r^{T_1}(s)\right)$$
$$\times \exp\left\{\int_0^{T_1} [a_r(s,T_2) - a_r(s,T_1)]\,[a_n(s,T_1)\rho_{nr} - \sigma_I(s)\rho_{rI} - a_r(s,T_1)]\,ds\right\} \tag{2.5.6}$$

Or

$$P_r(T_1,T_2) \mid \mathcal{F}_t = \frac{P_r(t,T_2)}{P_r(t,T_1)}\mathcal{E}\left(\int_t^{T_1} [a_r(s,T_2) - a_r(s,T_1)]\,dW_r^{T_1}(s)\right) \times e^{C(t,T_1,T_2)} \tag{2.5.7}$$

, where

$$C(t,T_1,T_2) = \int_t^{T_1} [a_r(s,T_2) - a_r(s,T_1)]\,[a_n(s,T_1)\rho_{nr} - \sigma_I(s)\rho_{rI} - a_r(s,T_1)]\,ds \tag{2.5.8}$$

Hence

$$E^{T_1}[P_r(T_1,T_2) \mid \mathcal{F}_t] = \frac{P_r(t,T_2)}{P_r(t,T_1)}e^{C(t,T_1,T_2)} \tag{2.5.9}$$

So we see that the expectation of the future real zero bond price under the nominal forward measure is equal to the current forward price of the real bond, multiplied by a correction factor. The factor depends on the volatilities and correlations of the nominal rate, the real rate and the inflation index. Applying (2.5.9) in (2.5.1) gives us

$$\mathbf{YYIIS}(t, T_{i-1}, T_i, \psi_i, N) = N\psi_i \left[P_n(t, T_{i-1}) \frac{P_r(t, T_i)}{P_r(t, T_{i-1})} e^{C(t, T_{i-1}, T_i)} - P_n(t, T_i) \right] \tag{2.5.10}$$

Straightforward calculations shows that the correction term can be explicitly computed as

$$\begin{aligned} C(t, T_{i-1}, T_i) = \sigma_r \beta_r(T_{i-1}, T_i) \bigg[\beta_r(t, T_{i-1}) \bigg(\rho_{r,I}\sigma_I - \frac{1}{2}\beta_r(t, T_{i-1}) \\ + \frac{\rho_{n,r}\sigma_n}{\kappa_n + \kappa_r} \left(1 + \kappa_r \beta_n(t, T_{i-1})\right) \bigg) - \frac{\rho_{n,r}\sigma_n}{\kappa_n + \kappa_r} \beta_n(t, T_{i-1}) \bigg] \end{aligned} \tag{2.5.11}$$

This term accounts for the stochasticity of real rates. Indeed it vanishes for $\sigma_r = 0$.

The time t value of the inflation linked leg is obtained by summing up the values of all payments.

$$\begin{aligned} \mathbf{YYIIS}(t, \mathcal{T}, \Psi, N) = N\psi_{\iota(t)} \left[\frac{I(t)}{I(T_{\iota(t)-1})} P_r(t, T_{\iota(t)}) - P_n(t, T_{\iota(t)}) \right] \\ + N \sum_{i=\iota(t)+1}^{M} \psi_i \left[P_n(t, T_{i-1}) \frac{P_r(t, T_i)}{P_r(t, T_{i-1})} e^{C(t, T_{i-1}, T_i)} - P_n(t, T_i) \right] \end{aligned} \tag{2.5.12}$$

where we set $\mathcal{T} := \{T_1, \cdots, T_M\}$, $\Psi := \{\psi_1, \cdots, \psi_M\}$, $\iota(t) = \min\{i : T_i > t\}$ and where the first cash flow has been priced according to the zero coupon inflation leg formula derived in (1.2.13). Speficially, at $t = 0$

$$\begin{aligned} \mathbf{YYIIS}(0, \mathcal{T}, \Psi, N) = N\psi_1 \left[P_r(0, T_1) - P_n(0, T_1) \right] \\ + N \sum_{i=2}^{M} \psi_i \left[P_n(0, T_{i-1}) \frac{P_r(0, T_i)}{P_r(0, T_{i-1})} e^{C(0, T_{i-1}, T_i)} - P_n(0, T_i) \right] \end{aligned} \tag{2.5.13}$$

The advantage of the Jarrow-Yildirim model is the simple closed formula it results in. However, the dependence on the real rate parameters, such as the volatility of real rates is a significant drawback, as it is not easily estimated.

2.6 Inflation Linked Cap/Floor

We recall that the inflation index, $I(t)$, is log-normally distributed under Q. Under the nominal forward measure, the inflation index $\frac{I(T_i)}{I(T_{i-1})}$ preserves a log-normal distribution. Thus, (1.2.23) can be calculated when we know the expectation of the ratio and the variance of it's logarithm. Using the fact that if X is a log-normally distributed random variable with $E[X] = m$ and $\mathrm{Std}[\ln(X)] = v$ then

$$E\left[[\omega\,(X-K)]^{+}\right] = \omega m\Phi\left(\omega\frac{\ln\frac{m}{K}+\frac{1}{2}v^2}{v}\right) - \omega K\Phi\left(\omega\frac{\ln\frac{m}{K}-\frac{1}{2}v^2}{v}\right) \tag{2.6.1}$$

The conditional expectation of $I(T_i)/I(T_{i-1})$ is obtained directly from (1.2.19) and (2.5.10)

$$E^{T_i}\left[\left.\frac{I(T_i)}{I(T_{i-1})}\right|\mathcal{F}_t\right] = \frac{P_n(t,T_{i-1})}{P_n(t,T_i)}\frac{P_r(t,T_i)}{P_r(t,T_{i-1})}e^{C(t,T_{i-1},T_i)} \tag{2.6.2}$$

Since a change of measure has no impact on the variance, it can be equivalently calculated under the martingale measure. By standard calculations it can then be shown that

$$\mathrm{Var}^{T_i}\left[\ln\left.\frac{I(T_i)}{I(T_{i-1})}\right|\mathcal{F}_t\right] = V^2(t,T_{i-1},T_i) \tag{2.6.3}$$

, where

$$\begin{aligned}
V^2(t,T_{i-1},T_i) &= \frac{\sigma_n^2}{2\kappa_n}\beta_n^2(T_{i-1},T_i)\left[1-e^{-2\kappa_n(T_{i-1}-t)}\right] + \sigma_I^2(T_i-T_{i-1})\\
&+ \frac{\sigma_r^2}{2\kappa_r}\beta_r^2(T_{i-1},T_i)\left[1-e^{-2\kappa_r(T_{i-1}-t)}\right]\\
&- 2\rho_{nr}\frac{\sigma_n\sigma_r}{(\kappa_n+\kappa_r)}\beta_n(T_{i-1},T_i)\beta_r(T_{i-1},T_i)\left[1-e^{-(\kappa_n+\kappa_r)(T_{i-1}-t)}\right]\\
&+ \frac{\sigma_n^2}{\kappa_n^2}\left[T_i-T_{i-1}+\frac{2}{\kappa_n}e^{-\kappa_n(T_i-T_{i-1})}-\frac{1}{2\kappa_n}e^{-2\kappa_n(T_i-T_{i-1})}-\frac{3}{2\kappa_n}\right]\\
&+ \frac{\sigma_r^2}{\kappa_r^2}\left[T_i-T_{i-1}+\frac{2}{\kappa_r}e^{-\kappa_r(T_i-T_{i-1})}-\frac{1}{2\kappa_r}e^{-2\kappa_r(T_i-T_{i-1})}-\frac{3}{2\kappa_r}\right]\\
&- 2\rho_{nr}\frac{\sigma_n\sigma_r}{\kappa_n\kappa_r}\left[T_i-T_{i-1}-\beta_n(T_{i-1},T_i)-\beta_r(T_{i-1},T_i)+\frac{1-e^{-(\kappa_n+\kappa_r)(T_i-T_{i-1})}}{\kappa_n+\kappa_r}\right]\\
&+ 2\rho_{nI}\frac{\sigma_n\sigma_I}{\kappa_n}\left[T_i-T_{i-1}-\beta_n(T_{i-1},T_i)\right] - 2\rho_{rI}\frac{\sigma_r\sigma_I}{\kappa_r}\left[T_i-T_{i-1}-\beta_r(T_{i-1},T_i)\right]
\end{aligned} \tag{2.6.4}$$

The quantities derived in this section then yields the Caplet/Floorlet price as

$$\begin{aligned}
&\mathbf{ILCFLT}(t,T_{i-1},T_i,\psi_i,K,N,\omega) =\\
&\omega N\psi_i P_n(t,T_i)\left[\frac{P_n(t,T_{i-1})}{P_n(t,T_i)}\frac{P_r(t,T_i)}{P_r(t,T_{i-1})}e^{C(t,T_{i-1},T_i)}\Phi\left(\omega d_1^i(t)\right) - K\Phi\left(\omega d_2^i(t)\right)\right]\\
&d_1^i(t) = \frac{\ln\frac{P_n(t,T_{i-1})}{KP_n(t,T_i)}\frac{P_r(t,T_i)}{P_r(t,T_{i-1})} + C(t,T_{i-1},T_i) + \frac{1}{2}V^2(t,T_{i-1},T_i)}{V(t,T_{i-1},T_i)}\\
&d_2^i(t) = d_1 - V(t,T_{i-1},T_i)
\end{aligned} \tag{2.6.5}$$

Just as for the year-on-year inflation swap, the price depends on the volatility of real rates. In the following sections we will present two market models that have been proposed as an alternative for valuation of inflation linked instruments. The models strive to arrive at valuation formulas , where the input parameters are more easily determined than in the short rate approach of the Jarrow-Yildirim model.

Chapter 3

Market Model I - A Libor Market Model for nominal and real forward rates

3.1 Year-On-Year Inflation Swap

By a change of measure, the expectation in (1.2.21) can be rewritten as

$$\begin{aligned} P_n(t, T_{i-1}) E^{T_{i-1}} \left[P_r(T_{i-1}, T_i) \middle| \mathcal{F}_t \right] &= P_n(t, T_i) E^{T_i} \left[\frac{P_r(T_{i-1}, T_i)}{P_n(T_{i-1}, T_i)} \middle| \mathcal{F}_t \right] \\ = P_n(t, T_i) E^{T_i} &\left[\frac{1 + \tau_i \cdot F_n(T_{i-1}, T_{i-1}, T_i)}{1 + \tau_i \cdot F_r(T_{i-1}, T_{i-1}, T_i)} \middle| \mathcal{F}_t \right] \end{aligned} \tag{3.1.1}$$

where τ_i denotes year fraction between T_{i-1} and T_i and $F_k : k \in \{n, r\}$ denotes the simply compounded forward rate. The expectation can be evaluated if we know the distribution of simply compounded nominal and real forward rates under the nominal T_i-forward measure. This inspired Mercurio[2] to choose them as the quantities to model, with the following dynamics under $Q_n^{T_i}$

$$\frac{dF_n(t, T_{i-1}, T_i)}{F_n(t, T_{i-1}, T_i)} = \sigma_{n,i} dW_{n,i}(t) \tag{3.1.2}$$

And under $Q_r^{T_i}$

$$\frac{dF_r(t, T_{i-1}, T_i)}{F_r(t, T_{i-1}, T_i)} = \sigma_{r,i} dW_{r,i}(t) \tag{3.1.3}$$

To obtain the dynamics of the real forward rate under $Q_n^{T_i}$, we compute the drift adjustment using Proposition (A.1.1) to find that under $Q_n^{T_i}$

$$\frac{dF_r(t, T_{i-1}, T_i)}{F_r(t, T_{i-1}, T_i)} = -\sigma_{r,i}\sigma_{I,i}\rho_{I,r,i} dt + \sigma_{r,i} dW_{r,i}(t) \tag{3.1.4}$$

where $\sigma_{n,i}$ and $\sigma_{r,i}$ are positive constants and $\rho_{I,r,i}$ is the instantaneous correlation between $I(\cdot, T_i)$ and $F_r(\cdot, T_{i-1}, T_i)$.

Since $I(t)P_r(t,T)$ is the price of the inflation linked bond, which is a traded asset in the nominal economy, it holds that the forward inflation index

$$\mathcal{I}(t,T_i) = I(t)\frac{P_r(t,T_i)}{P_n(t,T_i)} \tag{3.1.5}$$

is a martingale under $Q_n^{T_i}$, where it is proposed to follow log-normal dynamics

$$\frac{d\mathcal{I}(t,T_i)}{\mathcal{I}(t,T_i)} = \sigma_{I,i}dW_{I,i}(t) \tag{3.1.6}$$

where $\sigma_{I,i}$ is a positive constant and $W_{I,i}$ is a $Q_n^{T_i}$ brownian motion.

Mercurio noted that under $Q_n^{T_i}$ and conditional on $\mathcal{F}_t$ the pair

$$(X_i, Y_i) = \left(\ln\frac{F_n(T_{i-1},T_{i-1},T_i)}{F_n(t,T_{i-1},T_i)}, \ln\frac{F_r(T_{i-1},T_{i-1},T_i)}{F_r(t,T_{i-1},T_i)}\right) \tag{3.1.7}$$

is distributed as a bivariate normal random variable with mean vector and covariance matrix, respectively given by

$$M_{X_i,Y_i} = \begin{bmatrix}\mu_{x,i}(t)\\ \mu_{y,i}(t)\end{bmatrix}, \quad V_{X_i,Y_i} = \begin{bmatrix}\sigma_{x,i}^2(t) & \rho_{n,r,i}\sigma_{x,i}(t)\sigma_{y,i}(t)\\ \rho_{n,r,i}\sigma_{x,i}(t)\sigma_{y,i}(t) & \sigma_{y,i}^2(t)\end{bmatrix} \tag{3.1.8}$$

where

$$\begin{aligned}
\mu_{x,i}(t) &= -\frac{1}{2}\sigma_{n,i}^2(T_{i-1}-t), \quad \sigma_{x,i}(t) = \sigma_{n,i}\sqrt{(T_{i-1}-t)}\\
\mu_{y,i}(t) &= \left[-\frac{1}{2}\sigma_{r,i}^2 - \rho_{I,r,i}\sigma_{I,i}\sigma_{r,i}\right](T_{i-1}-t), \quad \sigma_{y,i}(t) = \sigma_{r,i}\sqrt{(T_{i-1}-t)}
\end{aligned}$$

We recall the fact that the bivariate density $f_{X_i,Y_i}(x,y)$ of (X_i,Y_i) can be decomposed in terms of the conditional density $f_{X_i|Y_i}(x,y)$ as

$$f_{X_i,Y_i}(x,y) = f_{X_i|Y_i}(x,y)f_{Y_i}(y)$$

where

$$\begin{aligned}
f_{X_i|Y_i}(x,y) &= \frac{1}{\sigma_{x,i}(t)\sqrt{2\pi}\sqrt{1-\rho_{n,r,i}^2}}\exp\left\{-\frac{\left[\frac{x-\mu_{x,i}(t)}{\sigma_{x,i}(t)} - \rho_{n,r,i}\frac{y-\mu_{y,i}(t)}{\sigma_{y,i}(t)}\right]^2}{2(1-\rho_{n,r,i}^2)}\right\}\\
f_{Y_i}(y) &= \frac{1}{\sigma_{y,i}(t)\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left[\frac{y-\mu_{y,i}(t)}{\sigma_{y,i}(t)}\right]^2\right\}
\end{aligned}$$

Noting that

$$\begin{aligned}
F_n(T_{i-1},T_{i-1},T_i) &= e^{X_i}F_n(t,T_{i-1},T_i)\\
F_r(T_{i-1},T_{i-1},T_i) &= e^{Y_i}F_r(t,T_{i-1},T_i)
\end{aligned}$$

, the expectation in (3.1.1) can be calculated as

$$\int_{-\infty}^{+\infty} \frac{\int_{-\infty}^{+\infty}(1+\tau_i F_n(t,T_{i-1},T_i)e^x) f_{X_i|Y_i}(x,y)dx}{1+\tau_i F_r(t,T_{i-1},T_i)e^y} f_{Y_i}(y)dy$$

$$= \int_{-\infty}^{+\infty} \frac{1+\tau_i F_n(t,T_{i-1},T_i)e^{\mu_{x,i}(t)+\rho_{n,r,i}\sigma_{x,i}(t)\frac{y-\mu_{y,i}(t)}{\sigma_{y,i}(t)}+\frac{1}{2}\sigma_{x,i}^2(t)[1-\rho_{n,r,i}^2]}}{1+\tau_i F_r(t,T_{i-1},T_i)e^y} f_{Y_i}(y)dy$$

$$= \left\{ \text{by } \mu_{x,i}(t) = -\frac{1}{2}\sigma_{x,i}^2(t) \text{ and variable substitution } z = \frac{y-\mu_{y,i}(t)}{\sigma_{y,i}(t)} \right\}$$

$$= \int_{-\infty}^{+\infty} \frac{1+\tau_i F_n(t,T_{i-1},T_i)e^{\rho_{n,r,i}\sigma_{x,i}(t)z-\frac{1}{2}\sigma_{x,i}^2(t)\rho_{n,r,i}^2}}{1+\tau_i F_r(t,T_{i-1},T_i)e^{\mu_{y,i}(t)+\sigma_{y,i}(t)z}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

so that

$$\begin{aligned} &\mathbf{YYIS}(t,T_{i-1},T_i,\psi_i,N) \\ &= N\psi_i P_n(t,T_i) \int_{-\infty}^{+\infty} \frac{1+\tau_i F_n(t,T_{i-1},T_i)e^{\rho_{n,r,i}\sigma_{x,i}(t)z-\frac{1}{2}\sigma_{x,i}^2(t)\rho_{n,r,i}^2}}{1+\tau_i F_r(t,T_{i-1},T_i)e^{\mu_{y,i}(t)+\sigma_{y,i}(t)z}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \\ &- N\psi_i P_n(t,T_i) \end{aligned} \tag{3.1.9}$$

Some care needs to be taken when valuing the whole inflation leg. We can't simply sum up the values in (3.1.9). To see this, note that by (3.1.5) and the assumption of simply compounded rates we have

$$\frac{\mathcal{I}(t,T_i)}{\mathcal{I}(t,T_{i-1})} = \frac{1+\tau_i F_n(t,T_{i-1},T_i)}{1+\tau_i F_r(t,T_{i-1},T_i)} \tag{3.1.10}$$

Thus, if we assume that $\sigma_{I,i}$, $\sigma_{n,i}$ and $\sigma_{r,i}$ are positive constants then $\sigma_{I,i-1}$ cannot be constant as well. It's admissable values are obtained by equating the quadratic variations on both side of (3.1.10). For instance, if nominal and real forward rates as well as the forward inflation index were driven by the same brownian motion, then equating the quadriatic variations in (3.1.10) yields

$$\sigma_{I,i-1} = \sigma_{I,i} + \sigma_{r,i}\frac{\tau_i F_r(t,T_{i-1},T_i)}{1+\tau_i F_r(t,T_{i-1},T_i)} - \sigma_{n,i}\frac{\tau_i F_n(t,T_{i-1},T_i)}{1+\tau_i F_n(t,T_{i-1},T_i)}$$

Mercurio applied a "freezing procedure" where the forward rates in the diffusion coefficient on the right hand side of (3.1.10) are frozen at their time 0 value, so that we can still get forward inflation index volatilities that are approximately constant. In the case where all processes are driven by the same brownian motion, equating the quadratic variations would yield

$$\sigma_{I,i-1} \approx \sigma_{I,i} + \sigma_{r,i}\frac{\tau_i F_r(0,T_{i-1},T_i)}{1+\tau_i F_r(0,T_{i-1},T_i)} - \sigma_{n,i}\frac{\tau_i F_n(0,T_{i-1},T_i)}{1+\tau_i F_n(0,T_{i-1},T_i)}$$

Thus, applying this approximation for each i, we can still assume that the volatilities $\sigma_{I,i}$ are all constant. The time t value of the inflation leg is then given by

$$\begin{aligned}
&\mathbf{YYIIS}(t,\mathcal{T},\Psi,N) = N\psi_{\iota(t)}\left[\frac{I(t)}{I(T_{\iota(t)-1})}P_r(t,T_{\iota(t)}) - P_n(t,T_{\iota(t)})\right]\\
&+N\sum_{\iota(t)+1}^{M}\psi_i P_n(t,T_i)\times\left[\int_{-\infty}^{+\infty}\frac{1+\tau_i F_n(t,T_{i-1},T_i)e^{\rho_{n,r,i}\sigma_{x,i}(t)z-\frac{1}{2}\sigma_{x,i}^2(t)\rho_{n,r,i}^2}}{1+\tau_i F_r(t,T_{i-1},T_i)e^{\mu_{y,i}(t)+\sigma_{y,i}(t)z}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz-1\right]
\end{aligned}\tag{3.1.11}$$

where we set $\mathcal{T} := \{T_1,\cdots,T_M\}$, $\Psi := \{\psi_1,\cdots,\psi_M\}$, $\iota(t) = \min\{i : T_i > t\}$ and where the first cash flow has been priced according to the zero coupon inflation leg formula derived in (1.2.13).

At $t = 0$ we get

$$\begin{aligned}
&\mathbf{YYIIS}(0,\mathcal{T},\Psi,N) = N\psi_1\left[P_r(0,T_1) - P_n(0,T_1)\right]\\
&+N\sum_{2}^{M}\psi_i P_n(0,T_i)\times\left[\int_{-\infty}^{+\infty}\frac{1+\tau_i F_n(0,T_{i-1},T_i)e^{\rho_{n,r,i}\sigma_{x,i}(0)z-\frac{1}{2}\sigma_{x,i}^2(0)\rho_{n,r,i}^2}}{1+\tau_i F_r(0,T_{i-1},T_i)e^{\mu_{y,i}(0)+\sigma_{y,i}(0)z}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz-1\right]\\
&=N\sum_{1}^{M}\psi_i P_n(0,T_i)\times\left[\int_{-\infty}^{+\infty}\frac{1+\tau_i F_n(0,T_{i-1},T_i)e^{\rho_{n,r,i}\sigma_{x,i}(0)z-\frac{1}{2}\sigma_{x,i}^2(0)\rho_{n,r,i}^2}}{1+\tau_i F_r(0,T_{i-1},T_i)e^{\mu_{y,i}(0)+\sigma_{y,i}(0)z}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz-1\right]
\end{aligned}\tag{3.1.12}$$

The price depends on the following parameters: the instantaneous volatilities of nominal and real forward rates and their correlations for each payment time T_i : $\{1 < i <= M\}$; and the volatilities of forward inflation indices and their correlations with real forward rates for each payment time $T_i : \{1 < i <= M\}$.

This formula looks looks more complicated than (2.5.12) both in terms of input parameters and the calculations involved. Even with approximations made, we fail to arrive at a closed-form valuation formula for a benchmark inflation derivative. And as in the Jarrow and Yilidrim model, the price depends on a number of real rate parameters that may be difficult to estimate.

3.2 Inflation Linked Cap/Floor

Applying iterated expectation on (1.2.23) we get

$$\begin{aligned}
&\mathbf{ILCFLT}(t,T_{i-1},T_i,\psi_i,K,N,\omega)\\
&=N\psi_i P_n(t,T_i)E^{T_i}\left[\left[\omega\left(\frac{I(T_i)}{I(T_{i-1})}-K\right)\right]^+\middle|\mathcal{F}_t\right]\\
&=N\psi_i P_n(t,T_i)E^{T_i}\left[E^{T_i}\left[\left[\omega\left(\frac{I(T_i)}{I(T_{i-1})}-K\right)\right]^+\middle|\mathcal{F}_{T-1}\right]\middle|\mathcal{F}_t\right]\\
&=N\psi_i P_n(t,T_i)E^{T_i}\left[\frac{1}{I(T_{i-1})}E^{T_i}\left[\left[\omega\left(I(T_i)-I(T_{i-1})K\right)\right]^+\middle|\mathcal{F}_{T-1}\right]\middle|\mathcal{F}_t\right]
\end{aligned}\tag{3.2.1}$$

It is clear the the evaluation of the outer expectation depends on if one models the forward inflation index(as presented in Market Model II), or the forward rates, which is the approach of Market Model I.

Assuming log-normal dynamics of the forward inflation index as defined in (3.1.6) and using that $I(T_i) = \mathcal{I}(T_i, T_i)$, yields the inner expectation as

$$E^{T_i}\left[[\omega\left(I(T_i) - KI(T_{i-1})\right)]^+ \middle| \mathcal{F}_{T-1} \right] = E^{T_i}\left[[\omega\left(\mathcal{I}(T_i, T_i) - K\mathcal{I}(T_{i-1}, T_{i-1})\right)]^+ \middle| \mathcal{F}_{T-1} \right]$$
$$= \omega\mathcal{I}(T_{i-1}, T_i)\Phi\left(\omega\frac{\ln\frac{\mathcal{I}(T_{i-1},T_i)}{K\mathcal{I}(T_{i-1},T_{i-1})} + \frac{1}{2}\sigma_{I,i}^2(T_i - T_{i-1})}{\sigma_{I,i}\sqrt{T_i - T_{i-1}}}\right)$$
$$- \omega KI(T_{i-1})\Phi\left(\omega\frac{\ln\frac{\mathcal{I}(T_{i-1},T_i)}{K\mathcal{I}(T_{i-1},T_{i-1})} - \frac{1}{2}\sigma_{I,i}^2(T_i - T_{i-1})}{\sigma_{I,i}\sqrt{T_i - T_{i-1}}}\right)$$

Hence

$$\begin{aligned}
&\mathbf{ILCFLT}(t, T_{i-1}, T_i, \psi_i, K, N, \omega) \\
&= \omega N\psi_i P_n(t, T_i)E^{T_i}\left[\frac{\mathcal{I}(T_{i-1}, T_i)}{\mathcal{I}(T_{i-1}, T_{i-1})}\Phi\left(\omega\frac{\ln\frac{\mathcal{I}(T_{i-1},T_i)}{K\mathcal{I}(T_{i-1},T_{i-1})} + \frac{1}{2}\sigma_{I,i}^2(T_i - T_{i-1})}{\sigma_{I,i}\sqrt{T_i - T_{i-1}}}\right)\right. \\
&\left. -K\Phi\left(\omega\frac{\ln\frac{\mathcal{I}(T_{i-1},T_i)}{K\mathcal{I}(T_{i-1},T_{i-1})} - \frac{1}{2}\sigma_{I,i}^2(T_i - T_{i-1})}{\sigma_{I,i}\sqrt{T_i - T_{i-1}}}\right)\middle| \mathcal{F}_t\right]
\end{aligned} \tag{3.2.2}$$

And by the definition of $\mathcal{I}(T_{i-1}, T_{i-1})$ in (3.1.5), and the choice to model simply compounded real and nominal forward rates, we note that

$$\frac{\mathcal{I}(T_{i-1}, T_i)}{\mathcal{I}(T_{i-1}, T_{i-1})} = \frac{1 + \tau_i F_n(T_{i-1}, T_{i-1}, T_i)}{1 + \tau_i F_r(T_{i-1}, T_{i-1}, T_i)} \tag{3.2.3}$$

, so that we get the Caplet/Floorlet price as

$$\begin{aligned}
&\mathbf{ILCFLT}(t, T_{i-1}, T_i, \psi_i, K, N, \omega) \\
&= \omega N\psi_i P_n(t, T_i)E^{T_i}\left[\frac{1 + \tau_i F_n(T_{i-1}, T_{i-1}, T_i)}{1 + \tau_i F_r(T_{i-1}, T_{i-1}, T_i)}\Phi\left(\omega d_1^i(t)\right) - K\Phi\left(\omega d_2^i(t)\right)\middle| \mathcal{F}_t\right] \\
&d_1^i(t) = \frac{\ln\frac{1+\tau_i F_n(T_{i-1},T_{i-1},T_i)}{K[1+\tau_i F_r(T_{i-1},T_{i-1},T_i)]} + \frac{1}{2}\sigma_{I,i}^2(T_i - T_{i-1})}{\sigma_{I,i}\sqrt{T_i - T_{i-1}}} \\
&d_2^i(t) = d_1^i(t) - \sigma_{I,i}\sqrt{(T_i - T_{i-1})}
\end{aligned} \tag{3.2.4}$$

We recall the assumption that nominal and real forward rates evolve according to (3.1.2) and (3.1.4) and utilize Mercurios freezing procedure described earlier, yielding constant forward inflation index volatilities volatilities. Again we use that, under these assumptions, the pair (3.1.7) is distributed as a bivariate normal random variable with mean vector and covarience matrix given by (3.1.8). And so we can evaluate the expectation in (3.2.4).

The dimensionality of the problem can be reduced by assuming deterministic real rates. As a consequence, the future rate $F_r(T_{i-1}, T_{i-1}, T_i)$ is simply equal to the current forward rate $F_r(t, T_{i-1}, T_i)$, so that we can write the Caplet/Floorlet price as

$$\begin{aligned}&\mathbf{ILCFLT}(t, T_{i-1}, T_i, \psi_i, K, N, \omega)\\&= \omega N \psi_i P_n(t, T_i) E^{T_i}\left[\frac{1 + \tau_i F_n(T_{i-1}, T_{i-1}, T_i)}{1 + \tau_i F_r(t, T_{i-1}, T_i)} \Phi\left(\omega d_1^i(t)\right) - K\Phi\left(\omega d_2^i(t)\right) \middle| \mathcal{F}_t \right]\end{aligned} \tag{3.2.5}$$

And since the nominal forward rate $F_n(\cdot, T_{i-1}, T_i)$ evolves as specified in (3.1.2), we have

$$\begin{aligned}&\mathbf{ILCFLT}(t, T_{i-1}, T_i, \psi_i, K, N, \omega)\\&= \omega N \psi_i P_n(t, T_i) \int_{-\infty}^{\infty} J(x) \frac{1}{\sigma_{n,i}\sqrt{2\pi(T_{i-1} - t)}} e^{-\frac{1}{2}\left(\frac{x + \frac{1}{2}\sigma_{n,i}^2(T_{i-1}-t)}{\sigma_{n,i}\sqrt{T_{i-1}-t}}\right)^2} dx\end{aligned} \tag{3.2.6}$$

where

$$\begin{aligned}J(x) :=& \frac{1 + \tau_i F_n(t, T_{i-1}, T_i)e^x}{1 + \tau_i F_r(t, T_{i-1}, T_i)} \Phi\left(\omega \frac{\ln \frac{1+\tau_i F_n(t,T_{i-1},T_i)e^x}{K[1+\tau_i F_r(t,T_{i-1},T_i)]} + \frac{1}{2}\sigma_{I,i}^2(T_i - T_{i-1})}{\sigma_{I,i}\sqrt{T_i - T_{i-1}}}\right)\\&- K\Phi\left(\omega \frac{\ln \frac{1+\tau_i F_n(t,T_{i-1},T_i)e^x}{K[1+\tau_i F_r(t,T_{i-1},T_i)]} - \frac{1}{2}\sigma_{I,i}^2(T_i - T_{i-1})}{\sigma_{I,i}\sqrt{T_i - T_{i-1}}}\right)\end{aligned}$$

The time 0 price of the Inflation Indexed Cap/Floor is then obtained by summing up the respective caplets/floorlets

$$\begin{aligned}&\mathbf{ILCF}(0, \mathcal{T}, \Psi, K, N, \omega) = \sum_{i=1}^{M} \mathbf{ILCFT}(0, T_{i-1}, T_i, \psi_i, K, N, \omega)\\&= \omega N \psi_1 \left[P_r(0, T_1)\Phi\left(\omega d_1^i(0)\right) - K P_n(0, T_1)\Phi\left(\omega d_2^i(0)\right)\right]\\&+ \omega N \sum_{i=2}^{M} \psi_i P_n(0, T_i) \int_{-\infty}^{\infty} J(0, x) \frac{1}{\sigma_{n,i}\sqrt{2\pi T_{i-1}}} e^{-\frac{1}{2}\left(\frac{x + \frac{1}{2}\sigma_{n,i}^2 T_{i-1}}{\sigma_{n,i}\sqrt{T_{i-1}}}\right)^2} dx\end{aligned} \tag{3.2.7}$$

The advantage of Market Model I, is that it is based on modeling observable quantities, i.e. the individual forward rates. A clear disadvantage in comparison with the Jarrow-Yildirim model is the , relatively, more complicated Caplet/Floorlet prices. Even though we made the unrealistic simplification that real rates are deterministic, we still end up with a "non closed-form" price formula.

Chapter 4

Market Model II - Modeling the forward inflation indices

4.1 Year-On-Year Inflation Swap

Both the Jarrow-Yildirim Model and Market Model I share the drawback that they depend on the volatility of real rates, which might be a difficult parameter to estimate. To remedy this, a second market model has been proposed by Mercurio[2] and Belgrade, Benhamou, and Koehler[4]. In Market Model I, Mercurio modeled the respective nominal and real forward rates for each forward date T_i. *The core property of Market Model II is the choice to model each respective forward inflation index* $\mathcal{I}(\cdot, T_i)$.

Using that $I(T_i) = \mathcal{I}(T_i, T_i)$ and that $\mathcal{I}(t, T_i)$ is a martingale under $Q_n^{T_i}$ we can write, for $t < T_{i-1}$

$$\begin{aligned}
\mathbf{YYIIS}(t, T_{i-1}, T, \psi_i, N) &= N\psi_i P_n(t, T_i) E^{T_i}\left[\frac{I(T_i)}{I(T_{i-1})} - 1 \middle| \mathcal{F}_t\right] \\
&= N\psi_i P_n(t, T_i) E^{T_i}\left[\frac{\mathcal{I}(T_i, T_i)}{\mathcal{I}(T_{i-1}, T_{i-1})} - 1 \middle| \mathcal{F}_t\right] \\
&= N\psi_i P_n(t, T_i) E^{T_i}\left[\frac{\mathcal{I}(T_{i-1}, T_i)}{\mathcal{I}(T_{i-1}, T_{i-1})} - 1 \middle| \mathcal{F}_t\right]
\end{aligned} \tag{4.1.1}$$

The dynamics of $\mathcal{I}(t, T_i)$ under $Q_n^{T_i}$ is given by (3.1.6). Applying the toolkit in proposition (A.1.1) yields the dynamics of $\mathcal{I}(t, T_{i-1})$ under $Q_n^{T_i}$ as

$$\frac{d\mathcal{I}(t, T_{i-1})}{\mathcal{I}(t, T_{i-1})} = \sigma_{I,i-1}\left[-\frac{\tau_i \sigma_{n,i} F_n(t, T_{i-1}, T_i)}{1 + F_n(t, T_{i-1}, T_i)}\rho_{I,n,i} dt + dW_{I,i-1}(t)\right] \tag{4.1.2}$$

where $\sigma_{I,i-1}$ is a positive constant, $W_{I,i-1}$ is a $Q_n^{T_i}$-Brownian motion with

$$dW_{I,i-1}(t) dW_{I,i}(t) = \rho_{I,i} dt$$

and $\rho_{I,n,i}$ is the instantaneous correlation between $\mathcal{I}(\cdot, T_{i-1})$ and $F_n(\cdot, T_{i-1}, T_i)$

We see that the dynamics of $\mathcal{I}(\cdot, T_{i-1})$ under $Q_n^{T_i}$ depends on the nominal forward rate $F_n(t, T_{i-1}, T_i)$. To simplify the calculation in (4.1.1), Mercurio proposed to freeze the drift in (4.1.2) at it's current time t value. By this freezing procedure $\mathcal{I}(T_{i-1}, T_{i-1})|\mathcal{F}_t$ is lognormally distributed also under $Q_n^{T_i}$. And since integration by parts on $d\frac{\mathcal{I}(t,T_i)}{\mathcal{I}(t,T_{i-1})}$ yields

$$\begin{aligned} d\frac{\mathcal{I}(t, T_i)}{\mathcal{I}(t, T_{i-1})} &= \sigma_{I,i-1}\left[\frac{\tau_i\sigma_{n,i}F_n(t, T_{i-1}, T_i)}{1 + F_n(t, T_{i-1}, T_i)}\rho_{I,n,i} + \sigma_{I,i-1} - \sigma_{I,i}\rho_{I,i}\right]dt \\ &+ \sigma_{I,i-1}dW_{I,i-1}(t) + \sigma_{I,i}dW_{I,i}(t) \end{aligned} \tag{4.1.3}$$

, freezing the drift at it's time-t value and noting the resulting log-normality of $\frac{\mathcal{I}(t,T_i)}{\mathcal{I}(t,T_{i-1})}$, enables us to calculate the expectation in (4.1.1) as

$$E_n^{T_i}\left[\frac{\mathcal{I}(T_{i-1}, T_i)}{\mathcal{I}(T_{i-1}, T_{i-1})}\middle| \mathcal{F}_t\right] = \frac{\mathcal{I}(t, T_i)}{\mathcal{I}(t, T_{i-1})}e^{D_i(t)}$$

where

$$D_i(t) = \sigma_{I,i-1}\left[\frac{\tau_i\sigma_{n,i}F_n(t, T_{i-1}, T_i)}{1 + \tau_iF_n(t, T_{i-1}, T_i)}\rho_{I,n,i} + \sigma_{I,i-1} - \sigma_{I,i}\rho_{I,i}\right](T_{i-1} - t)$$

Thus

$$\begin{aligned} \mathbf{YYIIS}(t, T_{i-1}, T_i, \psi_i, N) &= N\psi_iP_n(t, T_i)\left[\frac{\mathcal{I}(t, T_i)}{\mathcal{I}(t, T_{i-1})}e^{D_i(t)} - 1\right] \\ &= N\psi_iP_n(t, T_i)\left[\frac{P_n(t, T_{i-1})P_r(t, T_i)}{P_r(t, T_{i-1})P_n(t, T_i)}e^{D_i(t)} - 1\right] \end{aligned} \tag{4.1.4}$$

And we get the value of the inflation leg as

$$\begin{aligned} \mathbf{YYIIS}(t, \mathcal{T}, \Psi, N) &= N\psi_{\iota(t)}P_n(t, T_{\iota(t)})\left[\frac{\mathcal{I}(t, T_{\iota(t)})}{I(T_{\iota(t)-1})} - 1\right] \\ &+ N\sum_{i=\iota(t)+1}^{M}\psi_iP_n(t, T_i)\left[\frac{\mathcal{I}(t, T_i)}{\mathcal{I}(t, T_{i-1})}e^{D_i(t)} - 1\right] \\ &= N\psi_{\iota(t)}\left[\frac{I(t)}{I(T_{\iota(t)-1})}P_r(t, T_{\iota(t)}) - P_n(t, T_{\iota(t)})\right] \\ &+ N\sum_{i=\iota(t)+1}^{M}\psi_i\left[P_n(t, T_{i-1})\frac{P_r(t, T_i)}{P_r(t, T_{i-1})}e^{D_i(t)} - P_n(t, T_i)\right] \end{aligned} \tag{4.1.5}$$

where we set $\mathcal{T} := \{T_1, \cdots, T_M\}$, $\Psi := \{\psi_1, \cdots, \psi_M\}$, $\iota(t) = \min\{i : T_i > t\}$ and where the first cash flow has been priced according to the zero coupon inflation leg formula derived in (1.2.13). In particular at $t = 0$

$$\begin{aligned}
\mathbf{YYIIS}(0, \mathcal{T}, \Psi, N) &= N \sum_{i=1}^{M} \psi_i P_n(0, T_i) \left[\frac{\mathcal{I}(0, T_i)}{\mathcal{I}(0, T_{i-1})} e^{D_i(0)} - 1 \right] \\
&= N\psi_1 \left[P_r(0, T_1) - P_n(0, T_1) \right] \\
&+ N \sum_{i=2}^{M} \psi_i \left[P_n(0, T_{i-1}) \frac{P_r(0, T_i)}{P_r(0, T_{i-1})} e^{D_i(0)} - P_n(0, T_i) \right] \\
&= N \sum_{i=1}^{M} \psi_i \ P_n(0, T_i) \left[\frac{1 + \tau_i F_n(0, T_{i-1}, T_i)}{1 + \tau_i F_r(0, T_{i-1}, T_i)} e^{D_i(0)} - 1 \right]
\end{aligned} \tag{4.1.6}$$

This expression above has the advantage of using a market model approach combined with yielding a fully analytical formula. In addition, contrary to Market Model I, the correction term does not depend on the volatility of real rates.

A drawback of the formula is that the approximation used when freezing the drift may be rough for longer maturities. In fact, the formula above is exact only when the correlations between $\mathcal{I}(\cdot, T_{i-1})$ and $F_n(\cdot, T_{i-1}, T_i)$ are assumed to be zero so that the nominal forward rate is zeroed out from D_i.

4.2 Inflation Linked Cap/Floor

From (4.1.3) and again freezing the drift at it's time t value, we obtain

$$\ln \frac{\mathcal{I}(T_{i-1}, T_i)}{\mathcal{I}(T_{i-1}, T_{i-1})} \bigg| \mathcal{F}_t \sim N \left(\frac{\mathcal{I}(t, T_i)}{\mathcal{I}(t, T_{i-1})} + D_i(t) - V_i^2(t), V_i^2(t) \right) \tag{4.2.1}$$

where

$$V_i(t) := \sqrt{\left[\sigma_{I,i-1}^2 + \sigma_{I,i}^2 - 2\rho_{I,i}\sigma_{I,i-1}\sigma_{I,i} \right] [T_{i-1} - t]} \tag{4.2.2}$$

Choosing to model the forward inflation-index in (3.2.1) then yields

$$\begin{aligned}
&\mathbf{ILCFLT}(t, T_{i-1}, T_i, \psi_i, K, N, \omega) \\
&= \omega N \psi_i P_n(t, T_i) \left[\frac{\mathcal{I}(t, T_i)}{\mathcal{I}(t, T_{i-1})} e^{D_i(t)} \Phi \left(\omega \frac{\ln \frac{\mathcal{I}(t, T_i)}{K\mathcal{I}(t, T_{i-1})} + D_i(t) + \frac{1}{2}\mathcal{V}_i^2(t)}{\mathcal{V}_i(t)} \right) \right. \\
&\left. - K\Phi \left(\omega \frac{\ln \frac{\mathcal{I}(t, T_i)}{K\mathcal{I}(t, T_{i-1})} + D_i(t) - \frac{1}{2}\mathcal{V}_i^2(t)}{\mathcal{V}_i(t)} \right) \right] \\
&= \omega N \psi_i P_n(t, T_i) \left[\frac{1 + \tau_i F_n(t, T_{i-1}, T_i)}{1 + \tau_i F_r(t, T_{i-1}, T_i)} e^{D_i(t)} \Phi \left(\omega d_1^i(t) \right) - K\Phi \left(\omega d_2^i(t) \right) \right] \\
&d_1^i(t) = \frac{\ln \frac{1 + \tau_i F_n(t, T_{i-1}, T_i)}{K[1 + \tau_i F_r(t, T_{i-1}, T_i)]} + D_i(t) + \frac{1}{2}\mathcal{V}_i^2(t)}{\mathcal{V}_i(t)} \\
&d_2^i(t) = d_1^i(t) - \mathcal{V}_i(t)
\end{aligned} \tag{4.2.3}$$

where

$$\mathcal{V}_i(t) := \sqrt{V_i^2(t) + \sigma_{I,i}^2\,(T_i - T_{i-1})}$$

The price at time t of the Cap/Floor is obtained by summing up the individual caplets

$$\begin{aligned}
&\mathbf{ILCFL}(t, \mathcal{T}, \Psi, K, N, \omega)\\
&= \sum_{i=1}^{M} \mathbf{ILCFLT}(t, T_{i-1}, T_i, \psi_i, K, N, \omega)\\
&= \omega N \sum_{i=1}^{M} \psi_i P_n(t, T_i) \left[\frac{1 + \tau_i F_n(t, T_{i-1}, T_i)}{1 + \tau_i F_r(t, T_{i-1}, T_i)} e^{D_i(t)} \Phi\left(\omega d_1^i(t)\right) - K\Phi\left(\omega d_2^i(t)\right)\right]
\end{aligned} \tag{4.2.4}$$

As in the YYIS price (4.1.6), the Cap/Floor price depends on the instantaneous volatilities of forward inflation indices and their correlations, the instantaneous volatilities of nominal forward rates and the instantaneous correlations between forward inflation indices and nominal forward rates. And again, there is no dependency on the volatilities of real rates and the formula is analytic.

Chapter 5

Calibration

5.1 Nominal- and Real Curves

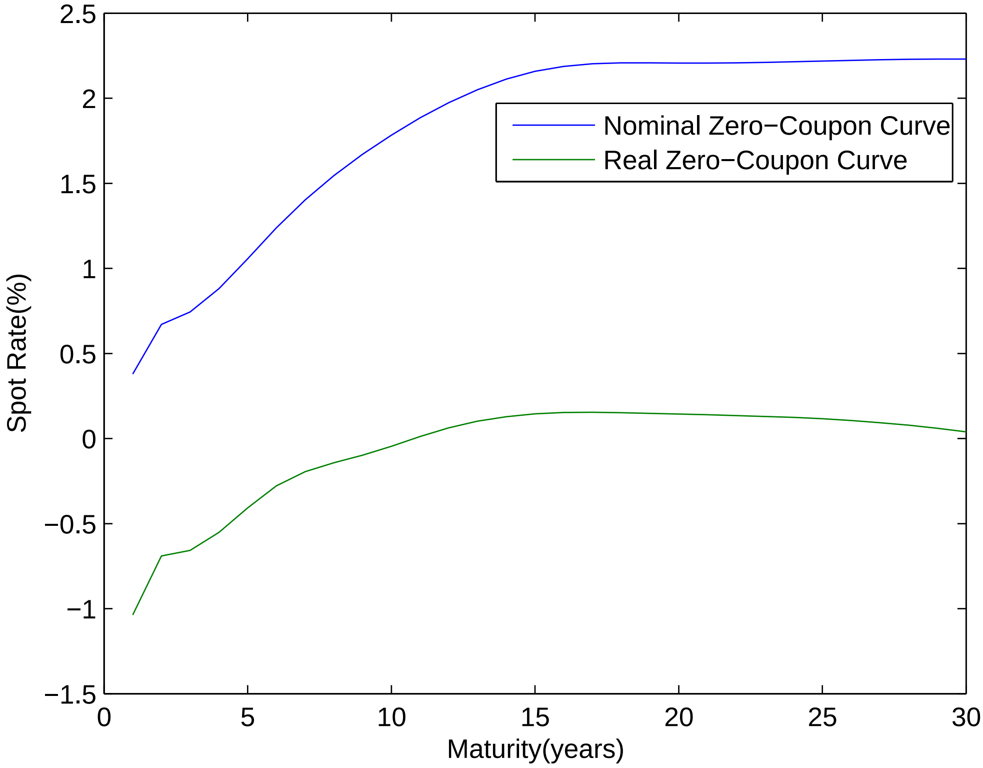

Figure 5.1: Calibrated Euro Zero-Coupon Curves, 13 jul-2012

We need to extract the nominal zero-coupon rates, $P_n(0, T_i)$ from the swap quotes, $S(T_i)$, that we showed in Figure 1.1. The fixed leg on the EUR denominated

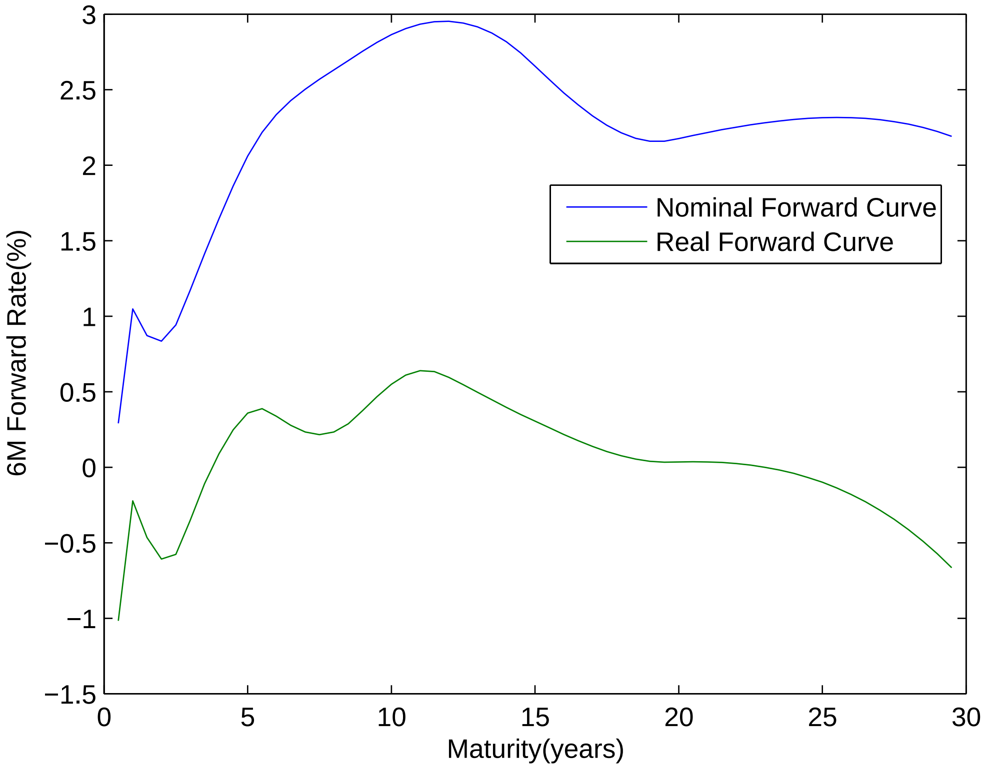

Figure 5.2: Calibrated Euro Forward Curves, 13 jul-2012

swaps rolls on a yearly basis, yielding the simple relation

$$S(T_i) = -\frac{1 - P(0,T_i)}{\sum_{j=1}^{i} P(0,T_j)} \tag{5.1.1}$$

, so that for all maturities $\{T_i, i > 0\}$, we can iteratively back out the zero-coupon rates as

$$\begin{aligned} P(0,T_i) &= \frac{1 - S(T_i)\sum_{j=0}^{i-1} P(0,T_j)}{1 + S(T_i)} \\ P(0,T_0) &= P(0,0) := 1 \end{aligned} \tag{5.1.2}$$

To obtain the real zero-coupon rates, we take the break-even inflation rates, $b(T_i)$, that we showed in Figure 1.1 and apply (1.2.16), i.e.

$$P_r(0,T_i) = P_n(0,T_i)(1 + b(T_i))^{T_i}$$

The resulting nominal and real *spot rates* are shown in Figure 5.1. The corresponding *forward rates* are displayed in Figure 5.2.

5.2 Simplified Jarrow-Yildirim Model

As a starting point for our calibration, we consider a simplistic model with the following evolution of the inflation index

$$\frac{dI(t)}{I(t)} = [r_n(t) - r_r(t)]\, dt + \sigma_I dW_I^Q(t) \tag{5.2.1}$$

where the nominal and real short rates, $r_n(.)$ and $r_r(.)$, are (unrealistically) assumed to be deterministic. And the inflation index volatility, σ_I , is constant. In this model, the Fisher equation is still preserved since.

$$E^Q\left[\frac{I(T)}{I(t)}\middle|\mathcal{F}_t\right] = e^{\int_t^T r_n(u)-r_u(u)du} \tag{5.2.2}$$

Note that, by Proposition (2.3.1) , this model is equivilant to the Jarrow-Yildirim model with $\sigma_n = \sigma_r = 0$. Since there is then a 1-1 correspondence between Year-On-Year Floor *price* and implied *volatility*, we can recover the implied Floor volatility surface, as shown found in Figure 5.3.

The Floorlet volatility surface is constructed as follows. First, we construct the Floorlet prices by bootstrapping the quoted Year-On-Year Floor prices. Since we are only dependent on the inflation index volatility parameter, we may then - for each Floorlet with expiry T_i and strike K_i - imply the corresponding inflation index volatility σ_{T_i,K_i}. The result is displayed in Figure 5.4.

Clearly, the skew shape of the Floor surface and the rifled wing shape of the Floorlet surface indicates that the assumption of a single constant volatility is not realistic. However, the simplistic model can still be of some use. When pricing Inflation Caps/Floors, it can be utilized as a volatility parameter(by interpolating in strike and expiry dimensions) to retrieve the appropriate volatility.

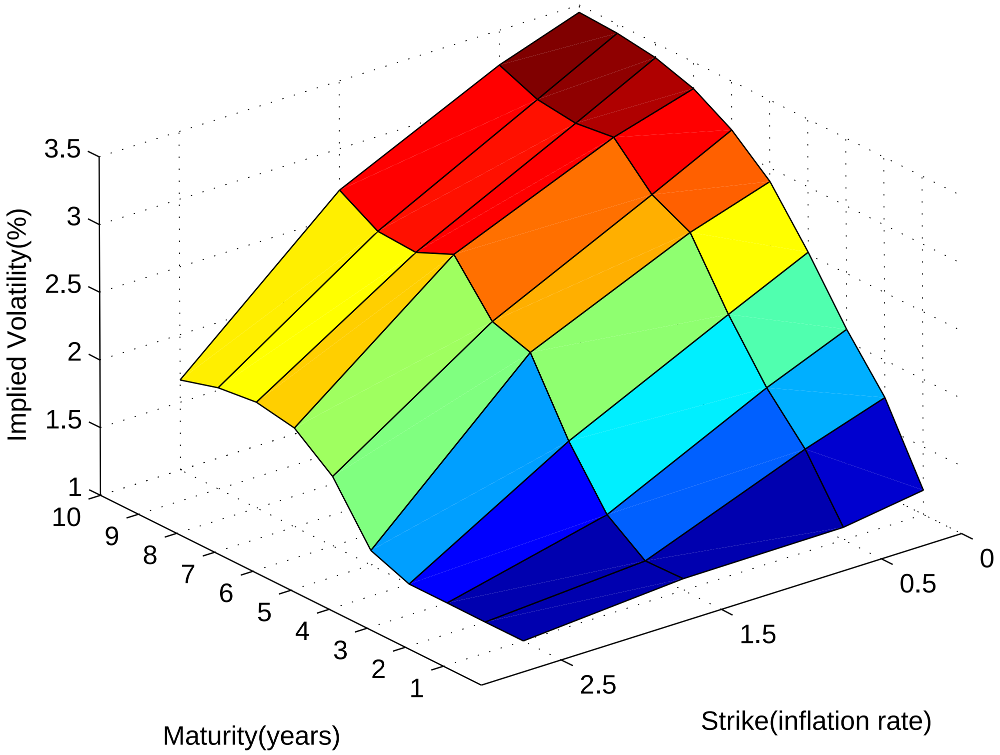

Figure 5.3: EUR Year-On-Year Inflation Floor Volatilities, 13 jul-2012

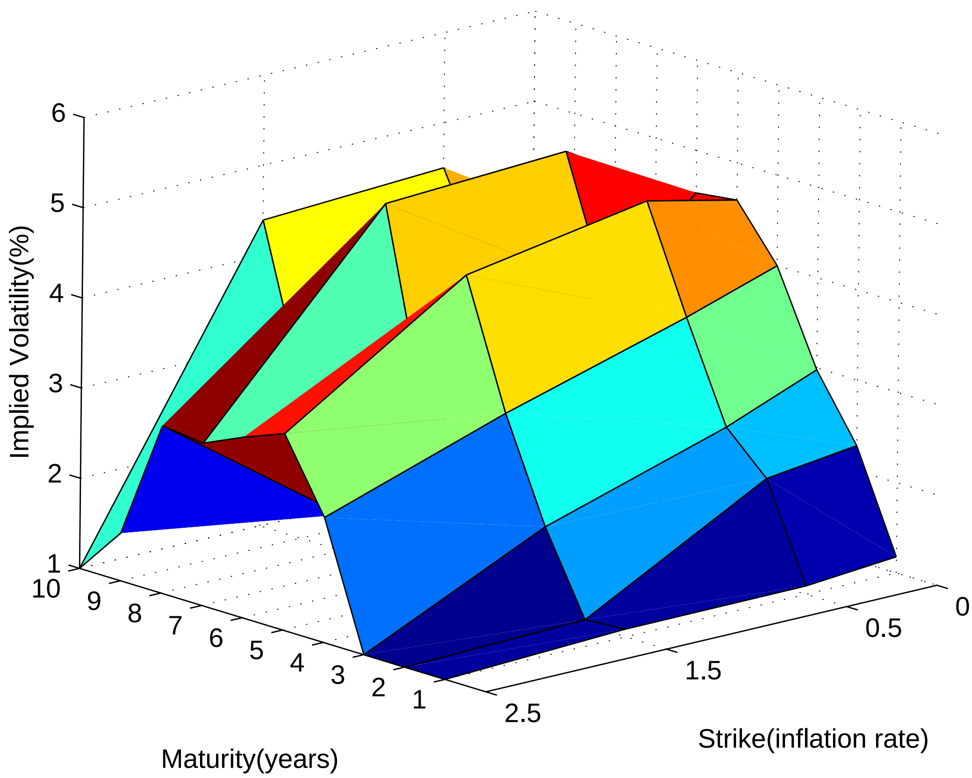

Figure 5.4: EUR Year-On-Year Inflation Floorlet Volatilities, 13 jul-2012

5.3 Jarrow Yildirim Model

5.3.1 Calibrating nominal volatility parameters to ATM Cap volatilities

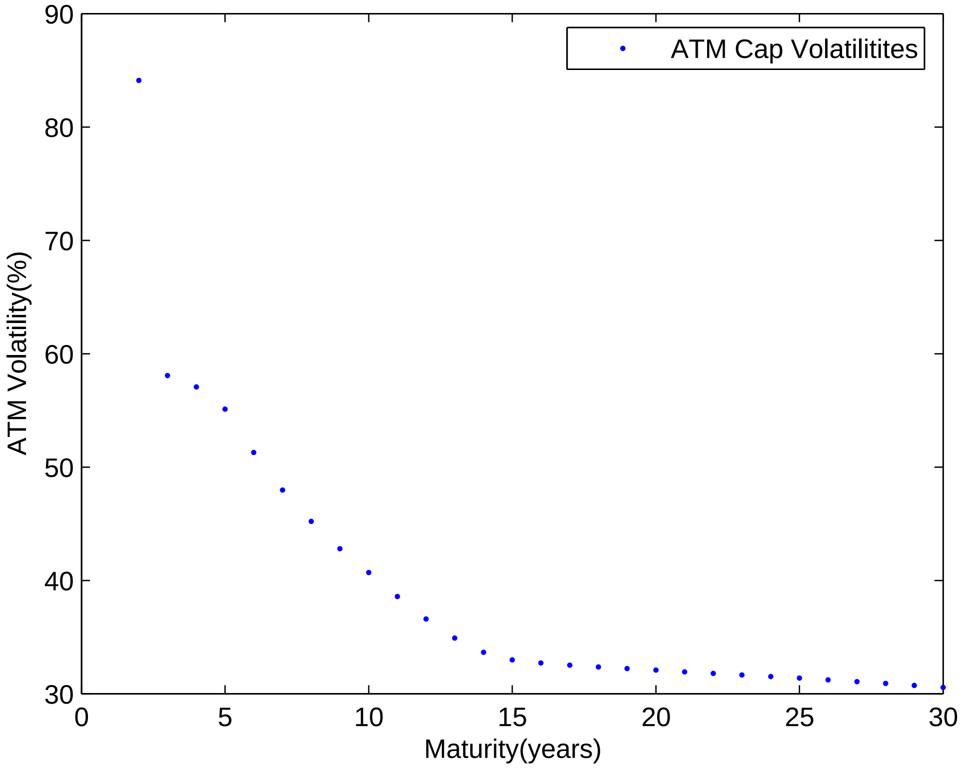

Figure 5.5: EUR ATM Cap Volatility Curve, 13 jul-2012

By the choice of nominal volatility function, pricing a nominal Cap under the J-Y model renders the well known Hull-White Cap/Floor valuation formula. We may then estimate the nominal volatility parameters with the following scheme. For each maturity T_i, we observe the ATM Cap (Black) volatility quote, σ_i^{ATM}, shown in figure 5.5, and the accompanying ATM strike level K_i^{ATM}. We can then fit the nominal volatility parameters κ_n, σ_n by performing a least squares optimization over

$$\mathbf{Cap}^{\text{Hull-White}}(t, T_i, \psi_i, K_i^{\mathrm{ATM}}, \sigma_n, \kappa_n) - \mathbf{Cap}^{\text{Black}}(t, T_i, \psi_i, K_i^{\mathrm{ATM}}, \sigma_i^{\mathrm{ATM}})$$

From the resulting theoretical Hull-White prices we then back out the *implied* Black volatility. The result is shown in Figure 5.6. The fit is not too bad in the long end of the curve. In the short end, we suffer from the limitations of our choice of nominal volatility function. The exponentially declining form of volatility can not recover the "hump" observed in the short end of the curve. However, note that although it was the choice of Jarrow and Yildirim , the J-Y framework is not limited to Hull-White term structures. We are free to choose other volatility functions for a better fit to Cap/Floor volatilities.

The implementation of nominal volatility structure calibration is a subject in itself and is beyond the scope of this thesis. We simply point out that we are free to choose a nominal volatility structure, other than that of Hull-White. For instance, had we set $\sigma_n(t,T) = \sigma$ then we would have rendered a Ho-Lee term structure.

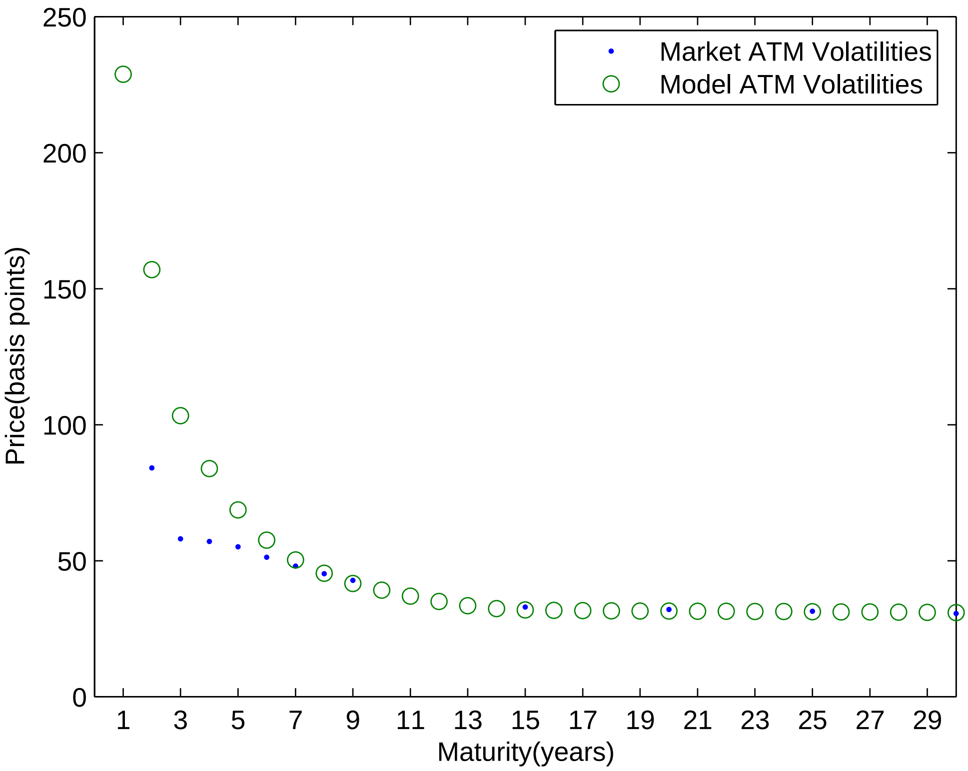

Figure 5.6: EUR Market vs Model ATM Cap volatilities, 13 jul-2012

5.3.2 Fitting parameters to Year-On-Year Inflation Cap quotes

The remaining parameters to estimate are $\{\kappa_r, \sigma_r, \sigma_I, \rho_{nr}, \rho_{Ir}\, \rho_{In}\}$. All these parameters enter the YoY inflation Cap/Floor valuation formula, so that we may attempt to calibrate to market prices. However, given the indicative shape of the volatility surface recovered in the previous section, we know that we cannot fit to the whole surface. Removing the two OTM contracts at strike 0 and 0.5 still results in high relative errors as shown in Figure 5.7. Excluding the longer dated contracts from the calibration (Figure 5.8), still results in a poor fit.

We conclude that we must restrict ourself to the (closest to) ATM contract and restrict the expiry dimension to get a reasonable fit, as shown in figure in 5.9. The fit in the expiry dimension can be improved by choosing more sophisticated volatility functions for the real rate and the inflation index. The presence of a "strike skew" however, makes calibration unfeasible for non ATM contracts.

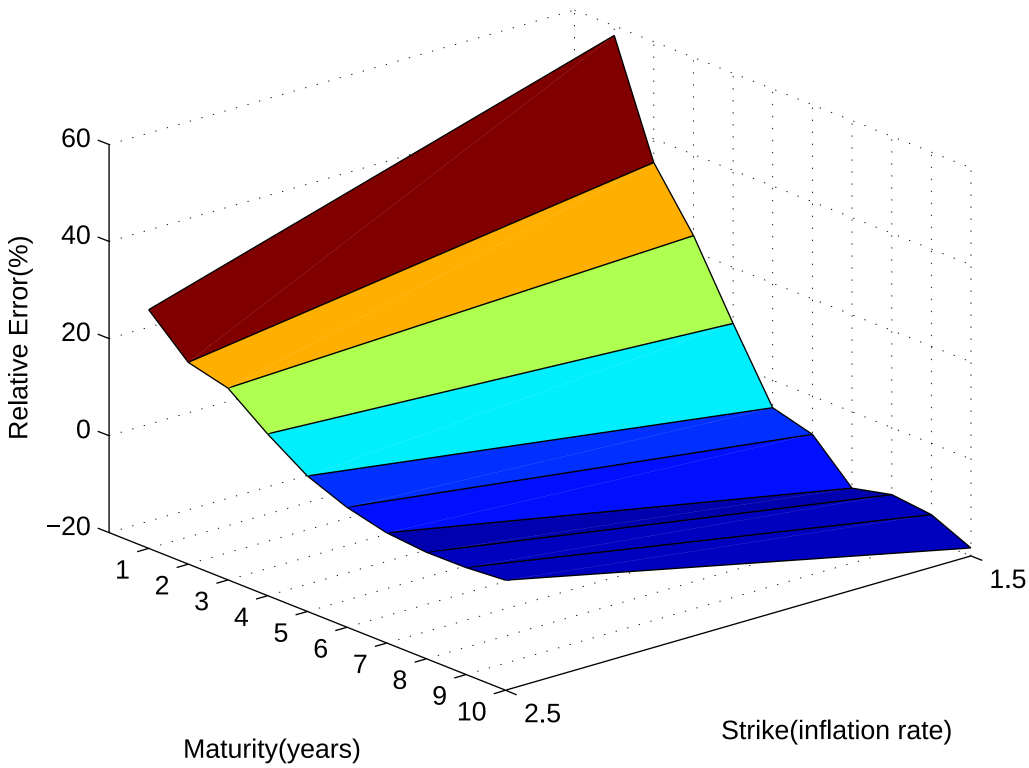

Figure 5.7: Relative Error, EUR Year-on-Year Inflation Floor Model Prices, 13 jul-2012

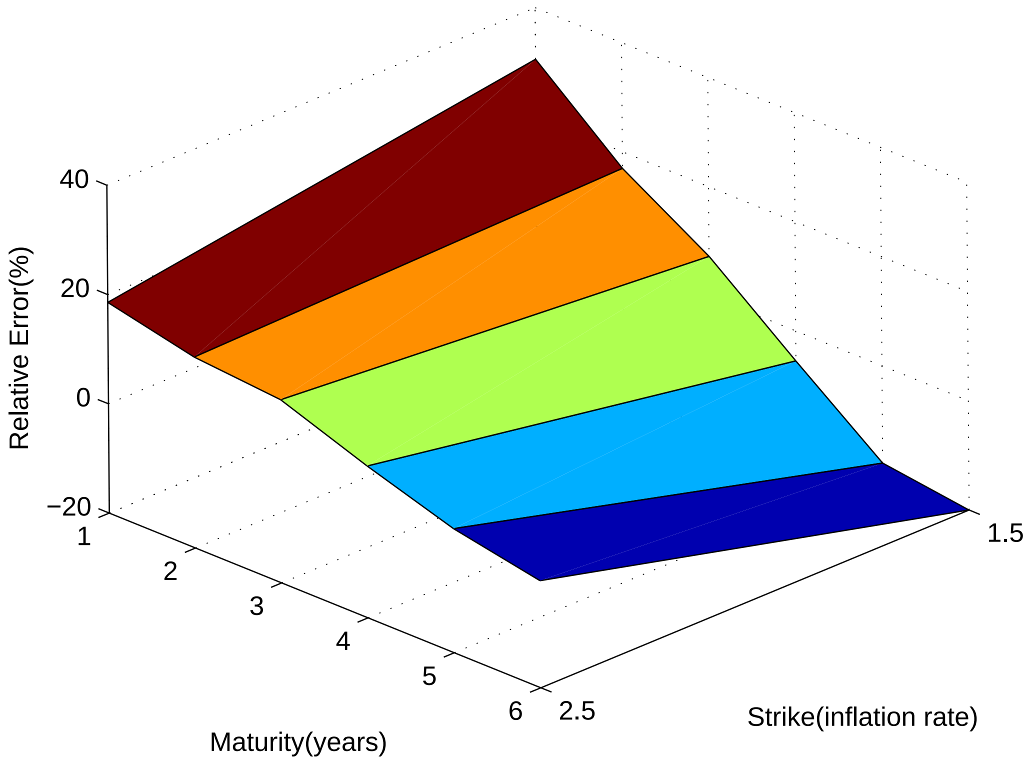

Figure 5.8: Relative Error, EUR Year-on-Year Inflation Floor Model Prices, 13 jul-2012

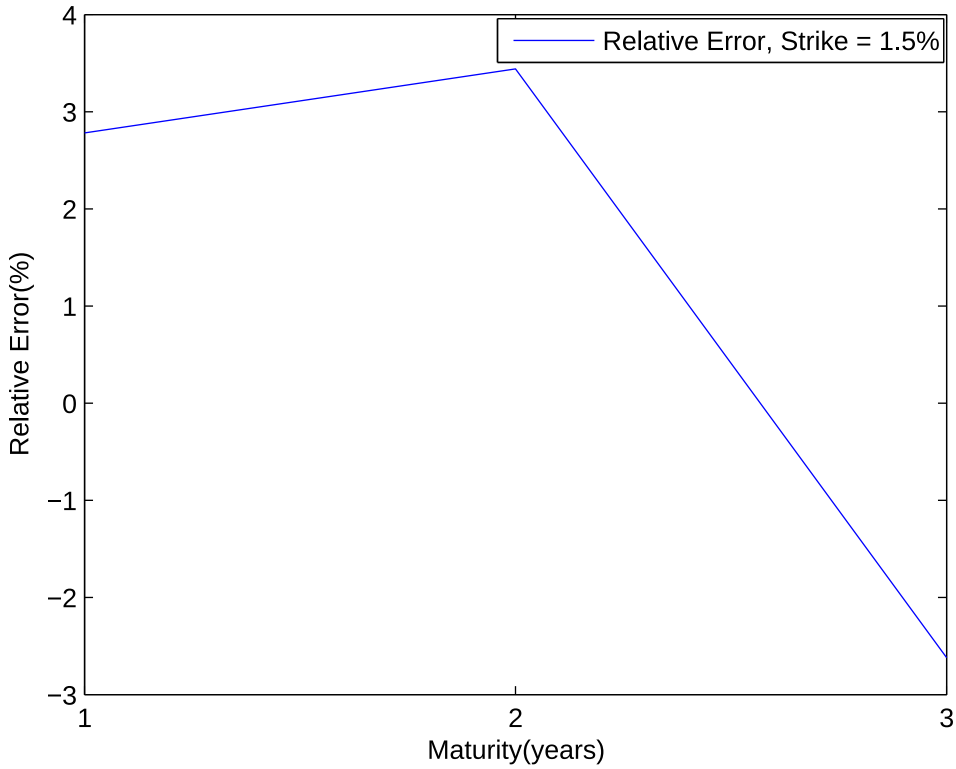

Figure 5.9: Relative Error, EUR ATM Year-on-Year Inflation Floor J-Y Model Prices, 13 jul-2012

5.4 Market Model II

5.4.1 Nominal volatility parameters

For each maturity T_i we need to estimate the volatility , $\sigma_{n,i}$, of the (simply compounded) nominal forward rate $F_n(\cdot, T_i)$. By the log-normal dynamics of $F_n(\cdot, T_i)$ we get an automatic calibration to the quoted (Black) cap volatility σ_i^{ATM}.

5.4.2 Fitting parameters to Year-On-Year Inflation Cap quotes

The remaining parameters to estimate, for each each maturity $\{T_i, i > 2\}$, are

$$\{\sigma_{I,i-1}, \sigma_{I,i}, \rho_{I,i}, \rho_{I,n,i}\}$$

The fitting procedure is run iteratively. That is, $\sigma_{I,1}$ is directly obtained from the 1-Year Floor, since it depends on no other unknown parameters. We then proceed to use least squares estimation to fit the rest of the parameters to the corresponding Floor prices. The resulting price error surface is plotted in Figure 5.10.

Since each expiry has it's own set of parameters, there is no need to restrict the number of contracts in the expiry dimension. The skew in the strike dimension however, results in a poor fit if we want to include non ATM contracts. From the figure, it's clear that if we assume constant volatility in the expiry dimension, then OTM contracts are underpriced and ITM contracts are overpriced. As in the J-Y model, we must restrict ourselves to (close to) ATM contracts to recover market prices. Or alternatively, take the "practitioners approach" and imply a set of parameters for each Expiry/Strike pair.

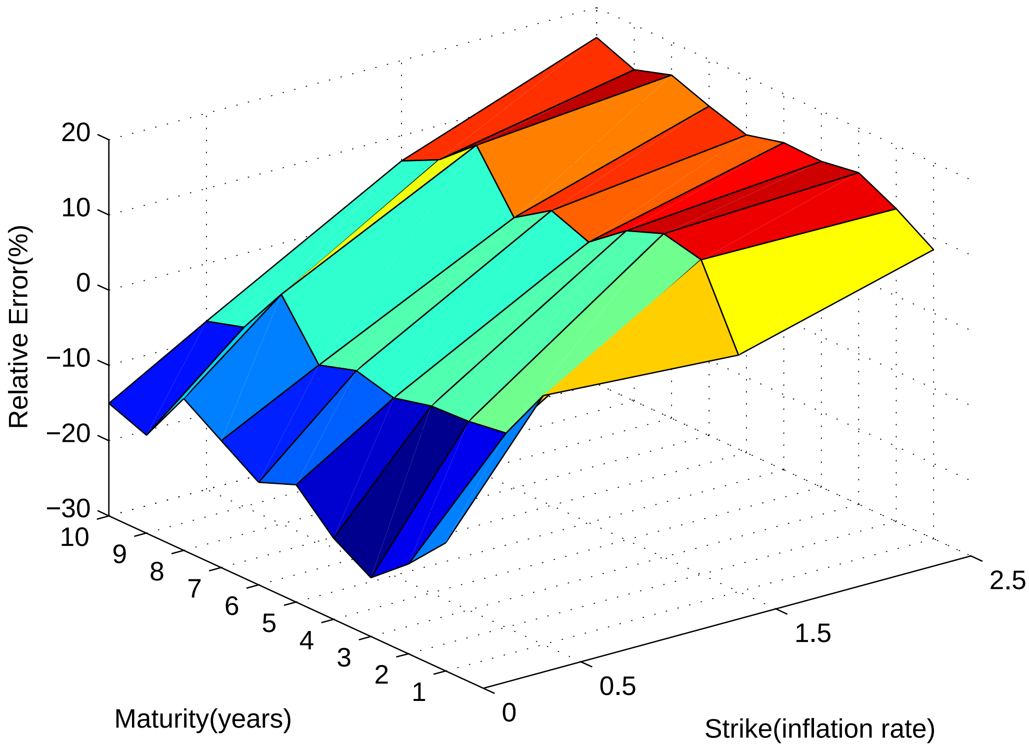

Figure 5.10: Relative Error, EUR ATM Year-on-Year Inflation Floor Market Model II Prices, 13 jul-2012

Chapter 6

Conclusions and extensions

6.1 Conclusions

In this thesis, we have presented the market for inflation derivatives and compared three approaches for pricing standard contracts.

The first approach is a HJM framework where we have set a Hull-White term structure both for the nominal and the real economy. The result is analytically tractable prices for Year-On-Year Inflation Swaps and Caps/Floors. A practical downside is that it requires real rate parameters that are not trivial to estimate. Furthermore, the model cannot be reconciled with the the full volatility surface of inflation Caps/Floors. That is, since the model does not account for the "inflation smile" it can only be calibrated to ATM contracts.

The second approach is a market model were the modeled quantities are the simply compounded nominal and real forward rates. The advantage of this approach is that is that it models observable quantities, i.e. the forward rates. The downside is that it leads to non-closed form prices of the standard contracts. And it still requires the estimation of real rate parameters. Finally, the forward rate is assumed to follow a Log-Normal distribution, which may not be a realistic assumption in the presence of negative real forward rates.

The third approach models the respective forward inflation indices. By usage of "drift freezing" approximations , this approach leads to closed form prices of the standard contracts. And there is no dependence on real rate parameters. Furthermore, the nominal volatility parameter is automatically calibrated to quoted nominal cap volatilities. And since each respective forward inflation index is modeled, adding contracts in the expiry dimension has no negative impact on calibration performance. Of the models evaluated, this approach seems the most promising. In the strike dimension however, the smile effect makes it difficult to reconcile OTM contracts.

6.2 Extensions

In light of the conclusions drawn so far, a natural next step is to attempt to take the inflation smile into account. There has been research in this area. Mercurio and Damiano[6] extend Market Model II by by a stochastic volatility framework with 'Heston' dynamics. They produce smile consistent closed-form formulas for inflation-indexed caplets and floorlets.

Taking a different approach, Kenyon[8] proposed that by the low inflation volatilities, it's natural to model the Year-on-Year inflation rate itself, with a normal distribution. He proceeds with proposing normal-mixture models and normal-gamma models to take the smile effect into account. The result is closed form price formulas that well recover the inflation smile.

As a final note, Mercurio and Damiano[7] developed a framework that leads to SABR-like dynamics for forward inflation rates, and closed-form prices for the standard contracts.

Appendix A

Appendix

A.1 Change of numeraire

The following proposition is taken directly from [3].

Proposition A.1.1 (A change of numeraire toolkit)
Consider a n-vector diffusion process whose dynamics under Q^S is given by

$$dX(t) = \mu_X^S(t) + \sigma_X(t) C dW^S(t)$$

where W^S is n-dimensional standard Brownian motion, $\mu^S(t)$ is a $n \times 1$ vector , $\sigma_X(t)$ is a $n \times n$ diagonal matrix and the $n \times n$ matrix C is introduced to model correlation, with $\rho := CC'$

Let us assume that the two numeraires S and U evolve under Q^U according to

$$dS(t) = (\cdots)dt + \sigma^S(t) C dW^U(t)$$
$$dU(t) = (\cdots)dt + \sigma^U(t) C dW^U(t)$$

where both $\sigma^S(t)$ and $\sigma^U(t)$ are $1 \times n$ vectors , W^U is n-dimensional standard Brownian motion and $CC' = \rho$. Then, the drift of the process X under the numeraire U is

$$\mu_X^U(t) = \mu_X^S(t) - \sigma_X(t) \rho \left(\frac{\sigma^S(t)}{S(t)} - \frac{\sigma^U(t)}{U(t)} \right)'$$

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