Annuity Divisor - Comparison Between Different Computational Methods

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Abstract

This thesis is focused on a vital component of the Swedish national pension system called Annuity Divisor which determines the annual pension amount. It is based on the life expectancy of the whole population and it also includes a set discount rate. There are however two different formulas used within the Swedish national pension system and these different methods are also based on different data, historical and prognostic. The political aspects regarding the Annuity Divisor are also considered.

The process from raw data until the final value is shown and compared with methods used in Finland, Norway and Poland. It is shown that the different formulas used in the Swedish national pension system yield similar results and that the main difference arises as a consequence of the different data used.

By using the Finnish formula for the Annuity Divisor, which is slightly easier, the result can be improved in respect to continuous discounting.

Acknowledgement

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Contents

1 Background

Historically the dominating national pension system were Defined Benefit (DB) schemes, it is still the most common form but this is changing in favour of the Defined Contribution (DC) schemes. The way the benefit levels are determined varies but in most cases DB schemes have a much weaker link between benefit levels and actual capital contributed as premiums and with life expectancy [1].

As of 2011 Sweden, Norway, Poland and Italy are the only OECD countries that have Notional/Non-Financial Defined Contribution (NDC) schemes. Portugal and Finland have DB schemes but the benefits will be reduced by a factor directly related to life expectancy [13].

The aim of this report is to study a component of the Swedish national pension system called Annuity Divisor (AD). The Swedish AD is compared with AD:s of the national pension systems of Norway, Finland and Poland. The different formulas used in order to compute the AD as well as the various ways used in order to generate the necessary input data are compared. The political aspects of the AD are also covered.

1.1 The Swedish Pension System

The Swedish national pension system consists of a non income related part and an income related part. The non income related part is a minimum level pension, which is financed by the national budget called Garantipension.

The income related pension is separated from the national budget and consists of two parts; the first part is a NDC scheme called Inkomstpension and the second part is a FDC scheme called Premium Pension. The premiums are distributed as; 86.5 % to the Inkomstpension and 13.5 % to the Premium Pension.

1.1.1 Inkomstpension

In the Inkomstpension system each person has an individual non financial account holding a pension balance (PB). This is a non financial or notional account in the sense that the capital is not locked to the individuals account but is being invested in buffer funds that then pay out the pensions to the pensioners. Current premiums (P) paid are basically being distributed to current pensioners. The pension balance for an individual is determined as

$$PB(t) = \begin{cases} (PB(t-1) + P(t)) \cdot \frac{I(t)}{I(t-1)} \cdot ACF(t) \cdot IGF(t) & BR(t) \geq 1 \\ (PB(t-1) + P(t)) \cdot \frac{I(t)}{I(t-1)} \cdot ACF(t) \cdot IGF(t) \cdot BR(t) & BR(t) < 1 \end{cases}$$

Here I(t) is the Income Index and $ACF(t)$ is an Administrative Cost Factor for the year t. $IGF(t)$ is an Inheritance Gain Factor that distributes capital to those born the same year as the deceased i.e to the same cohort. There is also a Balance Ratio (BR) that is active when liabilities of the Inkomstpension system are greater than its assets, i.e when BR is less than one. At the time of

retirement the pension balance PB is being divided by the AD which determines the annual pension amount for the first year as a pensioner

$$Pension(t) = \frac{PB}{AD}$$

The pension is then recalculated each year as

$$Pension(t) = \begin{cases} \frac{I(t)}{I(t-1)\cdot 1.016} \cdot Pension(t-1) & \text{for } BR(t) \geq 1 \\ \frac{I(t)}{I(t-1)\cdot 1.016} \cdot BR(t) \cdot Pension(t-1) & \text{for } BR(t) < 1 \end{cases}$$

Here there is a rate factor 1.016, this is due to that the AD has been assigned this rate in advance [2].

1.1.2 Premium Pension

In the Premium Pension each person's acquired capital is invested in funds of his or her choice and can then be withdrawn at the time of retirement in various forms. The pension balance therefore changes continually and there are also Administrative Costs (AC) and Inheritance Gain Factors applied once a year for the Premium Pension but these are not cohort specific.

$$PB(t) = PB(s) \cdot IGF(t) - AC(t) \qquad \text{for } s < t$$

At the time of retirement the accumulated capital is transformed into a annual pension amount

$$Pension(t) = \frac{PB(t) - AC(t)}{AD}$$

Here the pension for each year is calculated in the same way and the AD is recalculated for each year. The insured can choose between a fund insurance and a traditional insurance, it is only in the traditional insurance that the size of the AD becomes a liability for the insurer [2].

2 Life expectancy modelling

There are generally two ways of estimating life expectancy; one that is purely based on actual data and the other is a model which is fitted to data. There is also a combination of the two.

The most common way of displaying life expectancy is by the use of a so called *life table*. The life table shows how the size of a fictive cohort, l_x, declines with age. This is done by the use of a death risk, q_x, which is an estimate of the likelihood of death occurring in the age span of $[x, x+1)$. The initial size of the cohort, l_0, is usually set to 100,000 and l_x is determined by the product

$$l_x = l_0 \prod_{y=0}^{x-1} (1 - q_y)$$

The life expectancy at the age x is then approximately given by

$$e_x = \sum_{y=x}^{\infty} \frac{(l_y - l_{y+1})(\frac{(y+1)+y}{2})}{l_x}$$

This is under the assumption that deaths occur uniformly at each age, which for example is not the case for deaths occurring at the age of zero years. This is just an illustration of the formula, not the actual one that is being used.

There are two major types of life tables; cohort and period tables. A cohort table is a table in which the data is collected for each of the years the cohort has lived through until extinction, it therefore represents the true life expectancy of the cohort for a given age. In a period table, data is collected during the given period and the data is therefore a representation of simultaneous outcomes for all living cohorts during this period. Life expectancy based on a period table basically gives a representation of the conditions of the whole population of the country during that period or it can be thought of as the average life length of a fictive person living his or her entire life during the conditions at the specified period. A period is usually between one to ten years.

The other common method of estimating life length is by fitting a mathematical formula to the data by the use of for example the method of least squares. In these methods age can usually be a continuous input whereas in the previous method age is discrete. For period life tables a method of this kind is sometimes used for higher ages for a smoothing purpose since the number of observations decreases by age which makes the interpretation of the outcomes more uncertain.

The input data that is needed for the calculations are the population size and the number of births and deaths. Generally population size and the number of births are registered per year; population size is also specified for each age. The population size is usually measured at the end of the year so to get an estimate of the size during the year t a mean value of the size at the year $t-1$ and t is often used. The deaths are however ideally differentiated within the years by the use of *Lexis triangles*.

A Lexis triangle is used to determine for which cohort the deceased belonged to, since the person could have died before or after his or her birthday at the year for which the death occurred. In a *Lexis diagram* the vertical-axis represents the age x and the horizontal-axis represents the time t. A persons life is represented by a line that starts by birth ($\circ$) or by immigration ($\circ$) and ends in either emigration ($\bullet$) or death ($\times$), see *Figure* 1. The square that is formed by the intersection of the horizontal lines passing through x and $x+1$ and the vertical lines passing through t and $t+1$ is divided into two Lexis triangles; one upper and one lower by dividing the square by a line passing through (t, x) and $(t+1, x+1)$, see *Figure 2*. Ideally the risk exposure time can be determined by summing up the total length of the parts of the lines that are within the square or triangle, this is however usually not possible [3].

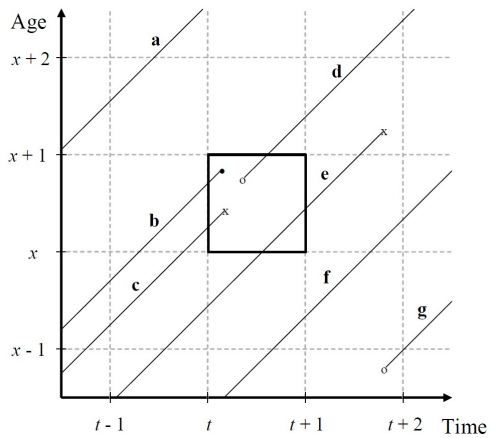

Figure 1: The Lexis diagram shows the life of individuals within the highlighted square at the age of x for the year t. *Source: mortality.org [3]*

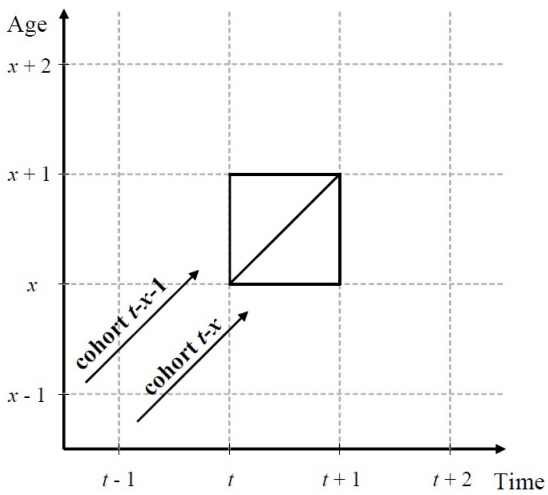

Figure 2: The figure shows how the two cohorts are separated between the upper and lower Lexis triangle in the year t. *Source: mortality.org [3]*

2.1 Formula based computation

Assume T is a continuous random variable with probability density function $f(t)$ and cumulative distribution function $F(t)$. Here T represents the event of death. Then

$$F(t) = P(T < t)$$

represents the probability of death occurring before time t. The opposite is referred to as the *survival function*

$$S(t) = P(T > t) = 1 - F(t) = \int_t^\infty f(x)dx$$

There is also the *hazard function* $\mu(x)$ which is the instantaneous rate of occurrence of the event T which is defined as

$$\mu(x) = \lim_{dt \to 0} \frac{P(t < T \leq t + dt | T > t)}{dt}$$

This is the probability that death occurs between time t and $t + dt$ given that it has not occurred before time t. This can also be expressed as

$$\mu(x) = \lim_{dt \to 0} \frac{\frac{P(t<T\leq t+dt)}{P(T>t)}}{dt} = \lim_{dt \to 0} \frac{f(t)dt}{S(t)dt} = \frac{f(t)}{S(t)} \tag{1}$$

and since

$$\frac{d}{dt}S(t) = -f(t)$$

equation 1 can be written as

$$\mu(x) = -\frac{d}{dt}log(S(t))$$

The survival function can therefore be obtained given the hazard function $\mu(x)$ as

$$S(t) = e^{-\int_0^t \mu(x)dx}$$

The expected remaining life length is thus the expected value of $S(t)$, expressed as

$$E[S(t)] = \int_0^\infty S(t)dt = \int_0^\infty e^{-\int_0^t \mu(x)dx} dt$$

3 Life table based computations

In all countries except Poland, the Statistics Agency is estimating the death risks and the Pensions Agency is using these estimations in order to determine the AD. In Poland this is all handled by the Statistics Agency. Original notation has been used in general.

3.1 Sweden

This section regards the computation of the AD in the Inkomstpension system where calculations are based on actual outcomes. First follows the regulations and then the formulas used.

3.1.1 Regulations by law [4]

1. The Annuity Divisor for the Inkomstpension scheme shall be the same for men and women.
2. The Annuity Divisor shall be determined in the way that the value of the pension payments to be made for the average life length of people in the same age as the insured is at the time of retirement will equal the accumulated capital.
3. An upcoming monthly payment of the Inkomstpension is expected to have a value which is the ratio of the initial monthly payment and a yearly rate factor of 1.016 up until the time of the upcoming monthly payment.
4. The number of upcoming pension payments shall be calculated with the guidance of the official statistics.
5. For an insured that has not reached the age of 65 years, the calculation shall be made with the guidance of official statistics over the life length of the population of Sweden during the five year period closest to the year the insured reached the age of 60 years. From the year the insured reaches the age of 65 years and from there on the calculation shall be done by the guidance of the statistics for the five year period closest to the year the insured reached the age of 64 years.

3.1.2 Definitions [2] [5]

The AD is defined as

$$AD_i = \frac{1}{12l_i} \sum_{k=i}^{r} \sum_{X=0}^{11} \left(l_k + (l_{k+1} - l_k)\frac{X}{12} \right) (1.016)^{-(k-i)} (1.016)^{-\frac{X}{12}} \tag{2}$$

As can be seen in equation (2) the AD basically represents the life expectancy for a person at the age i with the set discount rate of 1.6 % applied monthly. Linear interpolation [21] is used to determine the AD for withdrawal at the different months.

$$AD_{i,m} = AD_i + m\frac{AD_{i+1} + AD_i}{12}$$

Where

$$\begin{aligned} i &= \text{Retirement age } (61, 62, ...n) \\ m &= \text{The number of months after the month of birth.} \\ k - i &= \text{The number of years as a pensioner } (k = i, i+1, i+2, ...) \\ X &= \text{Months } (0, 1, ..., 11) \end{aligned}$$

Below follows the definitions of variables used in creating the life table which is used as an input in *equation* (2).

$F^{t,p}$ = The number of babies born during the period p with the most recent year being t.

$R_x^{t,p}$ = The risk exposure time at the age of x years for the period p with the most recent year being t.

$D_x^{t,p}$ = The number of deaths at the age of x years for the period p with the most recent year being t.

$d_x^{t,p}$ = The number of deaths at the age of x years conditional on that the person died after their birthday during the period p with the most recent year being t.

P_x^t = The population at the age of x years at the 31:st of December the year t.

$P_{x,m}^t$ = The mean population at the age of x at the year t.

p = The index p is the period which is set to five years.

3.1.3 Formulas [5] [6]

Here the average population $P_{x,m}^t$ at the age of x years at the year t is first determined in order to calculate the risk exposure time $R_x^{t,p}$.

$$P_{x,m}^t = \frac{P_x^t + P_x^{t-1}}{2}$$

$$R_x^{t,p} = \sum_{y=t-4}^{t} P_{x,m}^y$$

$$q_x^t = \begin{cases} \frac{D_0^{t,p}}{F^{t,p}} & \text{for } x \in \{0\} \\ \frac{D_x^{t,p}}{R_x^{t,p}+d_x^{t,p}} & \text{for } x \in \{1, ..., 90\} \\ \text{Based on A Generalized Perks Formula} & \text{for } x \in \{91, ..., \infty\} \end{cases}$$

For the ages of 91 and above a different method of determining the q_x^t values is used; it is called A Generalized Perks Formula for Old-Age Mortality. The Perks Formula is a generalized Gompertz Makehem formula. The Gompertz Makeham formula is defined as

$$\mu(x) = A + Be^{kx} \qquad (A \geq 0, \text{B and k} > 0)$$

Where $\mu(x)$ is the death intensity and A is an age independent component. Perks formula is defined as

$$\mu(x) = \frac{A + Be^{kx}}{1 + De^{kx}} \qquad (A \geq 0, \text{B and k} > 0)$$

In the case of $A = 0$ it can be shown that if individuals follow the Gompertz Makeham formula the total force of mortality for a heterogeneous population will be given by Perks formula on the condition that the coefficient B is gamma

distributed at birth. B is known as a frailty component and is assumed to be constant during an individuals life.

In the Gompertz Makeham formula the death intensity will increase continuously while it will approach a constant limit value in the Perks formula. Statistics Sweden believes that the death intensity does not increase as much as in the Gompertz Makeham formula but that it should neither reach a constant value so they use a Generalized Perks formula, this is done by moving the distribution of the B coefficient to the right i.e by the use of a shifted gamma distribution. This is shown by the use of the following theorem.

Theorem 1 *("The theorem of the frangible man")*

If the intensity functions of individuals follow the Makeham law:

$$\mu(x|z) = A + ze^{kx} \qquad (A \geq 0, k \geq 0)$$

and the frailty variable z has a shifted gamma distribution with density

$$g(z) = \begin{cases} \frac{b^a (z-c)^{a-1}}{\Gamma(a)} e^{-b(z-c)} & \textit{for } z > c \textit{ and } (a, b \textit{ and } c > 0) \\ 0 & \textit{for } z \leq c \end{cases}$$

where c is the lower bound of the gamma distribution. The total force of mortality $\mu(x)$ can be written:

$$\mu(x|z) = \frac{A + Be^{kx}}{1 + De^{kx}} + ce^{kx} \tag{3}$$

where

$$B = \frac{A + ak}{kb - 1} \textit{ and } D = \frac{1}{kb - 1}$$

This concludes the theorem.

By the theorem above and under the assumption that $A = 0$ which is said to be a reasonable approximation for high ages; equation (3) can be written as

$$\mu(x|z) = \left(c + \frac{\eta}{1 + \eta\alpha^2 \int\limits_{x_0}^{x} e^{kt} dt} \right) e^{kx} \tag{4}$$

where η is the mean of $U = Z - c$ and α is the relative standard deviation of U. To fit the model to data the following approximation of the intensities is used

$$\tilde{\mu}(x + 0.5) \approx -ln(1 - q_x) \qquad (x = 85, 86, ...)$$

The four variables of equation (4) cannot however be unambiguously determined because of the many combinations that will yield the same fit to data. Therefore the variables k and α were first determined by trying different combinations of values. It was found that the different set-ups yield similar fits to the data and

finally the values were set to $k = 0.12$ and $\alpha = 0.5$ as to be used as "universal" constants. What is left to determine is c and η, these values will depend on the chosen data, unlike k and α. This is done by the use of the least squares method, i.e by finding the values of c and α that minimizes the following expression

$$\sum_{x=85.5}^{\infty} w_x^2 \left(\tilde{\mu}(x) - \left(c + \frac{\eta_0}{1+\eta_0 I(x)} + \frac{\Delta\eta}{(1+\eta_0 I(x))^2} \right) e^{kx} \right) \tag{5}$$

where

$$w_x^2 = w_{x+0.5}^2 = \begin{cases} \frac{n_x(1-q_x)}{q_x} & n_x \geq 0 \\ 0 & n_x < 12 \end{cases}$$

$$\begin{aligned} n_x &= \text{The number of survivals at the age of x years.} \\ I(x) &= \alpha^2 \frac{e^{kx} - e^{85.5k}}{k} \\ \Delta\eta &= \eta - \eta_0 \\ \eta_0 &= \text{Initial guess of } \eta \end{aligned}$$

Here w_x is a weight factor which decreases with higher ages. With the estimations of c and η the intensity $\mu(x)$ can then be computed from *equation* (4) and then transformed to death risks as

$$\hat{q}_x = 1 - e^{-\hat{\mu}(x)}$$

The life table can now be constructed as

$$\begin{aligned} l_0 &= 100,000 \\ l_{x+1} &= l_x(1 - q_x) \end{aligned}$$

3.2 Norway

In the Norwegian life table calculation no smoothing function is used for higher ages so this is a pure data based method. The AD for each cohort is fixed and is determined the year before the cohort will turn 62.

3.2.1 Definitions [8]

The Annuity Divisor is defined as

$$AD_{K,A} = \frac{l_{K,A}}{\frac{1}{40} \sum_{i=27}^{66} l_{K,i}} \left(\sum_{x=A}^{\infty} 0.9925^{x-A} P_{K,A,x} \right) \tag{6}$$

for the retirement age $A \in \{62, 75\}$ and cohort $K \geq 1954$. Here 0.9925 is a discounting factor, which is comparable to a rate factor of 1.0076, by switching the sign in the exponent.

$$
\begin{aligned}
D_x^t &= \text{The number of deaths at the age of } x \text{ years at the year } t. \\
e_x^t &= \text{The number of deaths at the age of } x \text{ years conditional on} \\
&\quad \text{that the person died after their birthday at the year } t. \\
f_x^t &= \text{The number of deaths at the age of } x \text{ years conditional on} \\
&\quad \text{that the person died before their birthday at the year } t. \\
P_x^t &= \text{The population at the age of } x \text{ years at the end of the year } t. \\
P_{x,m}^t &= \text{The mean population at the year } t. \\
F^t &= \text{The number of babies born the year } t. \\
d_{t,x} &= \text{The death risk at the age of x years for the year } t.
\end{aligned}
$$

3.2.2 Formulas [7] [8]

In the Norwegian system the death intensities μ_x^t are first estimated and then transformed into death risks $d_{t,x}$. An average over a period is then used to determine $q_{K,x}$, the death risk of cohort K, which is used in constructing the life table.

$$
\begin{aligned}
P_{x,m}^t &= \frac{P_x^{t-1} + P_x^t}{2} \\
\mu_x^t &= \begin{cases} \frac{D_x^t}{0.25(2P_x^{t-1} + P_x^t + F_x^t - f_x^t)} & \text{for } x = 0 \\ \frac{D_x^t}{P_{x,m}^t + 0.25(e_x^t - f_x^t)} & \text{for } x \geq 1 \end{cases} \\
d_{t,x} &= 1 - e^{-\mu_x^t}
\end{aligned}
$$

Up until here is computed by Statistics Norway and the following is computed by the Norwegian Pensions Agency.

$$
\begin{aligned}
q_{K,x} &= \begin{cases} \frac{1}{2}(d_{K+x,x} + d_{K+x+1,x}) & \text{for } x \in \{0, ..., 59\} \\ \frac{1}{10} \sum\limits_{t=K+51}^{K+60} d_{t,x} & \text{for } x \in \{60, ..., \infty\} \end{cases} \\
l_{K,x} &= \begin{cases} 1 & \text{for } x = 27, \\ l_{K,x-1}(1 - q_{K,x-1}) & \text{for } x \in \{28, ..., \infty\} \end{cases} \\
P_{K,A,x} &= \frac{l_{K,x} + l_{K,x+1}}{2l_{K,A}} \qquad \text{for } x \geq A
\end{aligned}
$$

The AD can also be rewritten as

$$
AD_{K,A} = \frac{1}{\frac{1}{40} \sum\limits_{i=27}^{66} l_{K,i}} \left(\sum_{x=A}^{\infty} 0.9925^{x-A} \frac{l_{K,x} + l_{K,x+1}}{2} \right) \tag{7}
$$

Here it can be seen that denominator outside of the parenthesis will be larger than in the Swedish model since it is an average of 40 ages which will result in a smaller AD. This is an implicit inheritance gain distribution.

3.3 Finland

The method used in the Finnish system is also a pure data based method.

3.3.1 Definitions [9] [10]

The life length coefficient can be considered as an AD even though it is used in a slightly different context in the Finnish pension system; it is determined as

$$EAL^t = \sum_{x=62}^{n} 1.02^{-(x+0.5-62)} \frac{L_x^t}{l_{62}^t} \tag{8}$$

here 1.02 is a discounting rate factor. The Finnish statistical office uses a partial formula for estimating the q_x values. In *Figure* 3 this method can be visualized. Here the thin diagonal lines represents a time span that starts at the first of January whereas the thicker lines represents a time span that goes through the middle of the year representing a cohort under the assumption that births take place uniformly during a year. As can be seen within the square [a b c d] in *Figure* 3 there are two thick lines (i and j) between the year t and $t+1$, one within the upper triangle and one within the lower triangle representing two cohorts. This is a Lexis diagram but one where the vertical axis has been flipped.

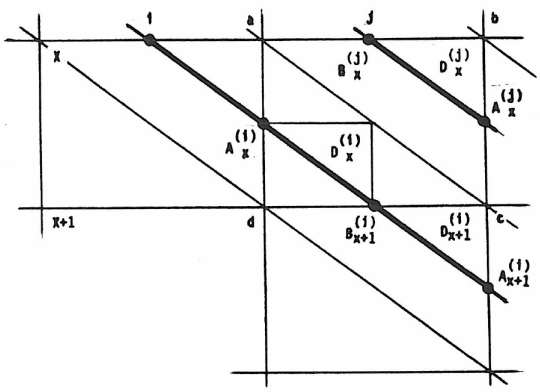

Figure 3: The figure shows two cohorts, the thick lines, in a Lexis diagram. *Source: Statistics Finland [10].*

$A_x^y =$ The population at the age of x years at the end of the year y.
$B_x^y =$ The population at the age of x years at the middle of the year y.
$E_x^y =$ The number of deaths at the age of x years conditional on that the person died after their birthday at the year y.
$D_x^y =$ The number of deaths at the age of x years at the year y.
$F_x^y =$ The net migration of people at the age of x years at the year y.
$p =$ The index p is the period which is set to five years.

3.3.2 Formulas [9] [10]

Here follows the formulas used in order to construct the life table. The indices i and j indicates the two cohorts seen in *Figure* 3. The index t indicates the year and $\{t, p\}$ indicates that the value is based on a period of p years with the most recent year being t.

$$A_x^{i,t} = \frac{1}{5} \sum_{y=t-5}^{t-1} A_x^y$$

$$A_x^{j,t} = \frac{1}{5} \sum_{y=t-4}^{t} A_x^y$$

$$A_x^{i,t} = A_x^{j,t-1}$$

$$B_0^{t,p} = \frac{1}{5} \sum_{y=t-4}^{t} B_0^y$$

$$D_x^{t,p} = \frac{1}{5} \sum_{y=t-4}^{t} D_x^y$$

$$E_x^{t,p} = \frac{1}{5} \sum_{y=t-4}^{t} E_x^y$$

$$F_x^{i,t} = \frac{1}{2} \left(A_{x+1}^{i,t} + D_x^{t,p} - E_x^{t,p} + E_{x+1}^{t,p} - A_x^{i,t} \right) = F_{x+1}^{i,t}$$

$$F_0^{j,t} = A_0^{j,t} + E_0^{t,p} - B_0^{t,p}$$

$$q_x^{i,t} = \frac{D_x^{t,p} - E_x^{t,p}}{A_x^{i,t} + \frac{1}{2} F_x^{i,t}}$$

$$
\begin{aligned}
q_x^{j,t} &= \frac{E_x^{t,p}}{A_x^{j,t} - \frac{1}{2}F_x^{j,t}} \\
q_x^t &= \begin{cases} q_x^{i,t} + q_x^{j,t} - q_x^{i,t}q_x^{j,t} & \text{for } x < 100 \\ 1 & \text{for } x = 100 \end{cases} \\
l_0 &= 100,000 \\
l_{x+1}^t &= (1 - q_x^t)l_x^t \\
L_x^t &= \frac{(l_x^t + l_{x+1}^t)}{2}
\end{aligned}
$$

Here $q_x^{i,t}$ represents the lower triangle and $q_x^{j,t}$ represents the upper triangle in *Figure* 3. As can be seen death risks are only used up until the age of 100 from where it is set to 1.

3.4 Poland

In the Polish system smoothing is used for all ages and with two methods.

3.4.1 Definitions [11]

In the Polish pension system the AD is the unisex life expectancy which is defined as

$$e_x = \frac{T_x}{l_x} \qquad x = 0, 1, 2, ..., 100 \tag{9}$$

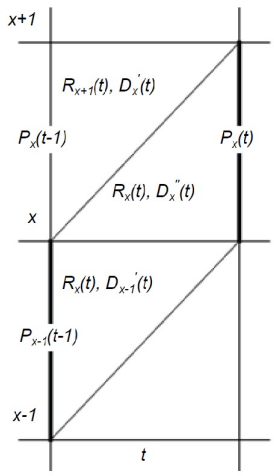

Figure 4: A graphical representation of the Polish method. *Source: Central Statistical Office [11].*

Below follows the variables needed, for a graphical representation see *Figure 4*.

$P_x(t)$ = The population at the age of x years at the end of year t.

$D_x^{'}(t)$ = The number of deaths in year t at the age of x, among people born in the year $t-x-1$.

$D_x^{"}(t)$ = The number of deaths in year t at the age of x among people born in in the year $t-x$.

$R_x(t)$ = The net migration of people at the age of x years.

3.4.2 Formulas

Here consideration is taken to migration as it was in the Finnish system. The principle is similar but the computation is slightly different.

$$R_x(t) = \frac{1}{2}\left(P_{x-1}(t-1) - P_x(t) - D_{x-1}^{"}(t) - D_x^{"}(t)\right) \qquad \text{for } 1 \leq x \leq 84$$

$$q_x^{'} = \frac{\sum_t D_x^{'}(t)}{\sum_t \left(P_x(t-1) - \frac{1}{2}R_{x+1}(t)\right)}$$

$$q_x^{"} = \frac{\sum_t D_x^{"}(t)}{\sum_t \left(P_x(t) + D_x^{"}(t)\frac{1}{2}R_x(t)\right)}$$

$$q_x = 1 - (1 - q_x^{'}(1 - q_x^{"})) \qquad \text{for } 0 \leq x \leq 84$$

The q_x values are then graduated by the use of weight factors

$A = [0.88571 \;\; 0.25714 \;\; -0.14286 \;\; 0.08571]$

$B = [0.25714 \;\; 0.37143 \;\; 0.34286 \;\; 0.17143 \;\; -0.14286]$

$C = [-0.08571 \;\; 0.34286 \;\; 0.48571 \;\; 0.34286 \;\; -0.08571]$

$D = [-0.09524 \;\; 0.14286 \;\; 0.28571 \;\; 0.33333 \;\; 0.28571 \;\; 0.14286 \;\; -0.09524]$

$E = [-0.09091 \;\; 0.06061 \;\; 0.16883 \;\; 0.23377 \;\; 0.25541 \;\; 0.23377 \;\; 0.16883 \;\; 0.06061 \;\; -0.09091]$

That are then applied accordingly

$$q_x = \begin{cases} A[q_1 \; q_2 \; q_3 \; q_4 \; q_5]^T & \text{for } x = 1 \\ B[q_1 \; q_2 \; q_3 \; q_4 \; q_5]^T & \text{for } x = 2 \\ C[q_1 \; q_2 \; q_3 \; q_4 \; q_5]^T & \text{for } x = 3 \\ D[q_{x-3} \; q_{x-2} \; q_{x-1} \; q_x \; q_{x+1} \; q_{x+2} \; q_{x+3}]^T & \text{for } x \in \{4, ..., 29\} \\ E[q_{x-4} \; q_{x-3} \; q_{x-2} \; q_{x-1} \; q_x \; q_{x+1} \; q_{x+2} \; q_{x-2} \; q_{x+4}]^T & \text{for } x \in \{30, ..., 84\} \end{cases}$$

This procedure is done three times. To determine probabilities of death among people older than 84 years a polynomial exponential function is fitted to the number of survivors

$$l_x = 100,000e^{(-b_0 - b_1 x - b_2 x^2 - \cdots - b_5 x^5)}$$

in points $x = 40, 45, ..., 85$ and then extrapolated for ages 85-120. The fitting is done by the use of the generalised least squares method (with application of Marquardt non linear optimization method). Finally the last necessary terms can be determined

$$
\begin{aligned}
l_0 &= 100,000 \\
l_x &= l_{x-1}(1 - q_{x-1}) && \text{for } x = 1, 2, ..., 120 \\
L_x &= \frac{l_x + l_{x+1}}{2} && \text{for } x = 1, 2, ..., 119 \\
T_x &= \sum_{y \geq x} L_y && \text{for} x = 0, 1, 2, ..., 100
\end{aligned}
$$

3.5 Comparison

The *AD* formulas used are similar, the major differences are the set discount rates, which varies from 0 to 2 % and that Sweden apply monthly discounting while Finland and Norway apply it once per year. Another difference is that in the Norwegian system the *AD* is in relation to an average of survivors of 40 different ages whereas in the rest the relation is only to survivors of the same age as the pensioner. This is a way of including the inheritance gain in the *AD*. In the Finnish and Polish system data is only included up until the age of 100 whereas no limit exists in the Norwegian and Swedish systems.

The key difference between all methods is how the death risk, q_x, is estimated. All countries differentiate deaths by cohorts in various ways but basically with the idea of Lexis triangles. In the Swedish, Finnish and Polish system the death risk is directly estimated whereas in the Norwegian system the death intensity, μ_x, is first estimated and then transformed into a death risk. In the Polish and Finnish system consideration is taken to the net migration during a year, the method is however slightly different from one another, whereas no consideration is taken in the Swedish and Norwegian system.

In the Swedish, Polish and Finnish systems data is gathered for a period before calculation of death risks are made whereas in the Norwegian system an average is taken of a number of one year estimations. Sweden and Poland use mathematical functions for smoothing purposes for the higher ages, the Swedish method A Generalized Perks Formula has a theoretical background while the exponential function used in the Polish system has no underlying theory.

Most computations have been conducted on data provided by *mortality.org* this is due to that the data for ages above the age of 100 is not available to the public from Statistics Sweden. By comparing the data with life tables provided by Statistics Sweden for years 2000-2011 there are only small differences in few cells. All computations have been made in MATLAB.

To determine how the different methods affect the *AD* all methods have been used on Swedish data. First a presentation is given of the values without any alteration which can be misleading since the rate of return varies and the usage in the different pension systems but it still provides some comparison, see *Table*

1. As can be seen the Polish *AD* is significantly larger, this is due to that no rate of return is being used and by the use of the exponential formula for higher ages.

Age	Finland	Norway	Poland	Sweden
61	17,82	18,69	24,76	18,83
62	17,28	17,92	23,97	18,24
63	16,74	17,15	23,20	17,66
64	16,20	16,38	22,44	17,07
65	15,66	15,61	21,70	16,49

Table 1: AD computed for the latest time period, i.e up until the year 2011, for various retirement ages and by the original methods stated by each country but based on Swedish data.

To remove the effect of the different discount rates the rate is set to the Swedish rate of 1.016 for Finland and Norway as well while Poland is discarded since it has no defined rate, see *Table* 2.

Age	Finland	Norway	Sweden
61	18,66	16,95	18,83
62	18,07	16,30	18,24
63	17,49	15,65	17,66
64	16,90	15,00	17,07
65	16,32	14,34	16,49

Table 2: AD computed for the latest time period, i.e up until the year 2011, for various retirement ages and by the original methods stated by each country but based on Swedish data and the discount rate factor is set to 1.016.

To get a better understanding of the effects of the different ways of estimating the death risk a combined life table for all methods can be seen in *Table* 3. Here the result from all official methods can be seen as well as the pure data based version of the Swedish and Polish methods referred to as *mod* in the table, i.e without the usage of a smoothing function for high ages. It can be seen that the death risk seems to be underestimated in the Polish method in respect to the others, this is due to the application of exponential function for ages 85 and above which is not suited for the Swedish data since Swedish people tend to live longer and by applying it for higher ages such as 90 and above yields results more in line with the others.

The different computational methods have been used to compute *AD* from 1975-2011 in order to see the general tendencies. In *Table* 4 a comparison between the strict data based methods are shown in reference to the Swedish method without the smoothing function for higher ages. The table shows the differences in percent, as can be seen the differences are rather small but the different methods seem to give either a positive or negative effect on the size of the *AD*. The effect tend to increase by retirement age which seems natural.

For a comparison of the effects of the application of the smoothing functions for higher ages used in Poland and Sweden see *Table* 5. Here the application age has been changed to 91 years for the exponential model used by the Polish Statistics

for a comparative reason. It should be noted that the estimated variables in the Generalized Perks Formula used here are not the official estimations used by the Swedish Statistics, the estimation varies some from the available estimates. For a comparison of the differences of the application of the GPF see *Table* 6.

In *Table* 7 the effect of the application of the GPF can be seen for each age and year. The effects are not significant but the application of the GPF does have a positive effect on the size of the AD which generally increases by retirement age and for each coming year. This is due to a systemic underestimation of the death risks in the ages of 91-100 according to Statistics Sweden.

To see the effect of the discretization used for the computation, the Swedish AD method has been computed for various number of discounting points per year. As a reference the AD has been computed with only one point in the middle of the year which is equivalent to the set-up used in the Finnish computation, see *equation* (8). This has been done for the most recent AD of 2011, see *Table* 8. The points are here symmetrically placed during a year, as can be seen the differences in reference to only applying the rate once decreases as the number of points increases. As the number of points goes to infinity this is equivalent to continuous compounding with the rate r determined as

$$e^r = 1.016$$

and with a uniform population decrease during the year. The 12 points officially used in the Swedish computation seems like a natural choice in respect to statement 3 in section 3.1.1 where it says that the rate should be taken in to account up until the month specified, however by using more points the statement should still be valid. For the effects over the period 1975-2011 see Appendix.

	Sweden		Poland		Norway	Finland
Age	l_x	l_x^{mod}	l_x	l_x^{mod}	l_x	l_x
0	100 000	100000	100000	100000	100000	100000
⋮	⋮	⋮	⋮	⋮	⋮	⋮
60	93 613	93613	93615	93615	93616	93614
61	93 054	93054	93057	93057	93057	93053
62	92 451	92451	92454	92454	92455	92449
63	91 776	91776	91779	91779	91781	91774
64	91 049	91049	91051	91051	91052	91044
65	90 196	90196	90198	90198	90198	90189
66	89 282	89282	89283	89283	89281	89273
67	88 327	88327	88326	88326	88324	88316
68	87 291	87291	87287	87287	87284	87277
69	86 179	86179	86173	86173	86169	86161
70	84 926	84926	84917	84917	84915	84904
71	83 601	83601	83590	83590	83588	83573
72	82 167	82167	82156	82156	82153	82134
73	80 580	80580	80568	80568	80564	80540
74	78 860	78860	78847	78847	78842	78811
75	76 977	76977	76964	76964	76958	76918
76	74 919	74919	74906	74906	74900	74848
77	72 698	72698	72685	72685	72680	72611
78	70 246	70246	70234	70234	70231	70143
79	67 571	67571	67560	67560	67558	67446
80	64 714	64714	64705	64705	64702	64565
81	61 573	61573	61569	61569	61564	61395
82	58 248	58248	58248	58248	58242	58038
83	54 667	54667	54669	54669	54665	54416
84	50 858	50858	50861	50861	50859	50559
85	46 876	46876	46947	46884	46881	46526
86	42 696	42696	42985	42707	42706	42286
87	38 402	38402	38960	38416	38419	37928
88	33 988	33988	34922	33996	34009	33442
89	29 632	29632	30925	29630	29654	29018
90	25 393	25393	27026	25382	25415	24717
91	21 357	21357	23281	21320	21377	20626
92	17 510	17551	19745	17485	17574	16777
93	14 082	14095	16465	14037	14124	13312
94	11 093	11013	13481	10950	11044	10238
95	8 547	8375	10823	8296	8405	7627
96	6 431	6186	8506	6111	6220	5501
97	4 719	4481	6535	4400	4510	3872
98	3 371	3134	4899	3055	3161	2615
99	2 340	2109	3577	2031	2133	1683
100	1 576	1388	2540	1314	1407	1053
101	1 028	878	1750	811	893	625
102	648	525	1168	476	538	347
103	393	314	753	282	325	194
104	230	183	468	157	191	103
105	129	108	280	87	112	55
106	69	55	161	43	60	25
107	35	28	88	20	30	10
108	17	16	46	10	17	5
109	8	9	23	6	9	2
110	3	7	11	4	7	2
111	1	4	5	2	5	1
112	0	4	2	2	5	1
113	0	3	0	0	4	0

Table 3: Combined life table for the period 2007-2011. Here the term *mod* indicates that the computation has been modified meaning that there has been no application of a smoothing function for higher ages.

	Poland				Norway				Finland			
Age	Min	Max	Mean	Median	Min	Max	Mean	Median	Min	Max	Mean	Median
61	-0,08	-0,01	-0,03	-0,03	0,00	0,03	0,01	0,01	-0,59	-0,43	-0,50	-0,50
62	-0,08	-0,01	-0,04	-0,04	0,00	0,03	0,01	0,02	-0,63	-0,45	-0,53	-0,53
63	-0,09	-0,01	-0,04	-0,04	0,00	0,03	0,02	0,02	-0,66	-0,47	-0,56	-0,56
64	-0,10	-0,01	-0,04	-0,04	0,00	0,04	0,02	0,02	-0,71	-0,50	-0,59	-0,59
65	-0,10	-0,01	-0,04	-0,04	0,00	0,04	0,02	0,02	-0,75	-0,53	-0,63	-0,63
66	-0,11	-0,01	-0,05	-0,05	0,00	0,04	0,02	0,02	-0,80	-0,56	-0,67	-0,67
67	-0,12	-0,01	-0,05	-0,05	0,00	0,04	0,02	0,02	-0,86	-0,59	-0,71	-0,72
68	-0,13	-0,01	-0,05	-0,05	0,00	0,05	0,02	0,03	-0,92	-0,63	-0,76	-0,76
69	-0,14	-0,01	-0,06	-0,05	0,00	0,05	0,02	0,03	-0,99	-0,67	-0,81	-0,82
70	-0,15	-0,01	-0,06	-0,06	0,00	0,05	0,03	0,03	-1,06	-0,71	-0,87	-0,87

Table 4: A comparison between the size of the *AD* determined by the Swedish model, see *equation* (2), but for input data calculated by the various pure data based methods normalized by the Swedish pure data based method. The differences are in %. The *AD* are from the period 1975-2011.

	Sweden				Poland			
Age	Min	Max	Mean	Median	Min	Max	Mean	Median
61	0,00	0,06	0,02	0,02	-0,04	0,21	0,07	0,08
62	0,00	0,06	0,02	0,02	-0,05	0,22	0,08	0,08
63	0,00	0,06	0,02	0,02	-0,05	0,23	0,08	0,09
64	0,00	0,07	0,02	0,02	-0,05	0,24	0,09	0,09
65	0,00	0,07	0,02	0,02	-0,06	0,26	0,09	0,10
66	0,00	0,08	0,02	0,02	-0,06	0,28	0,10	0,11
67	0,00	0,08	0,03	0,02	-0,07	0,30	0,11	0,12
68	0,00	0,09	0,03	0,03	-0,07	0,32	0,12	0,12
69	0,00	0,09	0,03	0,03	-0,08	0,35	0,13	0,13
70	0,00	0,10	0,03	0,03	-0,08	0,38	0,14	0,15

Table 5: A comparison between the size of the *AD* determined by the Swedish model, see *equation* (2), but for input data calculated by the application of the official Swedish model and the official Polish model but with application at the age of 91 normalized by the Swedish pure data based method. The differences are in %. The *AD* are from the period 1975-2011.

	Sweden Official				Sweden Non-Official			
Age	Min	Max	Mean	Median	Min	Max	Mean	Median
61	0,02	0,08	0,04	0,05	0,03	0,06	0,04	0,04
62	0,01	0,07	0,04	0,03	0,03	0,06	0,04	0,05
63	0,00	0,09	0,03	0,03	0,03	0,06	0,05	0,05
64	0,02	0,08	0,05	0,07	0,03	0,07	0,05	0,05
65	-0,01	0,07	0,03	0,05	0,03	0,07	0,05	0,05
66	-0,02	0,09	0,03	0,04	0,04	0,08	0,06	0,06
67	0,01	0,10	0,05	0,05	0,04	0,08	0,06	0,06
68	0,02	0,09	0,05	0,03	0,04	0,09	0,06	0,07
69	0,00	0,12	0,06	0,07	0,04	0,09	0,07	0,07
70	0,00	0,12	0,06	0,06	0,05	0,10	0,07	0,08

Table 6: A comparison between the size of the *AD* determined by the Swedish model, see *equation* (2), but for input data calculated by the application of the official Swedish model normalized by the Swedish pure data based method for official and non-official estimation. The differences are in %. The *AD* are from the period 2005-2011.

	Generalized Perks Formula Official Estimates						
	2005	2006	2007	2008	2009	2010	2011
61	0,021	0,023	0,019	0,048	0,062	0,046	0,076
62	0,016	0,029	0,029	0,014	0,032	0,075	0,060
63	0,004	0,001	0,047	0,029	0,042	0,033	0,089
64	0,022	0,023	0,067	0,051	0,068	0,070	0,077
65	-0,002	-0,010	0,019	0,047	0,069	0,047	0,061
66	-0,005	-0,024	0,044	0,019	0,040	0,092	0,059
67	0,027	0,011	0,025	0,064	0,103	0,053	0,098
68	0,022	0,024	0,033	0,019	0,069	0,089	0,071
69	0,033	-0,001	0,025	0,086	0,094	0,068	0,123
70	0,013	0,001	0,062	0,057	0,082	0,066	0,117

Table 7: The effect of the size of the *AD* by the application of the Generalized Perks Formula in reference to the pure data based method used in Sweden. The differences are in %. The *AD* are from the period 2005-2011.

	Number of points per year					
Age	4	12	52	365	1000	100.000
61	0,67	0,23	0,057	0,013	0,0087	0,0060
62	0,69	0,24	0,059	0,014	0,0090	0,0062
63	0,72	0,24	0,061	0,014	0,0093	0,0065
64	0,74	0,25	0,063	0,015	0,0097	0,0067
65	0,77	0,26	0,065	0,015	0,0100	0,0070
66	0,80	0,27	0,068	0,016	0,0104	0,0073
67	0,83	0,28	0,071	0,017	0,0109	0,0076
68	0,86	0,29	0,073	0,017	0,0114	0,0080
69	0,90	0,30	0,076	0,018	0,0119	0,0084
70	0,93	0,32	0,080	0,019	0,0124	0,0088

Table 8: A comparison of the effect of applying the discounting rate for a various number of times during a year in respect to applying it only once in the middle of the year. The comparison is for the *AD* of 2011. The differences are in %.

4 Prognostic Based Computations

4.1 Sweden [24] [23] [22] [2]

In the Premium Pension the following AD is used

$$AD_x = \int_0^\infty e^{-\delta t} \frac{l(x+t)}{l(x)} dt \tag{10}$$

where

$$\begin{aligned} l(x) &= e^{-\int_0^x \mu(t)dt} \\ \mu(x) &= a + be^{cx} \\ x &= \text{The exact time of retirement} \\ \delta &= \text{Interest rate} \end{aligned}$$

Here $\mu(x)$ is the Gompertz Makeham's Formula that was also included in the Generalized Perks Formula in the section 3.1.3. It consists of an age independent part a and an age dependent part b. This formula is a conditional life expectancy with continuously compounding discounting rate.

Here the parameters a, b and c are estimated in respect to the cohort which are closest to the age of 65 in a three year interval which for the period of 2010-2012 are persons born in 1946. The values remain the same during this period. This is done by the Swedish Pensions Agency based on the death rates projected by Statistics Sweden.

The future death rates are computed by the use of a Lee-Carter model defined as

$$log(m_x^t) = a_x + b_x k_t + \epsilon_x^t \tag{11}$$

this can be represented in matrix form as

$$M = A + BK + \epsilon \tag{12}$$

where

$$
\begin{aligned}
a_x &= \text{Age specific average term} \\
b_x &= \text{Age specific coefficient for the time trend} \\
\epsilon_x^t &= \text{Error term} \\
M &= \begin{bmatrix} log(m_x^t) & \cdots & log(m_x^{t+n-1}) \\ \vdots & \ddots & \vdots \\ log(m_{x+m-1}^t) & \cdots & log(m_{x+m-1}^{t+n-1}) \end{bmatrix} \\
A &= \begin{bmatrix} a_x & \cdots & a_x \\ \vdots & \ddots & \vdots \\ a_{x+m-1} & \cdots & a_{x+m-1} \end{bmatrix} \\
B &= \begin{bmatrix} b_x \\ \vdots \\ b_{x+m-1} \end{bmatrix} \\
K &= \begin{bmatrix} k_t & \cdots & k_{t+n-1} \end{bmatrix} \\
\epsilon &= \begin{bmatrix} \epsilon_x^t & \cdots & \epsilon_x^{t+n-1} \\ \vdots & \ddots & \vdots \\ \epsilon_{x+m-1}^t & \cdots & \epsilon_{x+m-1}^{t+n-1} \end{bmatrix}
\end{aligned}
$$

First the historical death rates are estimated as

$$m_x^t = \frac{D_x^t}{(P_{x-1}^{t-1} + P_x^t)/2}$$

where D_x^t is the number of deaths and P_x^t is the population at the age of x years at the year t. The death rates are then logarithmized and centered

$$\tilde{M} = M - \overline{M}$$

where $\overline{M}$ is the row-wise mean i.e for each age x. The singular value decomposition can now be applied to $\tilde{M}$ yielding

$$\tilde{M} = USV^T$$

here

$$
\begin{aligned}
U &= m \times m \quad \text{(unitary matrix)} \\
S &= m \times n \quad \text{(diagonal matrix)} \\
V &= n \times n \quad \text{(unitary matrix)}
\end{aligned}
$$

By centering, the first term A in (12) is now given by $\overline{M}$. What is left now is to estimate the vectors B and K. The largest value in the diagonal matrix S now contains the most 'information' about the matrix $\tilde{M}$, actually variance explained, and by using the corresponding vectors in U and V, $\tilde{M}$ can be estimated as

$$\tilde{M} \approx s_{1,1}\boldsymbol{u_1} \otimes \boldsymbol{v_1^T} = B \otimes K$$

Here $\boldsymbol{u_1}$, the first column vector of U, will define the relationship between the coefficients in B and $\boldsymbol{v_1}$, the first column vector of V, will define the relation among the time coefficients in K. There are a number of ways that yield the same result so the standard has been set such that

$$\sum_{k=1}^{m} b_k = 1$$

The time vector is then extrapolated under a linear assumption

$$\hat{k} = \frac{min(k_i) - max(k_i)}{n-1} \quad \text{for } i \in \{1, ..., n\}$$

The coefficient of the n:th year is therefore $n \cdot \hat{k}$. The death rates can then be extrapolated into the future. The death rates for the first year of the estimation, here 2012, needs to be determined before the extrapolation can be conducted. To transform the estimated death rates $\hat{m}_x$ into estimated probability of death $\hat{q}_x$ the following is done

$$\hat{q}_x^t = 1 - e^{-0.5(\hat{m}_x^t + \hat{m}_{x+1}^t)}$$

Estimations up until here are done by Statistics Sweden and the following is done by the Pensions Agency. The Lee Carter model was based on data for the period 1975-2011 and mainly in the age span of 50-100 years. These estimations are done every third year by Statistics Sweden and the Pensions Agency then base their estimations on the cohort that will be 65 at the end of the second year of this period. The previous estimations were done gender specific so to make them unisex the following is done

$$q_x^{uni} = \begin{cases} \frac{P_{x-3}^m q_x^m + P_{x-3}^f q_x^f}{P_{x-3}^m + P_{x-3}^f} & \text{for } x \in \{65\} \\ \\ \frac{l_x^m q_x^m + l_x^f q_x^f}{l_x^m + l_x^f} & \text{for } x \in \{66, ..., 90\} \end{cases}$$

where

$$\begin{aligned} P_x^m &= \text{The male population at the age of } x \text{ years} \\ P_x^f &= \text{The female population at the age of } x \text{ years} \\ q_x^m &= \text{The male death risk at the age of } x \text{ years} \\ q_x^f &= \text{The female death risk at the age of } x \text{ years} \\ l_x &= \begin{cases} P_{x-3}(1 - q_{x-1}) & \text{for } x \in \{65\} \\ l_{x-1}(1 - q_{x-1}) & \text{for } x \in \{66, ..., 90\} \end{cases} \end{aligned}$$

The reason of using P_{x-3} is that it is the current age of the cohort when the prognosis was made. Here the estimated death risk for each age corresponds to a different year, since the values are taken diagonally, see *Figure* 5, so if the year is t for age x the year will be $t+1$ for age $x+1$. The estimated death risks are then transformed into intensities as

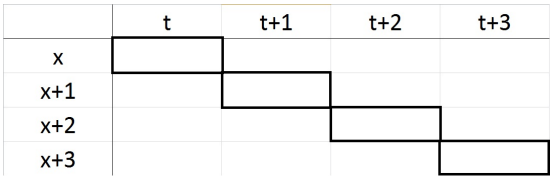

Figure 5: The black boxes indicates which death risk is chosen for each year.

$$\mu_{x+0.5} = log(1 - q_x)$$

With the intensities estimated the parameters a, b and c can finally be estimated by the method of least squares, i.e by minimizing the following expression

$$S = \sum_{x=65}^{90} \left(\mu_x - (a + be^{c(x+0.5)})\right)^2$$

The Makeham formula is used up until the age of 97 years and is then replaced by an inclining slope

$$\mu_x = \begin{cases} a + be^{cx} & \text{for } x < 97 \\ a + be^{c \cdot 97} + 0.001 \cdot x & \text{for } x \geq 97 \end{cases}$$

4.2 Comparison

For a comparison between the expected life length based on the life tables and the prognosticated estimations see *Table* 9. The prognostic expected life length is more than a year longer than the observed. This effect is caused mainly due to the prognosticated death rates but also because of the application of the Makeham's formula. To see the effect of Makeham's formula it has been fitted to period data in the same way that was used for the forecast, see *Table* 10. The application of the Makeham's formula give similar estimations as to the pure data based method and with the smoothing with the Generalized Perks Formula. The *AD* does not only depend on the expected life length but also on the distribution of survivors, because of the discounting rate.

By applying the yearly rate of 1.6 % to the Makeham estimation it can be seen in *Table* 11 that the result is fairly close to the official *AD* of the Inkomstpension system. The rate r used is determined as

$$r = log(1.016)$$

to be comparable for the continuous compounding.

Age	Observed	Prognostic
61	22,96	24,26
62	22,10	23,46
63	21,26	22,65
64	20,43	21,85
65	19,62	21,05
66	18,81	20,25
67	18,01	19,45
68	17,22	18,66
69	16,43	17,87
70	15,67	17,09

Table 9: A comparison between life lengths computed by the life table method (Observed) and the prognostic method for the year 2011 (the observed is based on the period 2007-2011).

Age	GPF	Pure	Makeham
61	22,96	22,94	22,93
62	22,10	22,08	22,09
63	21,26	21,24	21,24
64	20,43	20,41	20,41
65	19,62	19,60	19,58
66	18,81	18,79	18,77
67	18,01	17,99	17,96
68	17,22	17,20	17,16
69	16,43	16,41	16,37
70	15,67	15,65	15,60

Table 10: Conditional life expectancy for the period 2007-2011 estimated by Generalized Perks Formula (GPF), pure data based method and by Makeham's formula. Here GPF is applied according to 3.1.3 and the Makehams formula according to 4.1.

Age	Official	Makeham
61	18,83	18,78
62	18,24	18,19
63	17,66	17,61
64	17,07	17,02
65	16,49	16,43
66	15,91	15,84
67	15,33	15,25
68	14,74	14,66
69	14,16	14,07
70	13,58	13,48

Table 11: *AD* for the Inkomstpension system for the year 2011. The first column is the official *AD* whereas the second column is determined by the use of the Makeham function.

Another way of estimating the future death rates is by assuming that the current trend will continue and estimating the change rate r_x from the following expression

$$m_x^t(1+r_x)^\delta = m_x^{t+\delta}$$

This was the method previously used by Statistics Sweden. The method is rather intuitive and as an example is applied here, see *Table* 12, but instead by directly estimating the death rates as

$$q_x^t(1+r_x)^\delta = q_x^{t+\delta}$$

This was done for period data of five years with the reference period being 2007-2011 and then for period data in the past with five year interval from the latest year of the reference period. As a starting point the q_x^{2009} was used which was estimated as the q_x of the period 2007-2011 and then extrapolated by the estimated rate. The estimated q_x values were then fitted to a Makeham formula in the same way as the official method. The values are in line with the previous results.

	L-C	5	10	15	20	25
61	24,26	24,09	24,38	24,22	24,32	24,21
62	23,46	23,29	23,58	23,42	23,52	23,41
63	22,65	22,49	22,78	22,61	22,71	22,61
64	21,85	21,69	21,99	21,81	21,91	21,81
65	21,05	20,89	21,19	21,01	21,10	21,01
66	20,25	20,10	20,40	20,22	20,30	20,22
67	19,45	19,30	19,61	19,42	19,51	19,42
68	18,66	18,51	18,82	18,63	18,71	18,63
69	17,87	17,73	18,04	17,84	17,92	17,85
70	17,09	16,95	17,26	17,06	17,14	17,07

Table 12: Here the life expectancy for various ages is shown with two different methods. The first column correspond to the Lee Carter model, see *equation* 11, and the rest to the yearly rate change. The Makeham formula was then fitted to the data in the same way, see *equation* 4.1.

An alternative approach would be to look at a country with lower death rates such as Japan and estimate how many years ahead they are and then use these estimates. The already computed q_x values from *mortality.org* was used for this, which uses a somewhat different method than the ones in this report. To estimate how far ahead in years Japan is compared to Sweden a non-weighted least squares method was used for the ages 65 to 90 years i.e by finding δ that minimizes the following expression

$$S_\delta = \sum_{x=65}^{90} (q_{x,t}^{Swe} - q_{x,t-\delta}^{Jpn})^2$$

And also a weighted expression for ages 0 to 105 where each death risk is in relation to the number of survivals in the corresponding age

$$S_\delta = \sum_{x=0}^{105} (l_{x,t}^{Swe}(q_{x,t}^{Swe} - q_{x,t-\delta}^{Jpn}))^2$$

As can be seen in *Figure* 6 the two methods yields similar results and the death risk seem to continue to decrease faster in Japan than in Sweden. The Japanese

data can also be used to see if the current trend seems to continue. To estimate this, the yearly change was first estimated as

$$R^t = \frac{Q^{t+1}}{Q^t}$$

where

$$Q^t = [q_{65}^t \cdots q_{90}^t]^T$$
$$R_\delta^t = [r_{65}^t \cdots r_{90}^t]^T$$

The geometric mean can then be determined as

$$R^\delta = \left(\prod_{i=0}^{\delta-1} R^{t-i} \right)^{1/\delta}$$

Finally the non weighted mean for each δ was taken as

$$r_{segment} = \frac{1}{90 - 65 + 1} \sum_{x=65}^{90} r_x^\delta$$

The result can be seen in *Figure* 7, the yearly change rate seem to be rather constant during the period. It is hard to estimate the future Swedish death rates from this but it seems plausible that the current trend can continue since it has for Japan and the two curves behave similarly.

One approach would be to use the following relation $\hat{q}_{x,t}^{Swe} = q_{x,t-\delta}^{Jpn}$ in order to estimate future q_x values. This was done under the assumption that the linear trend that can be seen in *Figure* 6 continues into the future i.e by assuming that for each year the difference increases with 0.5 years.

$$\hat{q}_{t+n}^{Swe} = \begin{cases} q_{t-\delta_n}^{Jpn} & \text{for } \lfloor \delta_n \rfloor = \delta_n \\ \frac{q_{t-\lfloor \delta_n \rfloor}^{Jpn} + q_{t-\lfloor \delta_n \rfloor+1}^{Jpn}}{2} & \text{else} \end{cases}$$
$$\delta_n = \delta_0 + (n-1) \cdot 0.5$$

Here δ_0 was estimated to 13, by this approach the Japanese data could generate estimations for 22 years, after that the last estimation was extrapolated by the usage of the geometric mean for the latest 10 year period of Swedish data. The death risks were then fitted to a Makeham formula in the same manner as previously stated. The result can be seen in *Table* 13 in comparison with the Lee-Carter estimation. The estimations based on Japanese data yield a longer life expectancy than the estimations based on the Lee-Carter method. This method is almost completely based on actual data but with the underlying assumption that the relation between the Japanese and Swedish death risks continues in the same way as it has in the past.

	Lee-Carter	Japan estimation
61	24,26	24,43
62	23,46	23,62
63	22,65	22,81
64	21,85	22,01
65	21,05	21,21
66	20,25	20,43
67	19,45	19,65
68	18,66	18,88
69	17,87	18,13
70	17,09	17,38

Table 13: Comparison between the official Lee-Carter estimation and the estimation based on Japanese data.

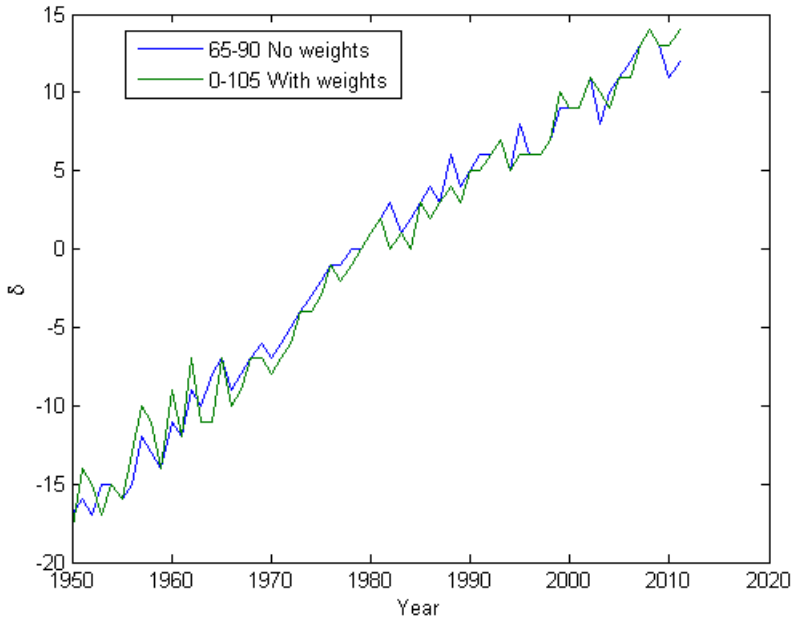

Figure 6: The plot shows the number of years ahead Japan is of Sweden represented as the δ from 1950 to 2011.

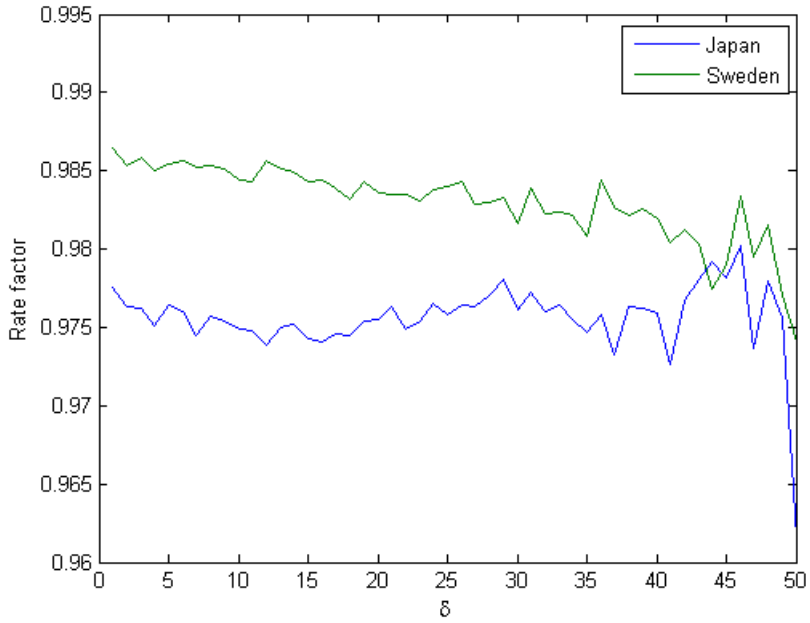

Figure 7: The plot shows the change rate factor of the death risks i.e by multiplying the current death risk with the rate factor will approximately give the next years death risk. The most recent year for Sweden and Japan is 2011 and 2009 respectively.

5 Politics

The guidelines of the current Swedish pension system were determined in 1994 and on January first 1999 the system was implemented. The system has however been updated since. The pension agreement is between five political parties with a strong majority. These parties have representatives in the *pension group* that initially had members from all seven parties [15]. In 1994 there was a coalition government led by Carl Bildt which was then proceeded by a social democratic government in October led by Ingvar Carlsson and later Göran Persson.

In the governments proposition in 1993/94 it was suggested that in order to determine the *AD*, consideration should be taken to the mortality during the years of acquisition and as a consequence the *AD* can be smaller than it would otherwise have been (this is similar to the implicit inheritance gain as was seen in the Norwegian *AD*). It was also suggested that income differences between men and women should be considered, which would also yield a lower *AD* as a consequence of the lower life expectancy and higher earnings of men [14].

It was suggested that the *AD* shall be determined on the basis of known circumstances and not on a prognosis. The pension group suggested that the definite *AD* should be determined at the age of 61 years but the government believed that it should be at the age of 65 years instead so that it would adjust faster to changes in life expectancy. The government believed that the best approach to adjusting to the changes in life expectancy would be to recalculate the *AD* for each year but shared the opinion with the pension group that this shall not be done since current pensioners are unable to compensate themselves of such a change. Coming generations however have the possibility of delaying their retirement it is said [14].

The discounting rate was suggested to be 1.5 %, the government believed that the growth rate of the economy could be faster but that the norm should be modestly chosen [14]. In the new governments proposition in 1997/98 it was suggested that the rate should be 1.6 % which Riksrevisionsverket, The Swedish National Audit Office, saw as a compromise between financial stability at national level and for the pensioner [16].

SACO, Swedish Academics Central Organization, and Arbetsgivarverket, governmental employers organization, were in favour of that the income differences between men and women were not taken in to account while LO, The Swedish Trade Union Confederation, and Riksförsäkringsverket believed that this should be taken in to account. TCO, The Swedish Confederation for Professional Employees, were in favour of the unisex *AD* but believed that consideration should be taken to that the average income for men is higher as well as that life expectancy is shorter. The consequence of not taking any consideration to the income differences is that the *AD* will be larger. Konjukturinstitutet National Institute of Economic Research, had an understanding for the governments suggestion of not taking the income differences between men and women into consideration but believed that it should be clearly stated that the *AD* will be larger as a consequence [12].

Finansinspektionen, The Swedish Financial Supervisory Authority, and Statskontoret, The Swedish Agency for Public Management, were in favour of having

a prognostic estimation of the life expectancy. Försäkringsförbundet, organization for several insurance companies, were against the fixed AD at the age of 65 and believed that it should be recalculated for each year [12].

The government argues that the effect of the higher average incomes and shorter life expectancy of men would lead to a surplus whereas the fixed AD at the age of 65 would lead to a deficit and that these effects would then completely or partially cancel out. From the individual's perspective the government believed that the AD shall be fixed since the individual has small means of affecting his or her financial situation [12].

The AD is based on national statistics which is not equivalent to the insured under the pension system. This is because not everyone have acquired capital on their personal account but the government argues that approximately 90 % of the population have so that this should be a good approximation [12].

Technical components such as the AD in the pension system are considered vital by Riksrevisionsverket for the long term survival of the system [12].

In an evaluation article written by LO in 2011 it is pointed out that the life expectancy is higher for people with higher education and also that it continues to increase faster compared to people with lower education [17].

In a report written in 2011 by Pensionsåldersutredningen, a commissioner group appointed by the government for evaluation of the retirement age, it is noted that there might exist a self selection bias in respect to when individuals choose to retire since people who believe that they have a shorter life expectancy might choose to retire early while those who believe they have a longer life expectancy might choose to work longer and that this could cause a strain on the pension system [18]. This was pointed out by Riksförbundet Pensionärsgemenskap, a non-profit pensioner organization, in the proposition in 1993/94 and that this should be considered when computing the AD [14].

In the governments proposition in 2001 regarding the automatic balancing mechanism it is assumed that the construction of the AD will eliminate about two thirds of the deficit risk caused by changes in life expectancy. The remaining risk is handled by the automatic balancing mechanism in a socially desirable way according to the government. In the use of non fixed AD the longevity risk is faced by the pensioners but with the application of the automatic balancing mechanism this risk is shared with the active. The alternative with non fixed AD and a sum-index instead of the income index are here considered but the government believed that such a system would essentially be a worse system since the fluctuation of the pension level in respect to the average income would have been greater. A sum-index would consider the total amount of capital contributed by the active working force whereas the income index only considers the average income. The Inkomstpension system, is by the government said to be the world's first financially stable distributional system [19].

6 Discussion

The major difference between the AD in the Inkomstpension and the AD in the Premium Pension is not the formula itself but the underlying death risks. In section 4.2 it is shown that the usage of Makeham's formula gives a very similar result as with the official method. The official method is not purely a data based method; it is using the Generalized Perks Formula for the calculation of the life table on which the calculation is conducted and the Generalized Perks Formula is also based on Makeham's formula. The usage of the Makeham's formula should not violate the conditions specified in 3.1.1 either since the formula will be based on official statistics of a five year period. So one should be able to use the same method in both systems but based on different death risks. This would not be fundamentally different from the current method but rather an application of a function at the age of 61 instead of 91.

The obvious advantage with a pure data based method as is being used in Finland and Norway is that the method is very transparent and does not depend on parameter estimations. One could also argue that application of smoothing functions have a small effect and will not have a big impact on the size of the pension.

The pension level is not guaranteed in the current system because of the balancing mechanism so there would not be a fundamental change if the AD was recalculated every year for pensioners. The advantage of recalculating the AD every year is that it can then be based on actual data and if trends shift in the future the AD will change with it unlike a prognostic based fixed AD. A disadvantage could be that pension levels risk getting too low as the AD increases over the year, this could partially be fixed by lowering the set rate of return. If a prognostic estimation were to be used for the AD in the Inkomstpension system the estimations should be done every year or it would otherwise seem unfair if the same estimation of future life expectancy were used for three cohorts. Since fairness between different cohorts is an important feature of the Inkomstpension system.

The current computational method of the AD in the Inkomstpension system could be changed to the Finnish set up with only one discounting point per year but without terminating the calculation at the age of 100. This formula is slightly easier and it also gives a better estimation in comparison with continuous discounting.

The different methods of estimating the death risks give rather similar results and only have a small impact on the size of the AD but the effect of the smoothing function for higher ages in the Swedish method seem to increase for each year.

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7 Appendix

Age	Min	Max	Mean	Median
61	0,009	0,011	0,010	0,010
62	0,009	0,011	0,010	0,010
63	0,009	0,011	0,010	0,010
64	0,010	0,012	0,011	0,011
65	0,010	0,012	0,011	0,011
66	0,010	0,013	0,012	0,012
67	0,011	0,014	0,012	0,012
68	0,011	0,014	0,013	0,013
69	0,012	0,015	0,013	0,013
70	0,012	0,016	0,014	0,014

Table 14: *AD* discounted 1000 times per year in reference to once in the middle of the year for the period 1975-2011. The differences are in %.