Master thesis

Analysis and comparison of capital allocation techniques in an insurance context

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Abstract

Companies issuing insurance cover, in return for insurance premiums, face the payments of claims occurring according to a loss distribution. Hence, capital must be held by the companies so that they can guarantee the fulfilment of the claims of each line of insurance. The increased incidence of insurance insolvency motivates the birth of new legislations as the European Solvency II Directive. Companies have to determine the required amount of capital and the optimal capital allocation across the different lines of insurance in order to keep the risk of insolvency at an adequate level. The capital allocation problem may be treated in different ways, starting from the insurance company balance sheet. Here, the running process and efficiency of four methods are evaluated and compared so as to point out the characteristics of each of the methods. The Value-at-Risk technique is straightforward and can be easily generated for any loss distribution. The insolvency put option principle is easily implementable and is sensitive to the degree of default. The capital asset pricing model is one of the oldest reliable methods and still provides very helpful intermediate results. The Myers and Read marginal capital allocation approach encourages diversification and introduces the concept of default value. Applications of the four methods to some fictive and real insurance companies are provided. The thesis further analyses the sensitivity of those methods to changes in the economic context and comments how insurance companies can anticipate those changes.

Keywords: Insurer balance sheet, Capital allocation, Surplus, Diversification, Value-at-Risk, Option pricing theory, Capital Asset Pricing Model, Marginal capital allocation, Default value.

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Chapter 1

Introduction

Since 1983, the number and cost of insurance insolvencies have dramatically increased. Twenty-two of the twenty-five largest insolvencies have occurred since 1983. It implies that the average guaranty fund assessments rose from 22 million dollars per year from 1969-1983 to 500 million dollars per year after 1983.

The subprime crisis has further highlighted the impact of the insolvency of insurance companies in today's financial world and thereby the need for legislation. In the framework of the European Solvency II Directive, coming into effect on January 2013, insurance companies have to determine their economic capital adjusted for the risk they incur. In the light of these new legislations, mathematical models of risk and procedure to determine levels of capital are necessary.

Purpose

Insurance companies, as banks, invest in derivative products to hedge risk and to match assets to liabilities. The initial capital of insurance industries comes from the policyholders that purchase policies protecting against unwilling financial incidents. Hence, insurance firms have to offer insurance with the highest guarantee for the claim's refund and with the smallest price. The insurance price covers the fair premium (covering just expected loss) and an extra safety loading. If insurers only charge the fair premium, insurance coverage would be costless on average. There are two issues. First, the economic cost of the firm's overall capital has to be determined. It is the capital that the firm has to hold so that the risk of insolvency stays minimal. Secondly, this capital has to be allocated across the different lines of insurance (pensions, car insurance, health insurance, etc).

Therefore insurance companies have to measure the economic profitability of the lines of business in order to maximize the market value of equity capital.
Equity is the residual amount of investor's capital in assets, after all liabilities are paid.

The project selection leads to the determination of the amount of the firm's equity capital that has to be assigned to each project undertaken by the company. The capital allocated to a line of business is used to absorb unexpected losses but is generally not

an end in itself. The companies try to determine which business units that are most profitable relative to risk in order to make decisions.
If the net income of a line of business is larger or equal than the cost of capital of this line multiplied by the capital allocated to this line, then the line of business is consistent with the goal of **value maximization**.

By correctly allocating capital to each line of business, a company is highly empowered to make the best strategic decisions. Capital allocation can be settled within a variety of risk measures or more elaborate models.
A **risk measure** is the quantification of the size of buffer capital that should be added to the position to provide a sufficient protection against undesirable outcomes.
A risk measure summarizes the information contained in the probability distribution to one number by considering what is important about the distribution from a specific prospect. The Value-at-Risk (VaR) and the insolvency put option will be considered in this thesis.

The Value-at-Risk of a position is the smallest amount of money that, if added to the position now and invested in a risk-free asset, ensures that the probability of a strictly negative value at time 1 is not greater than a specified small probability. This concept is frequently used but need very frequents data for accurate estimations.

The insolvency put option or the expected policyholder deficit (EPD) is the expected loss due to a specified probability of default of the firm. The concept has been proposed by Butsic (1994). It is closely related but more general than the VaR.

Allocating by a risk measure is straightforward but subjective. These measures ignore risks less severe than the critical probability selected. More elaborate capital allocation techniques have been suggested. This thesis considers the capital asset pricing method (CAPM) and the marginal capital allocation proposed by Myers and Read (MR).

The CAPM approach is one of the oldest financial theory techniques. It expresses the return on equity of a firm in a very simple way and provides a technique to decide the contribution of each line of business to the return on equity. This method is not the most accurate solution but is still used in practice as a very useful informer.

Marginal capital allocation refers to two different techniques; the first has been proposed by Merton and Perold (1993), and the second by Myers and Read (1999). Both methods calculate the change in required total capital by decreasing the expected loss from some business units. The result is expressed as a capital ratio. The main advantage of these techniques is that they recognize the benefits of diversification.
Diversification means reducing risk by engaging in a portfolio of business and not only in a single line firm. The MR approach is based on the option pricing theory and considers changes in an existing line of business and not by adding/withdrawing an entire business to the firm (as the Merton and Perold model does). It starts with the overall capital needed to keep the default cost low and, then, it allocates all the capital in an additive manner that directly reflects the individual contributions of each line to the overall capital requirement.

The aim of this thesis is to analyse and compare, based on the balance sheet of insurance companies, the performances of the four methods briefly introduced above.

The objective is not to cover all available methods. This thesis studies and evaluates some of the existing methods to implement capital allocation. All models have strengths and weaknesses. The objective is merely to point out some of the models characteristics that may be appraised if an insurance firm is considering what method to implement.

All methods can be applied to all risk elements and have the same intention: determining the capital that has to be allocated to each line of business in order to bear the probability of insolvency at an acceptable level. Each line of business is characterized by different statistical parameters calculated on basis of data collected internally by insurance companies. These characterizations are the inputs of the models.

Outline

The thesis is organized as follow. Chapter 2 displays a typical balance sheet for an insurance company and some important concepts from the field of predictive modelling. In Chapter 3 the techniques used to allocate the capital are introduced. The comparison analysis of the methods are presented in Chapter 4, where the results are illustrated through qualitative examples. Chapter 5 contains a concluding discussion together with suggestions for further research. At the end of the report, an appendix can be found. It includes a glossary of notation to help the reader follow the mathematics in this thesis.

Chapter 2

Assumptions and notations

Before studying the different methods, it is important to establish some notions and notations. This chapter presents a general balance sheet for an insurance company. Each of its component is introduced and explained. The assumptions made for the model are described and justified.

2.1 Insurer balance sheet

The idea of this work is to start from the total economic balance sheet of an insurance company, representing the market value over one period from time 0 to time T. The balance sheet in described in Table 2.1. The notations[1] have the next meaning : $V = V(0)$, $L = L(0)$, $V^1 = V(T)$, $L^1 = L(T)$ and the initial surplus $S = V - L$. A single period model is used ($T = 1$), time 0 is the time when a first policy is issued.

Balance sheet	Initial value	End of period payoff
Assets	V	V^1
Liabilities	L	L^1
Surplus	S	$V^1 - L^1$

Table 2.1: Insurer balance sheet over one period from time 0 to time T.

The initial market value of the firm assets is given by $V = \sum_i P_i + S$, where P_i are premiums collected at time 0 from policyholders for line i ($i = 1 \dots N$). The insurance company accepts, in return for premium, to underwrite the expected payments for each line of business L_i. The initial surplus S equals the initial value of the assets minus the present value of the liabilities. Thus, in order to hedge the lines liabilities by purchasing assets, the initial surplus has to be allocated across the lines of insurance.

[1] In this report, letters without exponents indicate the constant initial value of the quantities, while the uncertain future values are written with an exponent.

Assets and liabilities

The initial value of the assets (V) is made up of the capital of the shareholders and the fair market premium of the policyholders. This total amount is invested in a portfolio of assets within an investment policy. For the sake of simplicity, the entire portfolio is here considered as one single asset with payoff at the end of the period : $V^1 = VR_V$ where R_V is the return on asset.

The insurance firm also writes N lines of business with present value L_i. If $C_{i,k}$ is the amount that has to be paid out by the insurance company at the end of period k due to claims of the i^{th} line of business that have occurred before the end of period 1, then the liability's values of line i at time 0 and at time 1 are

$$L_i = \sum_{k=1}^{n} \mathrm{E}(C_{i,k}) e^{-r_k k}$$

and

$$L_i^1 = \sum_{k=1}^{n} \mathrm{E}(C_{i,k}|I_1) e^{-r_{1,k}(k-1)}.$$

In these formulas the parameter I_1 is the information available at time 1, $\mathrm{E}(C_{i,k}|I_1)$ is the conditional expectation of the cash flow and $r_{j,k}$ is the time j zero rate for a zero coupon bond maturing at time k with face value 1.

The PV(losses) refers to the present value of an uncertain cash flow occurring at a future time T in all the lines of business:

$$L = \sum_{i=1}^{N} L_i = \sum_{i=1}^{N} \sum_{k=1}^{n} \mathrm{E}(C_{i,k}) e^{-r_k k}.$$

The end-of-period total claim can be expressed in term of its return: $L^1 = LR_L = \sum_{i=1}^{N} L_i^1 = \sum_{i=1}^{N} L_i R_{L_i} = L\left(\sum_{i=1}^{N} x_i R_{L_i}\right)$ with $x_i = \frac{L_i}{L}$. The return on total loss R_L constructed so that $R_L = \sum_i x_i R_{L_i}$.

The value of liability L assumes that claims are paid, but insurance policies have payoff that depend on the insurer solvency. At the end of the period, $T = 1$, the asset value and the liabilities amounts are uncertain and two scenarios are possible:

i) $V^1 \geq L^1$. In that case the policyholders receive L^1; the insurance company receives the residual value (surplus) $V^1 - L^1$, and the firm is solvent.

ii) $V^1 < L^1$. In that case the firm cannot meet its obligations and is insolvent.

Equity and surplus

It is important to distinguish equity (E) from surplus (S). Equity is an output defined as the market value of the residual claim: $E = \max(0, V - L)$. Surplus is an input defined as the difference between the asset and the present value of the losses assuming no default: $S = V - L$. The two concepts are related

$$E = S + \max(0, L - V) = \begin{cases} 0 & \text{if default} \\ S & \text{if no default} \end{cases}$$

It should also be noted that surplus is costly. Capital allocated to secure losses against default is taxed. If the tax rate is t, then the return of 1€ of surplus is $(1-t)R_V$ and the return of 1€ invested in the financial market is R_V. The "cost" of 1€ of surplus is tR_V.

The surplus can also be expressed as the sum of line by line surplus contributions:

$$S = \sum_i L_i s_i,$$

where $s_i \equiv \dfrac{\partial S}{\partial L_i}$ is the marginal change in required total surplus in line i in response to a marginal increase in PV(losses).

The last concept that has to be defined is the aggregate surplus ratio s. It is the weighted average of the line by line surplus requirement:

$$s = \sum_i x_i s_i.$$

This implies $sL = \sum_i L_i s_i = S$ and $V = L + S = L(1+s)$. The last equation ties in that the insurer is solvent ($V > L$ because $s > 0$) at $t = 0$.

Premiums

Insurance companies hold an investment portfolio of securities and issue insurance products. The theory of insurance risk focuses on the underwriting activity and on the capital investment. To underwrite risks, the firm has to describe and anticipate the liabilities. Claims form a stochastic process in time (depending of the random sizes of claims and of the random number of claims).

At the start of the planning period the asset of the insurance company consists of the surplus plus the premium income amounting to P:

$$\text{At} \quad t = 0, \qquad V = S + P.$$

As long as no claim occurs, the surplus increases according to the premium income per time period. It is the fair premium income plus an extra loading to overcome the risk of insolvency. As the first claim arises, the surplus decreases by the amount of loss payment, and so on. The premium process is sketched in Figure 2.1. At the end of the period, the remaining paid premium (which have not been used to pay claims) are put into reserves. At the end of the next period, claims can be paid up to the value of (initial) equity capital plus accumulated surplus. It happens that, at a certain time τ, a large claim occurs resulting in a negative surplus for a certain time during which the company is insolvent.

To achieve a target probability of insolvency with a certain surplus, the company has to calculate the minimum premium for accepting the risks characterized by the loss distribution. The techniques discussed in this thesis have been proposed to solve this optimization problem. They are compatible with the goal of value maximization.

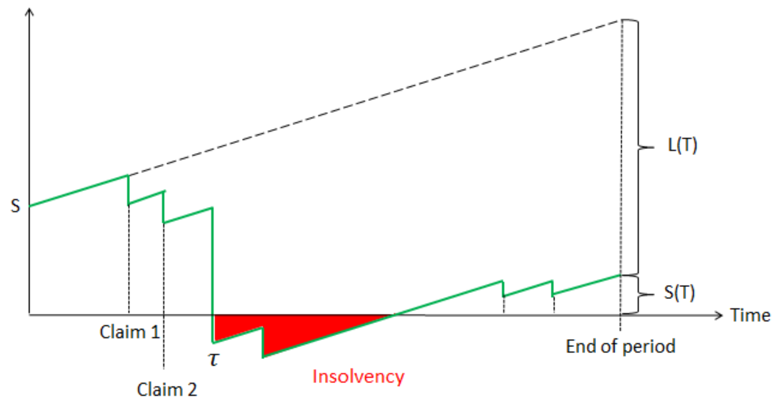


Figure 2.1: Development of premium income, claims, and surplus over time.

2.2 Model

Since claims on losses and return on asset cannot be explicitly traded, the value of the portfolio at time 1 is in reality not known. Therefore, a few estimations have to be made about the investment losses and the asset value. Assumptions are made about the distributions of the liabilities and asset to assess their future values.

Different distributions and/or numerical simulations can be chosen to simulate the uncertain future values of the losses and asset. The normal and lognormal distributions are well known and popular among financial engineers, which makes them a natural first choice for modelling the returns in the models.

The assumption that total losses and asset value are joint normal is a good simple solution. The normal distribution takes the shape of a "bell curve" and implies non-zero values. The main advantage is that the sum of random normal variables is also normally distributed. So, the losses by lines, the aggregate loss, the asset value and the surplus can together follow a normal distribution. This allows closed-form formulas to allocate capital. The main disadvantage is that their distributions will be symmetric. Moreover, the tails of the distribution are very thin and generate too few extreme values. The normal model cannot capture phenomena of joint extreme moves in several elements since simultaneous large values are relatively infrequent.

The joint lognormal distribution is a more appropriate choice because the distribution does not go below zero but has unlimited positive potential. Values are not symmetric and are right-skewed (the mean is greater than the median). The lognormal model is simple and more accurate. The distribution leads to frequent small gains and occasional large losses. Nevertheless, the sum of lognormal variables does not follow a lognormal distribution (so, if the distributions of the losses by lines are lognormal, the aggregate loss is not lognormally distributed and, inversely, if the distribution of the aggregate loss is lognormal, the losses by lines do not follow a lognormal distribution). It implies to resort to empirical methods to estimate some concepts such as the Value-at-Risk.

The normal and lognormal distributions are completely specified by their mean and standard deviation. The mean center the distribution at higher or lower values. The standard deviation changes the dispersion of the distribution. The volatilities, means and correlations between two random returns will be denoted by the Greek letters σ, μ and ρ. Returns are defined as the fraction between price at the end of a time horizon and initial price. For a security C (it could be the liability of a line of business or the asset value) with return R_C, it can be written $C^1 = CR_C$.

If the returns are assumed to be normally distributed, then the price at time 1 of the security can be expressed as

$$C^1 = CR_C = C(1 + \sigma_c X)$$

with $R_C \sim \mathbb{N}(1, \sigma_C^2)$ and $X \sim \mathbb{N}(0, 1)$. The mean and variance of the price at time 1 of the security C are $\mathrm{E}(C^1) = C$ and $\mathrm{Var}(C^1) = C^2\sigma_C^2$.

The choice of a lognormal distribution for the returns implies the following form for the price at time 1 of the security

$$C^1 = CR_C = Ce^{a+bX}$$

with $X \sim \mathbb{N}(0, 1)$. The constants a and b have to be defined so that the mean and variance of the price at time 1 are identical with those of the normal case in order to have comparable results. The conditions are $\mathrm{E}(C^1) = C = Ce^{a+\frac{b^2}{2}}$ and $\mathrm{Var}(C^1) = C^2\sigma_C^2 = C^2\left(e^{b^2} - 1\right)e^{2a+b^2}$. Setting $a = -\frac{1}{2}\ln(\sigma_C^2 + 1)$ and $b = \sqrt{\ln(\sigma_C^2 + 1)}$ leads to the following model

$$C^1 = CR_C = Ce^{-\frac{1}{2}\ln\left(\sigma_C^2+1\right)+\sqrt{\ln\left(\sigma_C^2+1\right)}X} = Ce^Y$$

with $X \sim \mathbb{N}(0, 1)$ and $Y \sim \mathbb{N}\left(-\frac{1}{2}\ln(\sigma_C^2 + 1), \ln(\sigma_C^2 + 1)\right)$.

In this work, the notations always refer to the mean and standard deviation of the distribution. The input parameters are σ_i and σ_V the volatility of the return on loss of the line of business i and of the return of the asset value, and, ρ_{ij} and ρ_{iV} the correlations between the returns of lines of business i and j and between the i^{th} line and the asset value. From these parameters, others can be deduced such as the volatility of the return of the total loss σ_L, the correlations between the return of the i^{th} line and the total loss ρ_{iL}.

Chapter 3

Theory

This chapter describes and explains the four techniques mentioned in the introduction: the VaR technique, the insolvency put option principle, the capital asset pricing model and the Myers and Read marginal capital allocation approach. It includes theoretical background, definitions of new concepts and derivations of the capital allocation principles.

3.1 Value-at-Risk

The first use of the Value-at-Risk (VaR) dates back from the second half of the 20^{th} century, but it has become more popular around 1995 with the Basel European convention. This new indicator has quickly been considered as a standard in the assessment of financial risks. The development was mostly due to J. P. Morgan.

The VaR is a risk measure that essentially depends of three elements: the distribution of investment loss of the portfolio, a level of confidence and a time period for the underlying asset. The concepts and properties used in the first part of this section have been broadly developed in [Hult et al. 2012]. The exceedance probability has been briefly introduced in [Cummins 2000].

3.1.a Definition and interpretation

The VaR at level $p \in (0,1)$ of a portfolio with value X at time 1 is defined as the smallest amount of money m (invested in a risk-free asset) that will be sufficient to cover potential loss at time 1 with probability of default of at least $1-p$.

$$\begin{aligned}
\mathrm{VaR}_p(X) &= \min\{m : P(mR_f + X < 0) \leq p\} \\
&\qquad \text{where } P(\cdot) \text{ refers to the probability} \\
&\qquad\qquad \text{and } R_f \text{ is the return of a risk-free asset} \\
&= \min\{m : P(K \leq m) \geq 1-p\} \ \text{ where } K = \frac{-X}{R_f}
\end{aligned}$$

$$= F_K^{-1}(1-p)$$
where $F_K(\cdot)$ is the distribution function of the variable K.

The variable K is interpreted as the portfolio loss, where positives values of K indicate losses and negative values indicate gain. The VaR_p will be negative if $X \geq 0$. For instance, a portfolio with a one period 5% VaR of 100€ will fall in value by more than 100€ (over one period) with a probability of at most 5%. Informally, it means that a loss of 100€ or more on this portfolio is expected to happen on 1 period in 20. The main disadvantage of the utilization of the VaR technique comes from the fact that VaR is only concerned about the frequency of shortfall but not with the size of the shortfall. Doubling the largest loss may not impact the VaR.

For an insurance firm, the value of the portfolio X is the net income $V^1 - L^1$. The random value $\frac{1}{R_f}(V^1 - L^1)$ can be rewritten $\omega^t Z$ with $\omega = \dfrac{1}{R_f}(V, \quad -L)^t$ and $Z = (R_V, \quad R_L)^t$. The return vector Z is a multivariate normal or lognormal random vector with mean μ and covariance matrix Σ. The vector μ is a vector of length $N+1$. The matrix Σ is a $(N+1) \times (N+1)$ positive definite symmetric matrix that can be developed as

$$\Sigma = \underbrace{\begin{pmatrix} \sigma_V & & & \\ & \sigma_1 & & \\ & & \ddots & \\ & & & \sigma_N \end{pmatrix}}_{\underline{\underline{\sigma}}} \underbrace{\begin{pmatrix} 1 & \rho_{1V} & \rho_{12} & \cdots & \rho_{1N} \\ \rho_{V1} & 1 & & & \vdots \\ \rho_{21} & & \ddots & & \\ \vdots & & & & \\ \rho_{1N} & & \cdots & & 1 \end{pmatrix}}_{\underline{\underline{\rho}}} \begin{pmatrix} \sigma_V & & & \\ & \sigma_1 & & \\ & & \ddots & \\ & & & \sigma_N \end{pmatrix},$$

where σ_V and σ_i are the variances of the returns on asset and liabilities, and ρ_{ij} the covariances between them ($\underline{\underline{\rho}}$ is the correlation matrix). The matrix can be decomposed as

$$\begin{aligned} \Sigma &= \underline{\underline{\sigma}} C' C \underline{\underline{\sigma}} && \text{with } C'C \text{ the Cholesky decomposition of } \underline{\underline{\rho}} \\ &= (C\underline{\underline{\sigma}})'(C\underline{\underline{\sigma}})' = A'A && \text{with } A = C\underline{\underline{\sigma}}. \end{aligned}$$

Normal Case

When the joint distribution of the asset value and liabilities losses is normal, the random vector $Z \sim \mathbb{N}_{N+1}(\mu, \Sigma)$ of multivariate normal density has an elliptical distribution and a stochastic representation $Z \overset{d}{=} \mu + AY$ with $Y \sim \mathbb{N}_{N+1}(0, I)$.

The elliptical distributions have the crucial property that the distribution of any linear combination of the components of the initial vector is known. It means that the portfolio can be completely characterized by its mean and covariance matrix. Then, because Y is spherically distributed and with Y_1 the first component of Y, the distribution of $\omega^t Z$ is given by[1]:

$$\omega^t Z \stackrel{d}{=} \omega^t \mu + \sqrt{\omega^t \Sigma \omega}\; Y_1 = g(Y_1).$$

The VaR can then be computed:

$$\begin{aligned}
\mathrm{VaR}_p(V^1 - L^1) &= F^{-1}_{\frac{-1}{R_f}(V^1-L^1)}(1-p) = F^{-1}_{-\omega^t Z}(1-p) = F^{-1}_{-g(Y_1)}(1-p) \\
&= -F^{-1}_{g(Y_1)}(p) \text{ because } F_{g(Y_1)}(\cdot) \text{ is continuous and strictly increasing}^2 \\
&= -g\left(F^{-1}_{Y_1}(p)\right) \text{ because } g(\cdot) \text{ is non decreasing and left continuous}^3 \\
&= -\omega^t \mu - \sqrt{\omega^t \Sigma \omega}\; F^{-1}_{Y_1}(p) = -\omega^t \mu - \sqrt{\omega^t \Sigma \omega}\; \Phi^{-1}(p). \qquad (3.1)
\end{aligned}$$

Lognormal Case

If the joint distribution of the asset value and liabilities losses is lognormal, the random vector Z has a stochastic representation $Z \stackrel{d}{=} e^{\mu + AY}$ with $Y \sim \mathbb{N}_{N+1}(0, I)$. But, the distribution of Z is not elliptical and $\omega^t Z$ is not a lognormal random variable. Therefore the previous derivation is no longer valid.

Since the random vector Z can easily be simulated, the simplest way to compute the VaR is numerically by using a Monte-Carlo simulation. If a sample $Z_1 \ldots Z_n$ of independent copies of Z is considered, the estimate of $\mathrm{VaR}_p(V^1 - L^1)$ is given by

$$\mathrm{VaR}_p(V^1 - L^1) = F^{-1}_{n,(-\omega^t Z)}(1-p) = \left(-\omega^t Z\right)_{[np]+1,n} \qquad (3.2)$$

where:

- $F^{-1}_{n,X}(p) = \min\{x : F_{n,X}(x) \geq p\}$ is the empirical quantile function of $F_{n,X}$,
- $[y]$ designates the largest integer smaller than y,
- $(-\omega^t Z)_{1,n} \geq \ldots \geq (-\omega^t Z)_{n,n}$ is the ordered sample.

The approximation is based on a Monte Carlo procedure as follows :

1. Generate $Y_k \sim \mathbb{N}_{N+1}(0, I)$,
2. Calculate $Z_k = e^{\mu + AY_k}$,
3. Compute $G_k = -\omega^t Z_k$ and repeat the procedure n times. The sample size n should be relative large to protect the accuracy.
4. Sort the G's by descending order; G_i is the i^{th} largest entrance in the set of repeated Monte Carlo simulations.
5. Output the $([np]+1)^{\text{th}}$ element of the ordered sequence.

[1]See proposition 9.3, Section 9.2 in [Hult et al. 2012].
[2]See proposition 6.4, Section 6.2 in [Hult et al. 2012].
[3]See proposition 6.3, Section 6.2 in [Hult et al. 2012].

3.1.b Exceedance probability

The VaR allocation principle is constructed through the use of the concept of exceedance probability. It is defined as the probability ε_i that loss at time 1 from a particular line of business i will exceed the expected loss of this lines plus the capital allocated to the line:

$$\varepsilon_i = P\left(L_i^1 > \mathrm{E}(L_i^1) + C_i\right).$$

Capital is then allocated so that the exceedance probabilities of each line are equal:

$$\varepsilon = \varepsilon_i = P\left(L_i^1 > \mathrm{E}(L_i^1) + C_i\right) = P\left(\frac{L_i^1}{\mathrm{E}(L_i^1)} > 1 + \frac{C_i}{\mathrm{E}(L_i^1)}\right) \quad \forall i = 1 \dots N$$

In this formula the parameter $1 + \dfrac{C_i}{\mathrm{E}(L_i^1)}$ is defined as the asset-to-liability ratio. Lines with higher risk will require more capital (relative to expected loss) to attain the exceedance probability target, which yields to a greater asset-to-liability ratio. The ratio can be interpreted as follow : if $1 + \dfrac{C_i}{\mathrm{E}(L_i^1)} = 1.4$ then 0.4€ has to be allocated to line i for each euro of liability.

There are two ways to address the allocation problem. First, the firm wants to achieve a specific level of protection (the same explicit VaR for each of its line of business) and, thereby, determine the total required capital. Each risk element is evaluated individually and then combined to provide the capital requirement of all risk elements. Secondly the total capital requirement ($\sum_i C_i$) is limited by the available surplus. The firm tries to achieve the smaller risk of insolvency, namely the smaller exceedance probability. It can then be formulated with the following optimization problem:

$$\begin{aligned} \underset{C_1 \dots C_n}{\text{minimize}} \quad & \varepsilon \\ \text{subject to} \quad & \varepsilon = P\left(\frac{L_i^1}{\mathrm{E}(L_i^1)} > 1 + \frac{C_i}{\mathrm{E}(L_i^1)}\right) \quad \forall i = 1 \dots N \\ & \sum_{i=1}^{N} C_i = \text{ capital requirement} \\ & 0 \leq \varepsilon \leq 1 \end{aligned}$$

Setting $X_i = \dfrac{L_i^1}{\mathrm{E}(L_i^1)}$ leads to the following reformulation of the first set of constraints

$$\varepsilon = 1 - P\left(X_i \leq 1 + \frac{C_i}{\mathrm{E}(L_i^1)}\right) = 1 - F_{X_i}\left(1 + \frac{C_i}{\mathrm{E}(L_i^1)}\right) \quad \forall i = 1 \dots N$$

where $F_{X_i}(\cdot)$ is the cumulative distribution of the variable X_i.

The capital required for each line of insurance is then a function of the exceedance probability ε:

$$C_i = \mathrm{E}(L_i^1)\left(F_{X_i}^{-1}(1 - \varepsilon) - 1\right).$$

If the returns are assumed to be normally distributed, then X_i also follows a normal distribution with $F_{X_i}(x) = \Phi\left(\frac{x-1}{\sigma_i/\mu_i}\right)$ and $\mathrm{E}(L_i^1) = L_i\mathrm{E}(R_{L_i}) = L_i\mu_i$. The required capital for each line is then

$$C_i = L_i\mu_i\left(\Phi^{-1}\left(\frac{-\varepsilon}{\sigma_i/\mu_i}\right) - 1\right). \tag{3.3}$$

In a similar way, lognormally distributed returns involve $\mathrm{E}(L_i^1) = L_i\mathrm{E}(R_{L_i}) = L_i e^{\mu_i + \frac{1}{2}\sigma_i^2}$, $F_{X_i}(x) = \Phi\left(\frac{\ln(x) + \frac{1}{2}\sigma_i^2}{\sigma_i}\right)$ and:

$$C_i = L_i e^{\mu_i + \frac{1}{2}\sigma_i^2}\left(\Phi^{-1}\left(\frac{\ln(1-\varepsilon) + \frac{1}{2}\sigma_i^2}{\sigma_i}\right) - 1\right). \tag{3.4}$$

3.2 Insolvency put option

The insolvency put option method works in the same way as the VaR but is more general. The VaR method considers the amount of loss that will be exceeded with a target probability, the insolvency put option method also considers the expected amount of loss.

The method is based on the option pricing theory which is explained in this section. Merton was the first to do the association between default and the exercising of a put option. But, the expected policyholder deficit (EPD) approach in a context of insurance insolvency has been developed by Butsic (1994). The required capital of the insurance firm is allocated across the lines in order to equalize the EPD of each line.

3.2.a Option pricing theory

An insurance contract can be looked upon as an option with a net value at the end of the period given by $V^T - L^T$ where V^T and L^T are random. At time T, the claims are divided between owners and insurance buyers:

$$\begin{aligned}
V^T &= \underbrace{\max\left(V^T - L^T, 0\right)}_{\text{owners}} + \underbrace{L^T - \max\left(L^T - V^T, 0\right)}_{\text{insurance buyers}} \\
&= L^T\left(\max\left(\frac{V^T}{L^T} - 1, 0\right) + 1 - \max\left(1 - \frac{V^T}{L^T}, 0\right)\right) \\
&= \begin{cases} (V^T - L^T) + L^T - 0 = V^T & \text{if } V^T > L^T \\ 0 + L^T - (L^T - V^T) = V^T & \text{if } V^T < L^T \end{cases}.
\end{aligned} \tag{3.5}$$

The first term $\max\left(\frac{V^T}{L^T} - 1, 0\right)$ corresponds to the payoff of a call option with strike price 1 and is denoted call$(\frac{V^T}{L^T}, 1)$. It stands for the right to the owners to buy the option at a predetermined price equal to 1. In the same way, the last term $\max\left(1 - \frac{V^T}{L^T}, 0\right)$ is equivalent to the payoff of a put option, denoted by put$(\frac{V^T}{L^T}, 1)$. It stands for the right to the owner to sell the option at a predetermined price 1. The previous relation (3.5) can then be expressed at time 0 as

$$V = L\left(\text{call}_0(\frac{V}{L}, 1) + 1 - \text{put}_0(\frac{V}{L}, 1)\right). \tag{3.6}$$

It is directly linked with the put-call parity relation. By taking into account that $V = S + P$, the relation (3.6) becomes:

$$\begin{aligned} P &= L\text{call}_0(\frac{V}{L}, 1) + L - L\text{put}_0(\frac{V}{L}, 1) - S \\ &= L - L\text{put}_0(\frac{V}{L}, 1). \end{aligned}$$

The step between the two last lines comes from the fact that, at $t = 0$, the payoff of call$_0(\frac{V}{L}, 1)$ is worth the surplus. The reduction $P = L - L\text{put}_0(\frac{V}{L}, 1)$ gives the value of the policyholders' claim. It is the difference between the present value of the liabilities if the probability of default is zero and the expected value that can be loss (expressed as a put option[4]).

3.2.b Expected policyholder deficit

The policyholder deficit is the difference between the amount the insurer is obligated to pay (to the insurance buyers) and the actual amount paid by the insurer. The expected policyholder deficit (EPD) can be determined from the probability distributions of the losses and asset.

The expected loss = E$(L) = \int_0^\infty xp(x)dx$ where $p(\cdot)$ is the density function of loss (x is positive because the insurance buyers will not pay the insurance company if no claim occurs). Then, if the asset is certain and the loss uncertain, the EPD is the expectation of loss exceeding asset:

$$EPD_L = \int_V^\infty (x - V)p(x)dx.$$

If the asset is uncertain and the loss certain, the EPD is the expectation of asset being less than the losses:

$$EPD_V = \int_0^L (L - y)q(y)dy$$

with $q(\cdot)$ the asset's density function.

These formulas can be applied if both asset and liability are uncertain. The results, for the cases of normally and lognormally distributed risk elements, are expressed below.

[4]The value of a put option is given by:

$$\text{put}\,(A, K, r, T, \sigma) = \Phi(-d_2)Ke^{-rT} - \Phi(-d_1)$$

with $d_1 = \dfrac{1}{\sigma\sqrt{T}}\left(ln\left(\frac{A}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T\right)$ and $d_2 = d_1 - \sigma\sqrt{T}$.

The derivations of these formulas can be found in Appendix A.2. These derivations and the theoretical background has been well explained in [Butsic 1999].

In the case of a joint lognormal returns distribution, the relation between the EDP ratio for each line of business and the capital allocated to this line is given by

$$\frac{EPD_i}{L_i} = \Phi\left(\frac{\sigma_i}{2} - \frac{1}{\sigma_i}\ln\left(1+\frac{C_i}{L_i}\right)\right) - \left(1+\frac{C_i}{L_i}\right)\Phi\left(-\frac{\sigma_i}{2} - \frac{1}{\sigma_i}\ln\left(1+\frac{C_i}{L_i}\right)\right). \quad (3.7)$$

In a similar way, when the returns are assumed to follow a normal distribution the relation is given by:

$$\frac{EPD_i}{L_i} = \sigma_i\phi\left(-\frac{C_i}{\sigma_i L_i}\right) - \frac{C_i}{L_i}\Phi\left(-\frac{C_i}{\sigma_i L_i}\right). \quad (3.8)$$

The previous equations give us a relation between the EDP ratio for each line of business and the capital allocated to this line. There are two different points of view to consider the problem as for the VaR technique. First, the firm wants to minimize the probability of deficiency but the total required capital ($\sum_i C_i$) cannot exceed the available capital. Surplus is then allocated in order to equalize the EPD of each line of insurance. Secondly, the firm looks for the amount of capital needed to achieve a specified EDP objective. The EPD are expressed as liability ratios to adjust the scale of different risk element sizes.

The formulas (3.7) and (3.8) are not invertible. The implementation of the insolvency put option method is therefore less trivial than the implementation of the VaR technique.

3.3 The capital asset pricing model.

The capital asset pricing model (CAPM) has been developed by William F. Sharpe and John Lintner during the sixties on the base of the earlier work by Harry Markowitz. This section has been constructed on the base of [Cummins 2000] and [Zweifel and Eisen 2012].

The results of the CAPM formula can be modelled through the security market line (SML). It graphs, for a given time, the market risk versus the return of the whole market. The slope of the SML is the market risk premium defined as the difference between the expected return of the market and the risk-free rate. It is a useful tool in determining whether a security offers a reasonable expected return for risk. A security with a high (positive) risk must achieve a high expected return in order to be profitable for the firm. Individual securities are plotted in the SML graph represented in Figure 3.1. Securities plotted above the line are undervalued because they yield a higher return for an equal amount of risk. In the same way, securities plotted below the line are overvalued because for a given amount of risk, they yield a lower return.

A second key concept is the capital market line (CML) shown in Figure 3.2. It shows the best reachable capital allocation by graphing the return of the complete

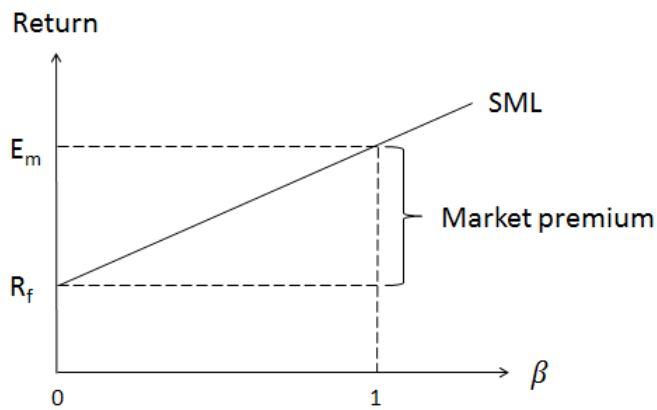


Figure 3.1: Security market line : $R = R_f + \beta(E_m - R_f)$. In equilibrium, individual securities and all portfolios lie on the SML.

market as a function of the portfolio's volatility. The line is formed by all the points included between the risk-free asset $(0, R_f)$ and the market portfolio (σ_m, E_m). The efficient frontier is the set of portfolios that achieves the minimum risk for a given expected rate of return. The slope of the CML equals the slope of the efficient frontier at the market portfolio.

The model rests on some assumptions listed in [Cummins 2000] and reported below. Except for the last assumption, they are not restrictive compared to the other models presented in this work.

- The investors are risk averse and select mean-variance diversified investments,
- The investors cannot influence prices (they are price takers),
- A risk-free asset exists and investors can lend/borrow unlimited amounts under the risk-free rate,
- There is no transaction or taxation costs and securities are infinitely divisible (the market is frictionless),
- All information is available at the same time to all investors,
- There are a "large number" of investors and securities,
- The returns are normally distributed.

The derivation of the CAPM formula in a general context is first presented. The use of this model for insurance firms is then elaborated. This section shows more specifically how to allocate capital across lines of insurance using the CAPM.

3.3.a The mean-variance diversification

The CAPM is based on the Markowitz diversification, also called mean-variance diversification. Suppose that an investor has a portfolio composed of N securities and an investment in a risk-free asset.

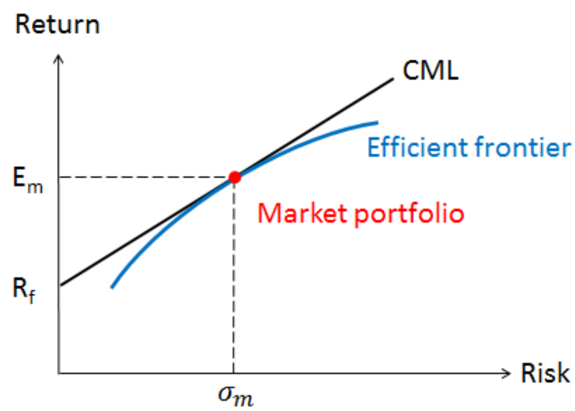


Figure 3.2: Capital market line : $R = R_f + \sigma \dfrac{E_m - R_f}{\sigma_m}$. The CML consists of the efficient portfolios.

The investor wants to minimize the risk of its portfolio (defined in terms of variances and covariances of returns) by achieving a target level of expected return for the entire portfolio. It can be formulated with the following optimization problem:

$$\begin{aligned} \underset{x_1 \dots x_n}{\text{minimize}} \quad & 2\sigma_m = 2\sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j C_{ij}} \\ \text{subject to} \quad & E_m = \sum_{i=1}^{N} x_i E_i + (1 - \sum_{i=1}^{N} x_i) R_f \end{aligned}$$

where :

- x_i : proportion of the portfolio invested in security i,
- R_i : return of security i,
- $C_{ij} = \text{cov}(R_i, R_j)$,
- $E_i = \text{E}(R_i)$: expected return of security i,
- R_f : return of a risk-free asset. It can be measured from the yield of the US Treasury bill for instance.
- $E_m = \text{E}(R_m)$: expected return of the market portfolio, defined as the level of expected return. It can be calculated, from representatives indices such as the S&P, as the ratio of the difference between the year-end index and the year-begin index on the year-begin index.

The solution is obtained by differentiating the Lagragian:

$$L(x_i, \lambda) = 2\sigma_m + \lambda' \left(E_m - \sum_{i=1}^{N} x_i E_i - (1 - \sum_{i=1}^{N} x_i) R_f \right)$$

in order to achieve first order conditions:

(A) $\dfrac{\partial L}{\partial x_i} = \dfrac{1}{\sigma_m} \sum_{j=1}^N x_j C_{ij} + \lambda'(-E_i + R_f) = 0 \quad \forall i = 1 \dots N,$

(B) $\dfrac{\partial L}{\partial \lambda'} = E_m - \sum_{i=1}^N x_i E_i - (1 - \sum_{i=1}^N x_i) R_f = 0.$

By multiplying each equation of the set of conditions (A) by x_i and summing it over all risky securities, we get

$$\frac{1}{\sigma_m} \underbrace{\sum_{i=1}^N \sum_{j=1}^N x_i x_j C_{ij}}_{\sigma_m^2} - \lambda' \left(\sum_{i=1}^N x_i E_i - R_f \sum_{i=1}^N x_i \right) = 0$$

which yields

$$\sigma_m = \lambda' \left(\sum_{i=1}^N x_i E_i + R_f (1 - \sum_{i=1}^N x_i) - R_f \right).$$

The condition (B) is then introduced to relate the standard deviation and the expected return of the portfolio

$$\sigma_m = \lambda' \left(E_m - R_f \right).$$

This relation can be rewritten $\frac{1}{\lambda'} = \lambda = \frac{E_m - R_f}{\sigma_m}$. The term $E_m - R_f$ is called the risk premium (see Figure 3.1) and λ is the market price of risk or the Sharpe ratio. This concept can be used for any security

$$\lambda_i = \frac{R_i - R_f}{\sigma_i} \tag{3.9}$$

or generalized to any portfolio of securities characterized by their mean μ and their covariance matrix Σ. It is then given by $\lambda(\omega) = \dfrac{\omega^t \mu - R_f}{\sqrt{\omega^t \Sigma \omega}}$. The efficient frontier is composed of the pairs $(\sigma_m(\lambda), \mu_m(\lambda))$ and the Sharpe ratio is the slope of the CML.

The Sharpe ratio is used to determine the level of risk of an investment compared to its potential for profit. A security has a good risk-adjusted performance if it has a great security's Sharpe ratio. This security generates thus a higher profitability compared to a risk-free investment. If $0 < \lambda < 1$, then the excess return relative to the risk-free rate is lower than the risk underwritten. A security with a negative Sharpe ratio is a security that would perform worse than a risk-free asset. The Sharpe ratio shows that a portfolio which can obtain high return is worthwhile if the additional risk associated to this portfolio is limited. The use of the Sharpe ratio as risk measure also assumes normally distributed returns for the investments.

The expected return of security i can be derived from condition (A) by considering the expression of the Sharpe ratio:

$$E_i = R_f + \frac{1}{\lambda \sigma_m} \sum_{j=1}^N x_j C_{ij} = R_f + \left(\frac{E_m - R_f}{\sigma_m} \right) \left(\frac{\sum_{j=1}^N x_j C_{ij}}{\sigma_m} \right).$$

This leads to

$$E_i = R_f + \beta_i(E_m - R_f), \text{ where } \beta_i = \frac{\sum_{j=1}^{N} x_j C_{ij}}{\sigma_m^2} = \frac{\text{cov}(R_i, R_m)}{\text{Var}(R_m)}. \tag{3.10}$$

The coefficient β is recognized as the regression coefficient of R_i and R_m. It is a measure of the sensitivity of a security to the systematic risk. The systematic risk is the risk inherent to the entire market that cannot be avoided with diversification, it affects all investments. For instance, inflation is a systematic risk. The coefficient β_i compares the market risk of security i with the risk of the rest of the market. Typically, the beta loss of the S&P index is equal to 1. A security with a high β has a rate of return that is highly dependent on the market rate of return. A security with a vanishing β is uncorrelated to the market.

Equation (3.10) is an equilibrium relation between risk and return that must hold for all traded securities. The first half of the formula (R_f) is the amount of compensation the investor needs for placing money in a risk-free asset over one period of time. The second half of the formula ($\beta_i(E_m - R_f)$) compensates the investor for taking risk.

3.3.b Insurance CAPM

Several articles such as [Cummins 2000] and [Zweifel 2012], develop models for pricing insurance contracts based on the CAPM. The derivation starts from the insurer balance sheet. The net income is equal to the sum of the investment income and the premium income:

$$\text{net income} = VR_V + PR_U, \tag{3.11}$$

where R_U is the rate of return on underwriting defined as the ratio $\dfrac{P - \text{E}(L)}{P}$. By dividing the net income by the equity capital (3.11) can be expressed as the expected return on equity

$$R_E = \frac{V}{E} R_V + \frac{P}{E} R_U = \left(1 + \frac{L}{E}\right) R_V + \frac{P}{E} R_U. \tag{3.12}$$

It expresses the expected return as a leverage of the rate of investment return and underwriting return. Moreover, $R_E = R_V + \frac{P}{E}\left(R_U + R_V \frac{L}{P}\right)$, the equity increases as long as $R_U + R_V \frac{L}{P} > 0$. If the firm does not subscript any insurance then $\frac{P}{E} = 0$ and $R_E = R_V$.

On the other hand, according to the CAPM relation (3.10), the equilibrium rate of return of the insurer's equity and asset are:

$$\text{E}(R_E) = R_f + \beta_E(E_m - R_f) \tag{3.13}$$

$$\text{E}(R_V) = R_f + \beta_V(E_m - R_f) \tag{3.14}$$

with $\beta_E = \dfrac{\text{cov}(R_E, R_m)}{\text{Var}(R_m)} = \frac{V}{E}\beta_V + \frac{P}{E}\beta_U$. The expression of β_E is due to the linearity of the covariance operator. Equation (3.12) can be inverted to obtain an expression for the return on underwritting: $R_U = \frac{E}{P} R_E - \frac{V}{P} R_V$. The expected return on underwriting follows from (3.13) and (3.14) and is given by

$$\begin{aligned}
\mathrm{E}(R_U) &= \frac{E}{P}\left(R_f + \beta_E(E_m - R_f)\right) - \frac{V}{P}\left(R_f + \beta_V(E_m - R_f)\right) \\
&= \frac{-L}{P} R_f + \left(\frac{E}{P}\beta_E - \frac{V}{P}\beta_V\right)(E_m - R_f) \\
&= \frac{-L}{P} R_f + \beta_U (E_m - R_f).
\end{aligned}$$

The last equation is also called the insurance CAPM. The return on underwriting, and thereby the value of the premium, has to be determined in line with the risk-adjusted capital return. The first term ($\frac{-L}{P} R_f$) corresponds to the interest credit for the use of policyholders funds and the second term ($\beta_U (E_m - R_f)$) to the insurer's reward for bearing the risk.

Keeping in mind that the rate of return on underwriting is $R_U = \dfrac{P - \mathrm{E}(L)}{P}$, the expected rate of return on underwriting and the beta loss of the underwriting can be expressed as

$$\mathrm{E}(R_U) = 1 - \frac{L R_L}{P} = 1 - \frac{L}{P} \sum_i R_{L_i} x_i = 1 - \frac{1}{P} \sum_i R_{L_i} L_i,$$

and

$$\beta_U = 1 - \frac{L}{P}\beta_L = 1 - \frac{1}{P} \sum_i L_i \beta_i$$

with β_i defined in (3.10). The required rate of return of each line of business follows then from the insurance CAPM and is given by

$$R_{L_i} = L_i R_f + \beta_i (E_m - R_f). \tag{3.15}$$

The capital is then allocated across the lines by considering the constraint on the available capital;

$$C_i = \frac{L_i R_{L_i}}{\sum_j L_j R_{L_j}} S. \tag{3.16}$$

3.4 The Myers-Read model

The Myers and Read article won the 2002 ARIA best paper prize and is since broadly discussed in the financial literature. It is an approach based on option pricing theory. The theory has been developed in Section 3.2.a and includes several techniques based on the option pricing model. The MR model is the continuation of several models that are already established; the Merton and Perold model for instance. The main difference is that the MR model allocates 100 percent of the capital.

The capital allocation is based on an incremental analysis using very small changes in the liability of each line. The marginal contributions to the global default risk vary across the lines. The capital is distributed among the individual lines of business of the firm at the margin based on these contributions.

The expansion of the model presumes here that the returns follow a lognormal distribution, but the method is applicable for both distributions (normal and lognormal). The expansion for the normal case is similar and the details are given in Appendix A.3.

3.4.a Default value

This method caters for a positive probability of insolvency. The possibility of default is therefore considered in the initial insurer balance sheet as in Table 3.1 It means that losses are accounted assuming that claims could not be paid. If that scenario occurs, the default value is positive[5]. The expected default value D is similar to the expected policyholder deficit defined in Section 3.2.

V = asset	L = PV(losses)
D = default value	E = equity

Table 3.1: Initial insurer balance sheet.

The default value is a function of L_i and is then affected by a marginal change in the PV(losses) of a single line of business. To evaluate this relation, the marginal default value $d_i = \dfrac{\partial D}{\partial L_i}$ is computed. The default value of the insurance company can then be expressed as the sum of the products of line by line liabilities and line of business default allocations

$$D = \sum_i L_i d_i = Ld.$$

The end-of-period default value and the payoff to equity are expressed as:

$$D^1 = \max\left(0, L^1 - V^1\right),$$

and

$$E^1 = V^1 - L^1 + D^1 = \max\left(0, V^1 - L^1\right)$$

So, the present value of the default option

$$D = PV\left(\max\left(0, L^1 - V^1\right)\right) = L \cdot PV\left(\max\left(0, \frac{L^1 - V^1}{L}\right)\right)$$

depends of the liabilities losses (L), the market value of asset (V) and their joint probability distribution. The variable $\frac{L^1-V^1}{L}$ follows a lognormal distribution: $\frac{L^1-V^1}{L} \overset{d}{=}$

[5]The fair price of an insurance is determined by the default value of this line of insurance and reflects its risk. The companies should have subscript an insurance guaranteeing the full payments of their claims. In doing so, some of the exposure to risk is transferred from the insurance company to the reinsurance company. From the insurance point of view, the payments to the reinsurance company could be seen as a liability and is treated as another line of business (thus included in L).

$e^{\mu+\sigma Z}$ with mean $\mu = \mathrm{E}\left(\ln\left(\frac{L^1-V^1}{L}\right)\right)$, volatility of the surplus-to-liability ratio $\sigma^2 = \mathrm{Var}\left(\ln\left(\frac{L^1-V^1}{L}\right)\right)$ and Z a standard normal variable.

The volatility of the default value is thus reduce to a single lognormal volatility defined by [Myers and Read 2001] as:

$$\sigma = \sqrt{\sigma_V^2 + \sigma_L^2 - 2\sigma_{LV}}, \tag{3.17}$$

depending on σ_V, σ_L, σ_{LV} the volatility of losses, asset and the covariance of losses and asset. These volatilities can be approximated by:

$$\begin{aligned} \sigma_L^2 &= \sum_i \sum_j x_i x_j \rho_{ij} \sigma_i \sigma_j = \sum_i x_i \sigma_{iL} & (3.18) \\ \sigma_{iL} &= \sum_j x_j \rho_{ij} \sigma_i \sigma_j \\ \sigma_{LV} &= \sum_i x_i \rho_{iV} \sigma_i \sigma_V = \sum_i x_i \sigma_{iV} & (3.19) \\ \sigma_{iV} &= \rho_{iV} \sigma_i \sigma_V \end{aligned}$$

with ρ_{ij} the correlation between log losses in two lines of insurance, ρ_{iV} the correlation between log loss in a line of insurance and log asset value, σ_{iL} is the covariance of log loss in the i^{th} line of business with log loss on the portfolio and σ_{iV} is the covariance of log loss in the i^{th} line of business with log asset value. In expression (3.17) of the volatility of the asset-to-liability ratio, the sign of the covariance is negative so that a positive correlation reduces the default value's volatility. If asset and total loss have the same trend, then an increase in the loss amount is compensated by an increase in the asset value.

Because the probability distribution of future losses' and asset's return is joint, the default value can be written $D = g(L, V, \sigma)$. Since $S = V - L$, the default value depends on the present value of future losses allied to the rate of surplus $D = f(L, S, \sigma)$. One compute the contribution of each line of business i to the company's default value:

$$d_i = \frac{\partial D}{\partial L_i} = \frac{\partial D}{\partial L}\frac{\partial L}{\partial L_i} + \frac{\partial D}{\partial S}\frac{\partial S}{\partial L_i} + \frac{\partial D}{\partial \sigma}\frac{\partial \sigma}{\partial L_i}.$$

Note that $L = \sum_i L_i$, $D = Ld$, $S = Ls$ and $L_i = x_i L$. Thus, $\dfrac{\partial D}{\partial L} = d$, $\dfrac{\partial L}{\partial L_i} = 1$, $\dfrac{\partial D}{\partial S} = \dfrac{\partial d}{\partial s}$, $\dfrac{\partial S}{\partial L_i} = \dfrac{\partial s}{\partial x_i}$, $\dfrac{\partial D}{\partial \sigma} = L\dfrac{\partial d}{\partial \sigma}$ and $\dfrac{\partial \sigma}{\partial L_i} = \dfrac{1}{L}\dfrac{\partial \sigma}{\partial x_i}$, giving:

$$d_i = d + \frac{\partial d}{\partial s}\frac{\partial s}{\partial x_i} + \frac{\partial d}{\partial \sigma}\frac{\partial \sigma}{\partial x_i}. \tag{3.20}$$

The decisive steps of the derivations of the surplus-to-liability ratio and of the volatility of the asset-to-liability ratio are set out below. They start back from the expansions of the aggregate surplus $s = \sum_j x_j s_j$ and of the volatilities (3.18) and (3.19):

$$\frac{\partial s}{\partial x_i} = L\frac{\partial s}{\partial L_i} = L\frac{\partial}{\partial L_i}\left(\sum_j \frac{L_j}{L} s_j\right) = L\frac{L\frac{\partial}{\partial L_i}\left(\sum_j L_j s_j\right) - \left(\sum_j L_j s_j\right)}{L^2} = s_i - s,$$

$$\frac{\partial \sigma}{\partial x_i} = L\frac{\partial \sigma}{\partial L_i} = \frac{L}{2\sigma}\left(\frac{\partial \sigma_L^2}{\partial L_i} + \frac{\partial \sigma_V^2}{\partial L_i} - 2\frac{\partial \sigma_{LV}}{\partial L_i}\right) = \frac{L}{\sigma}\left(\frac{1}{2}\frac{\partial \sigma_L^2}{\partial L_i} - \frac{\partial \sigma_{LV}}{\partial L_i}\right),$$

where

$$\begin{aligned}
\frac{\partial \sigma_L^2}{\partial L_i} &= \sum_j \sum_k \rho_{jk}\sigma_j\sigma_k \frac{\partial}{\partial L_i}\left(\frac{L_j L_k}{L^2}\right) \\
&= \sum_j \sum_k \rho_{jk}\sigma_j\sigma_k \frac{L^2 \frac{\partial (L_j L_k)}{\partial L_i} - (L_j L_k)(2L)}{L^4} \\
&= \frac{1}{L^2}\sum_k \rho_{ik}\sigma_i\sigma_k L_k + \frac{1}{L^2}\sum_j \rho_{ij}\sigma_i\sigma_j L_j - \frac{2}{L^3}\sum_j\sum_k \rho_{jk}\sigma_j\sigma_k L_j L_k \\
&= \frac{2}{L}\sigma_{iL} - \frac{2}{L}\sigma_L^2 = \frac{2}{L}\left(\sigma_{iL} - \sigma_L^2\right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \sigma_{LV}}{\partial L_i} &= \sum_j \rho_{jV}\sigma_j\sigma_V \frac{\partial}{\partial L_i}\left(\frac{L_k}{L}\right) = \sum_j \rho_{jV}\sigma_j\sigma_V \frac{L\frac{\partial L_k}{\partial L_i} - L_k}{L^2} \\
&= \frac{1}{L}\rho_{jV}\sigma_j\sigma_V - \frac{1}{L^2}\sum_k \rho_{kV}\sigma_k\sigma_V L_k = \frac{1}{L}\left(\sigma_{iV} - \sigma_{LV}\right).
\end{aligned}$$

Then $\frac{\partial \sigma}{\partial x_i} = \frac{1}{\sigma}\left((\sigma_{iL} - \sigma_L^2) - (\sigma_{iV} - \sigma_{LV})\right)$ and by making a substitution of these expressions into equation (3.20), the contribution of each line of business i to the company's default value is given by

$$d_i = d + \frac{\partial d}{\partial s}(s_i - s) + \frac{\partial d}{\partial \sigma}\frac{1}{\sigma}\Big((\sigma_{iL} - \sigma_L^2) - (\sigma_{iV} - \sigma_{LV})\Big). \qquad (3.21)$$

The first term represents an increase in the default value due to an increase in the present value of liabilities. The two last terms represent an increase/decrease in the company default value due to changes in the mix of insurance business.

3.4.b Allocation to lines of business

Model

Since the probability distributions of futures losses and asset is joint lognormal, the default value is a direct application of the Black-Scholes option pricing formula[6]. Indeed, the default value has the same payoff structure as a European put option on $L^1 - V^1$ with exercise price $(L - V)e^r$ which is the net payoff if both the asset and the liability get a position in a risk-free investment.

[6] The use of this model is used in many previous papers as in [Myers and Read 2001] and [Butsic 1999].

The default value can thus be expressed as:

$$D = L\Phi\{z\} - V\Phi\{z-\sigma\} = L\left(\Phi\{z\} - (1+s)\Phi\{z-\sigma\}\right)$$

where $\Phi\{\cdot\}$ is a cumulative distribution function of the standard normal distribution and $z = \frac{-1}{\sigma}\ln(1+s) + \frac{1}{2}\sigma$. The default value to liability ratio can then be computed:

$$d = \frac{D}{L} = \Phi\{z\} - (1+s)\Phi\{z-\sigma\}, \tag{3.22}$$

as well as the partial derivatives needed in (3.21):

$$\frac{\partial d}{\partial s} = -\Phi\{z-\sigma\} < 0 \text{ and } \frac{\partial d}{\partial \sigma} = \phi\{z\} > 0, \tag{3.23}$$

where $\phi\{\cdot\}$ is the probability density function of the standard normal distribution. The signs of the derivatives imply a decrease in d_i if the surplus requirement for line of business i increases or if line i's volatility (contingent of the risk) decreases. The higher σ_{iL}, the higher the marginal default value. The higher σ_{iV}, the lower the marginal default value. It means that if the trend of the liability of the line of business i has high correlation with the trend of the total loss ($\sigma_{iL} \nearrow$) then the line of business i has a higher probability of default. In the same way if the trend of the line i's liability has high correlation with the trend of the asset value ($\sigma_{iV} \nearrow$) then an increase in line i's liability will be compensated by an increase in the asset value. The probability of default of this line decreases.

Surplus requirements for the lines of business

The risk characteristics of a line (encompassing its variance and covariances) determine how the credit quality of the company as a whole is affected when the amount of business written in this line varies. Indeed, the change in the amount of business written in a single line alters the surplus requirement of the multi-line company. This relation is given through the marginal surplus of each line:

$$s_i = s + \left(\frac{\partial d}{\partial s}\right)^{-1}\left(d_i - d - \frac{1}{\sigma}\frac{\partial d}{\partial \sigma}\left((\sigma_{iL} - \sigma_L^2) - (\sigma_{iV} - \sigma_{LV})\right)\right).$$

There is a non-linear dependence between d_i and s_i. For instance, if a company supports a growth in its first line of business, this growth gives rise to a larger surplus requirement for this line ($s_1 \nearrow$) but only to a slight increase of the marginal default value d_1. This phenomenon occurs because a larger part of the capital is allocated to support the change in liability. In the same way, if a company has two lines of business i and j so that $s_i = s_j$, then it does not imply $d_i = d_j$.

An insurance company may engage different possible strategies to determine the capital requirement of its lines of business:

- uniform surplus: maintaining the same surplus-to-liability ratio for every line of business: $s_i = s \ \ \forall i$. The marginal default values by line are then known:

$$d_i = d + \left(\frac{\partial d}{\partial \sigma}\right) \frac{1}{\sigma} \left((\sigma_{iL} - \sigma_L^2) - (\sigma_{iV} - \sigma_{LV}) \right),$$

- uniform default value: maintaining the same default-value-to-liability ratio for every line of business: $d_i = d \ \ \forall i$. The surplus vary by line:

$$s_i = s \underbrace{- \left(\frac{\partial d}{\partial s}\right)^{-1} \left(\frac{\partial d}{\partial \sigma}\right)}_{>0} \frac{1}{\sigma} \left((\sigma_{iL} - \sigma_L^2) - (\sigma_{iV} - \sigma_{LV}) \right). \qquad (3.24)$$

It is important to keep in mind that even if the lines of business are considered individually, the insolvency will affect the multi-lines firm. If the company defaults on one policy, it defaults on all policies. Thereby the default risk of the company is more significant than the marginal default risk in a single line of business[7]. The contribution of each line of business to the company's default value should be the same; surplus should thus be allocated to lines of business in order to equalize the marginal default values (second strategy, equation (3.24)). In that case, the surplus adds up ($\sum\limits_i s_i x_i = s$) with no overlap of shortfall, because of (3.18) and (3.19).

3.4.c Simplification of the model

A simplification of the previous model has been proposed by [Bustic 1999]. The variables $\beta_i = \frac{\sigma_{iL}}{\sigma_L^2}$[8] and $\gamma_i = \frac{\sigma_{iV}}{\sigma_{LV}}$ are introduced in (3.24) and the relation becomes then

$$s_i = s - \left(\frac{\partial d}{\partial s}\right)^{-1} \left(\frac{\partial d}{\partial \sigma}\right) \frac{1}{\sigma} \left(\sigma_L^2(\beta_i - 1) - \sigma_{LV}(\gamma_i - 1) \right). \qquad (3.25)$$

Note that $\sum_i \beta_i x_i = 1$ and $\sum_i \gamma_i x_i = 1$ because of (3.18) and (3.19). The loss beta is an important determinant of the amount of capital allocated to a insurance line. Indeed, according to the authors, the action of the term $\sigma_{LV}(\gamma_i - 1)$ in the previous equation can be ignored since in practice the variance of the liability (σ_L^2) is large compared to the covariance between the asset and liability (σ_{LV}). This assumption will be tested in Chapter 4. Given this, (3.25) simplifies into:

$$s_i = s \underbrace{- \left(\frac{\partial d}{\partial s}\right)^{-1} \left(\frac{\partial d}{\partial \sigma}\right) \frac{\sigma_L^2}{\sigma}}_{Y \geq 0} (\beta_i - 1) = s + Y(\beta_i - 1).$$

[7]This conclusion has also been drawn by [Myers and Read 2001] and by [Cummins 2000].

[8]This expression of β is the same as the β defined in the CAPM (see Section 3.3.a).

If Y is constant, the surplus requirement for a line of business is a linear function of its loss beta. The variable Y has been defined as the capital allocation factor and is easily generated by industrial data. A line of business with a small loss beta will have a surplus requirement smaller than the marginal surplus for the entire business. It means that the line has a low covariance of loss and has low correlation with the other lines (σ_{iL} is small).

Chapter 4

Illustrations

This chapter uses the theory presented earlier in this report to provide indications and remarks about the accuracy and the relevance of the methods developed in Chapter 3. The goal is to illustrate and have a better understanding of the running process of those methods. The input data are the statistical parameters of the normal returns. From these, the parameters used for the lognormal returns are calculated as explain in Section 2.2. All the computations have been made using MATLAB.

4.1 Purposes and outline

As a first step, the four techniques will be applied to qualitative examples. These fictive insurance companies are not based on historical data but are realistic pricing exercises. Our techniques will be calibrated on a reference scenario and then some important risk parameters of the distribution of the losses and/or asset will be modified. It serves two purposes. On the one hand, it will allow to register how the methods are affected by changes in the parameters. Each change could be representative of a phenomenon occurring in today's economic world (politic crisis, natural catastrophe, etc). The examples confirm that the allocations add up under different assumptions about the returns distributions. The examples will also study if more knowledgeable methods (for instance, the MR method) are more efficient than older and more classic methods such as the CAPM. On the other hand, the section will give an overview of the sensitivity of the insurance firm's capital to changes in the economic context. It will provide a broad idea of how an insurance company can anticipate risks of insolvency.

The second step will consider a French insurance company to apply the models to a real case. The statistical parameters are then calculated using historical data.

Let us assume that the fictive company underwrites $N = 3$ lines of business, all having a present value of 100€ and a one-period volatility of $\sigma_i = 15\%$. All liabilities are supposed to be pairwise correlated with a coefficient of $\frac{1}{2}$. The company also underwrites a capital of 450€ in asset with the same one-period volatility as the liabilities ($\sigma_V = 15\%$) and which is positively correlated with the lines of insurance ($\rho_{iV} = 0.2\ \forall i$). The company is solvent at $t = 0$ with a surplus of 150€. The volatility of total losses and the covariance of losses and asset are calculated by the formulas (3.18) and (3.19):

	$p = 0.5\%$	$p = 1\%$	$p = 5\%$
Normal case	264	93	-373
Lognormal case	128	-13	-420

Table 4.1: VaR (€) of the portfolio for $p = 0.5\%$, $p = 1\%$, $p = 5\%$ and for the normal and lognormal distributions in the homogeneous case.

$\sigma_L = 12.25\%$ and $\sigma_{LV} = 0.45\%$. Further, a continuously compounded risk-free interest rate of 3% is assumed (it is the rate of US government bonds) and taxes are ignored. The returns of the scenario are assumed to be successively normally and lognormally distributed.

The four methods allocate 50€ of surplus to each line of business. So 150€ of capital is allocated to each line, this corresponds to one third of the total capital. It reflects the fact that lines with the same risk parameters are considered in the same way. The surplus is divided across the lines proportionally to their fraction of liabilities $\frac{L_i}{L}$. This result stands for both distributions.

Thereby all the lines characteristics (each computed with the formulas (3.18), (3.19), (3.10), (3.9) and (3.15)) are identical

$$\sigma_{iL} = 1.5\%, \quad \sigma_{iV} = 0.45\%, \quad \beta_i = 1, \quad \lambda_i = 0.4667, \quad R_{L_i} = 0.1 \qquad \forall i$$

A loss beta β equal to 1 means that the lines of insurance are representative stock; their movements are in the same direction and about the same amount as the movement of the financial market. The Sharpe ratio λ is positive but smaller than 1; the excess return of the liability relative to the risk-free rate is lower than the risk of the losses in these lines. In these conditions, the lines of insurance support the company.

The VaR of the portfolio for different levels of protection and for both distributions have been calculated. The results are summarized in Table 4.1.

For a level of protection of 5%, the insurer can be sure at 95% that a benefit of 373€ (or 420€ in the lognormal case) will occur at time 1. A negative VaR means that the value at the end of the period of the portfolio is positive. The insurer will have no loss. In the lognormal case, if the insurer wants to be sure at 99.5% that no loss will occur at the end of the period, he should invest at least 128€ more in a risk-free asset.

The homogeneous scenario is a simple confirmation of the suitability of the assumptions and working of the methods. The results are logical and bear out what could be expected. More interesting scenarios will now be considered. The chapter is organized as follows. A reference scenario is computed and then changes of different parameters are discussed. At the beginning, a change in the risk of one line of business will be considered to study the impact on the other lines of business. Secondly, the correlation coefficients will be modified to analyse the impact of diversification on the allocation of capital. Then, more elaborate cases will be considered including alterations in the present value of the liabilities, modifications in the asset value to consider negative surplus and inclusion of additional lines of business.

4.2 Scenarios

4.2.a Reference scenario

In this first scenario, the surplus-to-liability ratio is assumed to be 50%; the 300€ of liability are ensured by 450€ of asset. The input data is presented in Table 4.2. The covariance's (of each line's return with the returns on asset and on total losses) and standard deviations (of total losses and total losses with the asset) are calculated from these inputs according to the formulas (3.18) and (3.19) presented in Section 3.4.a. The example still considers 3 lines of business with the same liabilities. The volatility of the third line is now changed to 30%.

	Present Value of the liabilities and asset (€)	Standard deviation	Correlations		
			Line 1	Line 2	Line 3
Line 1	100	15 %	1	0.5	0.5
Line 2	100	15 %	0.5	1	0.5
Line 3	100	30 %	0.5	0.5	1
Asset	450	15 %	0.2	0.2	0.2
$\sigma_L = 16.56\%$		$\sigma_{iL} = (1.87,\ 1.87,\ 4.50)\%$			
$\sigma_{LV} = 0.60\%$		$\sigma_{iV} = (0.45,\ 0.45,\ 0.90)\%$			

Table 4.2: Summary of the data used for the reference scenario.

Table 4.3 summarizes the final and intermediate results obtained by the four methods. The results are computed using formulas presented in Chapter 3 and referred in the first column of the table. The results in the normal case are presented in the last column, those in the lognormal case are in the second to last column.

Focus first on the final allocation of capital. The two first lines of business get (in average) 133€ (or 29.5% of the capital) in the lognormal case and 139€ (or 31% of the capital) in the normal case. The last line gets (in average) 182€ (or 40.5% of the capital) in the lognormal case and 173€ (or 38% of the capital) in the normal case. The riskier line (3$^{\text{rd}}$ line) has the largest Sharpe ratio λ_3, loss beta β_3 and monthly return per unit of risk R_{L_3}. Therefore more capital is allocated to this line. All methods allocate the same amount of surplus to lines with the same volatility. The results are always similar across the methods but the remaining differences between the results are more noticeable in the lognormal case.

For both distributions and every level of protection, the amplitudes of the VaR are now greater than in the homogeneous case. Moreover, the amplitude of the VaR are larger in the lognormal case than in the normal case (especially when the level of protection p is small). This observation differs from the homogeneous case. It reflects the fact that the portfolio is riskier (because of the increase of σ_3). The VaR is still negative with a level of protection of 95%.

The allocation of surplus obtained with the VaR is represented in Figure 4.1. It represents for both distributions the exceedance probability curves. It is, for each line i and for a certain amount of capital x, the probability y that L_i^1 exceeds the expected value of L_i^1 plus the amount of capital x:

$$y = P\left(L_i^1 > \mathrm{E}(L_i^1) + x\right).$$

The sum of the capitals attributed to each line has to equal the surplus (dashed vertical lines on the graphs). This amount of surplus is divided into the lines in order to equalize the exceedance probabilities of each line (dashed horizontal lines on the graphs). The exceedance probability related to this allocation is small.

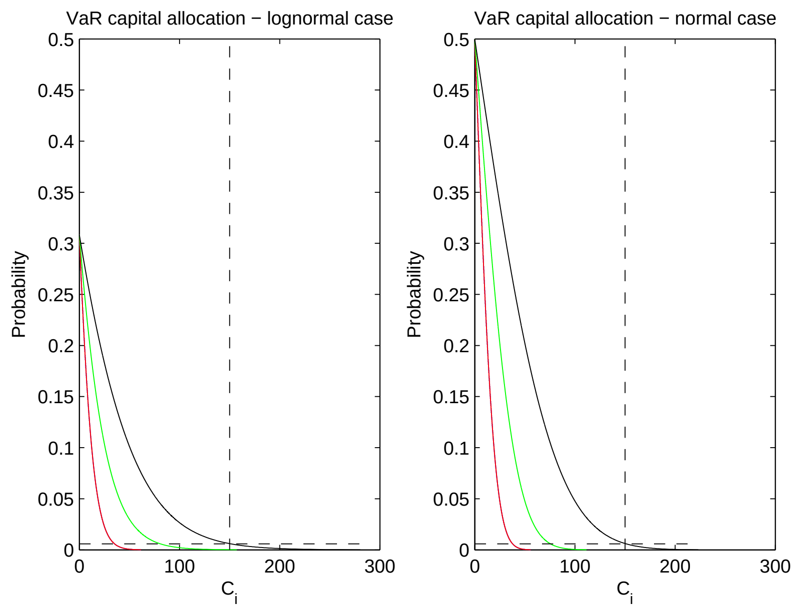


Figure 4.1: The surplus allocation of the reference scenario with the VaR technique is obtained through the exceedance probability curves (red lines : 2 first lines of business, green line : third line of business, black line : total capital, dashed lines : exceedance probability and total related surplus).

The allocation obtained with the CAPM is close to the allocation for the VaR and EPD methods. Moreover, the securities are fairly evaluated and fall exactly on the SML. All individual lines are below the CML. The SML, CML and the securities are plotted in Figure 4.2.

Focus then on the MR model and more specifically on the default value per euro of liability. In the lognormal case, the default value to liability ratio is $d = 0.16\%$ and the default value D is 0.48€. These values are larger in the normal case ($d = 0.19\%$, $D = 0.57$ €). It means that the price of a guarantee for the payments of the liabilities costs 0.16% (or 0.19%) of the total liability (300€). The second to last line of Table 4.3 presents the marginal surplus requirement for each line. For instance, in the lognormal case, a marginal increase of 1€ in the line 1's PV(loss) generates a change of 0.36€ in the required total surplus. The surplus increases from 150€ to 150.36€. Let us briefly analyse the value of the Greeks (Delta and Vega) for both distributions and the loss beta for the 3 policies. The amplitude of Vega is larger than the amplitude of

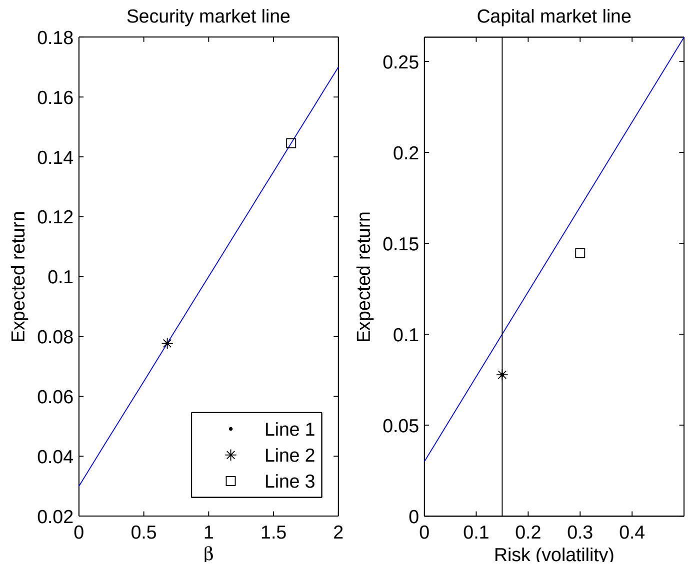


Figure 4.2: Reference scenario : the slope of the SML (left plot) is the risk premium and the black indicators are the couples (β_i, R_{L_i}). The slope of the CML (right plot) is the price of risk of an efficient portfolio and the black indicators are the couples (σ_i, R_{L_i}).

Delta. It means that changes in the volatilities have a bigger impact on the marginal default value than changes in the marginal surplus requirements. The loss beta for the three policies are 0.68, 0.68 and 1.64. The loss beta of the last line is greater than 1 ($\beta_3 = \frac{\sigma_{3L}}{\sigma_L^2} > 1$). It is representative of the volatility of that line of business: the trend of the 3$^{\text{rd}}$ line of business is parallel but larger than the trend of total losses.

Before closing this example, it is important to highlight the fact that the surplus allocations add up to 150€ for all the methods and that the simplification of the MR model leads to the same surplus allocation with an error of about 5%.

	Methods	Lognormal Results			Normal Results		
VaR							
	$p = 0.5\%$ (€)	606			395		
(3.1, 3.2)	$p = 1\%$ (€)	373			211		
	$p = 5\%$ (€)	-192			-290		
(3.3, 3.4)	Exceedance probability	5.10^{-3}			5.10^{-3}		
	Capital allocation (€)	135	135	181	138	138	176
EPD							
(3.7, 3.8)	ratio	2.10^{-3}			10^{-5}		
	Capital allocation (€)	131	131	187	138	138	175
CAPM							
(3.9)	Sharpe ratio				0.32	0.32	0.38
(3.10)	β				0.68	0.68	1.64
(3.15)	Expected return R_{L_i}				0.078	0.078	0.145
(3.16)	Capital allocation (€)				139	139	172
MR							
(3.22, 6.2)	Default/Liability value (d)	0.16%			0.19%		
(3.17)	Asset/Liability volatility (σ)	19.49 %					
(6.1)	Standard deviation of surplus (θ)				24.52%		
(3.23)	Delta $\left(\frac{\partial d}{\partial s}\right)$& Vega $\left(\frac{\partial d}{\partial \sigma}\right)$	-0.0147 & 0.0559			-0.0207 & 0.0499		
(3.24)	Marginal surplus requirement	36 %	36%	78%	41%	41%	68%
	Capital allocation (€)	136	136	178	141	141	168

Table 4.3: Summary of the results of the reference scenario (base-case correlations, same present values of liabilities). The number in brackets in the first column refer to the number of the equations used to compute the results.

4.2.b Change in the risk of the liabilities

The inputs stay as previously (see Table 4.2) except for the standard deviation of losses. The standard deviation of the third line's loss varies from 0 to 100%. The covariances and standard deviations are then recalculated for these inputs. The standard deviations of total losses (σ_L) and of total losses with the asset returns (σ_{LV}) vary from 0.09 to 0.37 and from 0.003 to 0.013 respectively. The covariances of line 3 σ_{3L} and σ_{3V} increase while the covariance's of line 1 and 2's losses with the total losses σ_{1L} and σ_{2L} decrease. The covariances of line 1 and 2's losses with the return on asset stay constant at 0.0045. When $\sigma_3 = 15\%$, the parameters coincide with those of the homogeneous case.

The changes in these standard deviations and covariance's have impacts on the capital allocation. The surplus allocation for the CAPM is close to the result for the VaR method. It is represented in the left plot in Figure 4.3 as a function of σ_3 that varies from 0 to 100%. The EPD surplus allocation is mostly similar to the MR allocation (the surplus changes almost linearly with σ_3), but the EPD allocation does not allocate negative surplus to lines 1 and 2 for too large values of σ_3. The surplus allocation resulting of the MR method can be seen in the right plot in Figure 4.3.

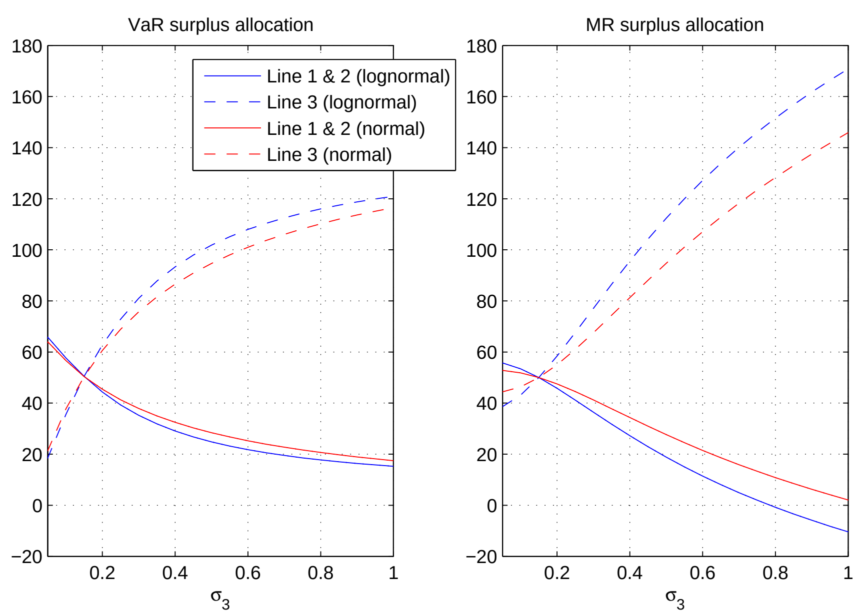


Figure 4.3: The lines' surplus are sensitive to changes in the risk of the liabilities. σ_3 modifies the the surplus allocation. Left plot : VaR technique, right plot : MR technique.

The surplus requirements of the two first lines of business vary with σ_3 but are always equal amongst themselves. The surplus s_3 increases with the volatility σ_3 (that confirms what have been said previously). The surplus requirement for the entire company stays constant at 0.5 but is reallocated from the two first lines to the third line. Indeed, the surplus allocated to lines 1 and 2 (solid lines in Figure 4.3) decrease as the surplus allocated to line 3 (dashed line) increases twice as fast. This shows that surplus is allocated in priority to lines of business with higher risk. The allocations add up to 150 € for both distributions, with the four methods and with all values of σ_3.

The surplus allocation obtained with the MR method differs severely from the three first methods for too large values of σ_3. Those differences are due to the fact that the MR method allocates negatives values to the first lines of business for $\sigma_3 \geq 0.8$. The MR method is the only method that considers the possibility of allocating negative surplus to a line so that the riskier line can have a higher surplus requirement.

Changes in the risk of the liabilities also have impact on different quantities of interest. An increase of σ_3 implies:

- an increase of the VaR for both distributions, especially for the lognormal distribution.

- an augmentation of the exceedance (VaR) and deficit (EPD) probabilities.

- an increase of β_3 and λ_3, and a decrease of $\beta_{1,2}$ and $\lambda_{1,2}$. The evolution of these parameters is shown in Figure 4.4. The loss beta are indicators of the lines' vulnerability to risk. For $\sigma_3 = 0$, the loss beta of line 3 is null. It means that the trend of line 3's loss is uncorrelated with the movement of total losses. When the value of β_i is in the interval $]0,\ 1[$, the trend of line i is less than the trend of total losses but they have the same direction. For $\beta_i > 1$, the trend of line i is more than the trend of total losses. So, for $\sigma_3 \geq 0.35$, the third line of business becomes the most influential parameter in the tendency of the total losses.

- a decrease of the efficiency of the portfolio. Indeed, as it can be seen in Figure 4.5, when $\sigma_3 > \sigma_1, \sigma_2$ the expected returns of the lines lie below the capital market line.

- an increase of the default value. For large values of σ_3, the default value is larger in the lognormal case. It may also be noticed that the growth of the default values are more significant for high values of the third standard deviation. The company has a higher risk of insolvency when the volatility of its lines return increase.

- an increase of the error between the MR results and the simplified MR results. The error is the same for both distributions but is more significant for line 3 than for the two first lines of business.

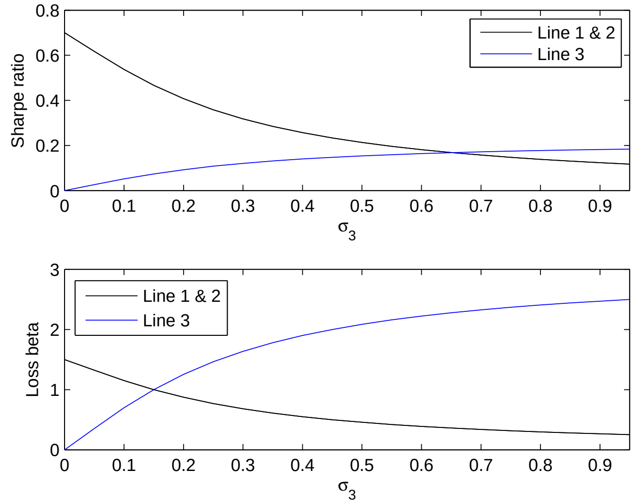


Figure 4.4: Evolution of the Sharpe ratios (upper plot) and beta losses (lower plot) of the three lines of business with σ_3.

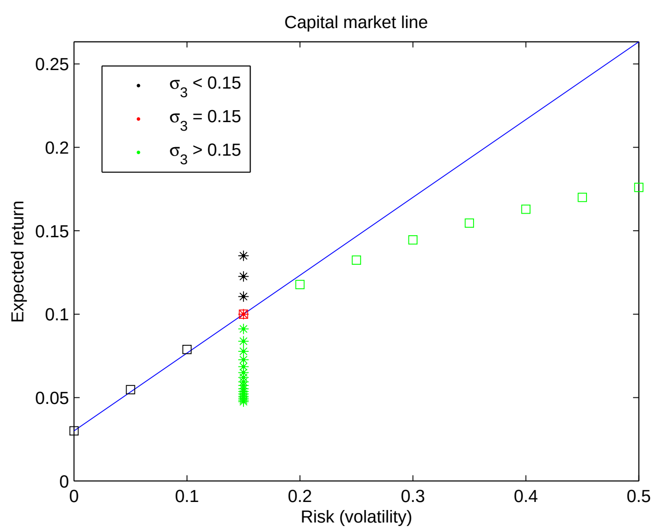


Figure 4.5: Capital market line. The star indicators are the couples (σ_1, R_{L_1}) and (σ_2, R_{L_2}). The square indicators are the couples (σ_3, R_{L_3}). The expected returns of the lines decrease when σ_3 increases.

4.2.c Diversification

The purpose of this scenario is to study the impact of diversification on the sharing of the surplus across the lines of business. Therefore two modifications will be considered: change in the correlations between the lines' returns and change in the correlations between the lines' returns and the asset's return.

To begin with, the same inputs as in the reference case are used except for the correlations across the lines' returns:

$$\rho_{ij} = \rho \in [0,\ 1] \quad \forall i,\ j = 1,2,3 \text{ and } i \neq j.$$

The standard deviation of total losses (σ_L) and the covariances of the different lines' losses with the total losses (σ_{iL}) increase linearly with the covariances.

The VaR and EPD allocation's formulas do not depend of the correlations between the lines of business. Therefore, even if the VaR of the entire portfolio increases with the correlations across the lines, the same proportion of surplus is allocated to each line for all the values taken by ρ. This brings forward a weakness of those two methods.

The surplus allocation resulting for the MR method is shown in the left plot in Figure 4.6. The default value of the firm increases slightly when the correlations across the liabilities rise, but it does not affect much the surplus requirement across the lines. Lines with higher risk (encompassing higher standard deviations of losses) still have higher surplus requirement. The CAPM surplus allocation is similar to the MR's allocation. However, it allocates between 1% and 5% less capital to the two first lines of business when the lines have low correlation. It comes from the fact that, when ρ is small, the largest determinants of the beta losses are the standard deviations of the lines.

Nevertheless, the changes in the correlations between the lines' returns have not a large impact on the surplus allocation. The next scenario will test different correlations between the lines' returns.

In the second modification, the correlations across the lines' returns go back as in the reference case and the correlation between the third line of business and the asset is modified:

$$\rho_{1V} = \rho_{2V} = 0.2 \quad \text{and} \quad \rho_{3V} \in [0,\ 1].$$

The standard deviation of total losses with the asset (σ_{LV}) and the covariances of the 3rd line's loss with the asset (σ_{3V}) increase linearly with ρ_{3V}.

The allocations of capital with the VaR and EPD techniques are again not modified by those changes. The variation of ρ_{3V} also has no impact on the allocation of capital for the CAPM.

The VaR of the portfolio decreases when the correlation between a line of business and the asset increases. For the same risk-taking a higher portfolio return is expected when ρ_{3V} is large. The portfolio becomes safer. When ρ_{jV} increases the surplus allocated to line j should decrease as the surplus allocated to the other lines should increase. A decrease of the default value is also expected. These assumptions are confirmed with the MR's surplus allocation which is shown in the right plot in Figure 4.6. For both distributions the surplus allocated to line 3 highly decreases when ρ_{3V}

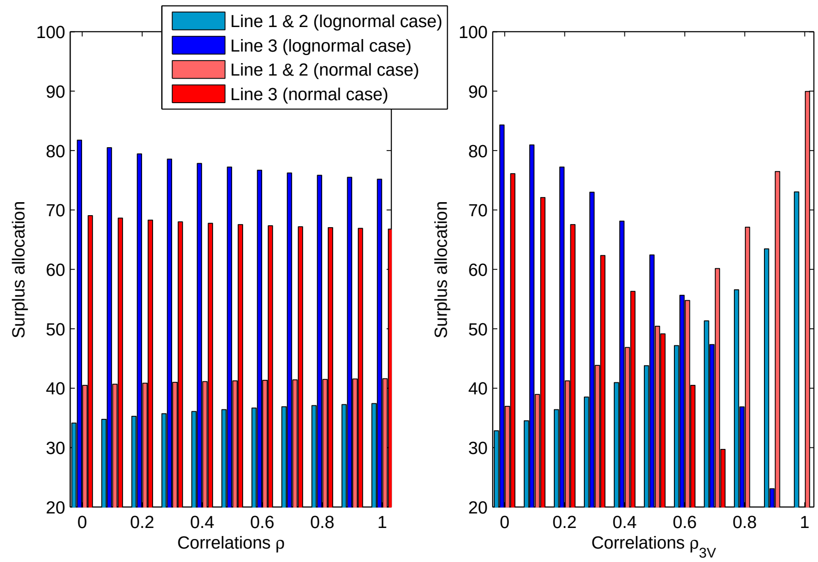


Figure 4.6: Diversification affects the surplus allocation. Variation of the MR surplus allocation with the correlations between the lines' returns (left plot) and with the correlation between line 3's return and the asset's return (right plot).

increases. The capital is reallocated in equal proportions to the two first lines of business. For $\rho_{3V} = 1$, no surplus is allocated to the third line. The default value decreases from 0.30% to 0.003% in the normal case and from 0.26% to 0.001% in the lognormal case.

The simplification proposed by [Butsic 1999] in Section 3.4.c is not robust to this change. Indeed, the simplification neglects the term $\sigma_{LV}(\gamma_{iV} - 1)$ which is affected by the change of the parameter ρ_{iV}:

$$\sigma_{LV}(\gamma_{iV}-1) = \sigma_{LV}(\frac{\sigma_{iV}}{\sigma_{LV}}-1) = \sigma_{iV}-\sigma_{LV} = \sigma_{iV}-\sum_j x_j\sigma_{jV} = \underbrace{\sigma_{iV}}_{\rho_{iV}\sigma_i\sigma_V}\underbrace{(1-x_i)}_{\geq 0}-\sum_{j\neq i} x_j\sigma_{jV}$$

The term $\sigma_{LV}(\gamma_{iV} - 1)$ increases when ρ_{iV} increases and the error on s_i becomes larger.

4.2.d Change in the present values of losses

The present values of the liabilities are now changed to $L_1 = 150$€, $L_2 = 120$€ and $L_3 = 30$€. The purpose of this scenario is to highlight the impact of changes in the present values of losses on the default value. The company has a smaller part of its total liability put in the riskier line (3$^{\text{rd}}$ line). The investment has been transferred to the two first lines which still have a standard deviation of 15 %. The risk relative to the portfolio is thus lower. Smaller VaR, exceedance and deficit probabilities relating to the VaR and EPD allocations could be expected.

The VaR of the portfolio for different levels of protection and for both distributions are summarized in Table 4.4. The VaR is smaller than in the reference case. It is in line with the predictions. Investing less capital in the riskier line is sensible.

	$p = 0.5\%$	$p = 1\%$	$p = 5\%$
Normal case	304	129	-349
Lognormal case	582	365	-195

Table 4.4: VaR (€) of the portfolio for $p = 0.5\%$, $p = 1\%$, $p = 5\%$ and for the normal and lognormal distributions with different present values of losses.

The resulting capital allocations also agree with these observations. With the four methods and for both distributions, less capital is allocated to the 3^{rd} line of insurance. This extra capital is allocated to the two first lines of business. More capital is reallocated to the first line because $L_1 > L_2$. The exceedance probability curves and the surplus allocation related to the VaR method are represented in Figure 4.7. The exceedance probabilities are smaller than in the reference case. The four methods (and the simplification of the MR method) give comparable results even if the VaR and EPD methods still allocate more capital to line 3 than the CAPM and MR methods.

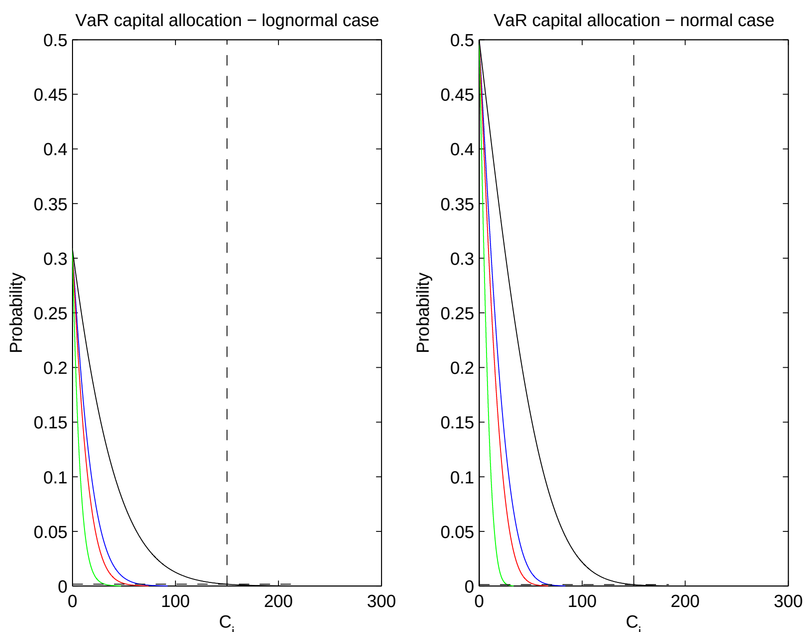


Figure 4.7: The VaR surplus allocations obtained through the exceedance probabilities with $L_1 = 150$€, $L_2 = 120$€ and $L_3 = 30$€ allocate less capital to line 3 (blue line : first line of business, red line : second of business, green line : third line of business, black line : total capital, dashed lines : exceedance probability and total related surplus).

This phenomenon can also been noticed through the loss beta, the default values and the Sharpe ratios. The loss beta are now (0.96, 0.69, 1.57) compared to (0.68, 0.68, 1.64) in the reference case. The same variations can be observed for the Sharpe ratios: $\lambda = (0.45,\ 0.42,\ 0.37)$. The default values are now $D = 0.24$€ (or $d = 0.08\%$) in the lognormal case and $D = 0.39$€ (or $d = 0.13\%$) in the normal case.

These values are smaller than for the reference scenario. The company has a lower probability of insolvency and therefore the cost of a guarantee for the payment of the liabilities is less expensive.

4.2.e More elaborate case

This scenario is widely different. Three lines of business are still considered but with different parameters. The present values of the liabilities stay as in the previous scenario while a more elaborate correlation matrix is now considered. Line 3 is isolated and the two other lines have high correlation. The data is summarized in Table 4.5. The standard deviation of loss is now 12.99% compared to 16.56% in the reference case. The results will be discussed here, but the table of results is further provided in Appendix A.4.

	Present Value of the liabilities and asset (€)	Standard deviation	Correlations		
			Line 1	Line 2	Line 3
Line 1	150	15 %	1	0.75	0
Line 2	120	15 %	0.75	1	0
Line 3	30	30 %	0	0	1
Asset	450	15 %	0.2	0.2	0.2
$\sigma_L = 12.99\%$		$\sigma_{iL} = (1.80,\ 1.74,\ 0.90)\%$			
$\sigma_{LV} = 0.49\%$		$\sigma_{iV} = (0.45,\ 0.45,\ 0.90)\%$			

Table 4.5: Summary of the data used for the more elaborate scenario.

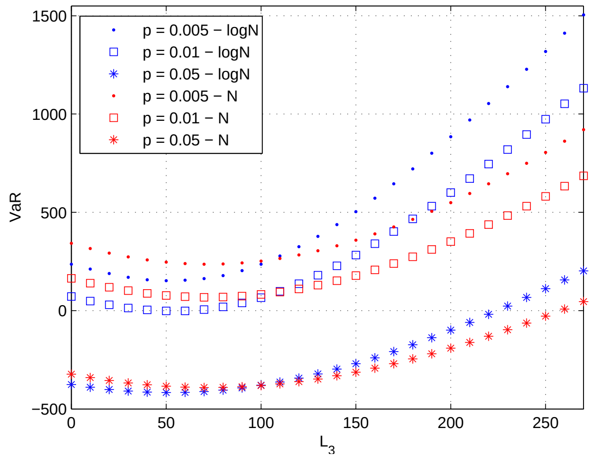
The surplus allocations of the VaR and of the EPD do not change from the previous scenario (see Section 4.2.d) since the correlations between the lines of business are not taking into account with these models. The VaR of the portfolio decreases compared to the previous scenario. Less capital has to be invested in a risk-free asset to avoid insolvency.

Line 3 is uncorrelated with the two other lines, its volume of liability is small compared to the total liability and β_3 is the lowest beta (the loss beta for the lines of business are $1.07,\ 1.03$ and 0.53). Therefore line 3 has the smallest surplus requirement even if it is the riskier line. As in the previous scenario, capital is transferred from line 3 to lines 1 and 2. The default value does not decrease much compared to the previous scenario. A change in the liability has a bigger effect on the default value of the company than a change in the correlations.

The robustness of the techniques to changes in the present values of losses with the more elaborate correlation matrix will now be studied. The proportion of liability of the third line of business $\left(\frac{L_3}{L}\right)$ varies from 0 to 0.9. The resulting amount of liability is equally distributed across the two other line of business to keep the total loss constant ($L = 300$€):

$$L_1 = 165 - 300\frac{x}{2}, \quad L_2 = 135 - 300\frac{x}{2}, \quad L_3 = 300x \qquad x \in [0, 0.9].$$

The fraction of liability invests in the third line is bounded at 0.9 to keep L_1 and L_2 positive. The other inputs stay as in Table 4.5. Before studying the surplus allocation, it is interesting to briefly have a look at the evolution of the VaR of the portfolio with the liabilities.



For both distributions, the VaR is more of less constant with the liabilities for $L_3 \leq 100$€. The VaR increases for $L_3 \geq 100$€ ($L_1 \leq 115$€, $L_2 \leq 85$€). The increasing of the VaR is more significant in the lognormal case and reaches values up to 1500€. The lognormal distribution better reflects the fact the portfolio is riskier when a larger fraction of the liability is invested in the riskier line.

The allocations of capital are about the same for the four methods and add up to 450€. The evolution of the proportions of surplus allocated to each line is almost linear. There is no sharp modifications; the changes are gradual. The methods are then robust to small changes in the present value of losses. They can be used in practice to price insurance. The capital allocated to the first line decreases from 250€ to 40€ as the capital allocated to line 2 goes from 200€ to 0€. The surplus is added to line 3; its surplus increases from 0€ (when $L_3 = 0$€) to 410€.

The default value, the exceedance and deficit probabilities related to the MR, VaR and EPD allocations increase when more capital is invested in the riskier line.

4.2.f Negative surplus

A negative surplus ($V < L$) should not occur often since insurance companies try to avoid this undesirable scenario. However, it is interesting to analyse how the four methods react to insolvency.

The resulting allocations of capital are not similar across the methods. The EPD and MR methods are alike and allocate more capital to the third line of business than to lines 1 and 2. It contrasts with the VaR and CAPM methods which allocate more capital to the safer lines when insolvency occurs.

The lines' surplus requirements do not depend of the asset value with the VaR and EPD methods. A negative surplus only implies changes in the total capital requirement constraint. The exceedance (or deficit) probability curves are thus identical. They are shown for both distributions and both methods in Figure 4.8. The results for a surplus of -50€ ($V = 250$ €) are put forward in these graphs and are summarized in Appendix A.4. The exceedance and deficit probabilities grew to high values.

The required expected returns given by the CAPM do not depend of the asset value. The portion of surplus allocated to line i is constant (the same as in the reference case).

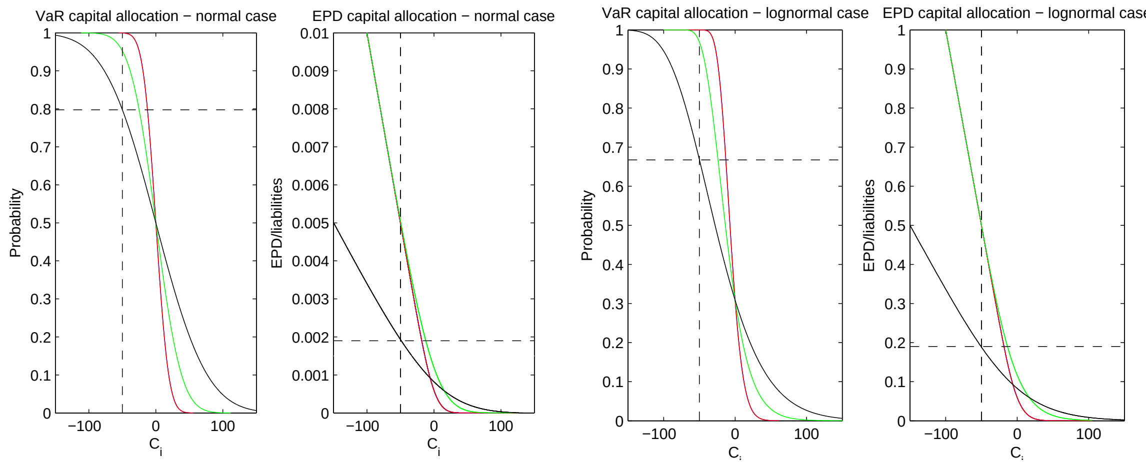


Figure 4.8: A negative surplus does not change the exceedance and deficit probability curves. The resulting surplus allocation, exceedance and deficit probabilities are different. (red lines : 2 first lines of business, green line : third line of business, black line : total capital, dashed lines : exceedance and deficit probability and total related surplus).

The allocated surplus varies then linearly with the value of the asset. It can be seen in the right plot in Figure 4.9. With a surplus of -50€ the CAPM allocates 87€ to the two first lines of business and 76€ to the third line. This is very close to the VaR capital allocation.

The optimal allocations of capital for the MR method are similar for both distributions (see left plots in Figure 4.9) and close to the allocations get by the EPD method. It differs from the CAPM when $V < L$. The MR method allocates more capital to the third line of business. Indeed for $S = -50$ €, the marginal surplus requirement of the lines are $s_1 = s_2 = -18\%$ and $s_3 = -14\%$. The evolution of the required surplus is linear and it grows quicker for line 3 than for lines 1 and 2.

When $V = 300$ the surplus allocated to the three lines of business is null with the CAPM while it is positive for line 3 and negative for lines 1 and 2 with the MR technique. It again highlights the fact that the MR method allocates negative surplus to less risky lines when the risk undertaken by the company is too high (negative surplus, high losses volatilities). It enables a better hedging of the riskier line to ensure a minimum default value.

The default values for both distributions increase when the asset value decreases. For an asset value of 250€ they are worth around 18%.

4.2.g Mix of insurance lines

The surplus allocation does not only depend of the risk characteristics of the lines of business and asset. It also depends of the mix of insurance lines written by the company. Methods have to be robust enough to changes in the mix of business in order to be useful for pricing insurance.

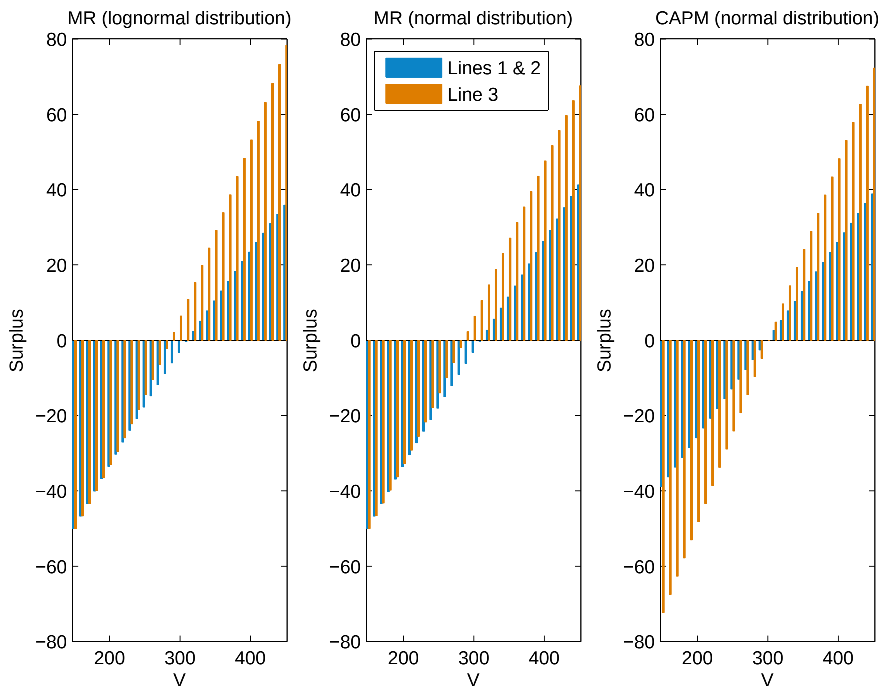


Figure 4.9: Modification of the allocations of surplus across the lines with the asset's value for the MR and CAPM methods.

This section supposes that the company has equal fractions of its total liability L in each of its N existing lines of business. Then, a $N+1^{st}$ line of business is added. The liabilities in the "old" lines go down from $\frac{L}{N}$ to $\frac{L}{N+1}$ as the liability in the "new" line increases from 0 to $\frac{L}{N+1}$. The total liability and asset value stay constant. To keep the scenario simple the input data is the same as in the reference case. All the lines (old and new) have a standard deviation of 15% except for the first (old) line which has a standard deviation of 30%.

The number of lines underwritten by the company goes from 3 to 20 so that the robustness of the methods to the addition of new lines can be evaluated. The left plot in Figure 4.10 shows how the surplus is divided into a growing number of lines. The results are similar for the four methods. The surplus stays constant at 150€. The first line ($\sigma = 30\%$) gets $\frac{2}{N+1}$ of the total surplus. The fractions of surplus allocated to the N other lines ($\sigma = 15\%$) are equal to $\frac{1}{N+1}$. The curves are hyperbolic and their slopes become smaller when N increases. These results suggest that it has less effect on the marginal surplus requirements of the lines to add a line of business to a 10-line company compared to a 5-line company. The marginal surplus requirements of lines of business are robust to the introduction of new lines of business if the company already has a certain degree of diversification.

The default value, exceedance and deficit probabilities decrease when the number N of lines of business decreases. An increasing of N also implies a decrease of the VaR as it can be seen in the right plot in Figure 4.10.

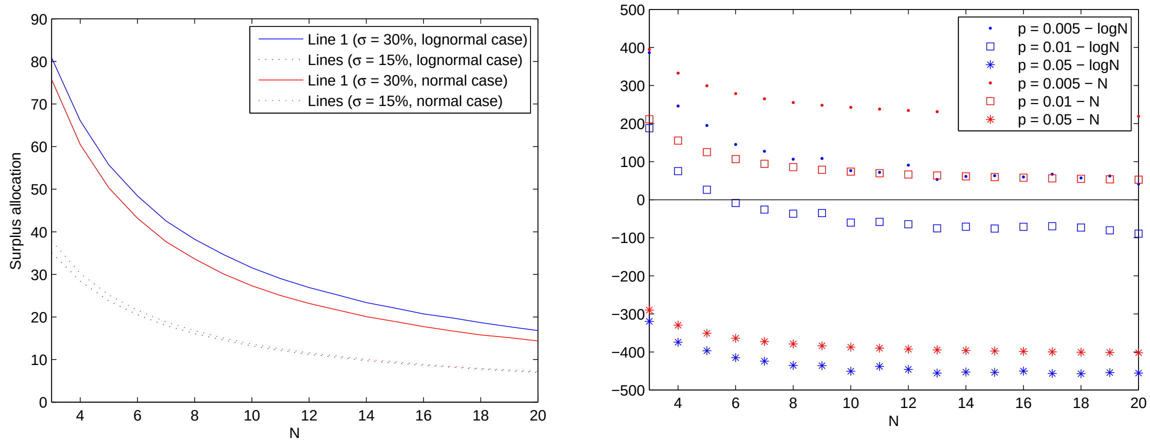


Figure 4.10: Diversification. Evolution of the surplus allocation across the lines (left plot) and of the VaR of the company (right plot) with the number N of lines underwritten by the company.

The portion of surplus allocated to the first line (the riskier) and the VaR have the same trend. For $N < 6$, the VaR decreases much more than for larger values of N and the portion of surplus allocated to line 1 decreases from 0.5 to 0.3. Whereas when N goes from 6 to 12, the portion of surplus $\frac{s_1}{S}$ goes down from 0.3 to 0.15. The VaR stays more of less constant for $N > 6$.

When N increases the covariances become an important determinant of the total losses variance. Since the liabilities are divided equally in each of the N lines, the total losses variance σ_L^2 can be expressed as

$$\begin{aligned}
\sigma_L^2 &= \sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j \rho_{ij}\sigma_i\sigma_j = \left(\frac{1}{N}\right)^2 \sum_{i=1}^{N}\sum_{j=1}^{N} \rho_{ij}\sigma_i\sigma_j \\
&= \left(\frac{1}{N}\right)^2 \sum_{k=1}^{N}\sigma_k^2 + \left(\frac{1}{N}\right)^2 \sum_{i=1}^{N}\sum_{j=1,\neq i}^{N} \rho_{ij}\sigma_i\sigma_j \\
&= \left(\frac{1}{N}\right)^2 N \cdot \text{ average variance } + \left(\frac{1}{N}\right)^2 N(N-1)\cdot \text{ average covariance} \\
&= \frac{1}{N}\cdot \text{ average variance } + \left(1-\frac{1}{N}\right)\cdot \text{ average covariance}
\end{aligned}$$

As N increases, the total losses is no longer affected by the average variance and tends to the average covariance. It explains why diversification cannot completely eliminate the risk of the portfolio.

Companies with 5 and 10 lines of business will now be consider to confirm the fact that companies with already a certain degree of diversification are robust to the introduction of new lines of business. It is supposed that the companies start with equal fractions of their total liability in each of the $N-1$ existing lines of business. The liability in the new line is then equal to zero. The fraction of liability in the new line will gradually increase from 0 to $\frac{1}{N}$. The surplus is being kept constant and the standard deviations of all the lines are put at 15%.

Both companies have the same trend (see Figure 4.11). When the liability of the new line becomes a larger fraction of the total liability, its surplus requirement rises while the surplus requirements of the existing lines fall much more slowly. The surplus requirement of the new line is much more sensitive to changes in the mix of business than the surplus requirement of the old lines. For the 5-line company, the marginal surplus requirement of the new line increases from 0 to 31% while it goes from 38% to 31% in each of the old lines. The differences are smaller for the 10-line company. The marginal surplus requirements for the new and old lines vary from 0 to 15 % and from 17% to 15%.

Companies with even a moderate degree of diversification are relatively robust to the introduction of new lines of business.

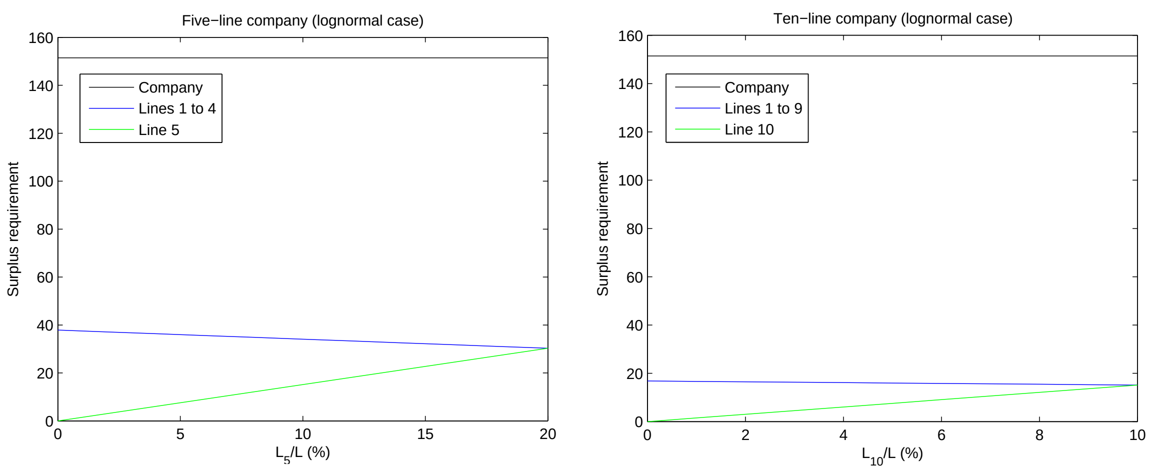


Figure 4.11: A new line of business is added to 5 and 10-line companies. The fraction of liability invested in the new line is gradually increased to evaluate the impact of diversification on the surplus requirement of the old and new lines.

4.3 Real case company

Data

Before concluding this section the performances of the methods will be tested with data coming from CNP Insurance. CNP Insurance is the leader insurer in France, with a net profit of € 951 million in 2012. The Group is also present in Europe and Latin America, and has strong activity in Brazil. It is the 4^{th} largest life insurance market in the world.

The data come from the CNP Insurance website and respect the International Financial Reporting Standards[1]. The insurance activities of CNP can be divided into

[1]The International Financial Reporting Standards (IFRS) are designed as a common global language for business affairs so that company accounts are understandable and comparable across inter-

three sectors: savings, personal insurance (including property and casualty damage, health insurance, loan insurance ...) and pensions. The liabilities by sectors are given as fractions of the total liability. The surplus reserve is also known. The data is available from 2006 till 2012 and are summarized in Table 4.6. From those samples, the returns of the three lines and of the asset are obtained. The returns are first assumed to be normally distributed and then lognormally distributed. In both cases, the standard deviations and the correlation matrix of the returns are estimated from the data. The graphs in Figure 4.12 show the histograms of the liabilities along with both density functions.

The normal distribution appears to be more appropriate to simulate the asset distribution than the lognormal distribution. Since the sizes of the samples are quite small, it is difficult to determine which distribution better fits the data. Additional data (they should be more and closer in time) is needed to carry out the techniques on reliable values of the statistical parameters.

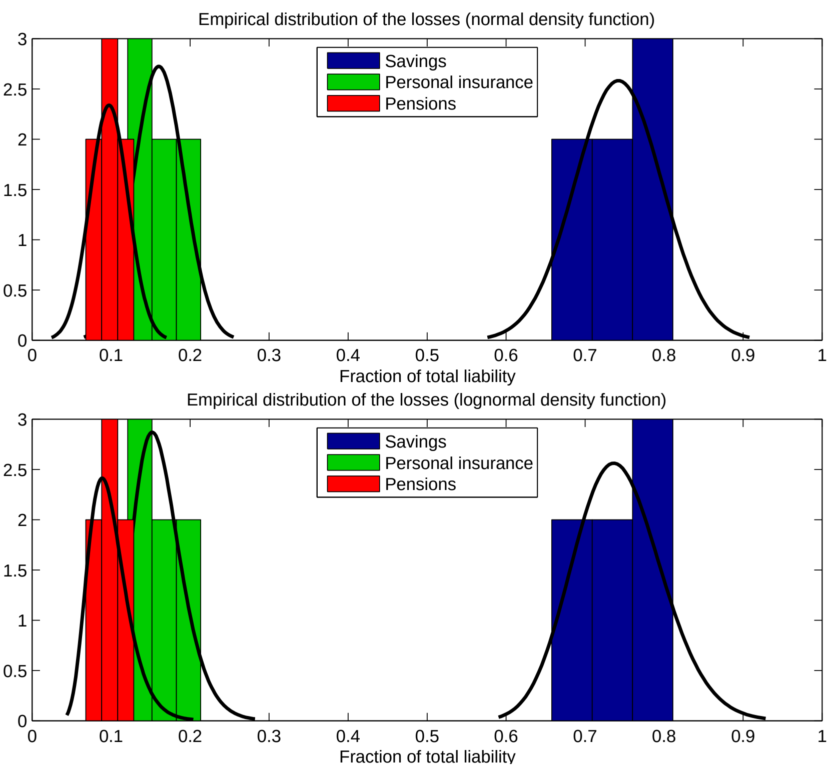


Figure 4.12: Distributions of the liabilities of the three lines of insurance fitted by a normal (upper plot) and a lognormal (lower plot) density function.

national boundaries (http: //en.wikipedia.org/wiki/International_Financial_Reporting_Standards).

Results

For both distributional assumptions, the savings and pensions insurances (line 1 and line 3) have large volatilities ($\pm 15\%$) compared to the volatility ($\pm 5\%$) of the personal insurance (line 2). The largest fraction of liability comes from the first line of insurance ($\frac{L_1}{L} = 66\%$) which is not correlated with the asset and with the other lines of insurance ($\rho_{1,2} = -0.10, \quad \rho_{1,3} = -0.37, \quad \rho_{1,V} = -0.15$). Therefore, as it has been said in the previous examples, a larger fraction of surplus should be allocated to line 1 than to the two other lines.

Line 2 has high correlation with the asset ($\rho_{2,V} = 0.34$) and is low risk. Line 3 has an high volatility but it contributes the least to the total liability ($\frac{L_3}{L} = 13\%$). The fraction of surplus allocated to those lines should therefore be relatively small.

The four methods (and the simplification of the MR model) are applied to CNP Insurance. The results are shown in Figure 4.13. All methods clearly allocate more capital to the first line of business, for both distributions.

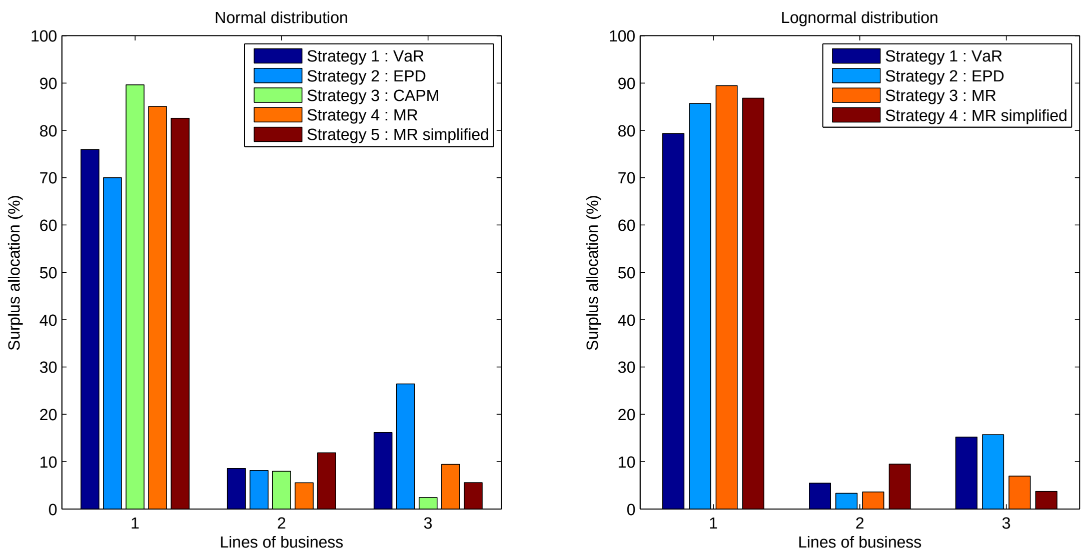


Figure 4.13: Percentage of surplus allocated to each line of business (line 1 : savings, line 2: personal insurance, line 3: pensions): normal distributional assumption (left plot) and lognormal distributional assumption (right plot).

The CAPM and the MR simplified methods are the only methods that allocate more capital to line 2 (σ_2 small, $\rho_{2V} > 0$) than to line 3 ($\frac{L_3}{L}$ small). Line 3 has a Sharpe ratio smaller than 1 ($\lambda = (4.32, \quad 2.94, \quad 0.37)$) and a negative loss beta ($\beta = (1.55, \quad 0.05, \quad -0.22)$). It means that the liability of line 3 moves in the opposite direction as the movement of the asset. Therefore the smaller proportion of capital is allocated to line 3 with methods using the loss beta to allocate capital (as the CAPM and the simplified MR model).

In most of the earlier scenarios, the risk of insolvency were higher when the returns were assumed to follow a lognormal distribution. These results still hold for the CNP Insurance. Less capital is allocated to lines 2 and 3 with the lognormal assumption, this capital is reallocated to the riskier line (line 1). The VaR model is the model for which the surplus allocation is the less sensitive to the choice of the distribution. It corroborates the observations made previously. The VaR method is easy to adapt to new assumptions about the returns distributions and looks to be robust to changes of distribution. These assumption is tested hereafter.

New distribution assumption

To test the robustness and the adjustment's easiness of the VaR method to new distributions, the data is now assumed to have returns following a non-standardized Student's t-distribution. This distribution is parametrized by its mean, variance and number ν of degrees of freedom. The t-distribution is like the normal distribution (symmetric and bell-shaped) but has heavier tails. When ν grows, the standard Student's t-distribution approaches the standard normal distribution. It is often used in finance to simulate heavy-tailed equity returns. As it can be seen in Figure 4.14, the validation is more successful and the fitted parametric distribution can be used as input for the VaR method.

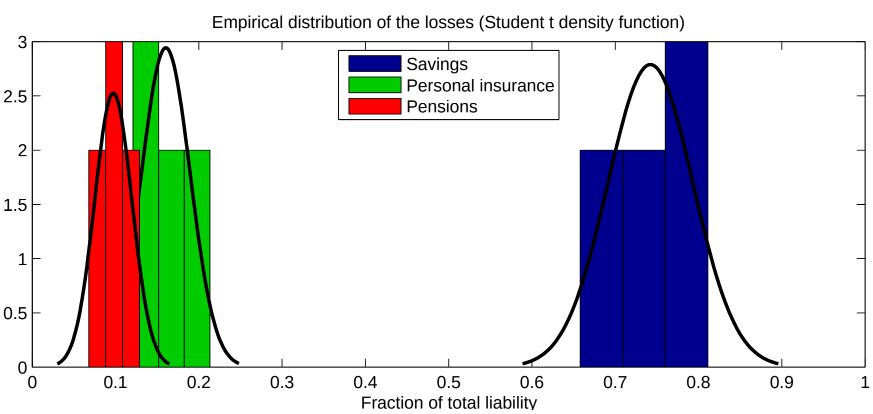


Figure 4.14: Distributions of the liabilities of the three lines of insurance fitted by a non-standardized Student's t-density function.

The lines' returns can then be written $R_{L_i} = \mu_i + \sigma_i X$ where the random variable X follows a standard Student's t-distribution with ν_i degrees of freedom. The set of parameters $(\mu_i,\ \sigma_i,\ \nu_i)$ are estimated from the data for all the lines and for the asset. The variable X has a mean equal to zero and a variance equal to $\dfrac{\nu}{\nu-2}$ (if $\nu > 2$).

The required capital for each line i is now given by:

$$\begin{aligned} C_i &= \mathrm{E}(L_i^1)\left(F_{X_i}^{-1}(1-\varepsilon)-1\right) \\ &= L_i\mu_i\left(t_{\nu_i}^{-1}\left(\frac{-\varepsilon}{\sigma_i/\mu_i}\right)-1\right). \end{aligned}$$

with $X_i = \dfrac{L_i^1}{\mathrm{E}(L_i^1)} = \dfrac{R_{L_i}}{\mathrm{E}(R_{L_i})} = 1 + \dfrac{\sigma_i}{\mu_i}X$ and $t_\nu\{\cdot\}$ the probability density function of the Student's t-distribution with ν degrees of freedom.

The fractions of surplus allocated to each line with the VaR method are summarized in Table 4.7 for the three distributional assumptions. The resulting optimal surplus allocation is close from the allocations get when the returns were assumed to be normally or lognormally distributed. It confirms that the VaR method is an effective solution to achieve the optimal capital allocation when the data cannot be clearly fitted by a distribution.

Time	Savings (%)	Personal insurance (%)	Pensions (%)	Surplus (billions €)
2006	81	12.2	6.8	2.2
2007	79.6	13.5	6.9	2.2
2008	73.7	16.1	10.2	2.2
2009	76.8	14.3	8.9	2.2
2010	73.8	16.3	9.9	2.9
2011	68.9	18.6	12.5	2.9
2012	65.8	21.3	12.9	3.4

Table 4.6: Summary of the data (liabilities by sectors and surplus) of CNP Insurance.

	Distribution assumptions		
	Normal	Lognormal	Student's t
Line 1	75.98 (%)	79.36 (%)	76.23 (%)
Line 2	8.55 (%)	5.45 (%)	8.51 (%)
Line 3	16.16 (%)	15.20 (%)	15.79 (%)

Table 4.7: Summary of the results of CNP Insurance with the VaR method for different distributional assumptions.

Chapter 5

Conclusion

The aim of my master thesis was to analyse and compare different existing methods to allocate the capital of an insurance company across its lines of insurance of business. This last chapter offers a summary of the results and draws up conclusions together with suggestions for further research.

5.1 Outcomes

The default value, exceedance and deficit probabilities are effective markers of the risk of insolvency incurred by an insurance company. However, the default value is more sensitive to changes in the different correlations. That exhibits the fact that the MR model takes diversification into account contrary to the three first methods. This is a major advantage of this method since the company as a whole is concerned by the solvency of each of its portfolios of contracts. The company's VaR can also be used as a marker for the firm's insolvency and is a good informer of the performance of the firm's portfolio.

The methods mostly tend to allocate the surplus across the lines of insurance similarly according to the input parameters. The returns' volatilities are always one of the main determinants of the capital requirements. The degree of correlation between the lines and the asset is also a critical factor to properly set capital levels.

The VaR method is easy to establish and has similar results for both distributional assumptions. Those results can thus be used as a trustful benchmark in the decision-making process. However, the VaR and EPD methods do not take the volatility of the asset return σ_V into account to determine the optimal capital allocation. It means that the allocation will not be reconsidered if some economics events were to affect the asset.

Capital allocations based on marginal default value differ from those based on risk measures. The VaR method, for instance, would allocate zero capital to a low-risk security. A marginal capital allocation procedure allocates negative capital to low-risk securities. Such securities are rewarded because they reduce the firm's default value. The beneficial performances of some contracts compensate the downside performances of others. It puts forward the advantages of diversification provided that the business lines of the portfolio have low correlation. The MR method is the only method sensitive to changes in the correlations between a line's and the asset return. Those changes have a significant impact on the surplus allocation. Moreover, lines of insurance which have returns highly correlated with the asset return are propitious for the companies. It seems therefore important to use models that take those parameters into account. The simplification of the MR model proposed by Butsic is useful if the set of data is incomplete or inaccurate. Nevertheless, it is not robust to extreme scenarios.

The CAPM provides reliable allocations and also offers interesting intermediate results. The beta losses and Sharpe ratio are helpful indicators of the lines vulnerability to systematic risk. But, it does not consider lognormally distributed returns which are often more realistic.

The choice of the distribution which fits the data better plays a key role in the allocation process. As it could be seen in Section 4.3 that considers real historical data, the choice of the appropriate distribution is not always straightforward and affects the resulting business operations.

5.2 Discussion: future and limitations

The framework is built around a concept of capital requirement. Firms are expounded on their balance sheet that may face changes in size and risk. These changes are not easy to predict or anticipate. An acceptable limit level of protection against the unwilling effect of insurer's insolvency is required by the policyholders and the authorities. Thereby firms raise capital to purchase assets and hedge liabilities. The inherent risk of each line of insurance has to be determined in order to divide the capital across the lines. Allocation is done in terms of the lines of insurance lending money to the asset division according to a pattern described by different techniques. Four techniques have been considered in this thesis. Each method has characteristics that make them better or worse in specific contexts.

The CAPM is a very old technique which still provide a good benchmark to compare other methods. The MR model is the most accurate and effective solution in the way that it takes diversification and most statistical parameters into account. Hence, looking just at the VaR may not be sufficient to have a appropriate capital allocation. However allocations will be sensitive to distributional assumptions. The VaR technique has the huge advantage that it is easily adaptable for other distributions used in finance and more specifically in an insurance context. For instance, the Poisson distribution models certain operational risk and can be used to simulate the number of claims in each period, the Beta distribution simulates recovery rates, etc.

A few recommendations about further research can be drawn.

- Considering multi-period claim on multiple-line firms will be more realistic and will need challenging reorganization of the methods.
- Implementing a gradual progress from the present allocation to the new allocation in case of rash changes in the allocation processes.
- Including a stochastic interest rate in the running of the methods.

Chapter A

Appendix

A.1 Glossary

This appendix provides detailed description of all the notations used in the paper.

Symbols	Definitions
D	default value
d	default value to liability ratio
C, C_i	required capital, required capital for line i
E	equity
E_m	expected return of the market portfolio
L	liability value
N	number of lines of business
P	premium
R_f	risk-free rate return
R_L, R_V, R_U	return on liability, asset and underwriting
S	surplus
s, s_i	aggregate surplus ratio, marginal surplus requirement of line i
V	asset value
x_i	portion of total liability of line i
Y	capital allocation factor
z	standard normal variable
β_i	loss beta for line i relative to all loses
γ_i	ratio of covariance of policy i to all assets to that of all all losses and assets
λ	Sharpe ratio
ρ_{ij}	correlation between policy i and policy j losses
ρ_{iL}	correlation between policy i losses and all losses
Σ	covariance matrix
σ	volatility of the surplus-to-liability ratio

σ_i	volatility of losses in policy i
σ_{iL}	covariance of log losses in policy i with log losses on the portfolio
σ_{iV}	covariance of log losses in policy i with log asset value
σ_L	volatility of losses
σ_{LV}	covariance of losses and asset
σ_V	volatility of asset
θ	normalized standard deviation of surplus
θ_i	normalized standard deviation of losses in policy i
θ_{iL}	normalized standard covariance of losses in policy i with losses on the portfolio
θ_{iV}	normalized standard covariance of losses in policy i with asset value
θ_L	normalized standard deviation of losses
θ_{LV}	covariance of losses and asset
θ_V	normalized standard deviation of asset
ε	exceedance probability
$\stackrel{d}{=}$	are equal in distribution

A.2 Expected policyholder's formulas in the normal and lognormal case

Normal case

If the joint probability distribution of losses and asset value is normal, the quantity of asset exceeding losses $V - L$ has a multivariate normal distribution with mean $\mu = \mathrm{E}(V - L)$ and standard deviation $\sigma = \mathrm{Var}(V - L)$. The EPD of the total portfolio is then given by

$$EPD = \mathrm{E}\left(\max(V - L, 0)\right) = \int_{-\infty}^{0} -zp(z)dz = \int_{-\infty}^{0} -z\frac{1}{\sqrt{2\pi\sigma^2}}e^{\frac{-(z-\mu)^2}{2\sigma^2}}dz$$

Setting $y = \dfrac{z-\mu}{\sigma}$ and making a substitution of integration variables leads to the following derivation

$$
\begin{aligned}
EPD &= \int_{-\infty}^{-\frac{\mu}{\sigma}} -(\sigma y+\mu)\frac{1}{\sqrt{2\pi\sigma^2}}e^{\frac{-y^2}{2}}\sigma dy = \int_{-\infty}^{-\frac{\mu}{\sigma}} -(\sigma y+\mu)\frac{1}{\sqrt{2\pi}}e^{\frac{-y^2}{2}} dy \\
&= -\sigma\int_{-\infty}^{-\frac{\mu}{\sigma}} \frac{y}{\sqrt{2\pi}}e^{\frac{-y^2}{2}} dy - \mu\int_{-\infty}^{-\frac{\mu}{\sigma}} \frac{1}{\sqrt{2\pi}}e^{\frac{-y^2}{2}} dy \\
&= \sigma\int_{-\infty}^{-\frac{\mu}{\sigma}} \phi'(y)dy - \mu\int_{-\infty}^{-\frac{\mu}{\sigma}} \phi(y)dy \\
&= \sigma\left(\phi(-\frac{\mu}{\sigma}) - \phi(-\infty)\right) - \mu\left(\Phi(-\frac{\mu}{\sigma}) - \Phi(-\infty)\right) \\
&= \sigma\left(\phi(-\frac{\mu}{\sigma}) - 0\right) - \mu\left(\Phi(-\frac{\mu}{\sigma}) - 0\right) = \sigma\phi(-\frac{\mu}{\sigma}) - \mu\Phi(-\frac{\mu}{\sigma}).
\end{aligned}
$$

But, the quantity of interest is the EPD ratio of one single line of business i. This is the expected loss L_i exceeding the capital C_i allocated to line i divided by L_i:

$$\frac{EPD_i}{L_i} = \sigma_i\phi(-\frac{c_i}{\sigma_i}) - c_i\Phi(-\frac{c_i}{\sigma_i}).$$

Lognormal case

If the joint probability distribution of losses and asset value is lognormal, the EPD of the total portfolio is still given by $EPD = \mathrm{E}\left(\max(V-L,0)\right)$. That is the expression of a call option with exercise price V and current stock price L. The value of the corresponding call option is then given by the Black's formula:

$$
\begin{aligned}
EPD &= L\Phi\left(\frac{\sigma}{2} - \frac{1}{\sigma}\ln\left(\frac{V}{L}\right)\right) - \Phi\left(-\frac{\sigma}{2} - \frac{1}{\sigma}\ln\left(\frac{V}{L}\right)\right)\left(\frac{V}{L}\right) \\
&= L\Phi\left(\frac{\sigma}{2} - \frac{1}{\sigma}\ln\left(1+\frac{C}{L}\right)\right) - \Phi\left(-\frac{\sigma}{2} - \frac{1}{\sigma}\ln\left(1+\frac{C}{L}\right)\right)\left(1+\frac{C}{L}\right).
\end{aligned}
$$

And the EPD ratio of one single line of business is given by:

$$\frac{EPD_i}{L_i} = \Phi\left(\frac{\sigma_i}{2} - \frac{1}{\sigma_i}\ln\left(1+\frac{C_i}{L_i}\right)\right) - \Phi\left(-\frac{\sigma_i}{2} - \frac{1}{\sigma_i}\ln\left(1+\frac{C_i}{L_i}\right)\right)\left(1+\frac{C_i}{L_i}\right).$$

A.3 Marginal capital allocation's formulas in the normal case

The same reasoning as in Section 3.4.b can be done if the joint probability distribution of losses and asset value is normal. In that case, the dependence of the default value with the distributions of the total losses and the asset is showed by the normalized standard deviation of surplus θ. This concept has a parallel interpretation as the the volatility of the asset-to-liability ratio σ defined earlier for the lognormal case.

$$\theta = \sqrt{\theta_L^2 + (1+s)\theta_V^2 - 2(1+s)\theta_{LV}} \tag{A.1}$$

depends on $\theta_V, \quad \theta_L, \quad \theta_{LV}$ the standard deviation of asset, losses, the covariance of losses and asset and on the aggregate surplus ratio s. The default value to liability ratio is then given by:

$$d = -s\Phi\left\{-z\right\} + \theta\phi\left\{z\right\} \tag{A.2}$$

where $z = \frac{s}{\theta}$. That gives for each line:

$$\begin{aligned} d_i &= d + \frac{\partial d}{\partial s}(s_i - s) \\ &+ \left(\frac{\partial d}{\partial \theta}\right)\frac{1}{\theta}\Big((\theta_{iL} - \theta_L^2) - (1+s)(\theta_{iV} - \sigma_{LV}) - (s_i - s)((1+s)\theta_V^2 - \theta_{LV})\Big), \end{aligned}$$

or

$$\begin{aligned} s_i &= s + \left(\frac{\partial d}{\partial s} + \frac{1}{\theta}\frac{\partial d}{\partial \theta}\Big((1+s)\theta_V^2 - \theta_{LV}\Big)\right)^{-1} \\ &\quad \left(d_i - d - \frac{1}{\theta}\frac{\partial d}{\partial \theta}\Big((\theta_{iL} - \theta_L^2) - (1+s)(\theta_{iV} - \sigma_{LV})\Big)\right), \end{aligned}$$

with $\dfrac{\partial d}{\partial s} = -\Phi\left\{-z\right\} < 0$ and $\dfrac{\partial d}{\partial \theta} = \phi\left\{z\right\} > 0$.

The two possible policies for the company in the normal cases are:

- uniform surplus that gives the marginal default value for each line:

$$d_i = d + \left(\frac{\partial d}{\partial \theta}\right)\frac{1}{\theta}\Big((\theta_{iL} - \theta_L^2) - (1+s)(\theta_{iV} - \sigma_{LV})\Big),$$

- uniform default value that gives the surplus requirement for each line:

$$\begin{aligned} s_i &= s - \left(\frac{\partial d}{\partial s} + \frac{1}{\theta}\frac{\partial d}{\partial \theta}\Big((1+s)\theta_V^2 - \theta_{LV}\Big)\right)^{-1} \\ &\quad \frac{1}{\theta}\left(\frac{\partial d}{\partial \theta}\right)\Big((\theta_{iL} - \theta_L^2) - (1+s)(\theta_{iV} - \theta_{LV})\Big). \end{aligned}$$

The previous equation (uniform default value) can be simplified to :

$$s_i = s \underbrace{- \left(\frac{\partial d}{\partial s} + \frac{1}{\theta}\frac{\partial d}{\partial \theta}\Big((1+s)\theta_V^2 - \theta_{LV}\Big)\right)^{-1}\left(\frac{\partial d}{\partial \theta}\right)\frac{\theta_L^2}{\theta}}_{Y \geq 0}(\beta_i - 1) = s + Y(\beta_i - 1).$$

with $\beta_i = \dfrac{\theta_{iL}}{\theta_L^2}$ and the assumption that $\theta_{LV} \ll \theta_L^2$.

A.4 Additional results

Elaborate scenario

This table summaries the results of the elaborate scenario (isolated line of business, different present values of liabilities).

Methods		Lognormal Results			Normal Results		
VaR	$p = 0.5\%$ (€)	176			273		
	$p = 1\%$ (€)	20			101		
	$p = 5\%$ (€)	-408			-367		
	Exceedance probability	6.10^{-4}			7.10^{-2}		
	Capital allocation (€)	226	181	35	213	175	57
EPD	ratio	9.10^{-4}			10^{-5}		
	Capital allocation (€)	225	180	45	188	158	105
CAPM	Sharpe ratio				0.50	0.48	0.12
	β				1.07	1.03	0.53
	Expected return R_{L_i}				0.105	0.102	0.067
	Capital allocation (€)				229	181	40
MR	Default/Liability value (d)	0.06%			0.12%		
	Asset/Liability volatility (σ)	17.17 %					
	Standard deviation of surplus (θ)				22.95%		
	Delta $\left(\frac{\partial d}{\partial s}\right)$& Vega $\left(\frac{\partial d}{\partial \sigma}\right)$	-0.0072 & 0.0299			-0.0147 & 0.0371		
	Marginal surplus requirement	54 %	52.5%	20%	52.7%	51.7%	27%
	Capital allocation (€)	231	183	36	229	182	38

Negative surplus scenario

This table summaries the results of the scenario considering negative surplus (base-case correlations, same present values of liabilities, $V = 250$€).

Methods		Lognormal Results			Normal Results		
VaR	$p = 0.5\%$ (€)	2279			1906		
	$p = 1\%$ (€)	2127			1170		
	$p = 5\%$ (€)	1562			1398		
	Exceedance probability	0.65			0.8		
	Capital allocation (€)	87	87	76	87.5	87.5	75
EPD	ratio	0.2			2.10^{-3}		
	Capital allocation (€)	82	82	86	81.5	81.5	87
CAPM	Sharpe ratio				0.50	0.48	0.12
	β				1.07	1.03	0.53
	Expected return R_{L_i}				0.105	0.102	0.067
	Capital allocation (€)				87	87	76
MR	Default/Liability value (d)	18.34%			18.44%		
	Asset/Liability volatility (σ)	19.49 %					
	Standard deviation of surplus (θ)				18.20%		
	Delta $\left(\frac{\partial d}{\partial s}\right)$& Vega $\left(\frac{\partial d}{\partial \sigma}\right)$	-0.7989 & 0.2340			-0.8201 & 026231		
	Marginal surplus requirement	-18 %	-18%	-14%	-18%	-18%	-14%
	Capital allocation (€)	82	82	86	82	82	86

A.5 References

Butsic, R. P., 1994, "Solvency Measurement for Property-Liability Risk-Based Capital Applications", The Journal of Risk and Insurance, 17: 438-477.

Bustic, R. P., 1999, "Capital allocation for property-liability insurers : a catastrophe reinsurance application", Casualty Actuarial Society, Baltimore Maryland.

Cummins, J. D. and Phillips, R. D., 1999, "Applications of Financial Pricing Models In Property-liability Insurance", The Handbook of Insurance Economics, Boston.

Cummins, J. D., 2000, "Assets pricing models and insurance ratemaking", invited paper, The Wharton School of the University of Pennsylvania, Philadelphia, USA.

Dhaene, J., Tsanakas, A., Valdez, E. A. and Vanduffel, S., 2010, "Optimal Capital

Allocation Principles", available at http://ssrn.com/abstract=1332264.

Hult, H., et al., 2012, "Risk and Portfolio Analysis: Principles and Methods", Springer Series in Operations Research and financial Engineering.

Merton, R. C. and Perold, A., 1993, "Management of Risk Capital in Financial Firms", Financial Services: Perspectives and Challenges, Harvard Business School Press, Boston.

Myers, S. C. and Read, J. A. Jr., 2001, "Capital Allocation for Insurance Companies", The Journal of Risk and Insurance, Vol. 68, No. 4, 454-580.

Sherries, M., 2004, "Solvency, Capital Allocation and Fair Rate of Return in Insurance", Faculty of Commerce and Economics, Sydney, Australia.

Zweifel, P. and Eisen, R., 2012, "Insurance Economics", Springer Texts in Business and Economics, DOI 10.1007/978-3-642-20548-4 6.

CNP assurances, 2013, "Investor credit update" http://www.cnp-finances.fr/sites/default/files/pdf/financement/2013/PitchbookCreditMarch2013.pdf