Analysis of Pension Strategies

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Abstract

In a time where people tend to retire earlier and live longer in combination with an augmented personal responsibility of allocating or at least choosing adequately composed pension funds, the importance of a deeper understanding of long term investment strategies is inevitably accentuated. On the background of discrepancies in suggested pension fund strategies by influential fund providers, professional advisers and previous literature, this thesis aims at addressing foremost one particular research question: How should an investor optimally allocate between risky and risk-less assets in a pension fund depending on age? In order to answer the question the sum of Human wealth, defined as the present value of all expected future incomes, and ordinary Financial wealth is maximized by applying a mean-variance and a expected utility approach. The latter, and mathematically more sound method yields a strategy suggesting 100% of available capital to be invested in risky assets until the age of 47 whereafter the portion should be gradually reduced and reach the level of 32% at the last period before retirement. The strategy is clearly favorable to solely holding a risk-free asset and it just outperforms the commonly applied "100 minus age"-strategy.

Keywords: Human wealth, Mean-variance, Stochastic dynamic programming.

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Chapter 1

Introduction

1.1 Background

The conditional life expectancy of a 65-year old person has been steadily increasing during the last century and is currently ca 84 years. Simultaneously, the corresponding average age of retirement has faced a downward trend and amounts to 62 years which has made the non-working period to a significantly larger portion of peoples lives[1] (see Figure 1.1). Furthermore, the opportunities for consumption and an active life-style in a modern society are more numerous and more expensive than ever. These changes in life expectancy and shifts in people's individual choices and behavior inevitably raises the pressure on retirees' financial situation. Private savings and pension funds have to carry bigger withdrawals and also for an extended period of time.

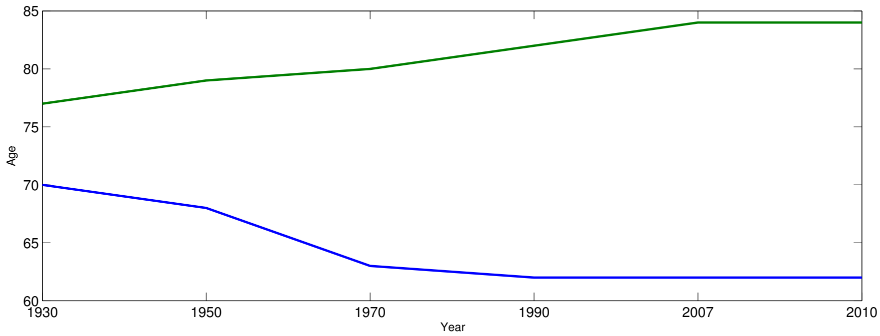

Figure 1.1: *Conditional life expectancy of a 65-year old (top curve) and average retirement age (bottom curve). Source: U.S. Department of Health and Human Services 2012.*

[1]The numbers represent the situation in United States whereas the average age of retirement in Europe and Sweden is 60.3 and 63.1, respectively. The conditional life-expectancy of a 65 year old Swedish person is ca 85 years.

Moreover, another transformation experienced during the last decades is an increased responsibility for individually financing one's pension. Both in terms of building up wealth throughout the working years and how to actively manage one's funds. In the United States, Social Security has been a gradually decreasing fraction of elderly citizens' total funding of retirement in favor for an increasing fraction of personal savings. Furthermore, In Sweden a considerable portion of the public pension is managed individually, each person has the opportunity to choose which funds to invest in. At the same time there is an abundance of different available funds which further elevates the requirements on individual's knowledge and competence to make adequate decisions in order to ensure a prosperous retirement.

1.2 Pension fund companies

A selection of companies offering so called "Target Date Funds", which alternates the shares invested in risky and risk-less assets depending on the investor's remaining time period until retirement, is; Vanguard "Target retirement" funds, Fidelity "Freedom Funds", T. Rowe Price Funds and TIAA-CREF Funds. These firms offer funds which are managed taking into account an investment horizon up to 40 years. By a closer inspection of the portfolio's content throughout the life-cycle from investment to retirement, one can conclude that the shares of risky assets gradually reduces whereas shares of risk-less assets experiences a corresponding increase. For example, the com-

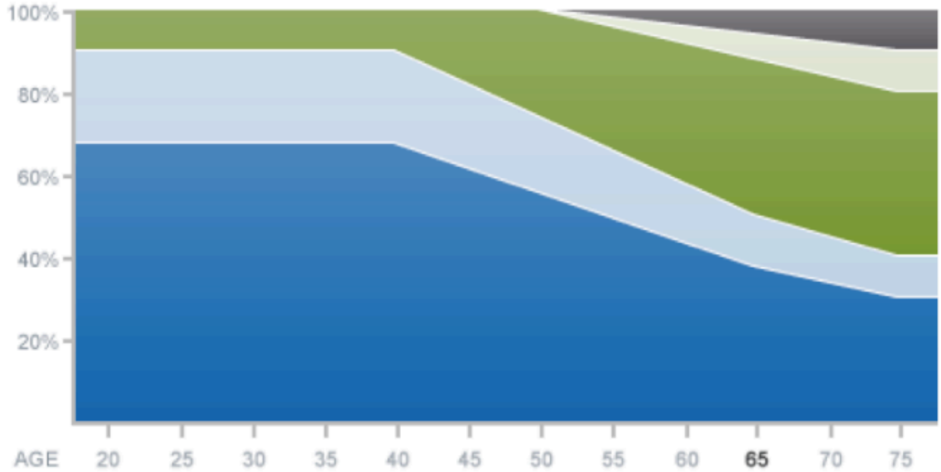

Figure 1.2: *The composition of a TIAA-CREF Target Date Fund of (starting from below) US Equity, International equity, Fixed Income, Short-Term Fixed Income and Inflaiton Protected Assets. Source: www.tiaa-cref.org.*

position of one of TIAA-CREF's Target Date-funds depicted in Figure 1.2 shows that ca 90% of the portfolio is invested in risky assets such as domestic and international equity and 10% in fixed-income until 25 years before retirement. Thereafter the portfolio's share of risky assets decreases linearly until the age of 65 whereas the fractions of secure assets; Fixed-income, Short-term Fixed Income and Inflation Protected Assets increase correspondingly.

Fidelity offers a similar target-date fund named "Freedom Fund" which is also constituted by 90% and 10% risky and risk-free assets, respectively, but only until 20 years before retirement. Subsequently these shares are alternated in favor for an increased fraction of Bond Funds and Short-term Funds as illustrated in Figure 1.3.

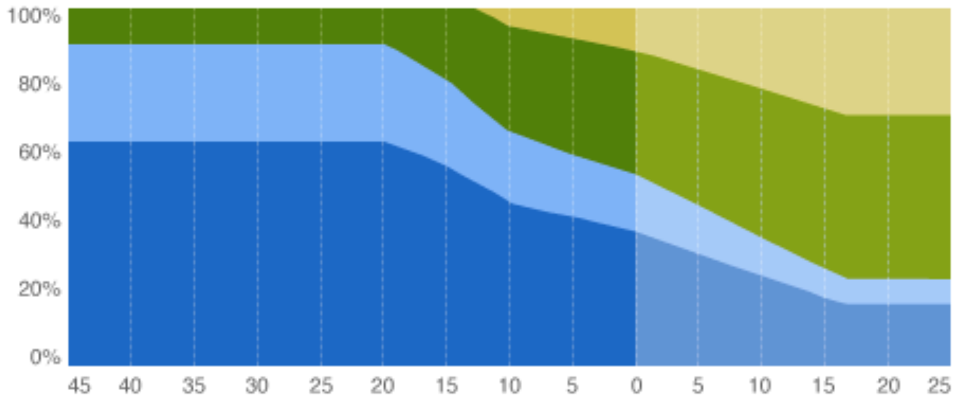

Figure 1.3: *The composition of a Fidelity Freedom Fund 2055 of (from below) Domestic Equity Funds, International Equity Funds, Bond Funds, Short-term Funds. On the x-axis, 0 represents the age of 65 and the number of years remaining until retirement and after retirement are shown to the left and right of 0, respectively. Source: www.fidelity.com*

There is a difference in how two specific Target Date-funds choose to allocate between risky and risk-less assets, respectively throughout the investment period. The transition period in which the shares of equities are exchanged for bonds and short-term funds starts when different amount of time remains until retirement in the two specific cases. Furthermore, whereas the TIAA-CREF's pension fund's transition into more secure assets is linear, the Fidelity fund indicate certainly a monotonically decreasing pattern of risky assets, however not linear. The two examples from Fidelity and TIAA-CREF shows that the allocation in pension funds can be managed differently and there seem to be no general and clear rule of how to optimally divide between risky and risk-less assets throughout a life-cycle.

1.3 Advices of professionals

John Ameriks and Stephen P. Zeldes [2] have studied how people, depending on their age, allocate their pension funds, and what decision rules that seem to be commonly applied. They are referring to one simple and dominating rule of thumb that is particularly popular and often advised. That is, having 100 subtracted by age, percentage points of investments in stocks and the remainder in secure assets. Hence that means starting off at a 75% share in in risky assets and then linearly reduce the portion reaching a level of 35% just before retirement, assuming starting and stop working at the age of 25 and 65, respectively.

A classical and influential work in the field of investment advices is *A random walk down Wall street* by Burton .G Malkiel in 1973. The author suggests that a 25 year old should hold 70% in risky assets and gradually decrease towards 30% by the age of 70 [9].

Chapter 2

Previous research

Several models and theories has been developed in order to deal with the issue of how to optimally allocate a pension fund between risky and risk-free assets. In 1969 Paul A. Samuelson [11], unlike how many portfolio allocation problem had been posed previously, formulated a multiple-period problem corresponding to the life-time consumption and investment decisions facing an investor. Samuelson aimed to maximize the sum of discounted utility of consumption for every period from $t = 0$ to T. The optimization problem was formulated accordingly,

$$\begin{aligned}
&\max_{\{C_t, w_t\}} \mathbf{\Sigma_t}_{t=0}^{T}(1+\rho)^{-t}U[C_t] \\
&\text{subject to} \\
&C_t = W_t - \frac{W_{t+1}}{(1+r)(1-w_t)+w_t Z_t},
\end{aligned} \tag{2.1}$$

where W_t, w_t, C_t and Z_t are the financial wealth, share in risky asset, consumption and a stochastic variable at period t, respectively. By solving the optimization problem through dynamic programming he came to the conclusion that an investor has the same relative aversion towards risk throughout all periods.Therefore should a person optimally invest exactly the same share of risky assets for every period, i.e. maintaining w_t constant (See [11] for more details).

As Samuelson formulated the optimization problem in a discrete timeframe, Robert C. Merton, also in 1969 [10], addressed the same problem in a continuous time setting. He reached the same conclusions suggesting a constant share of available capital should be devoted to asset classes exposed to risk. One of the driving factors behind the results yielded by Samuelson and Merton is the fact that only financial wealth was taken into account. A person's varying ability to recoup, through income from labor, at different phases of life was not considered.

Although, in 1992, Samuelson, Merton and Bodie [3] incorporated, except financial wealth, also labor income into the life-time model. In particular they studied how flexibility to different sources of earnings affected an optimal allocation strategy. They found that for instance younger people with more available job opportunities also could tolerate more risk in their pension savings.

Following in the path set by Samuelson, Merton and Bodie in 1992, Hanna and Chen [5] introduced human capital (or wealth) into their model for portfolio allocation. Defining human wealth as the present value of expected future earnings and combining it with financial capital, Hanna and Chen [5] let maximize the expected utility of their sum. They studied investment periods of 5 to 20 years and concluded that even relatively risk-averse investors should hold a rather aggressive share of risky assets when considering investment horizon of 20 years and more.

Chapter 3

Problem formulation

The increasing fraction of people's lives lived in retirement combined with an augmented responsibility of personally allocating or at least choosing funds with a adequate asset composition, accentuates the importance of deeper knowledge about pension fund investment strategies. Even among the most influential and dominating target-date fund providers there does not seem to be a complete consistency regarding what risk-level an investor should tolerate depending on age. Furthermore some of the most commonly applied strategies advised by professionals, for instance the "100 minus age"-strategy, might to some extent appear ad-hoc and lack a sound mathematical motivation.

Previous researchers have applied several approaches how to optimally allocate between risky and risk-free assets resulting in strategies suggesting everything from holding a constant share in risky assets throughout life to more realistic strategies advocating taking higher risks when younger and then gradually reduce the portion in favor for more secure assets when approaching retirement.

3.1 Research questions

In the light of the discrepancies in suggested strategies by professional advisers and leading pension fund firms, the variety of approaches employed in previous literature yielding varying optimal portfolio allocation decisions and most of all the increasing importance of a deeper understanding of pension fund strategies, the intention with this thesis is to elaborately analyze the decision of how to optimally allocate between risky and risk-free assets at different ages.

The all-embracing goal is to extend upon the literature and the analysis of asset allocation in pension funds. The aim is to analyze what factors drive the choice of division between risky and risk-less assets, in general and find a mathematical derivation of these decisions in particular. More concretely, the aim is to derive a life-cycle model that explains how people optimally should invest in asset-classes of different riskiness depending on their age and individual characteristics such as risk-aversion and source of labor income. The particular research questions to be addressed are:

1) How should an investor optimally allocate between risky and risk-less assets in a pension fund depending on age?

2) How does such a strategy perform compared to the "100 minus age"-strategy and solely holding secure assets?

3) How is an investor's portfolio value at the time of retirement affected by extreme events in the stock market?

Chapter 4

Method

4.1 Introducing human wealth

Managing a pension fund, in the end, comes down to optimally allocate an investor's initial capital in order to maximize its total return by the day of retirement while maintaining a adequate risk-level in accordance with the client's personal preferences. By solely having the financial wealth in focus, Samuelson and Merton [11], as mentioned above, found that adhering to a constant share in risky and risk-free assets throughout the whole investment period would be the best decision. However, from that point of view an important aspect is overlooked. At each period of portfolio reallocation, the present amount of capital is considered and possibly reinvested to match the desired shares of different asset classes. Although the current financial wealth is regarded, the investor's future financial capital, yet to be earned, is not taken into account. Intuitively, the magnitude of expected future earnings from labor income is likely to affect what risk-level, the investor could tolerate. In literature this part is (see for example Hanna and Shen, 1997 [5] and Hanna and Lee, 1995 [8]) often referred to as human wealth and its discounted present value can be mathematically defined as:

$$W_h = \sum_{i=x}^{y} E(L_i)e^{-ir_f} \tag{4.1}$$

where x is the year one starts working, y is the retirement age, L_i is the labor income for year i and r_f is the risk-free rate. Having a large amount of human wealth, i.e., expecting future incomes to be high, gives the investor flexibility to take higher risks since potential shortfalls in the stock market could be recovered more easily. At the same time a lower amount of human capital implies constraints on the tolerable risk-level due to limited amount of time to recoup. Therefore, human wealth is, in excess of financial wealth (later referred to as W_f), most definitely an important object for consideration when making a portfolio choice decision.

In one sense human wealth could be seen as a secure asset, e.g., a bond, that is expected to yield a relatively stable and predictable "return" (i.e. income). Such a standpoint can bring understanding to how a younger investor, whose accumulated total wealth (both financial and human capital) to a large extent already is constituted by "secure" human wealth, can bear more risky assets. Simultaneously, a person closer to retirement has throughout his working life transformed her human wealth into financial wealth and thus gradually reduced the secure fraction of the total wealth situation and henceforth has to compensate by investing more heavily in risk-less asset classes.

4.2 Mean-variance approach

Both financial and human wealth seem to be of significant importance for the portfolio allocation of a pension fund and the two parts will therefore be taken into account and considered as objects for optimization throughout a life-cycle. Although striving for a high degree of wealth, is undoubtedly one major aim, an investor's preference will also be assumed to include an aversion for risk. Hence an individual's allocation problem can be considered as a trade-off between expected return of both financial and human wealth and their associated risk - informally:

$$E[W_{f,i+1} + W_{h,i+1}] - const. * var(W_{f,i+1} + W_{h,i+1}) \tag{4.2}$$

A formal definition follows when appropriate definitions have been introduced. The above trade-off is apparently the foundation of a mean-variance approach that will initially be applied for addressing the portfolio choice problem.

4.2.1 Definitions

The model based on a mean-variance approach will include two investment opportunities, representing a risk-free and a risky asset. The first mentioned corresponds to a risk-less bond with a determined return given by,

$$R_B = e^{r_f}. \tag{4.3}$$

The price of the one risky asset available is assumed to follow a Geometric Brownian motion which is mathematically formulated as:

$$S_{i+1} = S_i e^{\mu_s - \frac{1}{2}\sigma_s^2 + \sigma_s Z_s} \tag{4.4}$$

where $Z_s \in N(0, 1)$ and μ_s and σ_s are drift and volatility parameters for the stock price, respectively.

Given the stochastic process for the risky asset, its return will be defined as stock price tomorrow divided by the stock price today,

$$R_{S,i+1} \equiv \frac{S_{i+1}}{S_i} = e^{\mu_s - \frac{1}{2}\sigma_s^2 + \sigma_s Z_s} \tag{4.5}$$

The yearly income from labor will be assumed to trend upwards over time but still be subject to random shocks affecting the magnitude of L_i, from year to year. Hence the stochastic process for L_i will also be a Geometric Brownian motion, although the corresponding parameters μ_L and σ_L will typically be lower compared to the case of the risky asset.

$$L_{i+1} = L_i e^{\mu_L - \frac{1}{2}\sigma_L^2 + \sigma_L Z_L}, \tag{4.6}$$

where $Z_L \in N(0,1)$. Thus, with the formulated stochastic process, the potential volatility of an individual's labor income is taken into account.
Income from employment will be one of the three sources, together with return from the risky and risk-less asset, contributing to building up an investor's financial wealth. Since its magnitude is stochastic, a return from labor income will be defined analogously to the stock return.

$$R_{L,i+1} \equiv \frac{L_{i+1}}{L_i} = e^{\mu_L - \frac{1}{2}\sigma_L^2 + \sigma_L Z_L} \tag{4.7}$$

In practice one can consider the labor income for next year as the return on "investing" the whole salary from last year, L_i, and being given $L_i R_{L,i+1}$ by the end of the following year.

Depending on the investor's kind of occupation there is a possibility for correlation between income from employment and the stock market. For example, consider a person employed in a large publicly owned corporation whose economic performance tend to co-vary with the stock market. In times of bad performance, the probability of loosing one's job increases simultaneously as the risky asset is more likely to yield a less positive or even negative return. That would put the investor in an unfavorable position whereas another person, for instance working for the public sector might face the same negative returns from the stock market although the risk of loosing employment is in essence unaffected. In order to take such a correlation into consideration in a portfolio allocation problem, the relationship between the two random variables Z_s and Z_L will be defined as,

$$Z_L = \rho Z_s + \sqrt{1-\rho^2} Z \tag{4.8}$$

where $\rho \in [-1, 1]$, $Z \in N(\mu, \sigma^2)$ and independent to Z_s and Z_L.

By applying (4.6), the income L_{x+t} at t years after the age of starting working, x, can be computed according to the following expression:

$$L_{x+t} = L_x[\prod_{n=1}^{t} e^{\mu_L - \frac{1}{2}\sigma_L^2 + \sigma_L Z_{L,n}}] \tag{4.9}$$

Furthermore, given the labor income at any period $x+t$, the human wealth consisting of the sum of all future discounted labor incomes can be computed by applying the following expression.

$$W_{h,x+t} = \sum_{k=t+1}^{y-x} [L_{x+k} e^{-(k-t)r_f}], \tag{4.10}$$

where y is the retirement age.

As mentioned above, the magnitude of the human wealth will gradually decrease as an individual approaches retirement since it is steadily exchanged for financial wealth. Furthermore (4.9) and (4.10) indicate that its value from one period to another evolves stochastically due to the stochastic development of labor income. Therefore it makes sense to define a return of human wealth. In fact human wealth, as part of an investor's total wealth, will be considered as one of the investment "opportunities", although the share in that asset class will be subject to a constraint presented later. The derivation of the return of the human wealth is however not as straightforward as for the risky asset or labor income.

Recalling (4.10), the human wealth at period $x+t$ and $x+t+1$ can be written according to,

$$\begin{aligned} W_{h,x+t} &= \sum_{k=t+1}^{y-x} [L_{x+k} e^{-(k-t)r_f}] \\ &= L_{x+t+1} e^{-1r_f} + L_{x+t+2} e^{-2r_f} + ... + L_y e^{-(y-x-t)r_f}. \end{aligned}$$

and

$$\begin{aligned} W_{h,x+t+1} &= \sum_{k=t+2}^{y-x} [L_{x+k} e^{-(k-t)r_f}] \\ &= L_{x+t+2} e^{-1r_f} + L_{x+t+3} e^{-2r_f} + ... + L_y e^{-(y-x-t-1)r_f}. \end{aligned}$$

Thus $W_{h,x+t+1}$ can be expressed in terms of $W_{h,x+t}$ and L_{x+t+1} by the relation:

$$W_{h,x+t+1} = e^{r_f} W_{h,x+t} - L_{x+t+1} = e^{r_f} W_{h,x+t} - L_{x+t} e^{\mu_L - \frac{1}{2}\sigma_L^2 + \sigma_L Z_{L,x+t+1}} \tag{4.11}$$

Henceforth, the return of human wealth from period $x+t$ to $x+t+1$ can be defined as (exchanging $x+t$ for i):

$$R_{W_h,i+1} \equiv \frac{W_{h,i+1}}{W_{h,i}} = e^{r_f} - \frac{L_i}{W_{h,i}} e^{\mu_L - \frac{1}{2}\sigma_L^2 + \sigma_L Z_{L,i+1}} \tag{4.12}$$

The interesting feature of the return of human wealth is that it evolves over time. An investor's labor income gradually increases throughout a life-cycle, whereas the human wealth simultaneously decreases which leads to the second term in (4.12) goes from almost negligible to most considerable.

4.2.2 Optimization problem

The trade-off problem sketched in (4.2), with the definitions made above, can now be formulated more formally. However, firstly the constitution of a person's total wealth and its transformation from one period to the next one, is studied in more detail since it will compose the constraints associated with the optimization problem. To begin with, the total wealth of a person is simply the sum of financial and human wealth, denoted:

$$V_i = W_{f,i} + W_{h,i} \tag{4.13}$$

The total wealth in the following period will be a function of the total wealth from the previous period, shares invested in available asset classes and their corresponding returns.

$$V_{i+1} = V_i(\alpha_{h,i} R_{W_h,i+1} + \alpha_{L,i} R_{L,i+1} + \alpha_{s,i} R_{s,i+1} + (1 - \alpha_{h,i} - \alpha_{L,i} - \alpha_{s,i}) R_B) \tag{4.14}$$

Where $\alpha_{h,i}$,$\alpha_{L,i}$,$\alpha_{s,i}\in [0,1]$ and corresponds to the shares of the total wealth V_i, invested in human wealth, labor and risky asset, respectively. Furthermore, the remainder $(1 - \alpha_{h,i} - \alpha_{L,i} - \alpha_{s,i})V_i$, is invested in the risk-free asset. Thus in this setting, the portfolio allocation problem does not only consider the two obvious asset classes, risky and risk-less asset but also "labor" and "human wealth". The share of these are however not open for the investor's own choice but must rather be exactly what is available. That is $\alpha_{L,i} V_i$ must equal L_i which in practice is equivalent to assuming the investor to continuing working during the following period and be given the income $L_i R_{L,i+1}$ which is added to the individual's financial wealth and thereby to the total wealth of next the period, V_{i+1}. This becomes the first constraint associated with the optimization problem,

$$\alpha_{L,i} V_i = w_{L,i} = L_i \tag{4.15}$$

where $w_{L,i}$ is introduced to ease the notation of the optimization problem stated below. Furthermore, in analogy with the reasoning regarding the first constraint, the second restriction declares that all available human wealth should be invested in the "human wealth asset" which will yield the return $R_{W_h,i+1}$. This is due to the fact that in reality one cannot choose how much to invest in human wealth. For example human wealth cannot instantly be realized into financial capital and invested in stocks for instance. There is no other choice than investing all of total human wealth into the human wealth "asset" and hope for a decent return. The constraint reads,

$$\alpha_{h,i} V_i = w_{h,i} = W_{h,i}. \tag{4.16}$$

In accordance with the introduced notation, the shares of total wealth invested in the risky and the risk-free asset are formulated as $\alpha_{s,i} V_i = w_{s,i}$ and $(1 - \alpha_{h,i} - \alpha_{L,i} - \alpha_{s,i}) V_i = w_{B,i}$, respectively.

The mean-variance approach aims to maximize the expected return while simultaneously minimizing the associated variance through choosing $w_{B,i}$, $w_{h,i}$, $w_{L,i}$ and $w_{s,i}$ adequately while satisfying all constraints.

$$\max_{w_{B,i}, w_{h,i}, w_{L,i}, w_{s,i}} \quad w_{B,i} R_B + w_{h,i} E[R_{W_h,i+1}] + w_{L,i} E[R_{L_h,i+1}] + w_{s,i} E[R_{s,i+1}] - \frac{\gamma}{V_i} \mathbf{w_i}^{\mathbf{T}} \mathbf{\Sigma_i} \mathbf{w_i}$$

subject to

$$w_{B,i} + w_{h,i} + w_{L,i} + w_{s,i} \leq V_i \tag{4.17}$$

$$w_{L,i} = L_i \tag{4.18}$$

$$w_{h,i} = W_{h,i} \tag{4.19}$$

Where γ is a coefficient capturing the investor's risk-aversion, $\mathbf{w_i} = [w_{h,i} \ w_{L,i} \ w_{S,i}]^T$ and Σ is the covariance matrix summarizing the variance and dependence between the risky assets,

$$\mathbf{\Sigma_i} = \begin{pmatrix} var(R_{W_h,i}) & cov(R_{W_h,i}, R_{L,i}) & cov(R_{W_h,i}, R_{S,i}) \\ cov(R_{W_h,i}, R_{L,i}) & var(R_{L,i}) & cov(R_{S,i}, R_{L,i}) \\ cov(R_{W_h,i}, R_{S,i}) & cov(R_{S,i}, R_{L,i}) & var(R_{S,i}) \end{pmatrix}$$

However, since the development of the return of human wealth will be dominated by the development of the ratio $\frac{L_i}{W_{h,i}}$ as indicated by (4.12), its covariation with the return of labor income and the risky asset is negligible and

$cov(R_{W_h,i}, R_{L,i})$ and $cov(R_{W_h,i}, R_{S,i})$ will therefore be set to zero. Furthermore, since the development of human wealth will be based upon simulations yielding the path of human wealth for one representative investor, the return will in practice be deterministic and hence also $var(R_{W_h,i})$ is set to zero. Hence the variance and covariance between return for risky asset and labor income will together with given level of labor income and human wealth (affecting constraints) drive the allocation into different assets.

4.2.3 Solving the mean-variance approach

In order to solve the problem formulated in the previous subsection, the expected returns, variances and covariances need to be derived. Regarding the expected returns, these are analytically derived for the risk-free bond, risky asset, labor income while the expected return of human wealth is computed with help from simulation.

Expected return of the risk-fee bond

$$E[R_B] = E[e^{r_f}] = e^{r_f} \tag{4.20}$$

Expected return of the risky asset (stock)

$$E[R_s] = E[e^{\mu_s - \frac{1}{2}\sigma_s^2 + \sigma_s Z_s}] = e^{\mu_s} \tag{4.21}$$

Expected return of labor income

$$E[R_L] = E[e^{\mu_L - \frac{1}{2}\sigma_L^2 + \sigma_L Z_L}] = e^{\mu_L} \tag{4.22}$$

Expected return of human wealth

The expected return of human wealth is also calculated based on (4.12). However, in order to achieve a solution and the development of the expected return for a representative investor, the numerical values of L_i and $W_{h,i}$ are computed via simulation. Using (4.6) iteratively for 10000 possible labor income paths (see Figure 4.1), the corresponding developments of human wealth are computed through (4.10) (see Figure 4.2). The mean value of these outcomes (see Figure 4.3) will serve as the path of human wealth for on representative individual (see Figure 4.4). Hence, the values of L_i and $W_{h,i}$ are given for each period i, and can be used when computing the expected return in (4.12).

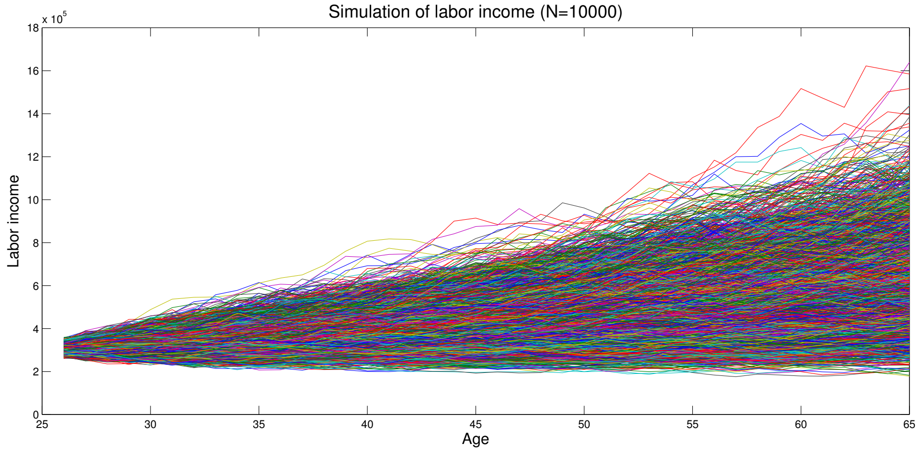

Figure 4.1: *Simulation of labor income based on (4.9). Number of simulations: 10000*

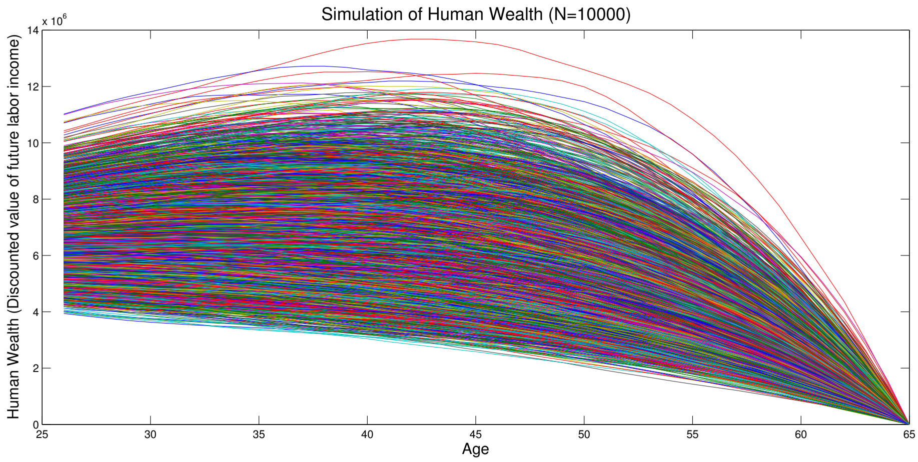

Figure 4.2: *Simulation of Human Wealth (i.e. discounted future labor incomes) based on (4.10) and the simulation outcome presented in Figure 4.1. Number of simulations: 10000*

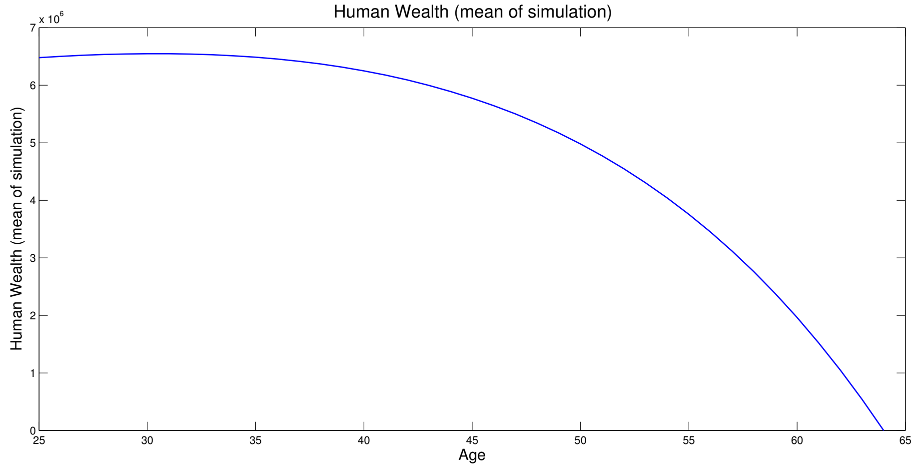

Figure 4.3: *The Human Wealth of one representative investor. Computed based upon the simulation results presented in Figure 4.2.*

Figure 4.4: *As Human Wealth decreases over a life-cycle as seen in Figure 4.3, the value of its expected return is also anticipated to diminish. The graph is computed by applying equation 4.12*

The expected return for human wealth is computed through (4.12) conditioned on the terms L_i and $W_{h,i}$ which are computed via simulation according to the procedure described above. The numerical values of L_i and $W_{h,i}$ will also serve as the basis for the constraints declared in (4.18) and (4.19).

Variance of risky asset return

$$var(R_s) = var(e^{\mu_s - \frac{1}{2}\sigma_s + \sigma_s Z_s}) = (e^{\sigma_s^2} - 1)e^{2\mu_s} \tag{4.23}$$

Variance of labor income return

$$var(R_L) = var(e^{\mu_L + \sigma_L Z_L}) = (e^{\sigma_L^2} - 1)e^{2\mu_L} \tag{4.24}$$

Covariance of risky asset and labor income returns

$$cov(R_s, i, R_{L,i}) = E[R_s R_{L,i}] - E[R_{s,i}]E[R_{L,i}] \tag{4.25}$$

With the derived expected returns, variances and covariances, the stated optimization problem associated with the mean-variance approach can be solved. Although the expected returns for human wealth are achieved via simulation, the quadratic feature of the optimization problem allows it to be solved analytically. The result-section presents the solution of the mean-variance problem, that is the optimum share invested in risky assets, for every period $i = 0, 1, 2...39$ in Figures 5.1 and 5.2.

4.3 Expected utility approach

Although the method based upon a mean-variance approach seems to give reasonable weights allocated to the risky asset in a pension fund depending on age, the method potentially lacks soundness in two aspects. First of all, even though it makes sense intuitively to maximize each individual period's difference between expected return and risk, it might not necessarily lead to the optimal value of the objective function in the end period. After all the investor's preferences does not consider neither the value of the pension fund nor its composition into different asset classes for any of the periods except for the last one, when the money are supposed to be withdrawn and finance retirement. Hence, there might be a theoretical possibility that choosing different allocations throughout the life-cycle, thus not necessarily maximizing the current period's portfolio value, would lead to a better value of the portfolio in the end. At least it is a issue that needs to be investigated. Secondly, optimizing particularly a mean-variance objective function over multiple periods, in general leads to non-convex formulations and might unable achieving a proper solution [1, 12].

In order to circumvent the potential difficulties arising when using a mean-variance approach a more sophisticated method will be applied. Instead of maximizing the difference between expected return and risk, the expected utility of an investor's total wealth in the last period will be maximized. Using a concave utility function convexity can be guaranteed (through minimizing the negative original objective function). Additionally, the concavity property, captures an investor's aversion for risk which is explicitly formulated in the objective function in the mean-variance method. Furthermore, a derivation based on stochastic dynamic programming deals with the first issue mentioned above and clarifies what optimization problem to be solved in each period. In particular a power utility function of the form,

$$u(x) = \frac{1}{\gamma} x^{\gamma}, \tag{4.26}$$

will be used in which $\gamma \in (0,1)$, represents the risk aversion coefficient. Since the investor's total wealth has to be taken into account, x constitute the sum of $W_{f,i}$ and $W_{h,i}$. Consequently, the experienced utility of an individual at a given period i is given by the expression,

$$\frac{1}{\gamma}(W_{f,i} + W_{h,i})^{\gamma}. \tag{4.27}$$

A person saving in a pension fund, intended to finance his retirement when the flow of labor income stops, ultimately wants to maximize the utility at the last period, say T. Due to the stochastic characteristic of the portfolio value at a future point in time, the expected value of the total wealth situation is to be maximized. Hence the objective function of an investor's optimization problem can be formulated as,

$$\max \frac{1}{\gamma} E[(W_{f,T} + W_{h,T})^{\gamma}]. \tag{4.28}$$

Constraints associated with the maximization problem are introduced in the following sections.

4.3.1 Total wealth

The objective for the investor is to maximize the utility of the total wealth, the sum of $W_{f,T}$ and $W_{h,T}$, at the last period T. Continuing using the notation V_i, representing the total wealth at any period i will ease the description of the stated problem in the proceding section. As an investor's life-cycle progresses, human wealth will sequentially be exchanged for financial capital. Since human wealth constitute the discounted value of all future labor incomes, going from one period to the next implies summing one less yearly income. Simultaneously this earned income is transformed from human to financial capital, enabled to be either consumed or invested. This will indeed affect the value of the investor's financial capital $W_{f,i}$ although the development of financial wealth is mostly dependent on returns of the risk and risk-free assets. The evolution of an individual's human, financial, and thereby, total wealth V_i for periods $i = 0, 1, 2, ..., T$ is described below.

$$W_{f,0} + W_{h,0} = V_0$$

$$
\begin{aligned}
W_{f,1} + W_{h,1} =& V_1 \\
=& V_0(\alpha_{h,0}R_{h,1} + \alpha_{L,0}R_{L,1} + \alpha_{s,0}R_{s,1} + (1 - \alpha_{h,0} - \alpha_{L,0} - \alpha_{s,0})R_B)
\end{aligned}
$$

$$
\begin{aligned}
W_{f,2} + W_{h,2} =& V_2 \\
=& V_1(\alpha_{h,1}R_{h,2} + \alpha_{L,1}R_{L,2} + \alpha_{s,1}R_{s,2} + (1 - \alpha_{h,1} - \alpha_{L,1} - \alpha_{s,1})R_B)
\end{aligned}
$$

$$
\begin{gathered}
. \\
. \\
.
\end{gathered}
$$

$$
\begin{aligned}
W_{f,T} + W_{h,T} =& V_T \\
=& V_{T-1}(\alpha_{h,T-1}R_{h,T} + \alpha_{L,T-1}R_{L,T} + \alpha_{s,T-1}R_{s,T} + (1 - \alpha_{h,T-1} \\
& - \alpha_{L,T-1} - \alpha_{s,T-1})R_B)
\end{aligned}
$$

Hence, one notices that the total wealth for one period is a function of the total wealth from the previous period, chosen shares in available assets and their corresponding returns. Thus, in one sense, the investment problem can be considered to consist of four different assets (human wealth, labor, risky and risk-free asset) with corresponding shares of total wealth, $\alpha_{h,i}$,$\alpha_{L,i}$, $\alpha_{s,i}$ and α_B (where $\alpha_B = 1-\alpha_{h,i}-\alpha_{L,i}, -\alpha_{s,i}$) although only two of them are open to the investor's own choice, namely $\alpha_{s,i}$ and α_B. This is a consequence of not being able to choose how much to invest in human wealth or labor. Instead these shares, $\alpha_{h,i}$ and $\alpha_{L,i}$ are predetermined in each period i according to,

$$
\alpha_{h,i} = \frac{W_{h,i}}{W_{h,i} + W_{f,i}} \tag{4.29}
$$

and

$$
\alpha_{L,i} = \frac{L_i}{W_{h,i} + W_{f,i}}. \tag{4.30}
$$

According to (4.29), all available human wealth should be invested in the "human wealth asset" which will yield the return $R_{W_h,i+1}$. This constraint is a key feature of the model in letting human wealth be considered as a risky asset yet something the investor is constrained to maintain holding as an investment. Likewise, (4.30) states that the labor income for period i should be invested in the "labor asset". In reality that is equivalent to assuming the investor to keep on working during the following period and be given the income $L_iR_{L,i+1}$ which is added to the individual's financial wealth.

4.3.2 Stochastic dynamic programming

With the introduced notation, the maximization problem stated in (4.28) can be rewritten as,

$$\max_{\alpha_{s,0},\alpha_{s,1},\ldots,\alpha_{s,T-1}} E[u(V_T)] = E[\frac{1}{\gamma}V_T^{\gamma}]. \tag{4.31}$$

Note that the only variable the investor needs to consider is α_s due to the fact that both $\alpha_{h,i}$ and $\alpha_{L,i}$ are constrained according to (4.29) and (4.30), respectively, and the share of available total wealth invested in the risk-free asset is simply given by $(1-\alpha_{h,i}-\alpha_{L,i}-\alpha_{s,i})$. In the optimization problem stated above it is also specified that every such $\alpha_{s,i}$ for every period $i = 0, 1, 2, .., 39$ will affect the total wealth and thereby the utility in the last period, T. In particular, the maximization problem in time period $T-1$ is studied in more detail.

At this stage every outcome from previous periods is known to the investor and only the decision regarding $\alpha_{s,T-1}$ remains. The optimization problem at this point in time is denoted as $J_{T-1}(V_{T-1})$.

$$\begin{aligned} J_{T-1}(V_{T-1}) &= \max_{\alpha_{s,T-1}} E_{T-1}[\frac{1}{\gamma}V_T^{\gamma}] \\ &= \max_{\alpha_{s,T-1}} E_{T-1}[\frac{1}{\gamma}(V_{T-1}(\alpha_{h,T-1}R_{h,T}+\alpha_{L,T-1}R_{L,T} \\ &+\alpha_{s,T-1}R_{s,T}+(1-\alpha_{h,T-1}-\alpha_{L,T-1}-\alpha_{s,T-1})R_B))^{\gamma}] \\ &=\frac{1}{\gamma}V_{T-1}^{\gamma}\max_{\alpha_{s,T-1}} E[(\alpha_{h,T-1}R_{h,T}+\alpha_{L,T-1}R_{L,T}+\alpha_{s,T-1}R_{s,T} \\ &+(1-\alpha_{h,T-1}-\alpha_{L,T-1}-\alpha_{s,T-1})R_B)^{\gamma}] \\ &=\frac{1}{\gamma}V_{T-1}^{\gamma}E[(\alpha^*_{h,T-1}R_{h,T}+\alpha^*_{L,T-1}R_{L,T}\alpha^*_{s,T-1}R_{s,T} \\ &+(1-\alpha^*_{h,T-1}-\alpha^*_{L,T-1}-\alpha^*_{s,T-1})R_B)^{\gamma})] \end{aligned} \tag{4.32}$$

Where the last equality comes from replacing "max" by instead using "$*$" on every α, indicating the optimal solution solving the maximization problem. In order to ease notation the following definition is made,

$$\begin{aligned} Q_{T-1} \equiv E[(\alpha^*_{h,T-1}R_{h,T}+\alpha^*_{L,T-1}R_{L,T}+\alpha^*_{s,T-1}R_{s,T}+(1-\alpha^*_{h,T-1}-\alpha^*_{L,T-1} \\ -\alpha^*_{s,T-1})R_B)^{\gamma})] \end{aligned} \tag{4.33}$$

and thus (4.32) can be written as,

$$J_{T-1}(V_{T-1}) = \frac{1}{\gamma}V_{T-1}^{\gamma}Q_{T-1}. \tag{4.34}$$

In a similar manner, at period $T-2$, every outcome prior and up to $T-2$ is known and the optimization problem can be formulated as stated below.

$$
\begin{aligned}
J_{T-2}(V_{T-2}) = &\max_{\alpha_{s,T-2}} E_{T-2}[\frac{1}{\gamma}V_{T-1}^{\gamma}Q_{T-1}] \\
&= \max_{\alpha_{s,T-2}} E_{T-2}[\frac{1}{\gamma}(V_{T-2}(\alpha_{h,T-2}R_{h,T-1} + \alpha_{L,T-2}R_{L,T-1} \\
&+\alpha_{s,T-2}R_{s,T-1} + (1-\alpha_{h,T-2}-\alpha_{L,T-2}-\alpha_{s,T-2})R_B))^{\gamma}Q_{T-1}] \\
&=\frac{1}{\gamma}V_{T-2}^{\gamma}Q_{T-1}\max_{\alpha_{s,T-2}} E[(\alpha_{h,T-2}R_{h,T-1} + \alpha_{L,T-2}R_{L,T-1} \\
&+\alpha_{s,T-2}R_{s,T-1}(1-\alpha_{h,T-2}-\alpha_{L,T-2}-\alpha_{s,T-2})R_B)^{\gamma}] \\
&=\frac{1}{\gamma}V_{T-2}^{\gamma}Q_{T-1}E[(\alpha_{h,T-2}^{*}R_{h,T-1} + \alpha_{L,T-2}^{*}R_{L,T-1} \\
&+\alpha_{s,T-2}^{*}R_{s,T-1} + (1-\alpha_{h,T-2}^{*}-\alpha_{L,T-2}^{*}-\alpha_{s,T-2}^{*})R_B)^{\gamma}]
\end{aligned} \tag{4.35}
$$

Analogously to (4.33), the following definition is made,

$$
\begin{aligned}
Q_{T-2} \equiv E[(\alpha_{h,T-2}^{*}R_{h,T-1} + \alpha_{L,T-2}^{*}R_{L,T-1} + \alpha_{s,T-2}^{*}R_{s,T-1} + (1-\alpha_{h,T-2}^{*} \\
-\alpha_{L,T-2}^{*}-\alpha_{s,T-2}^{*})R_B)^{\gamma}]
\end{aligned} \tag{4.36}
$$

Thus (4.35) can be written as,

$$
J_{T-2}(V_{T-1}) = \frac{1}{\gamma}V_{T-2}^{\gamma}Q_{T-1}Q_{T-2}. \tag{4.37}
$$

By recursively repeating the above stated procedure one finally arrives at the following expression,

$$
J_0(V_0) = \max_{\alpha_{s,0},\alpha_{s,1},\ldots,\alpha_{s,T-1}} E[u(V_T)] = \frac{1}{\gamma}V_0^{\gamma}\prod_{i=1}^{T-1}Q_i \tag{4.38}
$$

where Q_i equals,

$$
Q_i = E[(\alpha_{h,i}^{*}R_{h,i+1} + \alpha_{L,i}^{*}R_{L,i+1} + \alpha_{s,i}^{*}R_{s,i+1} + (1-\alpha_{h,i}^{*}-\alpha_{L,i}^{*}-\alpha_{s,i}^{*})R_B)^{\gamma}], \tag{4.39}
$$

or equivalently

$$
Q_i = \max_{\alpha_{s,i}} E[(\alpha_{h,i}R_{h,i+1} + \alpha_{L,i}R_{L,i+1} + \alpha_{s,i}R_{s,i+1} + (1-\alpha_{h,i}-\alpha_{L,i}-\alpha_{s,i})R_B)^{\gamma}] \tag{4.40}
$$

subject to

$$
\alpha_{h,i} = \frac{W_{h,i}}{W_{h,i}+W_{f,i}} \tag{4.41}
$$

$$\alpha_{L,i} = \frac{L_i}{W_{h,i} + W_{f,i}}. \tag{4.42}$$

Where the constraints (4.41) and (4.42) are recalled from Section 4.3.1. Since the underlying stochastic process defined by equations (4.4) and (4.6) is independent between periods a maximization of $E[u(V_T)]$ is achieved by optimally choosing $\alpha_{s,i}$ associated with Q_i for each period $i = 0, 1, ..., T-1$, individually.

4.3.3 Optimization problem

Hence from the derivation of the stochastic programming problem in Section 4.3, it's realized that the allocation problem comes down to solving an optimization problem for each period.

$$\max_{\alpha_{s,i}} E[(\alpha_{h,i}R_{h,i+1} + \alpha_{L,i}R_{L,i+1} + \alpha_{s,i}R_{s,i+1} + (1 - \alpha_{h,i} - \alpha_{L,i} - \alpha_{s,i})R_B)^\gamma]$$

subject to

$$\alpha_{h,i} = \frac{W_{h,i}}{W_{h,i} + W_{f,i}} \tag{4.43}$$

$$\alpha_{L,i} = \frac{L_i}{W_{h,i} + W_{f,i}} \tag{4.44}$$

The terms included in the constraints (4.43) and (4.44) are apparently known in the beginning of each period. However, like the procedure used in mean-variance approach, the numerical values for both L_i and $W_{h,i}$ will be compiled through simulation. Hence the situation of a representative investor will be determining these constraints.

4.3.4 Solving the expected utility approach

Due to the lack of a closed-form solution, the optimization problem stated in Section 4.3.3 is solved via simulation for all periods, $i = 0, 1, ..., T-1$. That is, given the values of L_i, $W_{h,i}$ and $W_{f,i}$, known in the beginning of each period, and 10000 outcomes of the stochastic variables, every α_s ranging from 0 to 1 is tested in order to find what choice yields the maximum utility.

Similarly to the method applied for the mean-variance approach, both L_i and $W_{h,i}$ are based on simulations as described in Section 4.2.3 and particularly in Figures 4.1 and 4.2. Hence the constraints stated in (4.43) and (4.44) equals the restrictions facing an representative investor.

The optimum share invested in the risky asset are computed following the described procedure above and results are presented in three different cases - two different risk-aversion coefficients and for one when correlation between risky asset and labor income is present. In particular Figures 5.3, 5.4 and 5.5 in Section 5.2 depicts the optimal strategies according to the expected utility strategy.

Chapter 5

Results

5.1 Mean-variance approach

Following the method described in Section 4.2 the optimal allocation strategy for a pension fund depending on the investor's age is computed. Using the mean-variance approach yields results for two different cases presented in figures 5.1 and 5.2. The outcomes are based on employing the parameters declared in the following table.

Parameters	r_f	μ_s	σ_s	$E[R_s]$	$\sigma(R_s)$	μ_L	σ_L	γ	ρ
Case 1	0.03	0.072	0.27	1.075	0.30	0.02	0.05	5.5	0
Case 2	0.03	0.072	0.27	1.075	0.30	0.02	0.05	5.5	0.5

Henceforth the risk-free rate is assumed to amount to 3% yearly, whereas having the $\mu_s = 0.072$ results in an expected annual return of 7.5% from the risky asset associated with a 30% standard deviation. Furthermore the model assumes that salaries appreciates with 2% per year which in practice translates to having incomes adjusted for inflation. The risk-aversion coefficient is assumed to be 5.5 and finally one notices that the correlation between the stochastic variables determining labor income and risky return, Z_L and Z_s according to (4.8) is set to zero and 0.5 in the two cases, respectively. That is covering two different scenarios in which a person works in a industry whose performance either is independent or covarying with the stock market.

5.1.1 Case 1

The graph depicted in Figure 5.1 shows what share, $\alpha_s \in [0, 1]$, of the financial capital to be held in the risky asset depending on the investor's age. Hence, until the age of 43 a person should hold 100% of available financial capital in the risky asset and thereafter gradually reduce the share in exchange for an appreciating portion in the risk-free asset. At the age

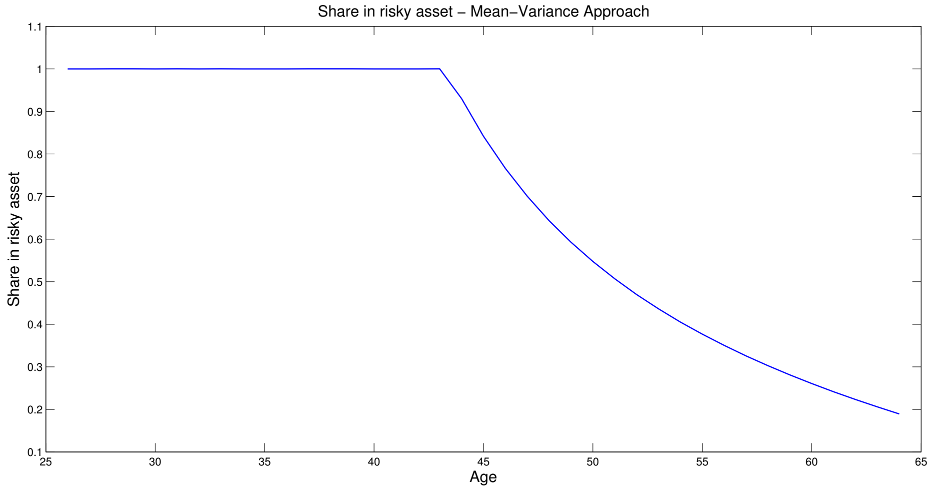

Figure 5.1: *The graph shows what share, α_s, of the available financial wealth that should be invested in risky assets. The investor should optimally invest 100% of financial wealth in risky asset until the age of ca 43. Thereafter the share of risky assets should gradually be exchanged for secure bonds. By the age of 64 (the last investment period before retirement)* $\alpha = 19\%$

of 52, 50% of the financial capital should be exposed to risk whereas the corresponding share at the age of 64 (the last allocation decision) is only 19%.

5.1.2 Case 2

In the second case it is assumed, as opposed to the first one, that there is a correlation between the return of the risky asset and the development of labor income. The blue graph in Figure 5.2 is exactly equivalent to the one presented in Case 1. However the green curve represents the optimal strategy given the presence of correlation. One apparent observation is that for any given age the share invested in risky assets is either equal or lower when correlation is present compared to when it is not. In particular from the age of 43, α_s is from 1.5 to 2.5 percentage points lower when the development of labor income is allowed to covary with the return of the stock market. On the background of viewing labor income, in a sense, as a risky asset, these result rhymes with intuition. Generally correlation between assets in a portfolio is bad and will lead to a more conservative holding of the assets. In the current case the share in "labor income" is fixed in accordance with constraint 4.18 which inevitably leads to the flexible α_s needs to be lowered. The results in Figure 5.2 captures this mechanism.

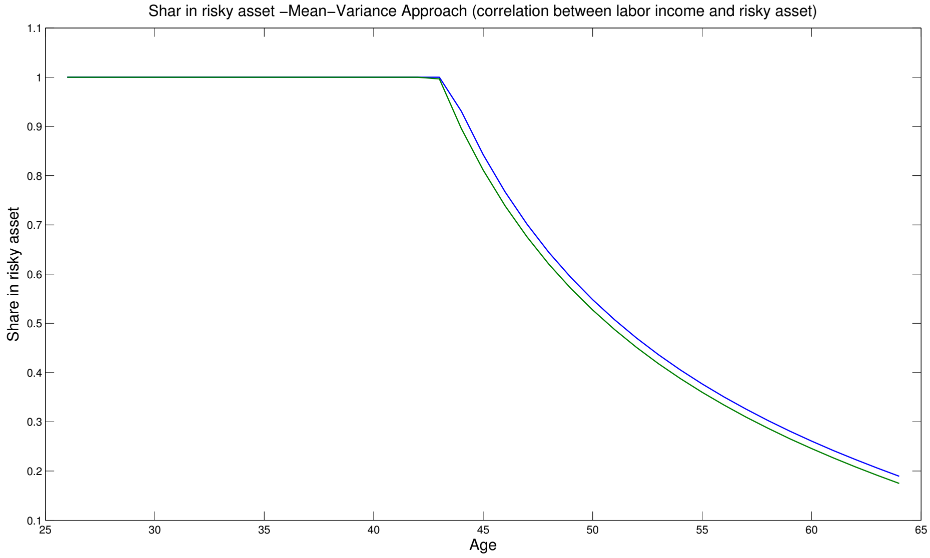

Figure 5.2: *The graph compares two strategies when correlation between development of labor income and return of risky asset is absent (top curve) and present (bottom curve). Thus it is noticed that correlation leads to a slightly more conservative share in risky assets for any given age.*

5.2 Expected utility approach

Although the mean-variance approach seem to lead to reasonable results, more reliability is assigned to the second method, using expected utility, due to its stronger mathematical soundness. In the following section, results from three different cases are presented, corresponding to employing different risk-aversion coefficients and the presence and absence of correlation. The parameters used in the three different scenarios are stated in the below table.

Parameters	r_f	μ_s	σ_s	$E[R_s]$	$\sigma(R_s)$	μ_L	σ_L	γ	ρ
Case 3	0.039	0.067	0.27	1.07	0.30	0.02	0.05	0.2	0
Case 4	0.039	0.067	0.27	1.07	0.30	0.02	0.05	0.5	0
Case 5	0.039	0.067	0.27	1.07	0.30	0.02	0.05	0.2	0.5

The chosen parameters are employed in a similar manner as described in the previous section, however there is a fundamental difference regarding the coefficient γ. In a mean-variance approach a higher γ corresponds to a higher aversion towards risk whereas the opposite holds when using the power utility function. Hence in Case 3 and 5 the investor is more risk-avert than in Case 4.

5.2.1 Case 3

In the first case when applying the expected utility approach there is no correlation between the risky asset and labor income development. The curve depicted in Figure 5.3 shows that the optimal portfolio strategy is to allocate 100% of financial capital to the risky asset until the age of 47 and then reduce the portion successively in favor for risk-free assets. For the last investment period (at age 64) the share of financial capital allocated to risky assets should optimally amount to ca 32% which is considerably higher compared to the corresponding share suggested by the mean-variance approach.

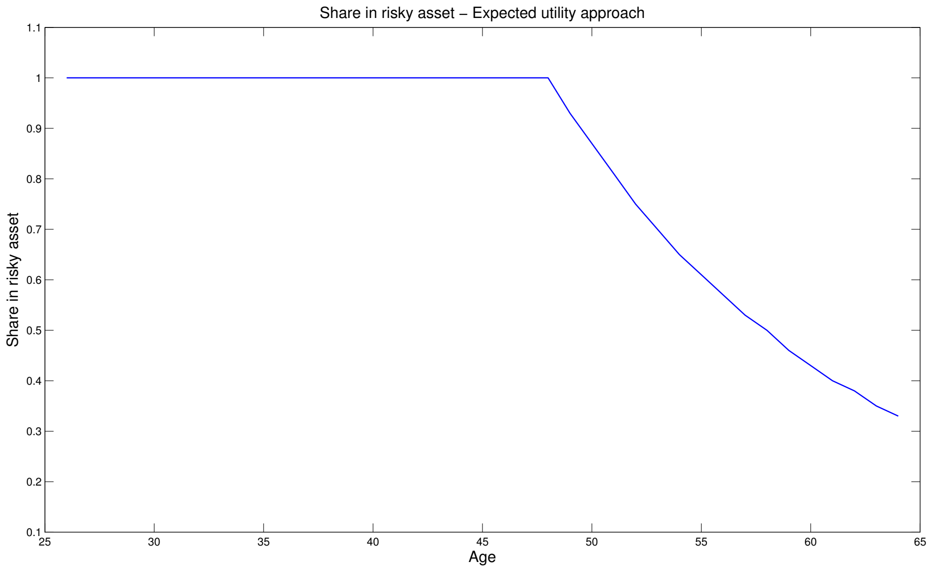

Figure 5.3: *The graph shows what share, α_s, of the available financial wealth that should be invested in risky assets. The investor should optimally invest 100% of financial wealth in risky asset until the age of ca 47. Thereafter the share of risky assets should gradually be exchanged for secure bonds. By the age of 64 (the last investment period before retirement)* $\alpha = 32\%$

5.2.2 Case 4

The γ-coefficient in the second case is increased from 0.2 to 0.5 which implies that the marginal utility associated with increasing total wealth is not decreasing as fast as compared to the first case. That is the investor is willing to accept more risk in favor for greater potential return, i.e. being less risk-averse. The change in γ is observed in Figure 5.4 where the optimal allocation strategy suggests that a 100% share should be held in the risky asset as long as until the age of 52. Even for the last period of allocation, the share still amounts to 42%. Thus it is noticed that for any given age the less risk-avert investor holds a considerable greater share of risky asset.

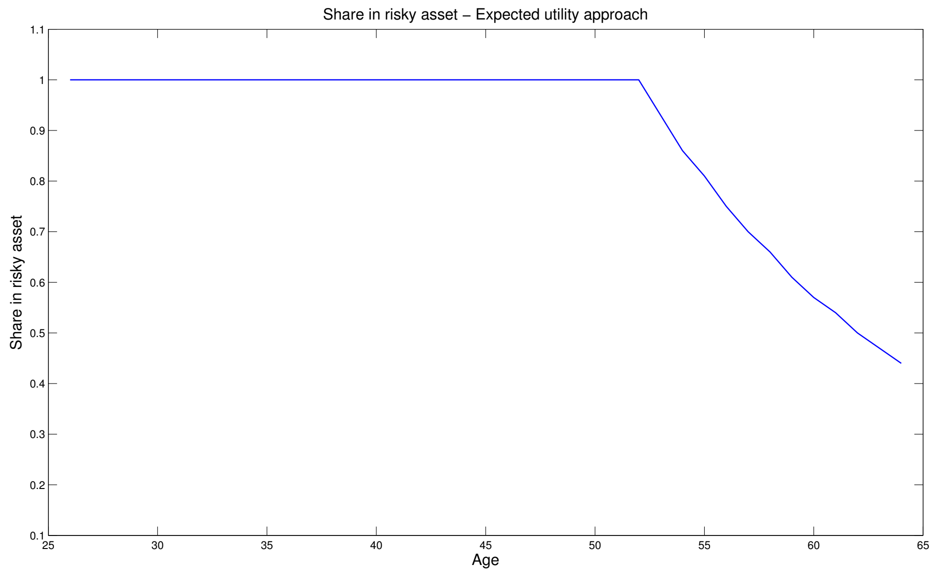

Figure 5.4: *The graph shows what share, α_s, of the available financial wealth that should be invested in risky assets. The investor should optimally invest 100% of financial wealth in risky asset until the age of ca 52. Thereafter the share of risky assets should gradually be exchanged for secure bonds. By the age of 64 (the last investment period before retirement)* $\alpha = 42\%$

5.2.3 Case 5

The deviation in optimal allocation strategy suggested in Case 3 and 4 illustrates the impact of different risk preferences. Both results are based on the assumption of no correlation between the development of labor income and the return of risky asset. However, the results shown in Figure 5.5 instead depicts what impact correlation between these two "returns" have on the optimal allocation strategy. The blue curve is equivalent to the one shown in Case 3 whereas the green one results from assuming $\rho = 0.5$. It is noticed that the presence of covariance implies that the optimal strategy is shifted slightly downwards, i.e for any given age the strategy assuming correlation holds a smaller share of risky assets. However the difference in α_s seem to amount to a mere 1% for the period 47 to 64 and hence the impact of covariance between the development of labor income and return on risky asset can be considered as relatively moderate.

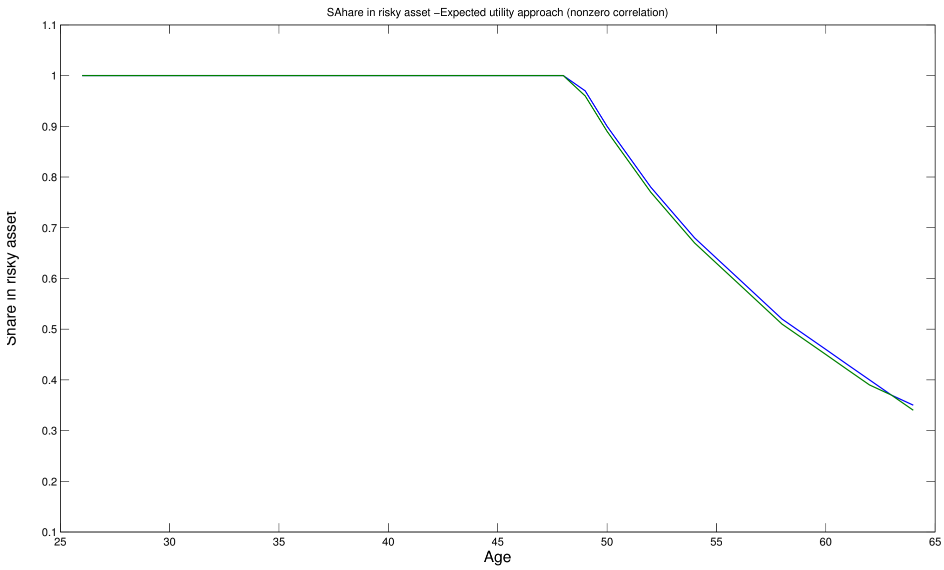

Figure 5.5: *The graph compares two strategies when correlation between development of labor income and return of risky asset is absent (top curve) and present (bottom curve). Thus it is noticed that correlation leads to a slightly more conservative share in risky assets for any given age.*

5.3 Strategy analysis

The results given by the mathematically most sound method, maximizing the expected utility of the last period, will be used as the benchmark strategy upon which further analysis and evaluation is performed. In particular, the strategy presented in Section 5.2.1 (Case 3) with associated parameters will be compared to two other strategies - "100 minus age" and "Only risk-free". The first one mentioned is a rule of thumb suggesting that 100 minus the investor's age (percentage points) of a pension fund should be invested in risky assets whereas "Only risk-free" simply sets α_s equal to zero for all investment periods.

First of all, in order to get a sense of how the human and financial wealth evolves throughout a life-cycle the benchmark strategy, onwards referred to as *Expected utility*-strategy, is employed to a series of stock return that exactly realizes as the expected returns for each period. Figure 5.6 illustrates how human wealth gradually decreases and eventually reaches zero in favor for an appreciating amount of financial capital. The human wealth curve is based upon the derivation explained in Section 4.2.3 and is equivalent to the one presented in Figure 4.3, thus the development of human wealth of one representative investor. The smoothness of the curve of financial develop-

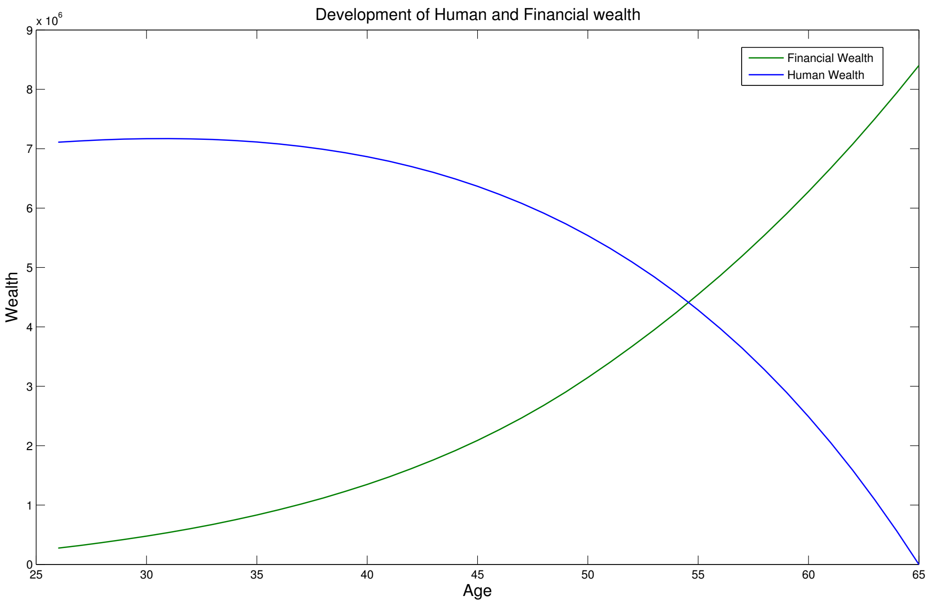

Figure 5.6: *The graph illustrates the development of financial (trending upwards) and human (trending downwards) wealth over a portfolio life-cycle.*

ment is simply due to the fact that only the expected returns are assumed to occur. In reality the shape would obviously capture the stochastic feature of the risky asset and thereby the development of the financial development. Nevertheless the graph shows the dynamics of total wealth and how its composition evolves over time.

In the remainder of this section, the performance of the "Expected utility"-strategy will be compared to the other two strategies outlined briefly above. The "100 minus age"-strategy allocates 75% of financial capital for the first period since the age of a person starting working is assumed to be 25 years old and finalizes with 36% in risky assets for the last investment period (at the age of 64). The "Risk-free"-strategy allocates all available financial capital to a secure asset. Figure 5.7 illustrates what share α_s each of these strate-

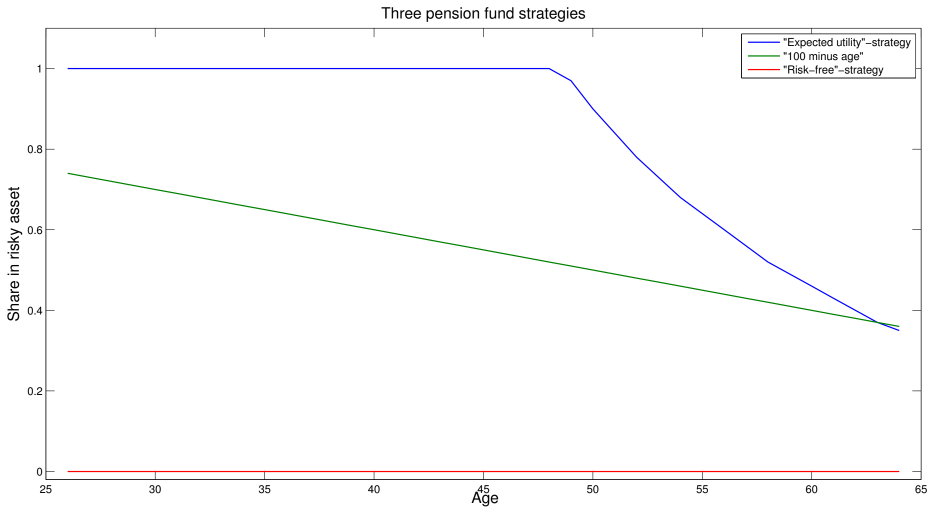

Figure 5.7: *The top, middle and bottom curve depicts "Expected utility"-, "100 minus age"- and "Risk-free only"-strategy, respectively.*

gies suggests to be allocated to the risky asset for a given age. Although the "Expected utility"-strategy almost exactly suggests the same share for the last period as do the "100 minus age"-strategy, the first mentioned does hold a considerably greater share of risky assets for every period prior to 64.

Any evaluation and comparison will be based upon a simulation of 10 000 000 different series of 39 returns of the risky asset. Thereby a distribution of the value of the portfolio at the last period, denoted as V_{40} and consequently also its corresponding utility $u(V_{40})$ is given. Comparing the mean value of V_{40} will give an indication of which strategy performs best. However their respective distribution of V_{40} might be associated with different standard deviations and therefore the most indicative and important measure for comparative purposes is $u(V_{40})$ which implicitly considers risk through its formulation.

The following three subsections will present the distribution (histograms) of $u(V_{40})$ and V_{40} for each one of the three strategies, respectively. Thereafter statistics are summarized and commented.

5.3.1 "Expected utility"-strategy

Figure 5.8 shows the distribution of the portfolio value at the last period based upon a simulation using 10^7 different outcomes whereas the histogram in Figure 5.9 illustrates the distribution of each portfolio value's associated utility.

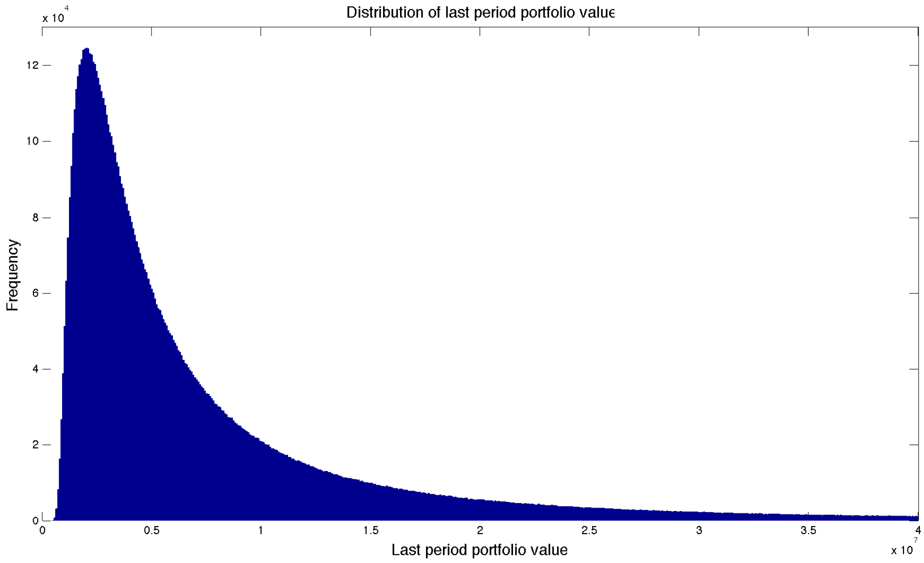

Figure 5.8: *Histogram of V_{40} when applying the "Expected utility"-strategy. Number of simulations: 10 000 000.*

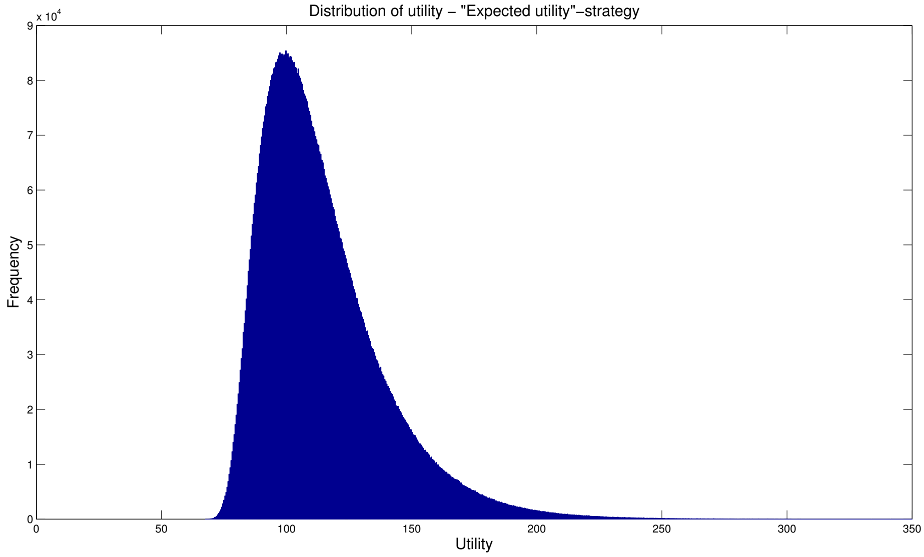

Figure 5.9: *Histogram of $u(V_{40})$ when applying the "Expected utility"-strategy. Number of simulations: 10 000 000.*

5.3.2 "100 minus age"

The corresponding distributions as for the "100 minus age"-strategy are presented in Figures 5.10 and 5.11. The histograms are based on the exactly same, 10^7, series of risky asset returns as used for the "Expected utility"-strategy.

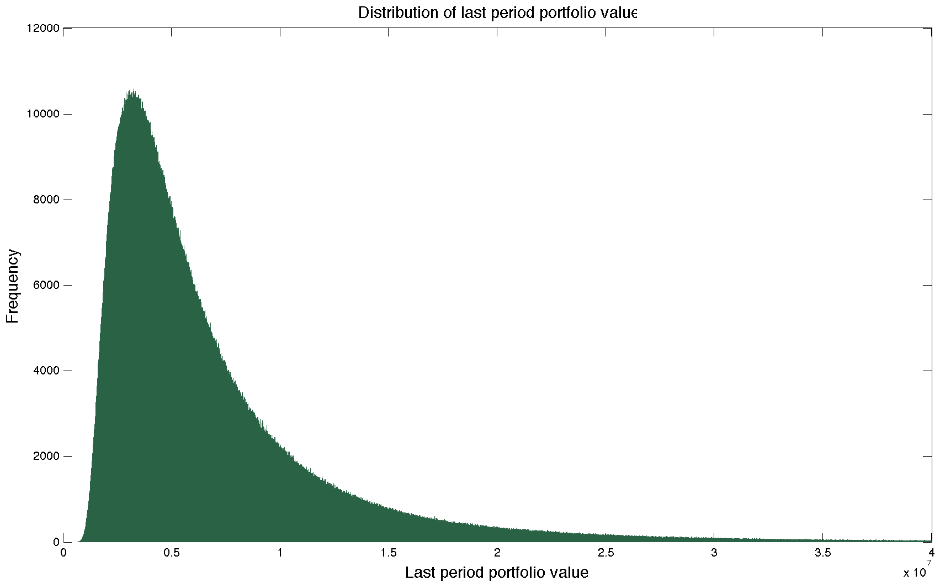

Figure 5.10: *Histogram of V_{40} when applying the "100 minus age"-strategy. Number of simulations: 10 000 000.*

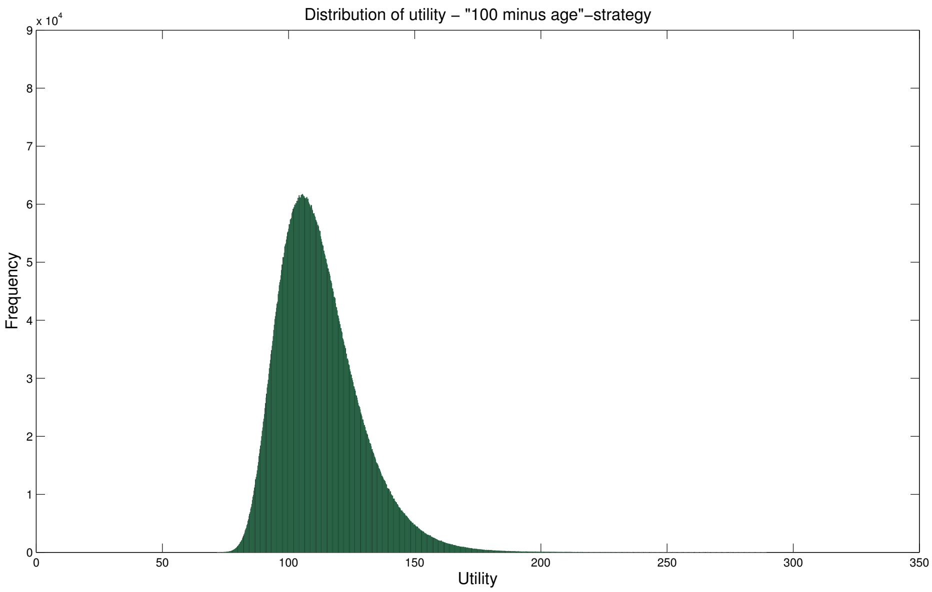

Figure 5.11: *Histogram of $u(V_{40})$ when applying the "100 minus age"-strategy. Number of simulations: 10 000 000.*

5.3.3 "Only risk-free"

Finally, for the sake of completion, the distributions associated with the "Only risk-free"-strategy are shown in Figures 5.12 and 5.13. Apparently both these histograms only shows one bar since all potential returns of the risky asset are redundant for the performance of the portfolio since 100% are invested in an asset without risk.

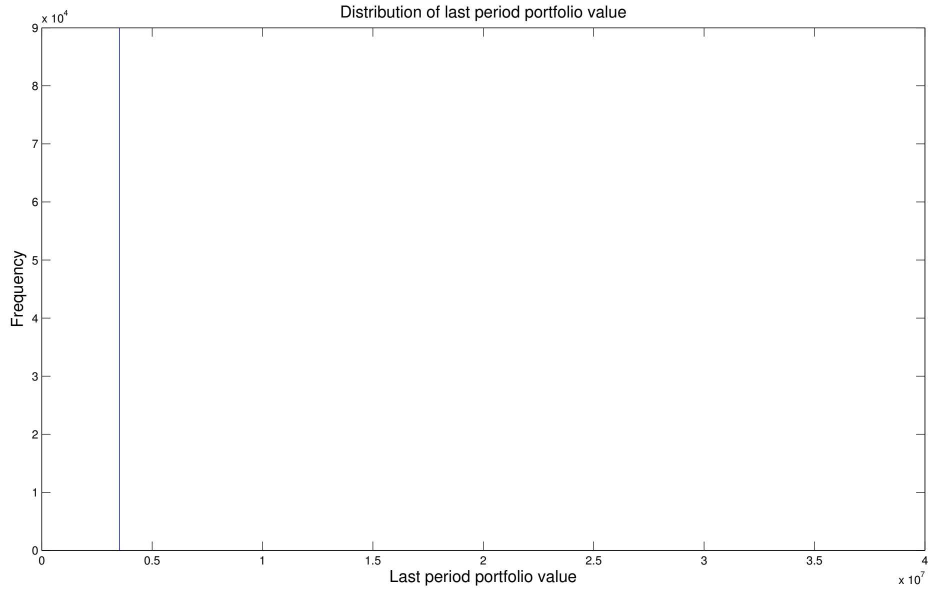

Figure 5.12: *Histogram of V_{40} when applying the "Risk-free only"-strategy. Number of simulations: 10 000 000.*

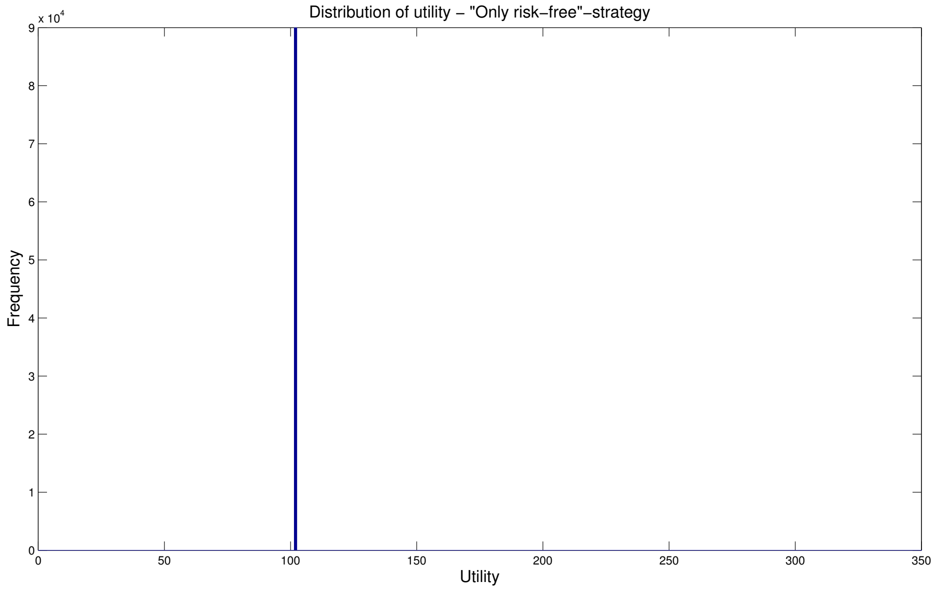

Figure 5.13: *Histogram of $u(V_{40})$ when applying the "Risk-free only"-strategy. Number of simulations: 10 000 000*

5.3.4 Summarizing statistics

There are some characteristic discrepancies of the three different sets of distributions. First of all one notices that solely holding the secure asset will give a portfolio value by the age of 65 amounting to ca 3.5 MSEK. However the derived strategy in this thesis, "Expected utility"-strategy, undoubtedly yields the highest mean value (based on the 10^7 outcomes) of the last period portfolio value.

Strategy	$\overline{V_{40}}$	Std. dev.	Skewness	$\overline{u(V_{40})}$	$VaR_{0.95}$
"Expected Utility"	10.57	237%	28.9	114.1	3.25
"100 minus age"	7.07	96.6%	6.0	112.3	4.00
"Only risk-free"	3.53	0%	-	102.0	3.53

The "Expected utility"-strategy will on average give a portfolio value of 10.6 MSEK at the age of 65 compared to 7.1 MSEK for the "100 minus age"-strategy. Hence holding a higher share of risky assets for a longer period of time as compared to the "100 minus age" seem to payoff rather significantly in the end although inevitably this comes with the cost of having a higher standard deviation of the portfolio value in the last period. However the measure of the utility of the last period portfolio value will capture the negative feature of risk due to the concavity of the utility function. All outcomes (out of the 10^7) with V_{40} falling short of the mean, $\overline{V_{40}}$, will be "punished" harder by the utility function than will a corresponding outcome of V_{40} realizing above mean, be "rewarded". That is for instance a 8.57 MSEK outcome of V_{40} will drag the mean of utility downwards more than what a 12.57 MSEK outcome will increase it. Hence a distribution of V_{40} with a large standard distribution will be unfavored by the utility function and therefore one can say that the mean of utility captures both the positive contribution of a high expected return but also the negative aspects of high risk. Therefore is $\overline{u(V_{40})}$ the most important measure for evaluating what strategy actually performs best. One notices that the "Only risk-free"-strategy achieves a decent mean utility score of 102 despite an expected portfolio value in the last period of a a mere 3.53 MSEK which can be explained by the utility function rewarding the lack of risk.

The "100 minus age"-strategy has a fairly high expected last period portfolio value with a relatively low standard deviation which yields a mean value of $u(V_{40})$ amounting to 112.3 Although the "Expected utility"-strategy seem to be the best performing with a mean utility score of 114.1 which is driven by the high expected portfolio value and dragged by the high standard deviation of the V_{40}-distribution. Furthermore one notices that the high standard deviation associated with the distribution of the last period portfolio value for the "Expected utility"-strategy is also reflected in the empirical vaule-at-risk measure with a value of 3.25 MSEK compared to 4.00 and 3.53 MSEK for the "100 minus age"- and "Only risk-free"-strategy, respectively. However it should once again be stressed that risk is captured in the mean utility value and that "Expected utility"-strategy clearly outperforms the "Only risk-free" whereas it just beats "100 minus age"-strategy.

5.4 Scenario analysis

In order to get a better understanding of how the "Expected utility"-strategy actually performs the following section will investigate what impact different scenarios in the stock market will have on the portfolio value. In particular the effect of a financial crisis occurring at three different stages of the portfolio life-cycle, will be studied. Firstly a financial crisis is assumed to take place at a early stage and it will be defined by having three consecutive periods with a return of the risky asset of ca negative 40%.

5.4.1 Early financial crisis

Figure 5.14 illustrates the average return of the risky asset based on a sample of 100 different series of returns. The small sample size is motivated by illustrative reasons only. With a small sample the heavily negative returns in the beginning of the portfolio life-cycle will show, yet not letting the stochastic characteristic of the process being faded by the average of a too large number of outcomes. For the actual results presented below, the computations are based on a sample size of 100 000.

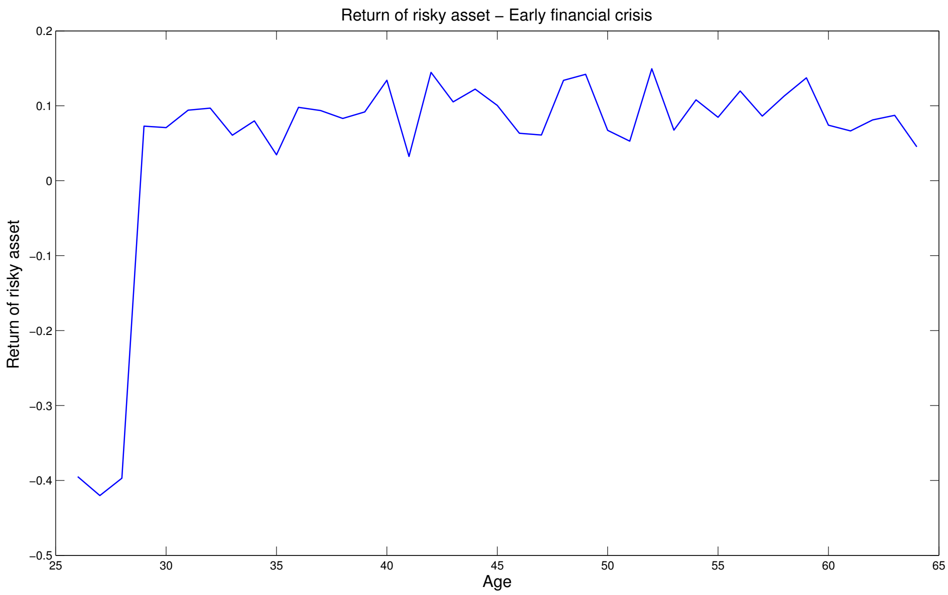

Figure 5.14: *The average return of the risky asset based on a sample of 100 simulations. The financial crisis occurs at a early stage. Note: comparative computations is instead based on 100 000 simulations*

5.4.2 Mid financial crisis

In the second scenario a similar financial crisis occurs for three periods when the investor has the age of 44, 45 and 46. Figure 5.15 illustrates in the same way as Figure 5.14 how the returns of the risky asset is typically just below 10% expect for three periods when they realize around -40%.

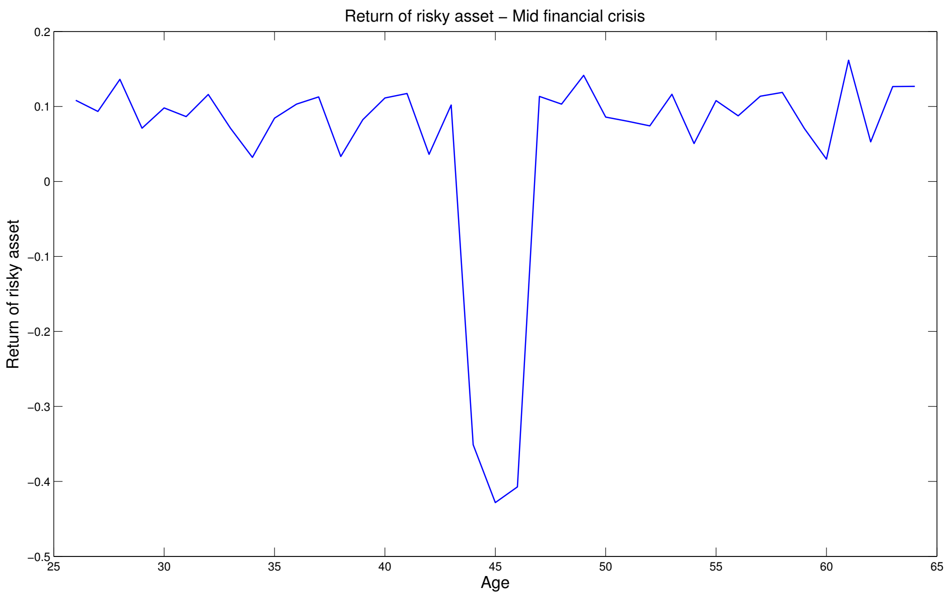

Figure 5.15: *The average return of the risky asset based on a sample of 100 simulations. The financial crisis occurs at a middle stage. Note: comparative computations is instead based on 100 000 simulations*

5.4.3 Late financial crisis

Finally the financial crisis is let to occur the very last three periods before retirement, i.e. when the investor is 63, 64 and 65 as depicted by Figure 5.16

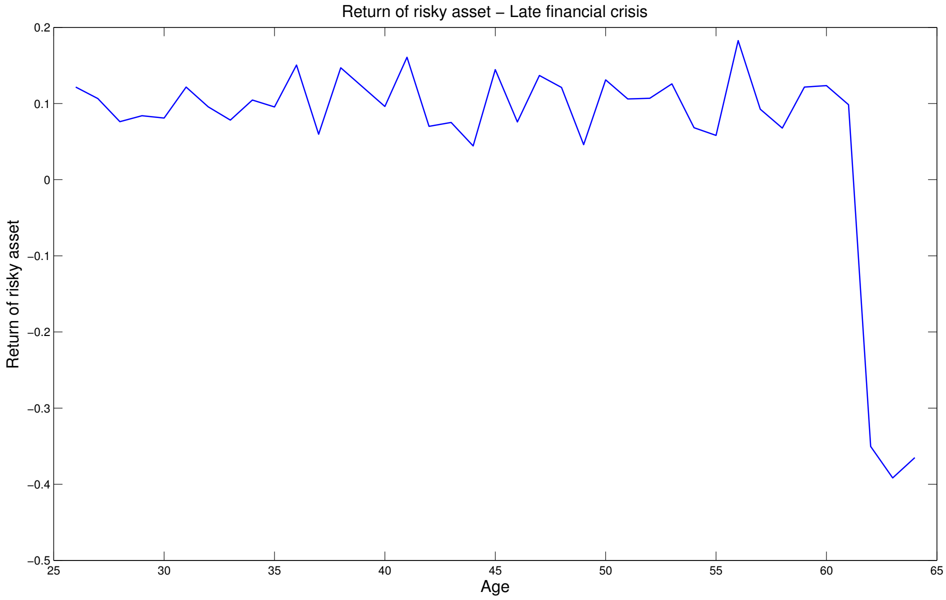

Figure 5.16: *The average return of the risky asset based on a sample of 100 simulations. The financial crisis occurs at a late stage. Note: comparative computations is instead based on 100 000 simulations*

The "Expected utility"-strategy is employed to the three different scenarios and the mean of the last period portfolio value together with its associated utility is computed and presented in the below table. One notices that experiencing a heavy recession in three periods occurring in the middle of the portfolio life-cycle (i.e. around the age of 45) will hurt the portfolio value at the last period severely and the most compared to the other two scenarios. In fact only 4.69 MSEK can be expected on average to be left for retirement corresponding to a utility score of 101.0.

Concerning the other two scenarios the difference in portfolio performance might be considered surprisingly small. Intuitively one might think that a portfolio would be most vulnerable to a financial crisis occurring close to retirement since the portfolio value should be at its most and thereby a highly negative return would have serious impact. This is of course generally the case but since the "Expected utility"-strategy only holds a relatively small share of risky assets close to retirement, the portfolio manages to cope the crisis fairly well. The Late financial crisis scenario leaves the investor to expect V_{40} and $u(V_{40})$ to be 10.1 MSEK and 112.5, respectively.

Financial crisis	V_{40} (MSEK)	$u(V_{40})$	$E[r_s]$	$\sqrt{var(r_s)}$	r_f
Early	10.4	115.5	0.07	0.35	0.035
Mid	4.69	101.0	0.07	0.35	0.035
Late	10.1	112.5	0.07	0.35	0.035

After all, the best bad scenario is to deal with a financial crisis at a early phase in the portfolio life-cycle. Even though the portfolio is 100% invested in risky assets and a three-period severe downturn will almost diminish available financial capital, there is still a vast amount of time to recoup and although the loss is tremendous in relative terms, it is not the case in absolute terms due the relatively small initial capital. That is, human wealth is strong at a early stage and will help the portfolio recover over the remainder of investment periods.

Consequently an investor is vulnerable to a crisis occurring in the middle of a portfolio-cycle but is saved by strong human wealth at the early phase and saved by the strategy, suggesting a conservative share in risky asset, approaching retirement.

Chapter 6

Conclusion

In a time where people tend to retire earlier and live longer, publicly provided resources for pension might not fully suffice and the need for privately financing one's retirement is inevitably increasing. Directly coupled with this trend comes an increased requirement of better understanding how to optimally manage a portfolio aimed for pension, not least in the aspect of choosing a proper risk level. In particular this thesis investigates what should be the optimally chosen share of risky assets composing a pension fund depending on the investor's age.

The concept of Human wealth is employed in order to address this question. It is defined as the sum of all discounted future incomes from labor and enables drawing a more covering picture of a person's whole financial situation. Instead of solely maximizing financial capital the objective is to optimize an individual's total wealth defined as the sum of both financial and human wealth. As the last mentioned part cannot be invested, since it is composed of expected not yet realized incomes, the investor is bound to allocate the financial capital. At a early phase financial wealth will constitute a small portion of total wealth allowing the investor to bear relatively more risk as opposed to approaching retirement when the financial part of total wealth is dominating, requiring a more conservative exposure towards risky assets. The dynamics in the composition of total wealth throughout a portfolio life-cycle will hence drive what share of riskiness an investor could tolerate.

In order to incorporate the concept of human and financial wealth mathematically a mean-variance approach is initially applied. By maximizing the difference between expected return of total wealth and its associated variance for every period $i = 1, 2, 3, ..., 39$ (assuming a working life of 40 years) a optimal strategy is given. It suggests that 100% of available capital should be invested in risky asset until the age of 43 whereafter the portion should be gradually exchanged for secure assets reaching a level of 19% in risky assets the last period before retirement.

Although the strategy seems to give reasonable results, the used method might be mathematically questionable in two aspects. Firstly, a utility function defined as the difference between expected return and variance might lead to a non-convex objective function and secondly it lacks a sound motivation of what optimization problem actually should be solved at each period. Therefore a more sophisticated method, maximizing the expected utility of the portfolio value at the last period, is employed. By using a power utility function convexity can be guaranteed and through stochastic dynamic programming a sound motivation of which optimization problem to be solved every period is achieved.

The second method provides a strategy which suggests holding all available capital in risky assets until the age of 47 and then successively reach a share of 32% by the age of 64 (the last investment period). Thus the expected utiltiy-method favors a slightly more aggressive strategy than the first method.

Furthermore both methods allow having a correlation between the return of risky assets and the labor income. When a covariance is present the strategy becomes slightly more conservative, i.e. for any given age the optimal share in risky assets is smaller. Consequently one can conclude that if a person works in a sector whose overall performance tend to correlate with the stock market he should decrease his share in risky assets or invest in asset classes that do not co-vary with the particular industry.

Moreover the strategy of the mathematically most reliable method is more thoroughly analyzed and it can be concluded to outperform holding only the risk-free asset or "100 minus age" percentage points in a risky asset - a rule of thumb for pension fund allocation rather commonly applied.

Finally the "Expected utility"-strategy's ability to cope with extreme stock market events such as a financial crisis occurring at a early, mid and late phase of the portfolio life-cycle, respectively, is investigated. Results show that a financial crisis around the age of 45 (out of the three scenarios) has the most severe impact on the last period portfolio value. However, a early and a late extreme stock market downturn is helped by a high level of human wealth and holding a relatively small share in risky assets, respectively.

Conclusively, the concept of human wealth combined with stochastic dynamic programming seem to provide a well-functioning strategy that allocates between risky and risk-free assets in a pension fund. Deeper analysis shows that it performs successfully compared to strategies often employed in practice and that it copes relatively well with extreme outcomes in the stock market. Hopefully can this thesis at least highlight the importance of considering one's full financial situation, including not solely currently available capital but also discounted future earnings, when choosing an appropriate risk-level for one's pension fund.

Chapter 7

Further research

Human wealth seems to be a useful concept for achieving a more all-embracing comprehension of an individual's total financial situation and thereby creating the dynamics driving a person's risk tolerance. The strategy based upon the incorporation of human wealth seems to result in both a reasonable and successful strategy that performs at least as well as existing commonly applied strategies. However, there is room for extending the model derived in this thesis in order to enable more interesting results.

First of all the model used in this thesis only considers two investment opportunities, one risky and one risk-free commodity. A more sophisticated and realistic model could consider several different sorts of risky assets. It would be particularly interesting to incorporate non-domestic assets such that currency risk has to be taken into consideration.

Moreover, the model used and analyzed emanates from the perspective of one representative investor based on a vast number of simulations. Although, this gives a realistic and applicable results to the broad mass, it would give deepened understanding if more "representative" investors' situations were analyzed. Thereby it would be possible to a greater extent derive taylor-made strategies for one particular individaul.

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