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A risk-transaction cost trade-off model for index tracking

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Abstract

This master thesis considers and evaluates a few different risk models for stock portfolios, including an ordinary sample covariance matrix, factor models and an approach inspired from random matrix theory. The risk models are evaluated by simulating minimum variance portfolios and employing a cross-validation. The Bloomberg+ transaction cost model is investigated and used to optimize portfolios of stocks, with respect to a trade off between the active risk of the portfolio and transaction costs. Further a few different simulations are performed while using the optimizer to rebalance long-only portfolios. The optimization problem is solved using an active-set algorithm. A couple of approaches are shown that may be used to visually try to decide a value for the risk aversion parameter λ in the objective function of the optimization problem.

The thesis concludes that there is a practical difference between the different risk models that are evaluated. The ordinary sample covariance matrix is shown to not perform as well as the other models. It also shows that more frequent rebalancing is preferable to less frequent. Further the thesis goes on to show a peculiar behavior of the optimization problem, which is that the optimizer does not rebalance all the way to 0 in simulations, even if enough time is provided, unless it is explicitly required by the constraints.

Keywords: Cross-validation, Distribution of eigenvalues, Factor models, Portfolio rebalancing, Risk estimation, Transaction costs

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Chapter 1

Introduction

The purpose of this thesis is to evaluate optimization routines for trading stock portfolios with a large and finite number of assets. Modern financial institutes use optimization routines when trading portfolios in order to balance transaction costs against deviation from target function. Transaction costs comprise of direct costs such as brokerage fees and taxes and of indirect costs such as market impact. The target function for a stock portfolio could be to minimize deviation from a benchmark index or to maximize projected excess return, the so called alpha.

Global stock indices have hundreds of assets, which creates the need for dimension reduction at optimization. A common method is to use factor models that reduce a portfolio's stock exposures to a few factor exposures. This thesis will use factor models and investigate how different models affect the optimization.

The the that will primarily be investigated are:

- Level of projected risk
- Proportion of systematic risk versus non-systematic risk
- Frequency and time horizon of rebalancing
- Assumptions in the transaction cost model
- Assumptions in the factor model

Deviation from the benchmark will be used as target function and indirect costs will be assumed according to market convention.

The outline of this thesis will be as follows. Chapter 2 starts by introducing the theoretical concepts that will be used in the thesis. Factor models are described and divided into 3 different categories; macroeconomic, statistical and fundamental factor models. Only statistical factor models are described closer. A way to use concepts from random matrix theory to remove eigenstates of the covariance matrix of returns that correspond to random noise is also described. Further the transaction cost function to be used in the optimization is described. In chapter 3 the thesis goes on to validate and test the performance of the different approaches to estimate the covariance matrix. The two methods used are a simulation of a minimum variance portfolio and a cross-validation. Chapter 4 describes the simulations and the results of these simulations. First the central optimization problem is stated. From this a transaction cost-risk frontier is generated. Further a simulation that rebalances the portfolio at fixed time intervals is simulated. Chapter 5 concludes the results of this thesis.

Chapter 2

Theoretical background

This chapter will cover the theoretical background that will be used to build, validate and simulate the model.

2.1 Factor models

The general specification of a factor model is of the form,

$$
\begin{aligned}
&R_{it} = \alpha_i + \beta_{i1} f_{1t} + \beta_{i2} f_{2t} + \cdots + \beta_{iK} f_{Kt} + \epsilon_{it} \\
&i = 1 \dots N, t = 1 \dots T,
\end{aligned}
$$

where R_{it} is the return of asset i at time point t, α_i are the intercepts, β_{ik} are the factor loadings, f_{kt} are the factor realizations and ϵ_{it} are the specific returns, that is the component of returns not explained by the k factors chosen. A more in depth discussion on factor models can be found in many books on portfolio management, among others in [7].

Factor models are typically divided into 3 types. Macroeconomic, fundamental and statistical factor models. A macroeconomic factor model uses observable financial time series as factors e.g. oil prices and interest rates. A fundamental factor model uses company characteristics as factors e.g. earnings and company size. A statistical factor model is based on the statistical characteristics of the price time series of the asset under consideration.

In order to use a factor model one must estimate the intercepts α_i and factor loading β_{ik}. Further the variances and covariances of the factor realizations f_{kt} and variances of the specific factors ϵ_{it} need to be estimated. The factor model assumes that the specific returns ϵ_{it} are uncorrelated across assets and time and uncorrelated with the factor realizations f_{kt}. Further it assumes that the specific returns have expected value 0. This process of model calibration will be examined more closely further down in the sections *Macroeconomic factor model* and *Statistical factor model*.

Let $\mathbf{R}_t$ be an N-dimensional column vector representing the N asset returns in each time step $t = 1, ..., T$. The sample covariance matrix of the returns is then given by

$$\Omega = \frac{1}{T-1} \sum_{t=1}^{T} (\mathbf{R}_t - \overline{\mathbf{R}})(\mathbf{R}_t - \overline{\mathbf{R}})'. \tag{2.1}$$

Where

$$\overline{\mathbf{R}} = \frac{1}{T} \sum_{t=1}^{T} \mathbf{R}_t$$

is the sample mean. The factor model may be used to estimate the sample covariance matrix by the covariance matrix of the asset returns modeled by the factor model. Let $\mathbf{F}_t$ be a K-dimensional column vector representing the K factor returns in each time step $t = 1, ..., T$. The factor return covariance matrix is then given by,

$$\Omega_f = \frac{1}{T-1} \sum_{t=1}^{T} (\mathbf{F}_t - \overline{\mathbf{F}})(\mathbf{F}_t - \overline{\mathbf{F}})', \tag{2.2}$$

where,

$$\overline{\mathbf{F}} = \frac{1}{T} \sum_{t=1}^{T} \mathbf{F}_t.$$

Further, the covariance matrix of the specific risks is given by

$$E = \begin{pmatrix} \sigma^2_{spec,1} & 0 & \cdots & 0 \\ 0 & \sigma^2_{spec,2} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \sigma^2_{spec,K} \end{pmatrix},$$

where $\sigma^2_{spec,k} = \text{Var}(\epsilon_{kt})$. Let the matrix of factor loadings be defined by

$$B = \begin{pmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1K} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{N1} & \beta_{N2} & \cdots & \beta_{NK} \end{pmatrix}.$$

The covariance matrix of the asset returns in the factor model, that is the factor model covariance matrix, is now given by

$$\Omega_{FM} = B\Omega_f B' + E.$$

This representation of the covariance matrix allows one to decompose the risk into two parts. The systematic risk, which is accounted for by the factors, and the non-systematic risk, also known as the idiosyncratic risk, which is not explained by the factors.

2.2 Macroeconomic factor model

A macroeconomic factor model uses observable economic and financial time series as factors. These factors may be oil prices, indices, interest rates, inflation rates. In this thesis a market model, with the SIX60 index as only factor will be evaluated. The factor loadings of the model will be estimated using time series regression. Another macroeconomic factor model one could consider is the Fama-French three-factor model with the 3 factors; the market, a factor based on market capitalization and a factor based on book-to-market ratio. The Fama-French model will not be considered in this thesis.

2.3 Statistical factor model

In a statistical factor model one may use the most important principal components of the return time series as dependent variables in a regression in order to determine the factor loadings in the model. The principal component weights are the eigenvectors of norm 1 of the sample covariance matrix Ω. They were obtained using MATLAB's built-in function `pca`. Let the N principal component weight vectors of dimension N be denoted by $\mathbf{p}_i, \ i = 1, ..., N$. The vectors $\mathbf{p}_i$ are pairwise orthogonal. The N principal components in each time step are now given by

$$f_{it} = \mathbf{p}_i' \cdot \mathbf{R}_t \qquad i = 1, ..., N.$$

Of these N principal component only a smaller subset will be chosen to accomplish a dimension reduction. A method for choosing how many of the most important principal components to include will be shown further down in the section *Random Matrix Theory*. Assume K principal components have been chosen. For each asset i a linear regression model is then given by

$$\begin{aligned}
R_{i1} &= \alpha_i + \beta_{i1} f_{11} + \beta_{i2} f_{21} + \cdots + \beta_{iK} f_{K1} + \epsilon_{i1} \\
R_{i2} &= \alpha_i + \beta_{i1} f_{12} + \beta_{i2} f_{22} + \cdots + \beta_{iK} f_{K2} + \epsilon_{i2} \\
&\vdots \\
R_{iT} &= \alpha_i + \beta_{i1} f_{1T} + \beta_{i2} f_{2T} + \cdots + \beta_{iK} f_{kT} + \epsilon_{iT}.
\end{aligned}$$

For each of the N assets a different set of $K + 1$ regression coefficients are obtained. That is the α_i:s and β_{ik}:s.

The factor model now reads,

$$\mathbf{R}_t = \boldsymbol{\alpha} + B \cdot \mathbf{F}_t + \boldsymbol{\epsilon}_t \qquad t = 1, ..., T$$

where

$$\begin{aligned}
\boldsymbol{\alpha} &= (\alpha_i, \alpha_2, \ldots, \alpha_N)' \\
\epsilon_t &= (\epsilon_{1t}, \epsilon_{2t}, \ldots, \epsilon_{Nt})'.
\end{aligned}$$

An estimation of the sample covariance matrix can now be obtained using the sample covariance matrix of the factor realizations according to (2.2), the regression coefficients and the squares of the standard errors of regression. This estimation is the covariance matrix of the returns in the statistical factor model and given by,

$$\Omega_{SFM} = B\Omega_f B' + E,$$

where E is given by the squares of the standard errors of regressions,

$$E = \begin{pmatrix} \hat{s}_1^2 & 0 & \cdots & 0 \\ 0 & \hat{s}_2^2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \hat{s}_N^2 \end{pmatrix}.$$

One may note that the portfolio risk is split into systematic and non-systematic risk. The term given by $B\Omega_f B'$ is the systematic risk and the term E represents the non-systematic risk. The non-systematic risk can in principle be diversified away completely.

2.4 Random Matrix Theory

Yet another approach to estimating the covariance matrix is to use results from random matrix theory to establish which eigenstates of the sample covariance matrix correspond to random noise.

This approach has some similarities to the earlier statistical factor model. The main difference being that a regression is not performed. A more thorough description of the approach can be found in [5]. One compares the distribution of eigenvalues of the sample covariance matrix with the distribution of eigenvalues of a sample covariance matrix from purely random noise. The sample covariance matrix of purely random noise is given by

$$\Omega_{noise} = \frac{1}{T} \cdot A \cdot A',$$

where each element of the $N \times T$-matrix A is given by an i.i.d $N(0, 1)$-random variable. The matrix Ω_{noise} is of Wishart type, of which many results are known. One known result is the distribution of eigenvalues in the limit when $N \to \infty$, $T \to \infty$ and the ratio $Q = \frac{T}{N}$ is held fixed. The distribution can be found in [5]. A plot of the limiting distribution with $Q = \frac{580}{1000}$ and a histogram of an outcome in the case with $N = 1000$ and $T = 580$ can be seen in Figure 2.1.

What one may notice is that there is a cut-off level for the size of the eigenvalues after which no more eigenvalues occur in the case of purely random noise. This is the fact that will be exploited to remove random noise

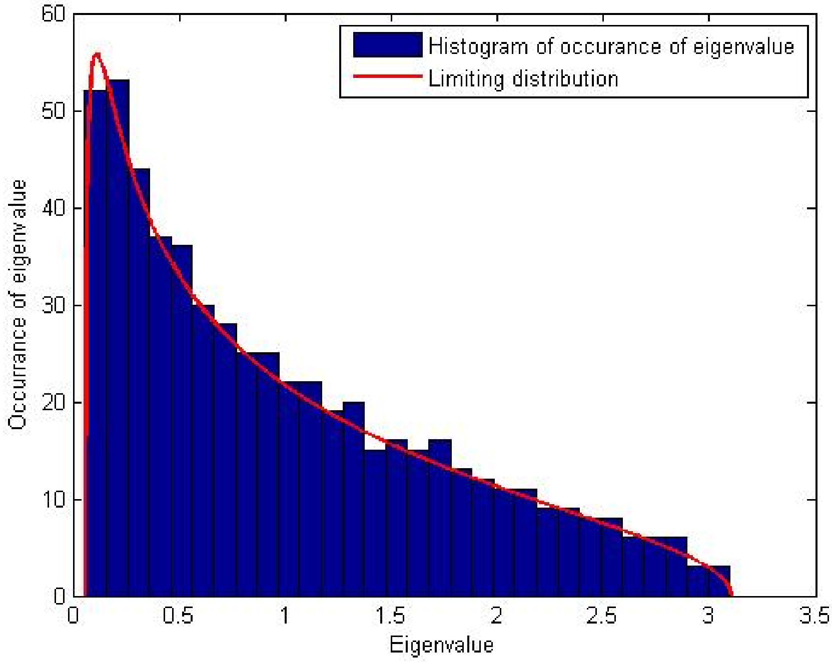
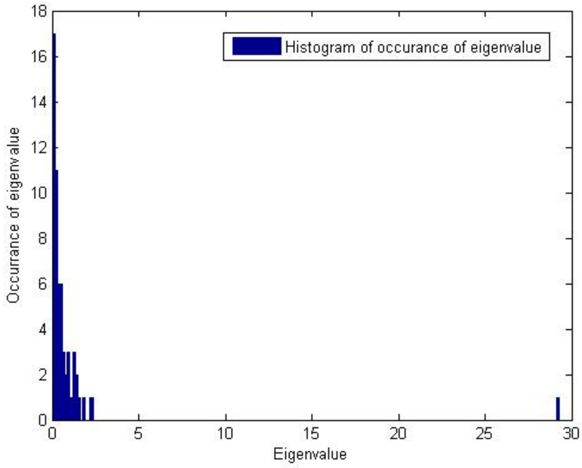
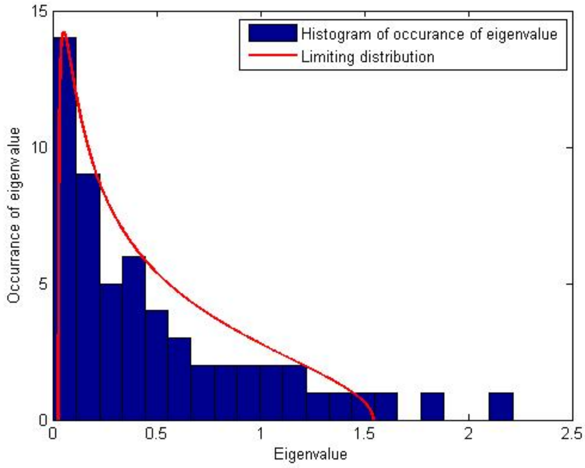


Figure 2.1: A plot and histogram of the limiting distribution and an outcome with finite N and T. Note that the y-axis only applies to the histogram and that the plot has been scaled to fit the histogram in the y-direction.



(a) Histogram of the eigenvalue distribution of the sample covariance matrix. One eigenvalue is considerably larger than the others.



(b) Histogram of the eigenvalue distribution of the sample covariance matrix with the largest eigenvalue removed. A scaled limiting distribution has been superimposed on the histogram.

Figure 2.2: Histograms of the sample covariance matrix

from a sample covariance matrix. In Figure 2.1 the cut-off occurs around $\lambda = 3$.

For a sample covariance matrix with $N = 100$ and $T = 58$ with real data the eigenvalue density in Figure 2.2 is obtained. The data used is described in the section *Data set* of chapter 3. One may notice that one eigenvalue is considerably larger than the others. This is the eigenvalue that corresponds to the market as a whole. One may also notice that only the 4 largest eigenvalues seem to be inconsistent with random noise. This would suggest that the choice of retaining these 4 largest eigenvalues to construct the filtered matrix should be a good idea. This method of determining how many eigenvectors or values to retain can be employed in the case of a statistical factor model also.

After the number of eigenstates to retain has been determined one needs to construct the estimated covariance matrix. Since the sample covariance matrix Ω is a symmetric matrix it allows us to express the covariance matrix in the following form

$$\Omega = PDP' = \begin{pmatrix} \mathbf{p}_1 & \cdots & \mathbf{p}_N \end{pmatrix} \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_N \end{pmatrix} \begin{pmatrix} \mathbf{p}_1' \\ \vdots \\ \mathbf{p}_N' \end{pmatrix}, \qquad (2.3)$$

where the $\mathbf{p}_i$:s are the eigenvectors of Ω represented as column vectors and the λ_i:s are the corresponding eigenvalues. We may remove the unwanted eigenstates by setting the corresponding eigenvalues to 0 in the above equation. However one does not wish to change the variance of the assets but only remove the unwanted correlation between assets. This means that we want the estimated covariance matrix and the sample covariance matrix to have the same diagonal. This will lead to the following representation of the estimated covariance matrix

$$\Omega_{RMT} = P_K D_K P_K' + E,$$

where the $N \times K$ matrix P_K has the K chosen eigenvectors as its column vectors and the $K \times K$ diagonal matrix D_K has the corresponding eigenvalues on its diagonal. This is exactly equivalent to setting the eigenvalues of the unwanted eigenstates to 0 in the equation (2.3). The diagonal matrix E contains stock-specific risk and is given by the **diagonal** of the matrix $\Omega - P_K D_K P_K'$.

One may notice that we have been able to divide the risk into a component representing the systematic risk and another representing the non-systematic risk, in a similar manner as in the statistical factor model. However we have not performed a regression in this case.

2.5 Rationale for reduction of dimension

The rationale for the reduction of dimension is that the sample covariance matrix can be filtered by removing the eigenstates that correspond to pure noise [5]. These are the eigenstates that have the lowest eigenvalues and thus the lowest risk. If these eigenstates were not filtered out a portfolio optimization process would be deceived into placing large weights into these perceived low-risk eigenstates, when in reality they are merely a product of an outcome from random noise. The dimension reduction can in fact be seen as a type of safeguard against overfitting. Other empirical studies show empirically that this idea of removing eigenstates corresponding to noise from the sample covariance matrix works well and is a good idea [2].

2.6 Transaction cost function

The modeling of transaction costs is a subject that has historically been frequently disregarded, in academia at least. It is not very difficult to see why this is. It is a highly complex problem that requires large amounts of very time-accurate trading data. The end results are often unreliable and in constant need for calibration. The data required for modeling and calibration are often not readily available. Recently however this subject has started receiving more attention from researchers. Perhaps due to a demand for such models within the trading community. According to researchers at Deutsche Bank [6], US large cap funds underperformed the S&P 500 by 40 bps each year on average over 5 years ending June 2008, while the average cost for US large cap trades during almost the same 5 year period was 23 bps. It is perfectly reasonable to think that an accurate understanding of transaction costs has a major effect on the performance of trading strategies.

In this thesis 3 types of transaction costs will be considered, commission, bid-ask spread and market impact.

$$\textit{transaction costs} = \textit{commission} + \textit{spread} + \textit{market impact}.$$

Commission is the simplest to model. It is simply a fixed percentage of the value of the trade. In reality there is also a small fixed amount paid to the custodian when assets are sold or bought. This fee is for most institutional investors small enough to be neglected in the transaction cost model.

The bid-ask spread can in principle be tricky to model. Since one will typically be forced to concede the entire spread if one is the initiator of a trade and none if one is not the initiator. A reasonable assumption would be to model the spread cost as half the percentage spread. It is assumed one does not know beforehand whether one will be forced to initiate the trade or if one can act as the passive party. One could therefore simply assume that on average half of the trades are initiated trades and the other half passive. Though another coefficient than 0.5 will be used in this thesis, which will be explained further down.

The market impact of trading itself is the really tricky part of the transaction cost function. In principle different definitions of market impact could be considered. In this thesis we use the definition of implementation shortfall as market impact given in [8], where it is defined as pre-trade price minus execution price.

Because the type of data needed to model and calibrate the market impact is not available to the author of this thesis, a review of available market impact functions was made to choose a suitable one. Many models are available, however many of them have constants that have not been calibrated, which is what is needed in this case. There is also a trade-off between complexity of the model and the ease with which it can be used in an optimization problem. The model that was chosen when taking into account these considerations was the Bloomberg+ model [6]. The Bloomberg+ model is a ready calibrated square-root model for market impact. The square-root model is well-known to people involved in estimating transaction costs. It states that the price impact in percent figures is proportional to the square root of the volume traded. More precisely

$$\textit{market impact} = \alpha \cdot \sigma \cdot \sqrt{\frac{V}{V_{daily}}},$$

where α is a numerical constant to be calibrated from market data, σ is the daily volatility of the asset, V is the traded volume and V_{daily} is the average daily volume of the asset. This model is justified using the inventory risk of a liquidity supplier in [7]. The Bloomberg+ model is given by

$$0.433 \cdot \frac{S}{P} + 0.353 \cdot \sigma \cdot \sqrt{\frac{V}{V_{daily}}},$$

where $\frac{S}{P}$ is the percentage spread. That is S is the spread and P the price of the asset, both in units of currency. One may note that the coefficient in front of the percentage spread is 0.433 and not 0.5. Presumably the researchers at Bloomberg have estimated this to give a better fit to trading data. This can also be seen in the performance comparisons of different models in [6]. The numerical constant α in the square-root model is seen to be 0.353 in the Bloomberg+ model. Now the final transaction cost function that will be used in the simulations may be written in its entirety.

$$\textit{transaction costs} = C + 0.433 \cdot \frac{S}{P} + 0.353 \cdot \sigma \cdot \sqrt{\frac{V}{V_{daily}}},$$

where C is the commission given as percentage figure.

The transaction cost function used later in the optimization will be

$$\begin{aligned}\varphi(\mathbf{w} - \mathbf{w}_{curr}) = \sum_{i=1}^{N} (C + 0.433 \cdot (\frac{S}{P})_i)|w_i - w_{i,curr}| + \\ 0.353 \cdot \sigma_i \cdot \sqrt{\frac{Val_{port} \cdot |w_i - w_{i,curr}|}{P_i \cdot V_{i,daily}}} \cdot |w_i - w_{i,curr}|,\end{aligned}$$

where $\mathbf{w}$ is vector of portfolio weights to be held after the transaction is complete, $\mathbf{w}_{curr}$ is the vector of portfolio weights currently held, Val_{port} is the value of the entire portfolio, P_i is the price of asset i, w_i is the i:th component of $\mathbf{w}$ and $w_{i,curr}$ is the i:th component of $\mathbf{w}_{curr}$. This function is of the form

$$\varphi(\mathbf{w} - \mathbf{w}_{curr}) = \sum_{i=1}^{N} \left(c_{1,i} \cdot |w_i - w_{i,curr}| + c_{2,i} \cdot |w_i - w_{i,curr}|^{\frac{3}{2}} \right),$$

where $c_{1,i}$ and $c_{2,i}$ are some non-negative constants. The optimization will be over the w_i:s and the domain formed by the w_i:s will be a convex set. Since the transaction cost function is a sum of functions of the form $|w_i - c|^p$ where $p \geq 1$ and c is some constant, it follows that the transaction cost function is a convex function. The fact that the transaction cost function is convex will be useful, since it will give a convex optimization problem which is easier to handle than a general optimization problem.

Chapter 3

Validation

This chapter will cover the validation of the the estimated covariance matrix. The validation of the transaction cost function may very well be an equally interesting and important endeavour as the validation of the estimated covariance matrix. This is however outside of the scope of this thesis. As mentioned earlier, the transaction cost function will be taken from the existing literature on the matter. First the data set used in the validation and simulation is described.

3.1 Data set

The data set used for the computations in this thesis were from the SIX60 index, which contains the 60 stocks with the highest turnover on the Stockholm Stock Exchange[1]. 756 data points consisting of daily price quotes between the dates $2010-01-01$ and $2013-01-01$ were chosen. The price quotes were transformed into 755 daily return. 2 of the stocks were left out from the data set due to incomplete time series. In total this gave 755 data points of daily returns on 58 assets from the SIX60 index.

3.2 Simulation of minimum variance portfolios

As a first approach to compare the performance of different estimations of the covariance matrices one may simulate minimum variance portfolios constructed using the different estimation approaches. The 3 different approaches to be compared are the ordinary sample covariance matrix given by (2.1), a covariance matrix obtained from a factor model using principal components as factors described in the section *Statistical factor model* and a filtered covariance matrix described in the section *Random Matrix Theory*. 3 simulations of 3 minimum variance portfolios are performed. The full sample used for the simulation consists of 755 time series of 58 different assets, as described in the section *Data set*. The full sample is common to all simulations. The simulations start at the time step $t = 100$ and estimates the covariance matrix by the 3 approaches, respectively. The 2 largest eigenstates are used for the factor model approach and the filtered covariance matrix. Then 3 separate fully invested minimum variance portfolios are formed and purchased, with an initial investment of 1 units of currency. The initial investment will not matter since the results will be given in relative figures. After 7 days 3 new fully invested minimum variance portfolios are formed with the money available after 7 day in each respective simulation. However new covariance matrices are estimated from the latest 100 time points. That is time points from $t = 8$ to $t = 107$. This process is repeated 93 times, successively moving forward.

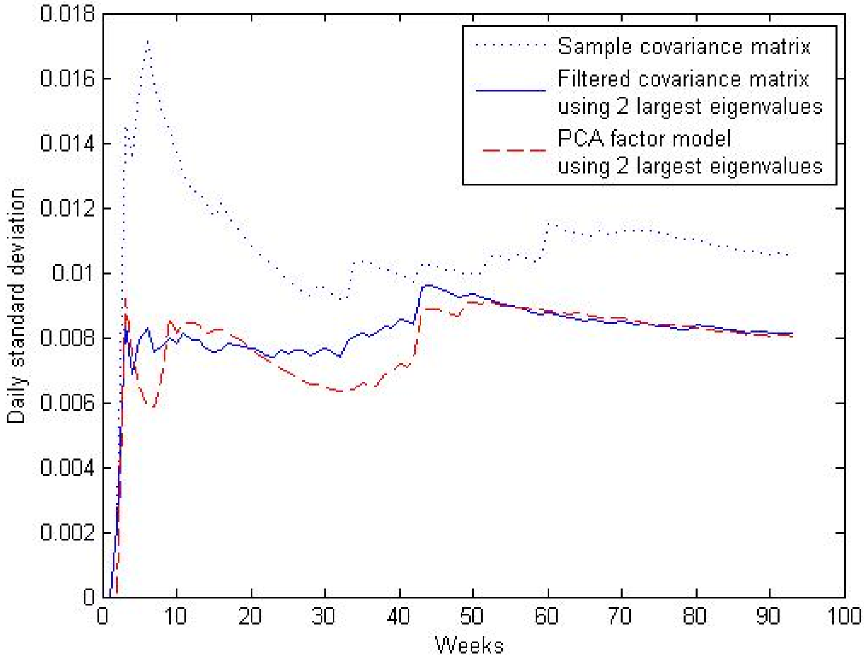


Figure 3.1: Simulation of 3 minimum variance portfolios. Weeks refer to a period of 7 trading days.

A graph of the evolution of the portfolio standard deviations can be seen in Figure 3.1. The 7-day standard deviations obtained from the simulation are rescaled to 1-day standard deviations.The standard deviations are the standard deviations that have accumulated so far in the simulation, that is using all the previous returns to calculate the accumulated standard deviation. One may notice that the filtered covariance matrix and the PCA factor model outperform an ordinary sample covariance matrix. As already stated earlier in the section *Rationale for reduction of dimension* this is because the the ordinary sample covariance matrix allows for portfolios that are erroneously perceived as very low risk. The result suggests that one should not use an ordinary sample covariance matrix to estimate the correlations and volatility of assets when better approaches are available. There is no clear difference between the PCA factor model and the filtered covariance matrix.

3.3 Cross-validation

An other aspect of the validation of the estimated covariance matrix concerns how stable the estimation is. Or in other words how well does an estimation of the covariance matrix perform on a data set independent of the data set used for the estimation. To shed some light on this matter an 8-fold cross-validation has been performed. Almost the entire data set with 752 time points was divided into 8 sections of equal size, according to the scheme in Figure 3.2. The original data set consisted of 755 time points, however the 3 last time points were omitted in order to facilitate the division of the data set into 8 equally large sections. 7 of the sections were used as the so called training set and 1 section as validation set. This procedure was repeated 8 times, each time with a different section as validation set. A more thorough description of cross-validation can be found in many books on data analysis [3]. A random long-only, fully invested portfolio with the constraint that no one asset weight is more than 10 percent was chosen. That is the portfolio weights, $\mathbf{w} = (w_1, ..., w_n)'$, had to fulfill

$$\begin{aligned} &\mathbf{w}' \cdot \mathbf{1} = 1 \\ &\mathbf{w} \geq \mathbf{0} \\ &w_i \leq 0.1 \qquad i = 1, ..., n, \end{aligned}$$

where vector inequalities are to be interpreted as elementwise inequalities and $\mathbf{0}$ and $\mathbf{1}$ are the vectors with zeros and ones for all elements. In each fold the training set was used to estimate a filtered covariance matrix with 4 eigenstates according to the scheme detailed in the section *Random Matrix Theory*. Then the predicted variance of the random portfolio was calculated according to

$$\sigma^2_{pred} = \mathbf{w}' \Omega_{train} \mathbf{w} \in \mathbb{R}.$$

The validation set was then used to compare what the actual variation of the returns of the same random portfolio was in the time period consisting of the validation set. 50 random portfolios were chosen and the 8 fold cross-validation was performed on each random portfolio. This resulted in 400 predictions of volatility and 400 accompanying realized volatilities. The

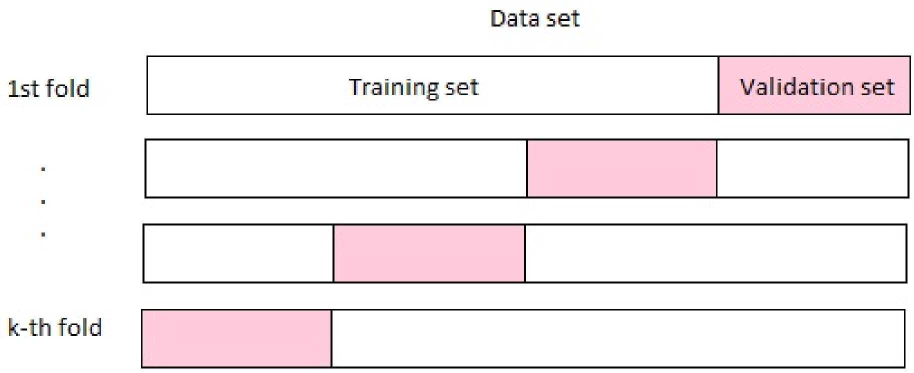


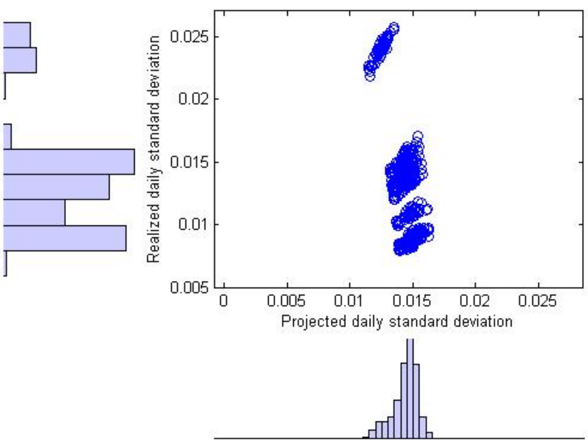
Figure 3.2: k-fold cross-validation.

results can be seen in Figure 3.3. The standard deviations are daily standard deviations.

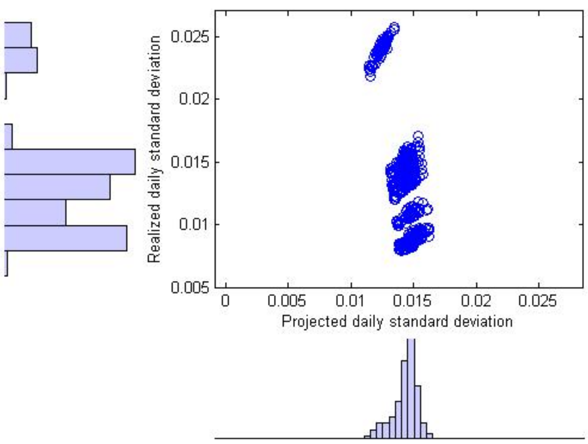
At the same time as the predicted variation of the random portfolio was calculated using the filtered covariance matrix, the variation was also predicted using the ordinary sample covariance matrix.

First one may observe that there is no big different between the predicted variances using the filtered covariance matrix and the ordinary sample covariance matrix. At first sight this may contradict the results in the section *Simulation of minimum variance portfolios*, where the filtered covariance matrix was shown to be superior to the ordinary sample covariance. However what the filtration does is to remove the possibility to invest in suspiciously low-risk portfolios arising from over-fitting. This does not contradict the fact that if we choose a portfolio at random both the filtered and the ordinary sample covariance matrix will typically yield about the same prediction and thus prediction error for the portfolio in question, which is what the results show.

The groupings or islands of data points in the scatter plots represent the 8 different folds. We note that the risk is underestimated considerably in one of the folds and overestimated in a few of them. A cross-validation was also performed for a single market factor model. The results were not noticeably different from the filtered matrix case. The fact that we obtain under- and overestimations occasionally is due to the heteroscedasticity of the returns time series, that is fluctuating volatility and correlation of the assets on the market.



(a) Scatter plot for the filtered covariance matrix.



(b) Scatter plot for the ordinary sample covariance matrix.

Figure 3.3: Results of 50 random portfolios used for the cross-validation.

Chapter 4

Simulations and results

This chapter contains the simulations and results of this study. The theory of chapter *Theoretical background* will be used. The central optimization problem that is solved in this section is

$$\begin{aligned} \underset{\mathbf{w}\in\mathbb{R}^n}{\text{minimize}} \quad & (\mathbf{w}-\mathbf{w}_{bench})'\Omega(\mathbf{w}-\mathbf{w}_{bench}) + \lambda \cdot \varphi(\mathbf{w}-\mathbf{w}_{curr}) \\ \text{subject to} \quad & \mathbf{w}' \cdot \mathbf{1} \leq 1 \\ & \mathbf{w} \geq \mathbf{0}, \end{aligned}$$

where $\varphi(\mathbf{w})$ is the transaction cost function described in the section *Transaction cost function*, $\mathbf{w}$ is the weights to be minimized, $\mathbf{w}_{bench}$ is the benchmark weights, $\mathbf{w}_{curr}$ is the current portfolio weights, Ω is the estimated covariance matrix and λ is an inverse risk aversion coefficient. The reason for having $\mathbf{w}-\mathbf{w}_{curr}$ as the argument of $\varphi(\cdot)$ is that this vector contains the weights that represent the assets that need to be either bought or sold. The vector $\mathbf{w}-\mathbf{w}_{bench}$ contains the active weights. The constraints represent the fact that buying for more cash than is available is not allowed and that short-selling is not allowed.

4.1 Optimization algorithm

The optimizer that was used to find the optimum of the considered problem was Matlab's optimizer *fmincon*, which allows for constrained nonlinear multivariable functions which is the case under consideration. The optimization algorithm that was chosen was the *active-set* algorithm, which uses a sequential quadratic programming method. The main advantage of this algorithm is ease of use. For example no gradient of the objective function needs to be supplied by the user.

One may note that the constraints are affine functions and the objective function is a convex function thus the problem is a convex optimization problem. The short proof of the following theorem can be found in [4].

Theorem 4.1 *Let x_{opt} be a local minimizer of a convex optimization problem. Then x_{opt} is also a global minimizer.*

This theorem allows one to conclude that the minimum that is found is truly a global minimum.

4.2 Transaction cost-risk frontier

By considering the inverse risk aversion parameterλ as a parameter that can vary one may construct a transaction cost-risk frontier analogous to Markowitz's efficient frontier in the mean-variance framework. $\mathbf{w}_{curr}$ was set to the zero vector and the benchmark and assets that were used are the ones described in the section *Data set*. This describes the situation where the portfolio consists of no assets and a re-balancing is needed to approach the benchmark. The cash amount available for investment is set to 10 billion SEK. At a first glance it may seem unnecessary to set a cash amount available for investment since we are dealing with relative returns, that is percentage changes, however one should note that the transaction cost function is very much dependent on the **number** of stock purchased, this is because of the market impact part of the transaction costs. The transaction cost-risk frontier was generated with values of λ ranging from 0 to 0.2, see Figure 4.1. A filtered covariance matrix with 4 eigenstates was used to estimate the covariance matrix. Note that it is the active risk that is being considered in this transaction cost-risk frontier.

Each time one wishes to re-balance one's portfolio it may be a good idea to generate such a transaction cost-risk frontier in order to visually inspect what the added cost for a lesser risk would be. 3 trajectories similar to the transaction cost-risk frontier were generated using a PCA factor model with 4 eigen-vectors. The 3 trajectories corresponding to the systematic risk, non-systematic risk and the total active risk can be seen in Figure 4.2.

An important fact to note is that the risks are given as standard deviations. This implies that one may not simply add the systematic and non-systematic risks and expect to obtain the total active risk. Rather, under the assumption that the systematic and non-systematic risks are independent, we have the formula

$$\sigma_{total}^2 = \sigma_{systematic}^2 + \sigma_{non-systematic}^2$$

where the σ:s are standard deviations. This leads to the conclusion that the contribution of the non-systematic risk to the total active risk is miniscule compared to the systematic risk. Which one may also notice from comparing the frontiers for total active risk and systematic risk and observing that they almost coincide. Intuitively this makes sense since the optimization is trying to achieve the benchmark, a reasonably well diversified portfolio.

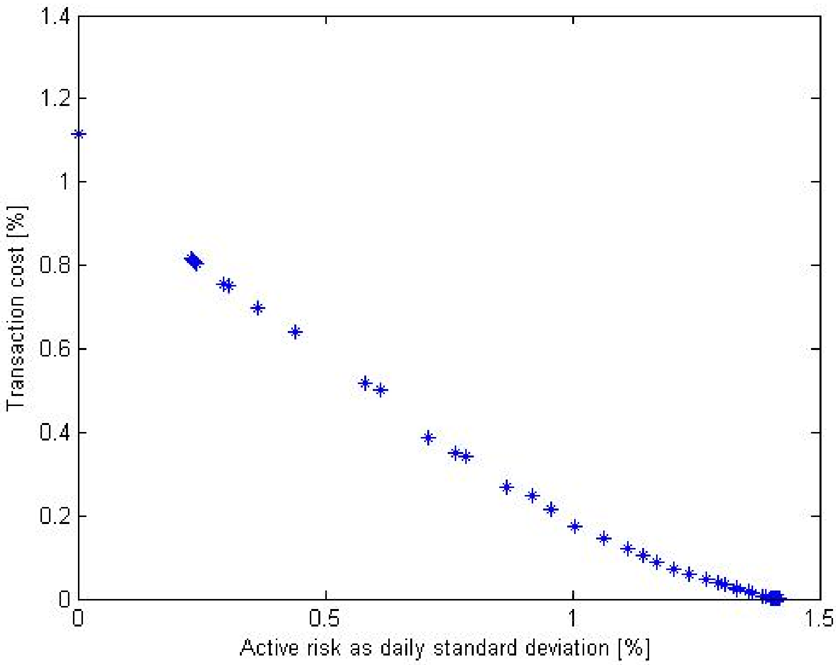


Figure 4.1: Transaction cost-risk frontier. $0 \leq \lambda \leq 0.2$. Portfolios below the frontier are not possible to achieve.

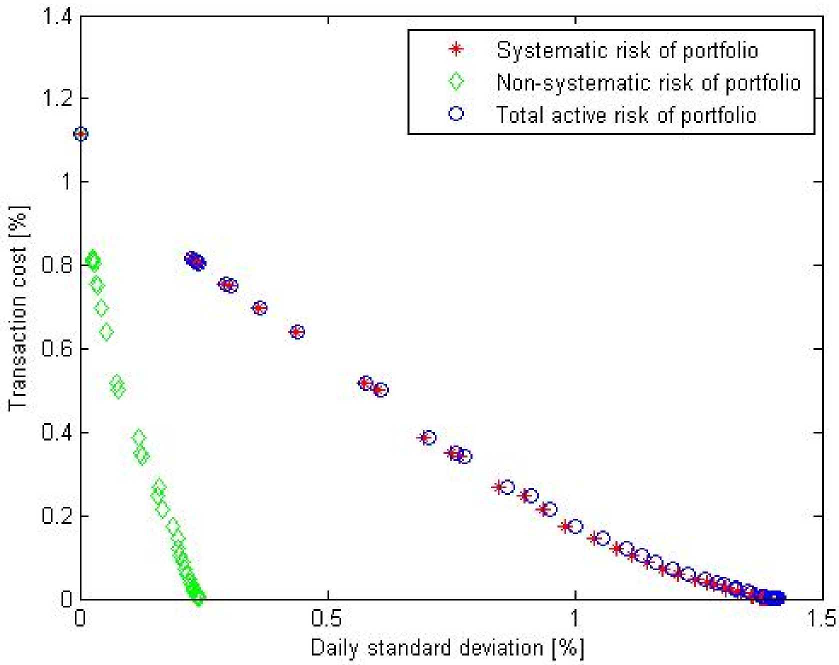


Figure 4.2: Transaction cost-risk frontier corresponding to systematic risk, non-systematic risk and the total active risk. $0 \leq \lambda \leq 0.2$.

Therefore one might expect to obtain a reasonably well diversified portfolio from the optimization. It is a well-known fact that non-systematic risk can be diversified away.

4.3 Rebalancing simulation

A simulation for different values of λ and different frequencies of rebalancing was performed using the data set described in the section *Data set*. The covariance matrix used in the optimization problem was the one described in the section *Random Matrix theory*, with 4 retained eigenstates. As already mentioned previously, the data set consisted of 755 daily returns over a two year period. The simulation was started at day 30 in order to have some data to estimate a covariance matrix when the simulation was started. Thus 725 days were simulated for different λ and frequencies of rebalancing. The rebalancing frequencies were, every 250,100,50 and 20 days.

The objective of the simulation was to see how the active risk and transaction costs are affected by the choice of rebalancing frequency and lambda. The simulation was started with a fully invested portfolio that perfectly matched the benchmark. The initial invested capital was chosen to 10 billion SEK. In reality a deviation from the benchmark weights, that is active risk, occurs when the portfolio obtains more cash from dividends or investments, or when the composition of the benchmark changes. In order to simulate dividends and further investments in the portfolio a cash amount was given to the simulated portfolio every day. The amount chosen was 10 percent of 10 billion SEK divided by 252, which is the approximate number of trading days. This is approximately equivalent to a cash addition of 4 bps of the initial investment a day.

A benchmark portfolio was also simulated alongside the portfolio tracking the benchmark, in order to measure the realized active risk. The benchmark portfolio was given an equally large initial investment of 10 billion SEK and the same cash daily cash amount was invested in the benchmark each day as for the tracking portfolio. However the benchmark was not made to incur any losses due to transaction costs.

The realized active risk was measured as a daily standard deviation. That is it was given by the sample standard deviation of the difference between the daily return of the tracking portfolio and the daily return of the benchmark portfolio. The results can be seen in Figure 4.3.

One may notice that for high λ, that is large active risk, the trajectories for different rebalancing frequencies follow each other very closely. Of course this is to be expected, since one is rebalancing one's portfolio with a very small amount. For lower values of λ one may notice that more frequent rebalancing will lower the transaction costs. This is also what one would expect, since the transaction cost function is sensitive to the trading volume.

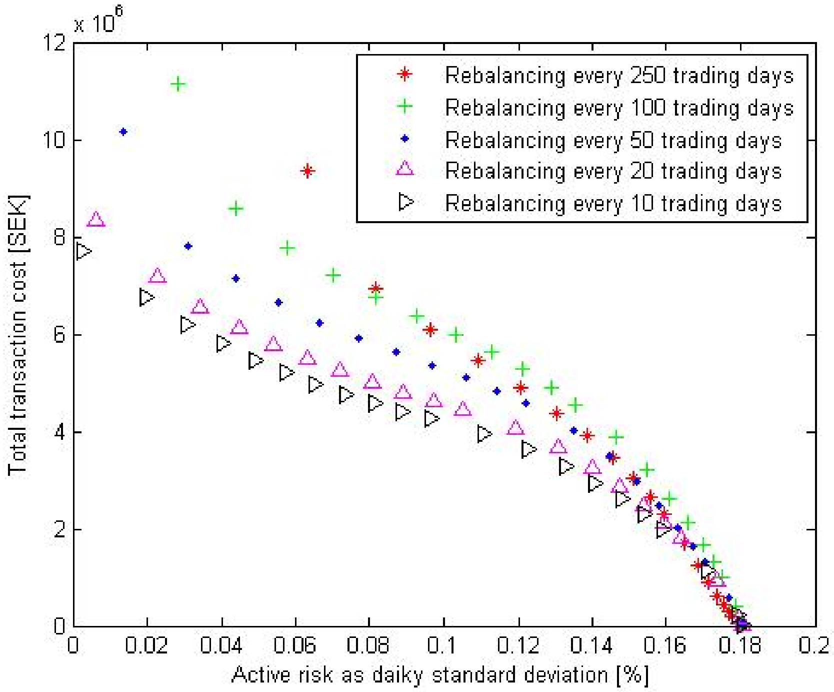
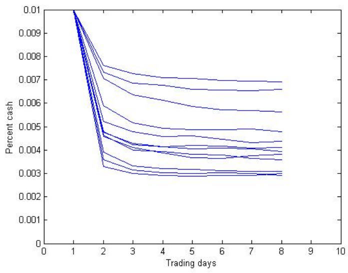


Figure 4.3: Active risk and the total transaction cost as a function of rebalancing frequecy and λ. λ is the parameter generating the parameter curves. $0 \leq \lambda \leq 0.05$.

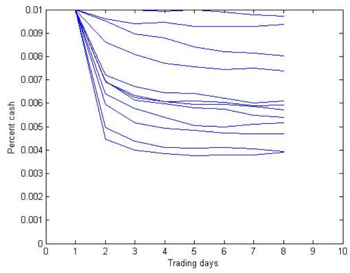
More frequent rebalancing reduces the need to trade large volumes.

4.4 Rebalancing a larger investment

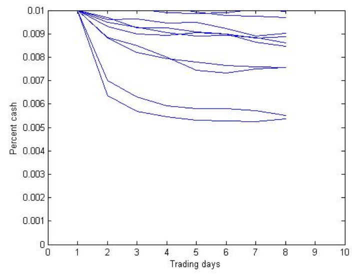
Another question of interest is, in which way and how fast does the optimization problem rebalance a portfolio after it receives a larger cash amount. This case was also simulated to shed some light on the matter. The simulation was performed on the same data set as previously. Time points with a distance of 50 days between successive time points was chosen. For each time point the portfolio was given a cash amount of 1 and 5 percent ($\alpha = 0.01$ and 0.05) of the value of the portfolio which was 10 billion SEK. That is the portfolio was given cash amounts of 100 million and 500 million SEK. This was done for 3 different values of the inverse risk aversion coefficient λ. The portfolio always consisted of the benchmark portfolio before it was given the cash amount. The results can be seen in Figure 4.4. Surprisingly one may notice that the optimizer does not create a fully invested portfolio. The reason for this is that there is always a cost for trading even very small amounts. The market impact part of the transaction cost function my converge towards 0 as the number of stocks traded decreases, however the commission and spread part of the transaction cost function remain the same (in relative figures). This means that the optimizer will stop trading after it has managed to decrease the active risk of the portfolio to a low enough level. Another thing that was observed is that the optimizer tends to invest in high beta stocks in order to achieve high correlation with the benchmark without having to buy too much. An approach to remove this behavior could be to require that be fully invested, which was not required in these simulations.



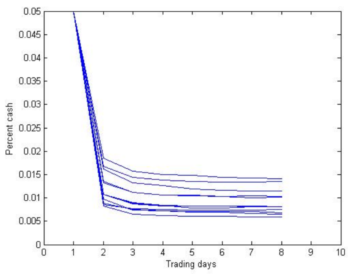
(a) The rebalancing trajectories for $\alpha = 0.01$ and $\lambda = 0.0005$.



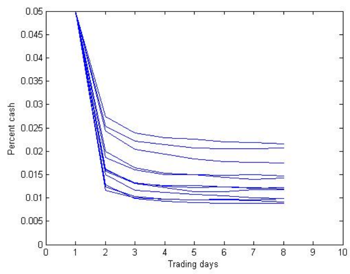
(b) The rebalancing trajectories for $\alpha = 0.01$ and $\lambda = 0.001$.



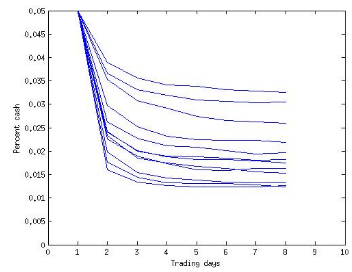
(c) The rebalancing trajectories for $\alpha = 0.01$ and $\lambda = 0.002$.



(d) The rebalancing trajectories for $\alpha = 0.05$ and $\lambda = 0.0005$.



(e) The rebalancing trajectories for $\alpha = 0.05$ and $\lambda = 0.001$.



(f) The rebalancing trajectories for $\alpha = 0.05$ and $\lambda = 0.002$.

Figure 4.4: The rebalancing trajectories for different cash amounts α and inverse risk aversion coefficients λ.

Chapter 5

Conclusions

The results of this thesis suggests that it does matter how one chooses to estimate the risk in ones model if one is going to use the risk estimate for optimization purposes. The ordinary sample covariance matrix was shown not to perform as well as the other models investigated. There does not seem to be a big difference between the two approaches called Random Matrix Theory and PCA Factor Model. However if one is using the risk estimate for risk projection purposes the results suggest that it does not matter which of the risk estimates investigated in this thesis one chooses. It is shown that many of the eigenstates of the ordinary sample covariance matrix correspond to the eigenstates of a covariance matrix obtained from pure noise. A few of the eigenstates corresponding to the largest eigenvalues can be considered meaningful and can in some cases be interpreted as explicit factors driving the market, e.g. the market factor or a grouping of the market by sector.

It was also shown that the optimization problem of balancing risk against transaction costs with the Bloomberg+ model as model for transaction costs leads to a convex optimization problem. This optimization problem is readily solvable using standard optimization tool kits in Matlab.

It was shown in the rebalancing simulation that for the rebalancing frequancies that were simulated a more frequent rebalancing will lead to lower transaction costs for the same level of active risk. This is because one is loosing trading volume for every trading day that the portfolio has an active risk and is not being rebalanced. That is the portfolio manager wishes to keep his or her participation rate as low as possible in order to obtain lower transaction costs.

The transaction cost model behaved a little bit differently from what one might suspect before a closer examination. The results suggest that it will make the optimizer invest a lot on the first day after receiving a larger cash amount, and much less on the following days. It will also not invest all of the received cash into stocks but rather try to reduce the active risk by investing in high beta stocks. This may not always be the behavior one is trying to achieve.

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