Kungliga Tekniska Högskolan

Master Thesis in Financial Mathematics

Analysis of Copula Opinion Pooling with Applications to Quantitative Portfolio Management

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Abstract

In 2005 Attilio Meucci presented his article *Beyond Black-Litterman: Views on Non-Normal Markets* which introduces the copula opinion pooling approach using generic non-normal market assumptions. Copulas and opinion pooling are used to express views on the market which provides a posterior market distribution that smoothly blends an arbitrarily distributed market prior distribution with arbitrarily chosen views. This thesis explains how to use this method in practice and investigates its performance in different investment situations. The method is tested on three portfolios, each showing some different feature. The conclusions that can be drawn are e.g that the method can be used in many different investment situations in many different ways, implementation and calculations can be made within a few seconds for a large data set and the method could be useful for portfolio managers using mathematical methods. The presented examples together with the method generate reasonable results.

Keywords: copula opinion pooling, copulas, t copula, views, tail dependence, maximum-likelihood, VaR, ES, mean-ES trade-off, portfolio theory, portfolio management, allocation, kernel estimation, correlation.

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Tyresö, May 24, 2015
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1 Introduction

This section explains the background, purpose and delimitations of the thesis.

1.1 Background

In the 1950s Harry Markowitz introduced his work on portfolio theory which is based on means and variances of asset returns computed from a given reference econometric model. His theory can be used to weight a portfolio of assets giving maximum return for a given risk level. Markowitz was an innovator in this field of financial mathematics and his findings are still taught at universities and give a good introduction to portfolio theory. However, using only means and variances for allocation have been criticised by practitioners because of e.g. the fact that variance is not the best measure of risk, a normal market distribution is needed and history most often will fail to predict the future.

In 1992 Fischer Black and Robert Litterman presented a framework for portfolio optimization using more inputs such as investor's market views and the level of confidence in these views to obtain the market distribution. The original Black-Litterman model is based on a number of normality assumptions which is not necessary a suitable choice of distribution. Both the market prior distribution and the views are assumed to be normally distributed. These assumptions are most often too strong because the distribution of the market may have fat tails, skewness and tail dependence, i.e. not normally distributed.

In 2005 Attilio Meucci presented his article *Beyond Black-Litterman: Views on Non-Normal Markets*, see [1], which extends the Black-Litterman methodology using generic non-normal market assumptions. Copulas and opinion pooling are used to express views on the market instead of views on parameters as in the Black-Litterman model. This approach provides a posterior market distribution that smoothly blends an arbitrarily distributed market prior distribution with arbitrarily chosen views. Meucci's copula opinion pooling (COP) will be further investigated in this thesis.

1.2 Purpose

The purpose of this thesis is to explain, analyze and apply the COP approach introduced by Attilio Meucci and see if it can be useful in portfolio management. This may interest portfolio managers using mathematical methods in their work but also financial mathematicians interested in more knowledge.

1.3 Delimitations

To be able to investigate COP one often has to fit multivariate distributions to given data. There are several methods to do this but within this thesis we have chosen to use one particular method. It is not evaluated if this method is the best to use in the situations we consider but the purpose of this thesis is to explain and present COP and the conclusions are not affected by the chosen method.

To make the analysis easier and to focus on the purpose of the thesis the number of assets used in the portfolio management examples are limited to at most five. In a real investment situation the number of assets could of course be much larger. Other assumptions made will be stated when necessary.

1.4 The Big Picture

Before introducing the COP approach we put the method into a bigger perspective in order to understand why this method may be useful and how we should use it. Briefly, the COP approach is a method for an investor to manipulate a probability distribution in line with forecasts. In this thesis we will include the COP approach in portfolio management and to get the big picture we first go through the needed steps according to [2].

First off we need to decide what kind of portfolio and market we want to model, which assets to include and the investment horizon. For a pension fund portfolio manager holding different kinds of bonds the market is modelled as a set of bonds and the investment horizon is often several years. A so called day-trader who wants instant profit can trade in many different assets but will prefer high volatility, which for instance can be found on smaller stock markets with high risk. A day-trader has an investment horizon of a day, a couple of hours or even shorter than that. We will use a set of N securities and an investment horizon τ. The known price of the securities at some time t is given by $\mathbf{p}_t$ meaning that we want to know the multivariate distribution of the prices at time $t+\tau$, $\mathbf{p}_{t+\tau}$, to be able to calculate the returns. The only information we have to model the future is the past and our own knowledge.

All markets have some characteristic that repeat itself through time which is called invariant. The invariant of a market can be used to model a set of independent and identically distributed (i.i.d.) random variables describing the market. In this work we will later on consider three different kinds of portfolios, namely a bond portfolio, an equity portfolio and a currency portfolio. For equities and currencies the typical invariant is return and for bonds the invariant is change in yield.

When we know which invariant to use we need to estimate the multivariate market distribution $\mathbf{X} = (X_1, \ldots, X_N)'$ from an invariant data set. One strength with the COP approach is that we do not have to assume some market distribution, we will use a generic one which we estimate non-parametrically. The main idea of the COP approach is that an investor can include experience at this stage to manipulate the market distribution. When the distribution of the invariant is modelled and the investor has included views the distribution is translated into a distribution of returns, if necessary.

To optimize a portfolio the investor has to decide which risk level and risk type to consider, how much money to invest and possible other constraints. When all this is done the optimal allocation can be computed according to the investor's specifications.

1.5 Outline

The rest of the thesis is divided into the following six sections:

- *Theoretical Background* explains and introduces the theory needed to fully understand the applications and analysis presented.
- *The General Theory of Copula Opinion Pooling* introduces the model and explains what happens when using COP.
- *Method, Copula Opinion Pooling in Practice* presents how COP will be applied in this thesis.
- *Data* presents the used data and where to find it.
- *Applications and Analysis* is the heart of the thesis and involves analysis of COP itself together with portfolio management cases.
- *Conclusions* sums up the thesis and states the conclusions.

2 Theoretical Background

In this section some theory needed to fully understand the following sections is presented. It is assumed that the reader has basic knowledge of mathematical statistics and financial mathematics. Some theory may be new and some may need to be freshened up.

2.1 Copulas

Joint distribution functions for a random vector implicitly contains both a description of the marginal distributions and of their dependence structure. Copulas provide a description of this dependence structure.

Consider a multivariate model $\mathbf{X} = (X_1, \ldots, X_N)$ where the components does not have a trivial dependency structure and marginal distribution functions $F_1, \ldots, F_N$. Given the vector $\mathbf{U} = (U_1, \ldots, U_N)$ where $U_i \sim U(0, 1),\ i = 1, \ldots N$, the quantile transform says that we can express $\mathbf{X}$ as

$$\mathbf{X} = (F_1^{-1}(U_1), \ldots, F_N^{-1}(U_N)).$$

$\mathbf{X}$ inherits the dependence among its components from $\mathbf{U}$. The distribution function C of a random vector $\mathbf{U}$ whose components are uniformly distributed on (0,1) is called a copula

$$C(u_1, \ldots, u_N) = P(U_1 \leq u_1, \ldots, U_N \leq u_N), \quad (u_1, \ldots, u_N) \in (0, 1)^N.$$

Let $(X_1, \ldots, X_N)$ be a random vector with distribution function F where $F_k(x) = \mathrm{P}(X_k \leq x)$ is continuous for every k. The probability transform says that the components of the vector $\mathbf{U} = (U_1, \ldots, U_N) = (F_1(X_1), \ldots, F_N(X_N))$ are uniformly distributed on $(0, 1)$. By definition, the distribution function C of $\mathbf{U}$ is a copula,

$$\begin{aligned} C(F_1(x_1), \ldots, F_N(x_N)) &= P(U_1 \leq F_1(x_1), \ldots, U_N \leq F_N(x_N)) \\ &= P(F_1^{-1}(U_1) \leq x_1, \ldots, F_N^{-1}(U_N) \leq x_N) \\ &= F(x_1, \ldots, x_N), \end{aligned}$$

which we call the copula of $\mathbf{X}$. Let us have a look at two examples of copulas.

Gaussian Copula

The copula $C_{\mathbf{R}}^{Ga}$ of a $N-$dimensional standard normal distribution, with linear correlation matrix $\mathbf{R}$ is defined as

$$C_{\mathbf{R}}^{Ga}(\mathbf{u}) = P(\Phi(X_1) \leq u_1, \ldots, \Phi(X_N) \leq u_N) = \Phi_{\mathbf{R}}^{N}(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_N)),$$

where $\Phi_{\mathbf{R}}^{N}$ is the distribution function of $\mathbf{X} \sim N_N(\mathbf{0}, \mathbf{R})$ and the copula $C_{\mathbf{R}}^{Ga}$ is the distribution function of the random vector $(\Phi(X_1), \ldots, \Phi(X_N))$ where Φ is the univariate standard normal distribution function.

Student's t Copula

The copula $C_{v,\mathbf{R}}^{t}$ of a $N-$dimensional standard Student's t distribution with $\nu > 0$ degrees of freedom and linear correlation matrix $\mathbf{R}$ is defined as

$$C_{\nu,\mathbf{R}}^{t}(\mathbf{u}) = P(t_\nu(X_1) \leq u_1, \ldots, t_\nu(X_N) \leq u_N) = t_{\nu,\mathbf{R}}^{N}(t_\nu^{-1}(u_1), \ldots, t_\nu^{-1}(u_N)),$$

where $t_{v,\mathbf{R}}^{N}$ is the distribution function of $\mathbf{X} \sim t_N(\mathbf{0}, \mathbf{R}, \nu)$. The copula $C_{v,\mathbf{R}}^{t}$ is the distribution function of the random vector $(t_v(X_1), \ldots, t_v(X_N))$ where t_v is the univariate standard Student's t_v distribution function.

2.2 Tail Dependence

For the pair (X_1, X_2) of random variables with equally distributed components the coefficient of upper tail dependence is defined as

$$\lambda_U = \lim_{x \to \infty} P(X_2 > x | X_1 > x), \tag{1}$$

and analogously lower tail dependence is defined as

$$\lambda_L = \lim_{x \to -\infty} P(X_2 \leq x | X_1 \leq x). \tag{2}$$

Suppose we want to simulate from a bivariate distribution (X_1, X_2) where X_1 and X_2 have some common distribution function F. If the dependence structure can be described by the Gaussian copula with correlation parameter ρ the upper tail dependence can be written as

$$\lambda_U = \lim_{x \to \infty} P(X_2 > x | X_1 > x) = \lim_{x \to \infty} P(\Phi^{-1}(F(X_2)) > \Phi^{-1}(F(x)) | \Phi^{-1}(F(X_1)) > \Phi^{-1}(F(x))), \tag{3}$$

and as $x \to \infty$, $\Phi^{-1}(F(x)) \to \infty$, so we can write

$$\lambda_U = \lim_{z \to \infty} P(\Phi^{-1}(F(X_2)) > z | \Phi^{-1}(F(X_1)) > z). \tag{4}$$

Under the Gaussian copula the pair $(\Phi^{-1}(F(X_1)), \Phi^{-1}(F(X_2)))$ has a bivariate normal distribution and λ_U is in fact equal to zero in this case, see [3] proposition 9.5. The symmetry of elliptical distributions gives that $\lambda_U = \lambda_L$.

If the dependence structure can be described by the t copula with ν degrees of freedom and correlation parameter ρ the upper tail dependence is

$$\lambda_U = \lim_{x\to\infty} P(X_2 > x | X_1 > x) = \lim_{x\to\infty} P(t_\nu^{-1}(F(X_2)) > t_\nu^{-1}(F(x)) | t_\nu^{-1}(F(X_1)) > t_\nu^{-1}(F(x))), \tag{5}$$

and if we set $z = t_\nu^{-1}(F(x))$ and again use the symmetry we may write

$$\lambda_U = \lambda_L = \lim_{z\to-\infty} P(t_\nu^{-1}(F(X_2)) \leq z | t_\nu^{-1}(F(X_1)) \leq z). \tag{6}$$

Under the t copula with ν degrees of freedom the pair $(t_\nu^{-1}(F(X_1)), t_\nu^{-1}(F(X_2)))$ has a Student's t distribution with ν degrees of freedom and since the t_ν-distribution is regularly varying at $-\infty$ with tail index $\alpha = \nu$ we may write

$$\lambda_U = \lambda_L = \frac{\int_{(\pi/2-\arcsin\rho)/2}^{\pi/2} \cos^\alpha t \,\mathrm{d}t}{\int_0^{\pi/2} \cos^\alpha t \,\mathrm{d}t}, \tag{7}$$

see [3] proposition 9.6. In Table 1 coefficients of tail dependence for the bivariate t copula are shown.

$\nu\backslash\rho$	-0.5	0	0.5	0.9	1
2	0.06	0.18	0.39	0.72	1
4	0.01	0.08	0.25	0.63	1
10	0.00	0.01	0.08	0.46	1
∞	0	0	0	0	1

Table 1: Coefficients of tail dependence for the bivariate t copula according to [4].

The t distribution when $\nu \to \infty$ is equal to the normal distribution and the last row in Table 1 shows what was claimed above, namely that the tail dependence of the Gaussian copula is zero. To illustrate tail dependence we show scatter plots with samples from two bivariate distributions with standard normal margins and $\rho = 0.9$. The difference between the distributions is that they have different dependence structures. The left plot in Figure 1 has a Gaussian copula and the right has a t copula with $\nu = 4$.

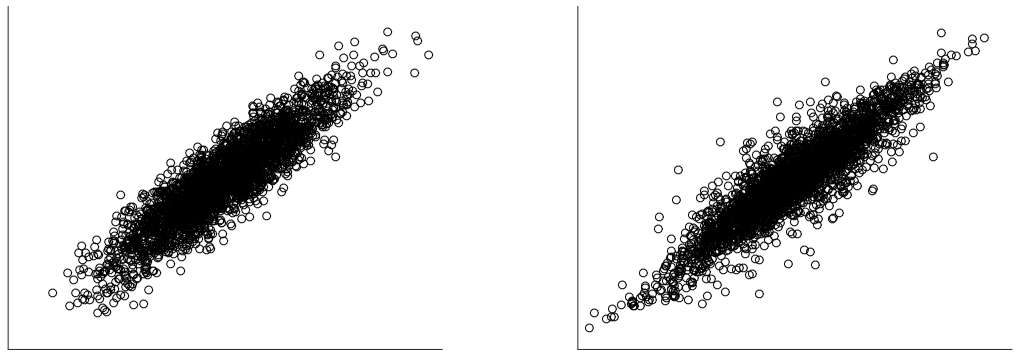

Figure 1: *Left plot*: Gaussian copula. *Right plot*: t_4 copula.

The tail dependences are easy to recognize, the Gaussian copula means zero upper and lower tail dependence and the t_4 copula with $\rho = 0.9$ means an upper and lower tail dependence of 0.63.

2.3 Maximum-Likelihood Estimation

Let $X_1, X_2, \ldots, X_N$ be random variables with joint probability function

$$f_{\boldsymbol{\theta}}(x_1, x_2, \ldots, x_N) = f(x_1, x_2, \ldots, x_N|\boldsymbol{\theta}),$$

where $\boldsymbol{\theta}$ is a vector of unknown parameters. Given observed data $X_1 = x_1, X_2 = x_2, \ldots, X_N = x_N$, the maximum-likelihood estimation of $\boldsymbol{\theta}$ is the $\boldsymbol{\theta}$ that maximizes the probability of the observed data.

If X_i for $i = 1, \ldots, N$ are independent and identically distributed the likelihood simplifies to

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{N} f(x_i|\boldsymbol{\theta}).$$

The logarithm of the likelihood is an increasing function so to simplify the maximization we will maximize the log likelihood

$$\ln(L(\boldsymbol{\theta})) = \sum_{i=1}^{N} \ln(f(x_i|\boldsymbol{\theta})).$$

2.4 Estimating the Parameters of a Copula

Now we will combine the theory of copulas and maximum-likelihood mentioned above to estimate the parameters of a copula. The following method is suggested by [5] to fit a copula to multivariate data.

Consider a multivariate distribution $\mathbf{X} = (X_1, \ldots, X_N)'$ with cumulative distribution functions (cdfs) $F_1, \ldots, F_N$ and a copula with parameter setup $\boldsymbol{\theta}$. We have a data set of T independent and identically distributed joint observations of $\mathbf{X}$ which form a $T \times N$ panel χ. From this panel we want to estimate the parameters of the copula.

A very general way to do this is to avoid choosing explicit parametric marginal distributions and instead use empirical distributions from which we will create a pseudo sample of the copula. We need empirical estimates $\hat{F}_1, \ldots, \hat{F}_N$ of the N marginal distributions and we suppose that $(\mathbf{x}_1, \ldots, \mathbf{x}_T)$ is a sample of N-dimensional i.i.d. vectors with $\mathbf{x}_i = (x_{1,i} \ldots, x_{N,i})$, $i = 1, \ldots, T$. We transform the vectors into pseudo observations

$$\hat{\mathbf{U}} = (\hat{F}_1(x_{1,i}), \ldots, \hat{F}_N(x_{N,i})), \quad i = 1, \ldots, T, \tag{8}$$

and get the parameters by solving

$$\text{maximize} \quad \ln(L(\boldsymbol{\theta})) = \sum_{i=1}^{T} \ln(C(\hat{\mathbf{U}}_i | \boldsymbol{\theta})). \tag{9}$$

Because of the fact that empirical distributions are used this method is called Canonical Maximum Likelihood.

2.5 Kernel Density Estimation

Let $X_1, \ldots, X_N$ be an i.i.d. sample from a distribution with pdf $f(x)$. To get a smooth estimate $\hat{f}(x)$ we can use a kernel smoother which will get rid of bumps and discontinuity of the estimated pdf. The kernel estimator of a pdf is defined as

$$\frac{1}{Nh} \sum_{i=1}^{N} K\left(\frac{x - X_i}{h}\right), \tag{10}$$

where $K(\cdot)$ is the kernel function satisfying $\int_{-\infty}^{\infty} K(x)dx = 1$ and is often symmetric around zero. This guarantees that the estimated $\hat{f}(x)$ is a density function, see [6]. h is the smoothing parameter and controls the smoothness of $\hat{f}(x)$. A larger h will generate a smoother $\hat{f}(x)$. For example the kernel function could be Gaussian, $K(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, which will ensure derivatives of all orders. The biggest drawback of the kernel approach occurs when the distribution has long tails and h is small, then there will be some noise in the tails. If h is to big, information in the main part of the distribution may disappear.

2.6 The Empirical Distribution

Consider a sample of independent and identically distributed vectors $\mathbf{X}_1, \ldots, \mathbf{X}_J$ with the common unknown distribution function $F(\mathbf{x}) = P(\mathbf{X} \leq \mathbf{x})$, where $\mathbf{X}$ is an independent copy of $\mathbf{X}_k$ and $\mathbf{X} \leq \mathbf{x}$ holds component-wise. An empirical distribution made from the sample $\mathbf{x}_1, \ldots, \mathbf{x}_J$ of random vectors $\mathbf{X}_1, \ldots, \mathbf{X}_J$ can be modelled by assigning probability weights $1/J$ to each observation.

We get

$$F_J(\mathbf{x}) = \frac{1}{J} \sum_{k=1}^{J} I\{\mathbf{x}_k \leq \mathbf{x}\}, \tag{11}$$

where $F_J(\mathbf{x})$ is the empirical distribution and I is the indicator function. For the vectors $\mathbf{X_1}, \ldots, \mathbf{X_J}$ we get the random counterpart of the empirical distribution as

$$F_{J,\mathbf{X}}(\mathbf{x}) = \frac{1}{J} \sum_{k=1}^{J} I\{\mathbf{X}_k \leq \mathbf{x}\}. \tag{12}$$

For a sequence $Z_1, \ldots, Z_J$ of independent copies of a random variable Z, where $E[Z]$ exists finitely, the (strong) law of large numbers gives

$$\frac{1}{J} \sum_{k=1}^{J} Z_k \to E[Z], \tag{13}$$

with probability 1 as $J \to \infty$. If we set $Z_k = I(\mathbf{X}_k \leq \mathbf{x})$ this gives $E[Z_k] = P(\mathbf{X}_k \leq \mathbf{x}) = F(\mathbf{x})$ and the law of large numbers says that $\lim_{J\to\infty} F_{J,\mathbf{X}}(\mathbf{x}) \to F(\mathbf{x})$. So if our sample is large enough the empirical distribution $F_{J,\mathbf{X}}(\mathbf{x})$ is a good approximation of the actual distribution function. Approximating distributions like this is called Monte Carlo simulation and will be used later on.

2.7 Selecting Sample Size

As we saw in the previous section increasing the sample size J will make the empirical distribution converge to the real distribution. Since we are dependent of computers in our calculations we will have to keep the sample J as low as possible without the result being suffering. Through graphical analysis we can decide if a sample is big enough and one way to do this is to use a qq plot and compare the empirical cumulative distribution function with the theoretical one. A qq plot is a probability plot which plots the quantiles of two probability distributions against each other. The theoretical distribution is not known so to simulate this a sample distribution with e.g. 10^7 samples could be used, which is much bigger than J. If the empirical distribution is qq plotted against the theoretical and they form a line with slope 1, it is a good fit. If we have two data sets of the same size one orders each set in increasing order and plot them against each other. If the data sets are of different sizes interpolated quantile estimates are used so that quantiles corresponding to the underlying distribution can be made.

2.8 Non-Normal Distributions

In finance one often makes assumptions regarding probability distributions, e.g. returns are often considered to be normally distributed. But in the real world there are far more non-normal than normal distributions and this holds for the financial world too. Non-normal distributions can be described by skewness and kurtosis. The skewness sk of the continuous probability distribution of the random variable X is defined as

$$sk = \frac{\mathrm{E}[(X-\mu)^3]}{\sigma^3}, \tag{14}$$

and for a sample of size J the skewness is

$$sk = \frac{1}{J-1}\sum_{i=1}^{J}\frac{(x_i-\mu)^3}{\sigma^3}, \tag{15}$$

where μ is the expected value, σ is the standard deviation and J is the number of data points. The skewness measures the asymmetry of the distribution and $sk \neq 0$ means that the distribution is asymmetric. The normal distribution has a skewness of zero. In Figure 2 we graphically present skewness.

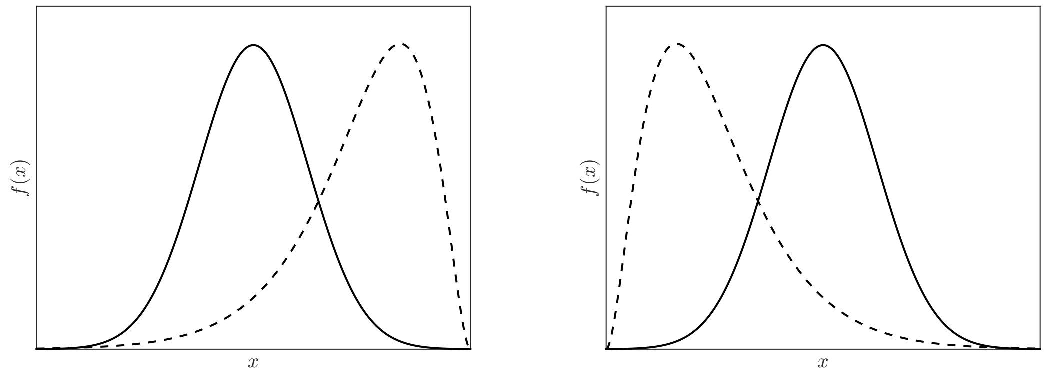

Figure 2: *Left picture*: skewed left. *Right picture*: skewed right. The dashed curves are skewed distributions and the full curves are normally distributed with $sk = 0$.

Negative skewness means left skewed and positive skewness means right skewed. Assume that the probability function $f(x)$ represents the return of a long investment, then investors would prefer a left skewed distribution due to higher probability of a high return.

The kurtosis ku of the continuous probability distribution of the random variable X is defined as

$$ku = \frac{\mathrm{E}[(X-\mu)^4]}{\sigma^4}, \tag{16}$$

and for a sample of size J the kurtosis is

$$ku = \frac{1}{J-1}\sum_{i=1}^{J}\frac{(x_i-\mu)^4}{\sigma^4}. \tag{17}$$

Kurtosis says whether the probability of unusual events is higher or lower compared to the normal distribution which has $ku = 3$. Kurtosis is graphically shown in Figure 3.

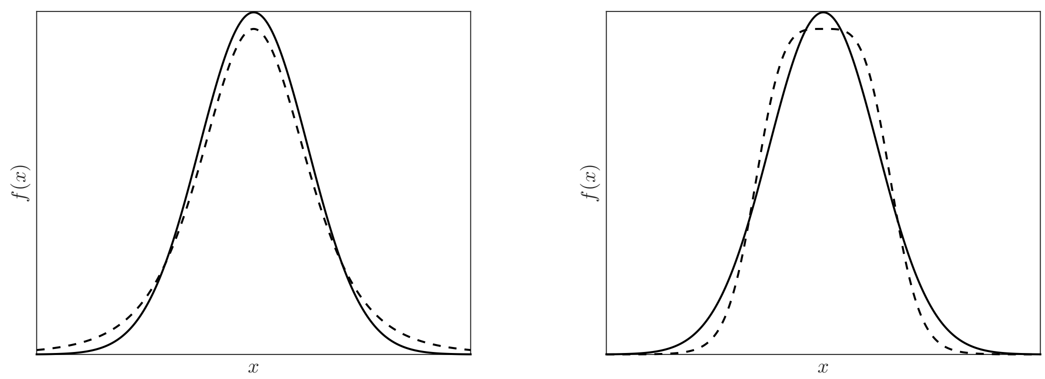

Figure 3: *Left picture*: $ku > 3$. *Right picture*: $ku < 3$. The dashed curves are distributions with $ku \neq 3$ and the full curves are normally distributed with $ku = 3$.

$ku > 3$ indicates a fat tailed distribution meaning that the probability of extreme events is greater than for a normal distribution. Assuming we have a distribution of returns, then an investor who wants to be more certain about the outcome would prefer low kurtosis. High kurtosis means higher risk for losses but also higher probability for high returns.

2.9 Investor's Views and Strategies

We will now have a look at some investment views and strategies. As mentioned before we will consider portfolios of bonds, equities and currencies. A bond portfolio manager will most often have views on the yield curve. To better understand the reasoning we will first show some bond characteristics.

Bonds are traded on yield or price. In the Swedish bond market praxis is to express the level of a bid or offer in terms of yield, but many other markets use price. Most bonds pay coupons at fixed predetermined times to the holder and if they do not, they are called zero-coupon bonds. We define the price of a bond as

$$p = \sum_{i=1}^{N} c_i e^{-r_i t_i} + K e^{-r_i t_N}, \tag{18}$$

where K is called the nominal amount which the holder receives at the time of maturity, t_N, c_i are the coupons payed out at the times t_i and r_i are the different zero rates for $i = 1, \ldots, N$.

Yield to maturity of a bond is defined as the interest rate y that solves the equation

$$p = \sum_{i=1}^{N} c_i e^{-yt_i} + K e^{-yt_N}, \tag{19}$$

where p is the market implied price in (18). The yield curve is built up by yields from bonds with different maturities. For illustration we in Figure 4 show an example of a linearly interpolated Swedish government yield curve built up by the $2, 5, 10, 15, 20$ and 25 year yields as of September 2014.

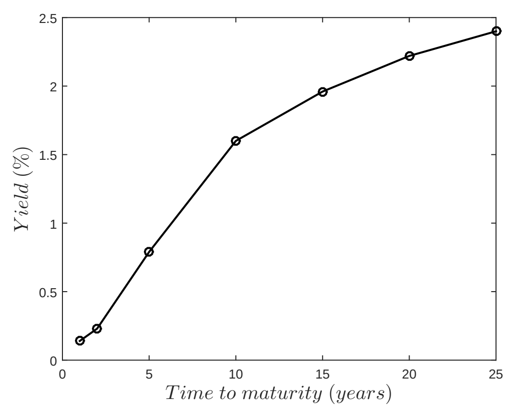

Figure 4: Swedish government yield curve as of September 2014.

Another term often used when talking about bonds is duration. The duration of a bond is defined as

$$D = \frac{\sum_{i=1}^{N} t_i c_i e^{-yt_i} + t_N K e^{-yt_N}}{p}, \tag{20}$$

which means that duration is expressed in years and it is a weighted average of the coupon dates where the weight for each coupon date is the present value of the coupon payment divided by the present value of all payments [7]. Duration is a measure of sensitivity of the price to changes in yield and higher duration means higher interest rate risk. So for example a 25 year bond will have higher duration than a 1 year bond meaning it is more risk-full, which feels intuitive. Approximately it holds that

$$\frac{\Delta p}{p} \approx -D \Delta y, \tag{21}$$

which is a first order approximation of changes in the bond price meaning that duration for bonds corresponds to delta for derivatives. A second order approximation of bond price changes is

$$\frac{\Delta p}{p} \approx -D\Delta y + \frac{1}{2}C(\Delta y)^2, \tag{22}$$

where $C = \frac{d^2p}{dy^2}$ is the convexity of the bond, corresponding to gamma for derivatives.

Let us now consider some different views and strategies for bonds. *Note: when describing some scenarios in this section the reader may ignore financial jargon and focus on the concluding parts. The scenarios are included to give more vivid examples and are inspired by* [8]. Imagine a Swedish investor who has a directional view on rates. Let us assume the market is pricing a 50% probability for a 10 basis points (bps) repo rate cut, meaning that rates should go down some bps in case of a cut. The meaning of a rate cut is to get a rebound in growth and an increased inflation which leads to higher long-end yields. In case of no cut, rates should go up due to the current pricing and with time the probability of rate cuts should decrease due to a better business cycle. This scenario also leads to higher long-end yields. Because of this the investor believes rates will go up in the future and as (18) and (19) indicates, bond prices go down when yields go up and vice versa. The investor should therefore short bonds with longer maturity, e.g. short the 10 year government bond.

Another example is a view on curvature, meaning that the yield curve should either be steeper or flatter. An investor faces the following situation, the 2 year government yield is very low which eventually will lead to higher economic activity and inflation. The yield curve should be much steeper due to the level of front-end yields. The investor choose to focus on the slope of the yield curve between 2 and 10 years believing that the 2 year yield will decrease and the 10 year yield will increase, i.e. a steeper curve. The view of the investor is illustrated in Figure 5.

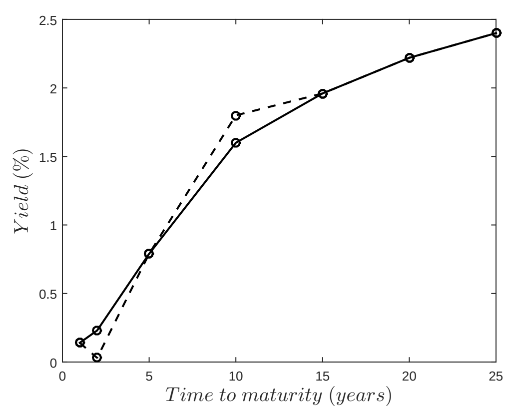

Figure 5: Current yield curve (full line) and the investor's forecast (dashed line).

The investor may also have a strategy on duration. Consider an investor who holds a portfolio with an average duration of 15 years, i.e. it contains mostly long-dated bonds. Now the investor believes the yield curve will parallel shift up 20 bps in a short period of time shown in Figure 6.

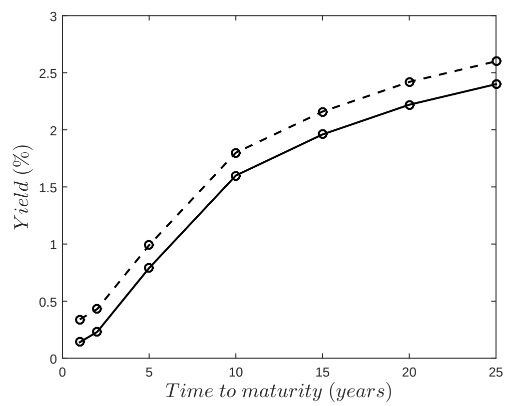

Figure 6: Current yield curve (full line) and the investor's forecast (dashed line).

Due to the high duration of the current portfolio the investor will loose a lot of money if the parallel shift takes place. Hence, the investor should short duration to hedge against rising rates.

For equity and currency portfolios one can of course have a bunch of different strategies as well. We will consider outright equity and currency portfolios which means we will not involve derivatives. This leads to quite simple strategies as e.g. *stock A will increase 5-10 % in 3 months* or *an investment in currency A will outperform an investment in currency B by 10 % given an investment horizon of 6 months.*

2.10 Risk Measures

Two commonly used risk measures are presented below.

2.10.1 Value-at-Risk

The value-at-risk (VaR) is a widely used risk measure. VaR at level $p \in (0, 1)$ of a portfolio with value X at time 1 is defined as

$$\mathrm{VaR}_p(X) = F_L^{-1}(1-p) = min\{m : P(mR_0 + X < 0) \leq p\}, \tag{23}$$

where R_0 is the risk-free percentage return of an asset. $\mathrm{VaR}_p(X)$ can be interpretted as the smallest value m such that the probability of the discounted loss $L = -X/R_0$ being at most m is at least $1 - p$.

Consider a sample $L_1, ..., L_n$ of independent copies of the discounted loss L, the empirical estimate of $\mathrm{VaR}_p(X)$ is then given by

$$\widehat{\mathrm{VaR}}_p(X) = L_{[np]+1,n}, \tag{24}$$

where $L_{1,n} \geq ... \geq L_{n,n}$ is the ordered sample and $[np]$ is the integer part of np.

2.10.2 Expected Shortfall

The expected shortfall can be interpreted as the expected loss that is incurred when VaR is exceeded. This risk measure considers the average VaR values beyond the level p and is defined as

$$\mathrm{ES}_p(X) = \frac{1}{p}\int_0^p \mathrm{VaR}_u(X)du. \tag{25}$$

This can be estimated as

$$\widehat{\mathrm{ES}}_p(X) = \frac{1}{p}\sum_{k=1}^{[np]}(\frac{L_{k,n}}{n} + (p - \frac{[np]}{n})L_{[np]+1,n}), \tag{26}$$

where $L_{1,n} \geq ... \geq L_{n,n}$ and $[np]$ is the integer part of np.

2.11 Portfolio Theory

In this section we go through some basic portfolio theory and end up with the allocation method we will use to further apply COP.

2.11.1 Mean-Variance Optimization

The classical mean-variance portfolio optimization by Harry Markowitz is defined as

$$\text{maximize} \quad \mathbf{w}^T\boldsymbol{\mu} - \frac{c}{2V_0}\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}, \tag{27}$$

$$\text{subject to} \quad \mathbf{w}^T\mathbf{1} \leq V_0,$$

where $c > 0$ is a constant, $\mathbf{w}^T = [w_1, \ldots, w_N]$ is the initial value of the position in the N assets respectively, V_0 is the initial capital, $\boldsymbol{\mu}$ is the vector of expected returns and $\boldsymbol{\Sigma}$ is the covariance matrix. This is a convex optimization problem with solution given by

$$\mathbf{w} = \frac{V_0}{c}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \frac{\max(\mathbf{1}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - c, 0)}{\mathbf{1}^T\boldsymbol{\Sigma}^{-1}\mathbf{1}}\mathbf{1}),$$

where the constant c depends on the risk aversion of the investor [3]. However, this approach only depends on means and variances which will fail to describe all possible probability distributions. We look at a simple example where an investor wants to build a portfolio of three stocks, A, B and C. The historical expected return and covariance matrix for a 6 month investment are

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_A \\ \mu_B \\ \mu_C \end{pmatrix} = \begin{pmatrix} 1.10 \\ 1.15 \\ 1.05 \end{pmatrix},$$

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_A^2 & \sigma_A\sigma_B\rho_{AB} & \sigma_A\sigma_C\rho_{AC} \\ \sigma_B\sigma_A\rho_{BA} & \sigma_B^2 & \sigma_B\sigma_C\rho_{BC} \\ \sigma_C\sigma_A\rho_{CA} & \sigma_C\sigma_B\rho_{CB} & \sigma_C^2 \end{pmatrix} = \begin{pmatrix} 0.2^2 & 0.04 & 0.024 \\ 0.04 & 0.25^2 & 0.0375 \\ 0.024 & 0.0375 & 0.3^2 \end{pmatrix}.$$

The optimal solution using $c = 1$ and $V_0 = 1$ is

$$\mathbf{w} = \begin{pmatrix} -0.88 \\ 2.74 \\ -0.86 \end{pmatrix},$$

meaning that we should buy B and short stock A and C. Solving the problem for different values of c and plotting the pairs $(\sigma(c), \mu(c))$ where $\sigma(c) = \sqrt{\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}}$ and $\mu(c) = \mathbf{w}^T\boldsymbol{\mu}$ gives the efficient frontier which is a set of optimal portfolios giving the highest expected return for some risk level. The efficient frontier for the example above is shown in Figure 7.

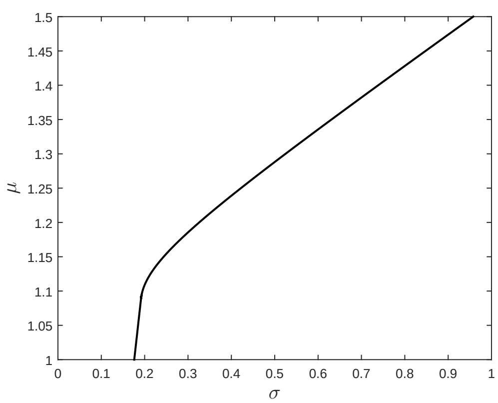

Figure 7: Efficient frontier for $0 < c \leq 35$.

This optimization procedure assumes normal markets and does not account for market asymmetries and extreme events.

2.11.2 Optimization in Non-Normal Markets

In many situations the actual distribution of the portfolio we want to optimize is non-normal which requires a method outside the Markowitz framework. Consider the returns $\mathbf{R}$ for a portfolio of N assets. An optimization method that accounts for asymmetries and extreme events is the swapped mean-ES trade-off defined as

$$\begin{gathered} \text{maximize} \quad \mathbf{w}^T \boldsymbol{\mu}, \\ \text{subject to} \quad \mathrm{ES}_p(\mathbf{w}) \leq \mathrm{ES}^*, \quad \mathbf{w}^T \mathbf{1} \leq V_0, \end{gathered} \tag{28}$$

where $\boldsymbol{\mu} = \mathrm{E}[\mathbf{R}]$ and ES^* is some desired highest level of expected shortfall of the resulting portfolio. Other constraints such as long only positions could also be included. The optimal solution to (28) giving the portfolio with relative weights $\mathbf{w}$, span an efficient frontier which can differ from the one from the mean-variance trade-off. A linear programming approach to solve the mean-ES trade-off defined as

$$\begin{gathered} \text{minimize} \quad \mathrm{ES}_p(\mathbf{w}), \\ \text{subject to} \quad \mathbf{w}^T \boldsymbol{\mu} \geq \mathbf{r}, \quad \mathbf{w}^T \mathbf{1} \leq V_0, \end{gathered} \tag{29}$$

is suggested by [9], which is similar to (28) but the conditions are switched. The MATLAB function called PORTFOLIOCVAR solves (29) as a non-linear function. In [10] it is shown that the efficient frontiers of the optimization problems (28) and (29) are equal, i.e it is possible to use PORTFOLIOCVAR to get the efficient frontier of (28).

3 The General Theory of Copula Opinion Pooling

This section explains the theory behind the COP approach.

First we need a representation of the financial market we want to model. Consider the multivariate distribution $\mathbf{M} \sim (X_1, X_2, \ldots, X_N)'$ of e.g. returns on a set of N securities. $\mathbf{M}$ could also represent a non-linear model of risk-factors, times to default or other random variables in the market. $\mathbf{M}$, with cdf $F_{\mathbf{M}}$, is called the prior market distribution. The twist with the copula opinion pooling method is that the prior distribution can be modified according to the practitioner's own forecasts, here called subjective views. $K \leq N$ views can be stated on linear combinations of the market $\mathbf{M}$ and are represented by the pick matrix $\mathbf{P}$ with size $K \times N$ where the rows of the pick matrix determines the weight of each view. The fact that $\mathbf{P}$ may not be a square matrix is a problem and to solve this $N - K$ additional linear combinations without views are also added through a matrix $\mathbf{P}^{\perp}$ of size $(N-K) \times N$. $\mathbf{P}^{\perp}$ is a basis for the null space of $\mathbf{P}$. The resulting view-adjusted market coordinates are therefore defined as

$$\bar{\mathbf{P}} = \begin{pmatrix} \mathbf{P} \\ \mathbf{P}^{\perp} \end{pmatrix}, \tag{30}$$

which is of size $N \times N$ and invertible. This means that the market-implied distribution of the views is

$$\mathbf{V} = \bar{\mathbf{P}}\mathbf{M}, \tag{31}$$

and due to the fact that K views are stated the distribution function for the views is

$$\hat{F}_k(v) = P_{views}\{V_k < v\}, \quad k = 1, \ldots, K. \tag{32}$$

The corresponding prior distribution function is

$$F_k(v) = P_{prior}\{V_k < v\}, \quad k = 1, \ldots, K, \tag{33}$$

and the opinion pooling approach gives the resulting posterior distribution function

$$\tilde{F}_k(v) = c_k \hat{F}_k + (1 - c_k) F_k, \quad k = 1, \ldots, K, \tag{34}$$

where $c_k \in [0, 1]$ is the confidence level of each view, e.g $c_1 = 0.3$ means a 30 percent confidence in view 1. If the confidence in the views are zero the posterior distribution is equal to the distribution induced by the market and if the confidence level is one the posterior distribution is fully affected by the views.

Now when we know the posterior distribution of each view we want to determine the joint distribution of the views. The joint distribution includes information from the posterior marginal distributions of the views and the co-dependence structure implied by the prior market distribution. The copula of the posterior distribution of the views is inherited from the prior copula

$$\begin{pmatrix} C_1 \\ \vdots \\ C_k \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} F_1^{-1}(V_1) \\ \vdots \\ F_k^{-1}(V_k) \end{pmatrix}, \tag{35}$$

and the posterior joint distribution of the views is defined as

$$\begin{pmatrix} V_1 \\ \vdots \\ V_k \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} \tilde{F_1}^{-1}(C_1) \\ \vdots \\ \tilde{F_k}^{-1}(C_k) \end{pmatrix}. \tag{36}$$

We can now define the posterior distribution of the market $\tilde{\mathbf{M}} \sim \tilde{f}_{\mathbf{M}}$ as

$$\tilde{\mathbf{M}} \stackrel{d}{=} \bar{\mathbf{P}}^{-1}\mathbf{V}, \tag{37}$$

where the first K entries of $\mathbf{V}$ are the posterior distribution in (36) and the rest are the same as the prior. From the prior distribution $\mathbf{M}$ the COP approach generates the posterior distribution in (37) reflecting the practitioner's views. When (37) is calculated one can use it in e.g. portfolio optimization or forecasting.

4 Method, Copula Opinion Pooling in Practice

This section shows how to use the COP approach in practice. The five steps are inspired by *Beyond Black-Litterman in Practice: a Five-Step Recipe to Input Views on Non-Normal Markets*, see [11].

This first step is actually not part of the COP approach itself but it is needed to generate a market prior distribution. How this market prior distribution is obtained is up to the practitioner to decide and is not the main focus in this thesis. However, this is one way to do it.

To represent a prior market distribution for a market consisting of N assets we first need a data set of T observations from the multivariate distribution forming the panel χ with size $T \times N$. The prior market distribution will be simulated using Monte Carlo simulation and in order for the simulated sample to inherit the dependence structure from the data set we first have to decide upon a copula to use and find the parameter setup $\boldsymbol{\theta}$ of that copula. To get a reasonable understanding of the dependence structure, bivariate scatter plots of the data set can be used. To transform the data into a copula scale we estimate the cdf using Gaussian kernel estimation. This is easily done in MATLAB with the function KSDENSITY with input CDF. Fitting the chosen copula to the data can be done by the copula canonical maximum-likelihood maximizer COPULAFIT and when the parameters are obtained we get a random sample of size J from the copula using COPULARND. This sample of size J is then transformed back to the original data set scale using KSDENSITY with input ICDF and we have the prior market distribution M of size $J \times N$

$$\mathbf{M} \sim F_{\mathbf{M}} \Longleftrightarrow M. \tag{38}$$

To ensure the market distribution is a suitable simulation of the data set we can use pdf plots and scatter plots. The pdf plots will show if the marginal distributions fit to the marginals from the data set and the scatter plot will show if the dependence structure is the same.

The first actual step of the COP approach is to rotate the market into the coordinates of the views, which is done by a matrix multiplication

$$V = M\bar{\mathbf{P}}'. \tag{39}$$

This means that V is a $J \times N$ matrix where each row is an independent joint draw from the market expressed in the coordinates of the views.

The second step is to compute the views' prior cdfs and the market-implied prior copula. In order to do this we sort the first K columns of (39). The columns of V are rearranged in ascending order and this new panel is called W where $W_{1,k} \leq W_{2,k} \leq \cdots \leq W_{J,k}$. The prior cdfs (33) of the views is approximated by its empirical counterpart

$$F_k(W_{j,k}) = \frac{j}{J+1}. \tag{40}$$

One more panel is needed and is called C. This panel represents the normalized ranking of each entry of V within its column, i.e. if $V_{2,3}$ is the fourth smallest simulation in column 3, then $C_{2,3} = \frac{4}{J+1}$. The rows of the panel C is the empirical representation of the copula in (35)

$$\textit{copula } \mathbf{C} \Longleftrightarrow \textit{ranking } C. \tag{41}$$

The views can be modelled by any distribution and we will use some different. Here we choose to present how to use the uniform distribution. The views can then be modelled as

$$\hat{F}_k(v) = \begin{cases} 0 & v \leq a_k \\ \frac{v-a_k}{b_k-a_k} & v \in [a_k, b_k] \\ 1 & v \geq b_k \end{cases}, \tag{42}$$

where $k = 1, 2, \ldots,$ number of views and $[a_k, b_k]$ is the forecasted interval for view k.

The third step is to compute the posterior marginal cdf of each view as

$$\tilde{F}_{j,k} = c_k \hat{F}_k(W_{j,k}) + (1 - c_k)\frac{j}{J+1}. \tag{43}$$

The joint posterior distribution of the views is obtained by linear interpolation as follows. We define the linear interpolator of a function $\mathbf{y}_{grid} = f(\mathbf{x}_{grid})$ as

$$x \mapsto y = \text{interp}(x; \mathbf{x}_{grid}, \mathbf{y}_{grid}).$$

To get a $J \times K$ matrix $\tilde{V}$ of joint scenarios from the posterior we compute

$$\tilde{V}_{j,k} = \text{interp}(C_{j,k}; \tilde{F}_{\cdot,k}, W_{\cdot,k}), \tag{44}$$

which we then combine together with the last $N - K$ columns from (39) to compute the posterior market distribution as

$$\tilde{M} = \tilde{V}\bar{\mathbf{P}}'^{-1}. \tag{45}$$

COP is easy to use and the above calculations with $J = 500000$ is done within a few seconds due to effective matrix calculations.

5 Data

The data used to evaluate the COP approach is presented in this section.

5.1 Bond Portfolio

We will consider a bond portfolio consisting of four Swedish government bonds with different maturities, namely 1,2,5 and 10 years. A quarterly investment horizon is used and the data ranging from 1996 to 2014 is taken from the financial database *Nordea Analytics*. The data is bootstrapped because of the fact that time to maturity changes every day and at all times there will therefore not exist e.g. bonds with a maturity of 30 years. The time series is shown in Figure 8.

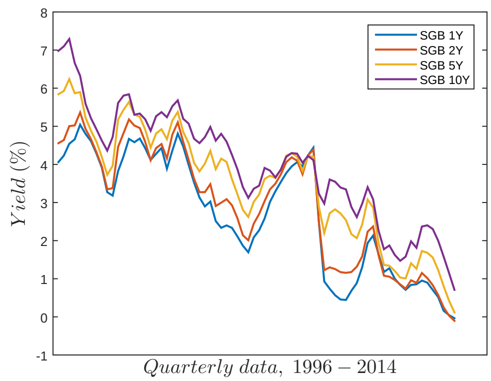

Figure 8: Yield time series of Swedish government bonds.

As stated earlier the invariant for bonds is change in yield and therefore this time series is needed and shown in Figure 9.

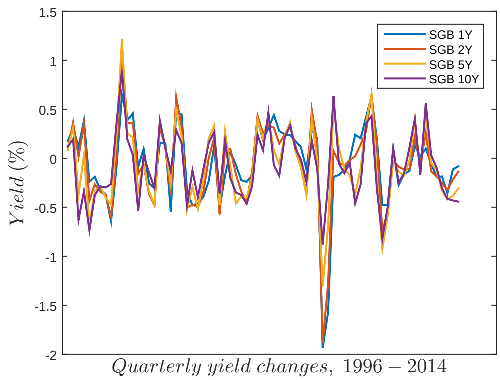

Figure 9: Quarterly yield changes time series of Swedish government bonds.

The resulting data set consists of 73 quarterly joint yield changes. As can be seen in Figure 8 and 9 there is a decreasing trend during the chosen data period and therefore it is likely that the resulting density functions that will be used later on will indicate high probabilities of negative yield changes. This trend will probably not continue in the same way for such a long period because then borrowing will lead to huge profits, which feels intuitively wrong. If this is a good description of what will happen in the future we do not know and it is not important for the purpose of this thesis. We will not make any adjustments to the underlying data set and if an investor thinks the underlying data set will give weird results the following sections will show that it is possible to manipulate the distributions, using COP.

5.2 Equity Portfolio

We will also consider an equity portfolio consisting of four stocks included in OMXS Large Cap, namely Nordea, Astra Zeneca, Telia Sonera and ABB. The stocks are from four different markets to ensure a diversified portfolio. Monthly historical stock prices for the period November 2004 to November 2014 was taken from the webpage of *OMX*. The historical prices are shown in Figure 10.

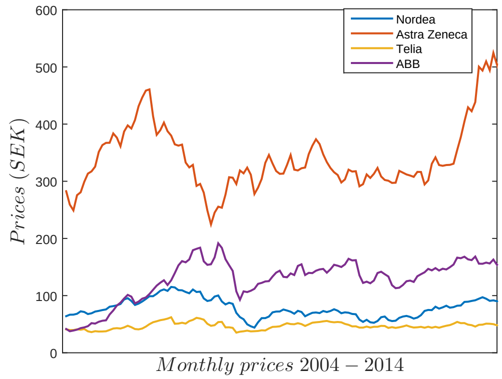

Figure 10: Time series of stock prices.

The invariant we will use for the stocks is monthly returns which are shown in Figure 11.

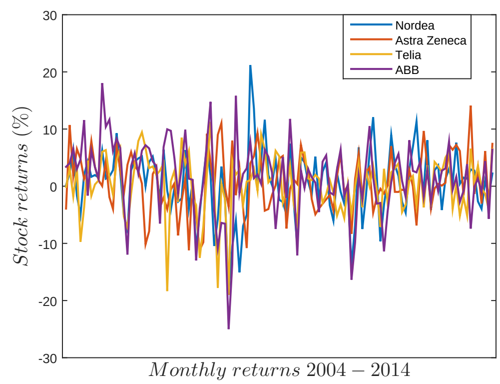

Figure 11: Time series of stock returns.

The resulting data set consists of 120 monthly joint returns.

5.3 Currency Portfolio

At last we will have a look at a portfolio of currency crosses namely EURSEK, USDSEK, EURUSD, NOKSEK and EURGBP. We will consider a quarterly investment horizon and the data spans from 1990 to 2015. Average quarterly prices are shown in Figure 12.

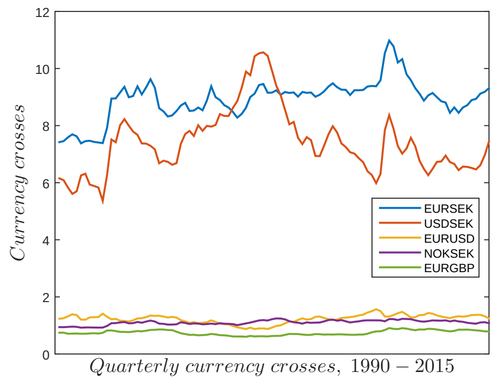

Figure 12: Time series of currency crosses.

The invariant we will use is quarterly returns, shown in Figure 13.

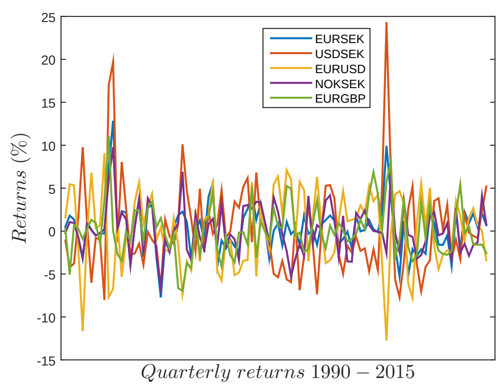

Figure 13: Time series of quarterly currency returns.

The resulting data set consists of 99 quarterly joint returns.

6 Applications and Analysis

This section demonstrates how to apply COP on the three portfolios we consider together with a portfolio management approach. The bond and equity portfolios will be used to fully illustrate COP using uniformly distributed views, which probably is not the best choice of distribution in reality but it is a good way to show how the method works. The views used in the currency portfolio will be more realistic.

6.1 Bond Portfolio

As stated earlier in the section *Theoretical Background* there are several investment strategies available considering bonds. We could of course just test some random strategy and evaluate it but in order to get some structure and evaluate the COP approach in a fair way we can use e.g. consensus forecasts. The problem is that there are no consensus available for bonds with different maturities. To solve this problem we notice that in the Swedish bond market there are futures contracts on the 2, 5 and 10 year bonds which means that you as an investor are able to bet on the future yield of these bonds. This means that the future priced in yields is a good market consensus estimate of future yields and we will use these implied changes as our lower limit in the investor's views. So in our bond portfolio the investor will have a view on the yield curve based on the market consensus on that particular day and this view will then affect the market prior distribution. Let us have a look at the case we will consider.

6.1.1 Views

An investor decides to make an investment in Swedish government bonds on the 18th of September 2014 using the COP approach and the swapped mean-ES trade-off. The investor is not allowed to buy bonds with duration higher than 12 so the portfolio will consist of 1,2,5 and 10 year bonds which all have duration less than 12. The idea is not to hold the bonds until maturity which counts as a risk-free investment due to the fact that the Swedish government has highest possible credit rating, the investment horizon is set to be one quarter, i.e. three months. The investor wants to earn money on changes in yields and will use priced in yields to express the views. According to Table 2 the market believes that the 2 year yield will increase 2 bps, the 5 year yield will decrease 5.2 bps and the 10 year yield will rise 9.1 bps in three months.

	2Y	5Y	10Y
Current yield (%)	0.226	0.787	1.596
Future yield (%)	0.246	0.735	1.687
Change (%)	0.020	-0.052	0.091

Table 2: Current and future yields.

This means a flatter curve between 2 and 5 years and a steeper curve between 5 and 10 years. The investor believes in more extreme scenarios than the implied scenario, namely 2 bps more extreme in each view. In the data a yield of 1 % is expressed as 0.01. The pick matrix becomes

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{46}$$

and the uniformly distributed views are expressed as

$$\hat{F}_k(v) = \begin{cases} 0 & v \leq a_k \\ \frac{v-a_k}{b_k-a_k} & v \in [a_k, b_k] \\ 1 & v \geq b_k \end{cases}, \tag{47}$$

for $k = 1, 2, 3$, $\mathbf{a} = 10^{-4} \begin{pmatrix} 2 \\ 5.2 \\ 9.1 \end{pmatrix}$ and $\mathbf{b} = 10^{-4} \begin{pmatrix} 4 \\ 7.2 \\ 11.1 \end{pmatrix}$. The confidence level will be set to 0.3 for each view as default but to investigate the model's sensitiveness and behaviour we will test other values as well. Notice that the first column in the pick matrix consists of zeros which means that no view is stated on the 1 year bond. The orthonormal basis for the null space of $\mathbf{P}$ is set to

$$\mathbf{P}^{\perp} = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}, \tag{48}$$

and the resulting view-adjusted market coordinates are

$$\bar{\mathbf{P}} = \begin{pmatrix} \mathbf{P} \\ \mathbf{P}^{\perp} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{49}$$

6.1.2 The Market Prior

The data set of changes in Swedish government bonds presented earlier can beyond the time series also be presented using histograms. Figure 14 shows histograms for the 1,2,5 and 10 year bonds together with kernel estimations of the probability functions.

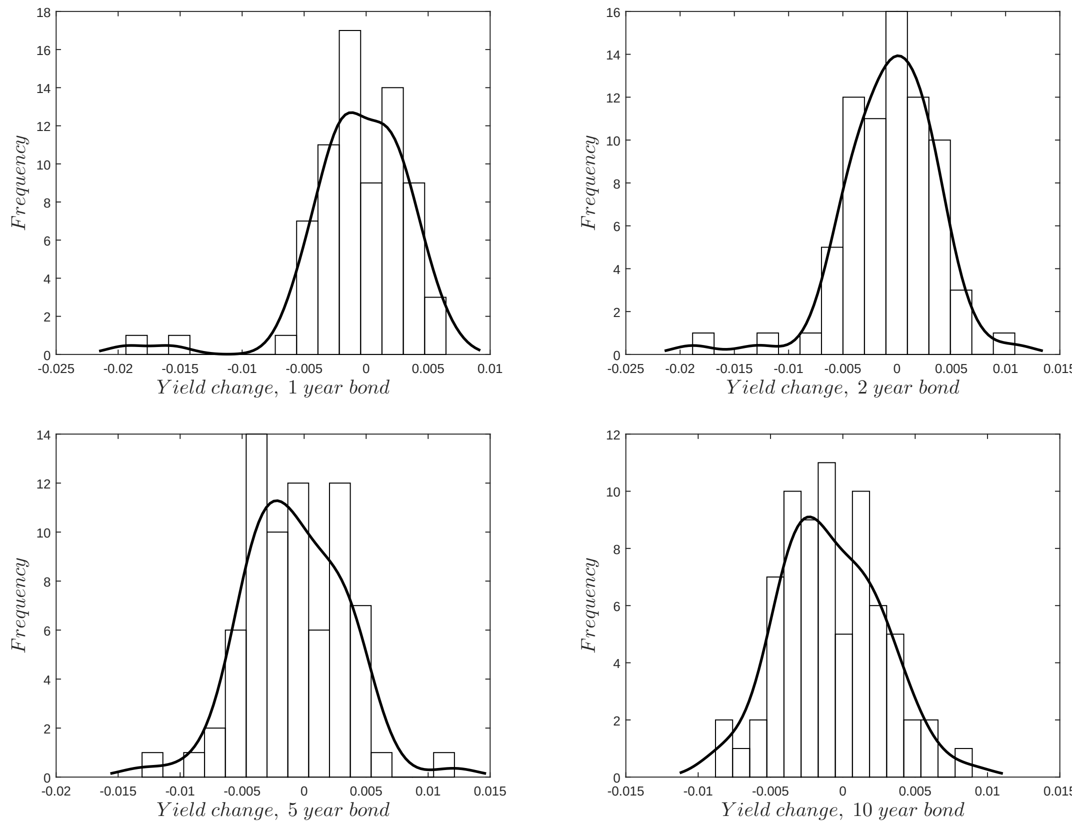

Figure 14: Histograms with kernel estimations.

As shown we have different marginal distributions for each bond. The dependence structure is also of great importance and to get an indication we study the scatter plots in Figure 15.

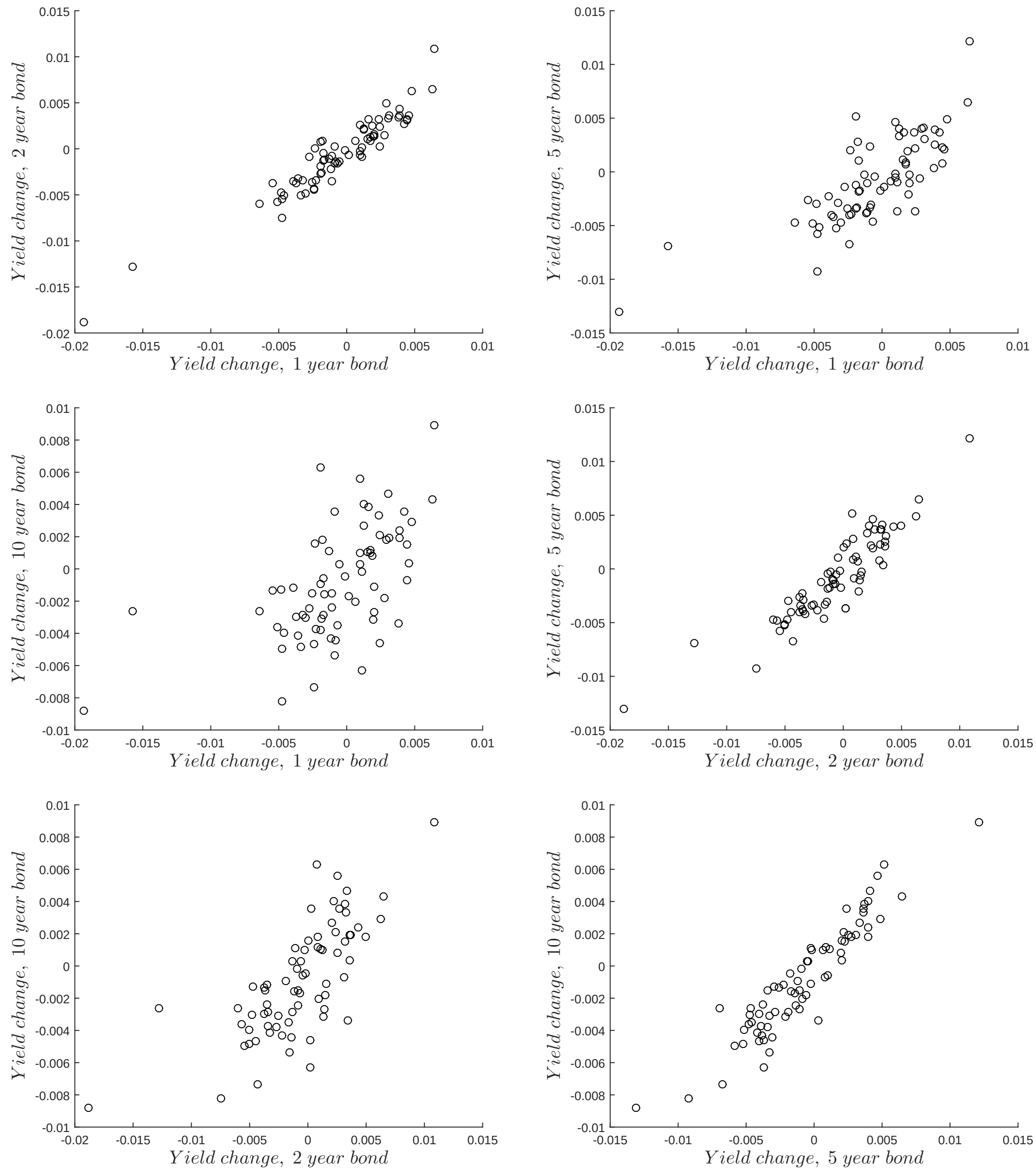

Figure 15: Bond scatter plots.

The data set is quite small and the dependence structure is hard to decide upon but some of the scatter plots may indicate some upper and lower tail dependence between the different bonds and strong positive correlations. It is reasonable to believe that the data has elliptical dependence and to inherit this we chose to fit a t copula to the data set which gives the correlation matrix

$$\mathbf{R} = \begin{pmatrix} 1 & 0.95 & 0.82 & 0.67 \\ 0.95 & 1 & 0.93 & 0.80 \\ 0.82 & 0.93 & 1 & 0.95 \\ 0.67 & 0.80 & 0.95 & 1 \end{pmatrix}, \tag{50}$$

and the degrees of freedom $\nu = 11.3$. Then the prior market distribution is obtained by $J = 500000$ Monte Carlo simulations generating the market prior distribution M of size $J \times N$. This sample size is motivated by the qq plot of the empirical marginal distribution of the 5 year bond against the real distribution shown in Figure 16. The real distribution is approximated by a sample of 10^7. The qq plot form an almost perfect line with slope one

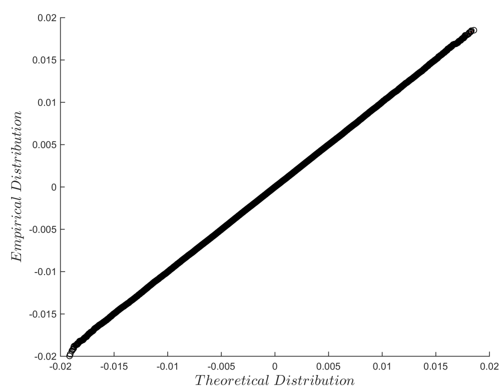

Figure 16: Bond qq plot.

which means that the sample size $J = 500000$ is large enough. A smaller sample size would work but the prior distribution is only created once and stored for further calculations. In Table 3 we show the mean, standard deviation, skewness and kurtosis of the resulting marginal distributions of the market prior distribution. Table 3 shows that the marginal

	1Y	2Y	5Y	10Y
μ ($\times 10^{-4}$)	-5.60	-6.36	-7.81	8.55
σ ($\times 10^{-2}$)	0.44	0.46	0.44	0.38
sk	-1.44	-0.81	0.060	0.17
ku	7.60	7.79	3.68	2.98

Table 3: Prior market statistics.

distributions of the 5 and 10 year bonds are nearly normal, their skewness are nearly 0 and their kurtosis are close to 3. Both the shorter bonds are left skewed and heavy tailed. All this was also indicated by the data set illustrated in Figure 14 and the probability density functions for the final prior distribution, shown in Figure 17, also agree with Table 3. However, Figure 17 also shows that the small deviations from $sk = 0$ and $ku = 3$ make

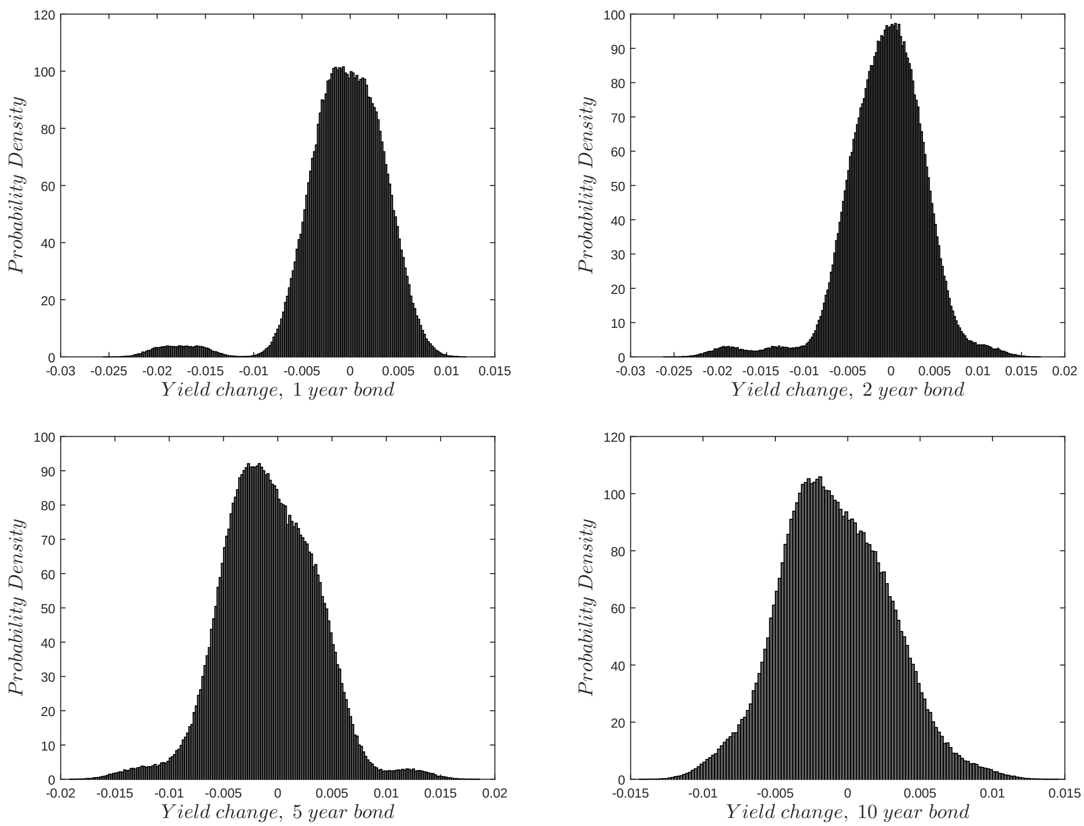

Figure 17: Bond probability plots.

the 5 and 10 year distributions to differ from normal distributions. The shapes of the probability densities can be compared to the fitted kernel densities in Figure 14 and as expected they fit.

6.1.3 The Market Posterior

To see how the views affect the prior distribution we plot the pdfs of the prior and posterior distributions in Figure 18.

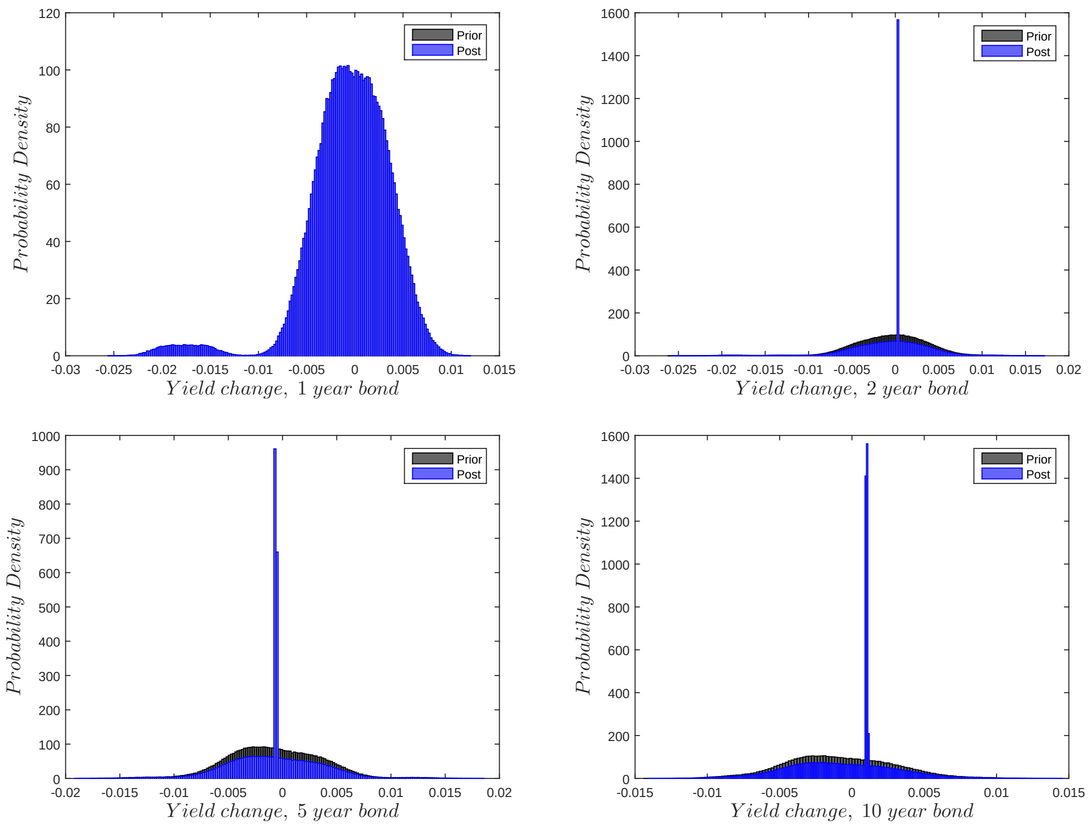

Figure 18: Probability plots of prior and posterior.

Notice that the prior distribution for the 1 year bond cannot be seen in Figure 18 due to the fact that no view were stated on that bond and therefore the distributions are equal and the posterior lies above the prior. The views affect the other distributions quite significantly, despite the confidence level of 0.3. Equally distributed spikes at the small intervals of the views can be seen in the view-affected plots.

It is of big interest to see if the dependence structure is maintained for the posterior distribution. To investigate this, we use scatter plots of the prior and posterior distributions, shown in Figure 19 and 20. We only plot a smaller sample of 5000 so that the dependence structure is possible to see.

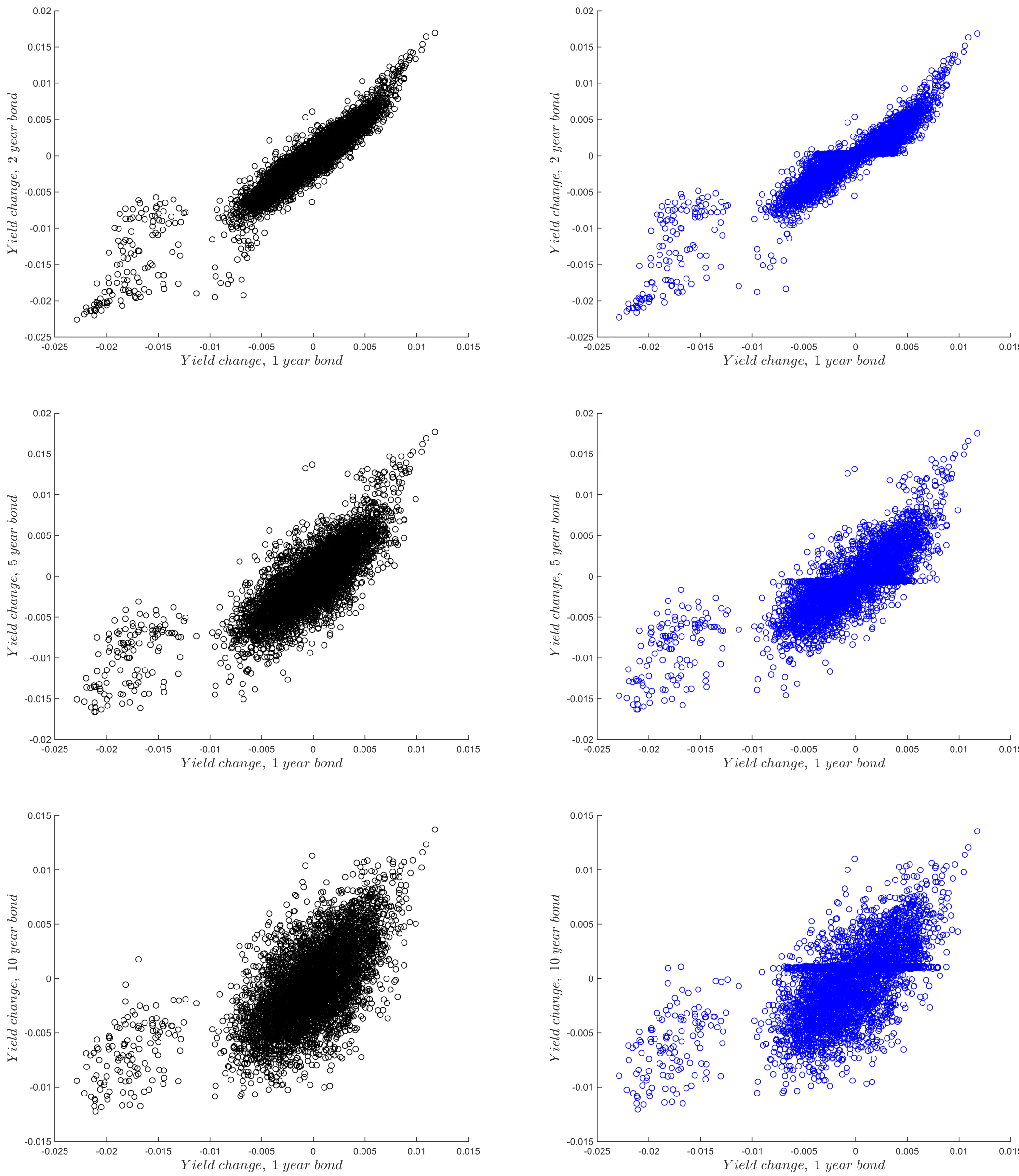

Figure 19: Bond scatter plots. *Left* in black: prior. *Right* in blue: posterior.

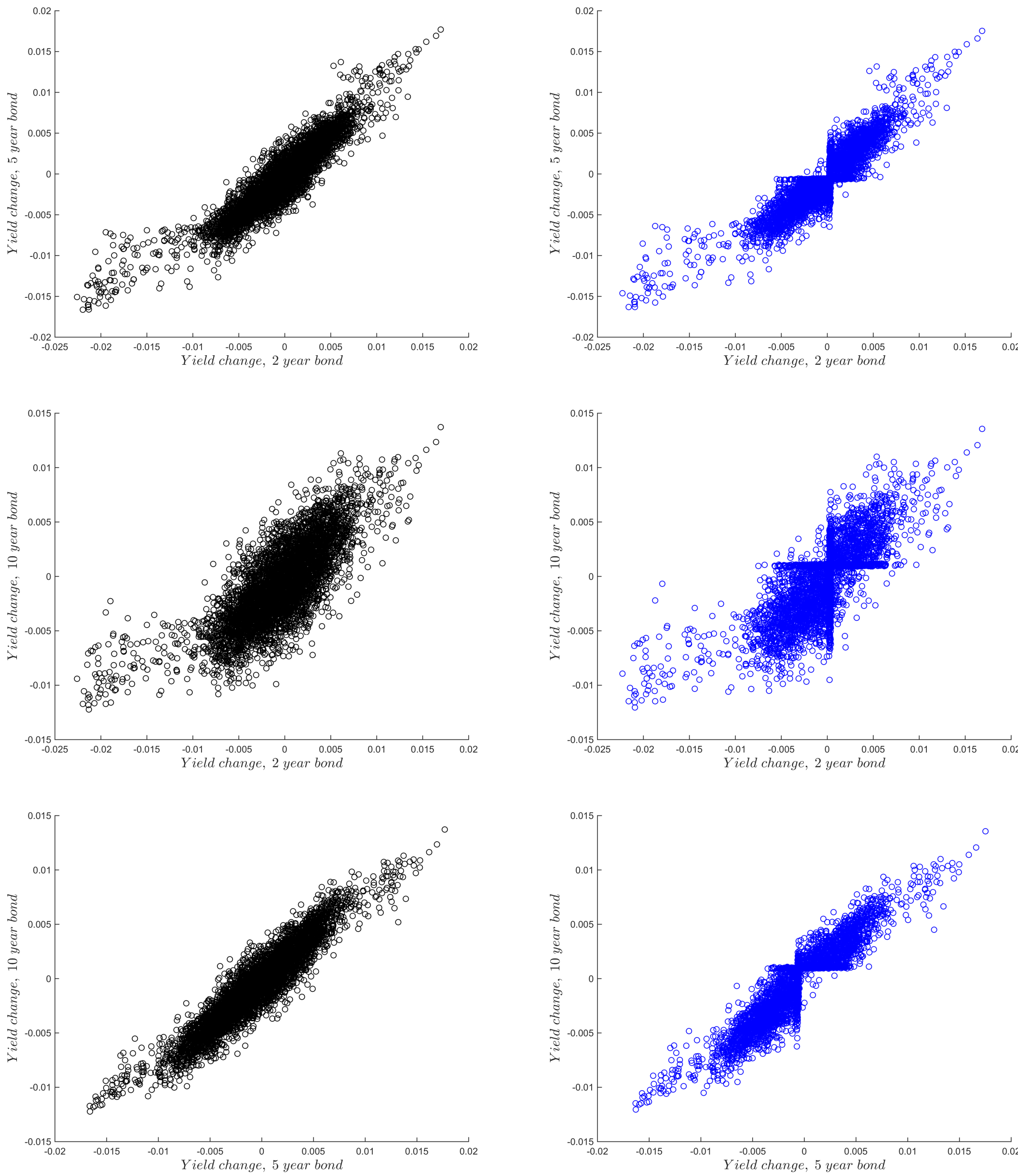

Figure 20: Bond scatter plots. *Left* in black: prior. *Right* in blue: posterior.

In Figure 19 and 20 the views are shown to the right in blue as a pattern, perpendicular to the axis of the corresponding bond. To put numbers on the dependence structure we now fit a t copula to the posterior distribution to be able to compare with the prior correlation matrix and the degrees of freedom. The new correlation matrix is

$$\mathbf{R_{post}} = \begin{pmatrix} 1 & 0.95 & 0.82 & 0.67 \\ 0.95 & 1 & 0.93 & 0.80 \\ 0.82 & 0.93 & 1 & 0.95 \\ 0.67 & 0.80 & 0.95 & 1 \end{pmatrix}, \tag{51}$$

i.e. exactly the same as the prior correlation matrix. The degrees of freedom is slightly different with $\nu_{post} = 9.5$. No new pattern is perpendicular to the axis of the 1 year bond in the posterior plots, again because no view is expressed on that bond.

6.1.4 Portfolio Allocation

To investigate how COP affects portfolio allocation we will first consider the prior distribution and then our view-adjusted distribution together with the swapped mean-ES trade-off. The prior and posterior are distributions of yield changes and to translate these yield changes into returns we use the second order approximation stated earlier, namely

$$\frac{\Delta p}{p} \approx -D\Delta y + \frac{1}{2}C(\Delta y)^2,$$

where D, p and C are collected from the financial platform *Bloomberg*. The net returns are then calculated as

$$R = \frac{p + \Delta p}{p} - 1.$$

In this case the optimization problem becomes

$$\text{maximize} \quad \mathbf{w}^T \boldsymbol{\mu},$$

$$\text{subject to} \quad \mathrm{ES}_{0.05}(\mathbf{w}) \leq \mathrm{ES}^*, \quad \mathbf{w}^T\mathbf{1} \leq 1, \quad w_k \geq 0, \quad k = 1, \ldots, 4\ ,$$

i.e. the investor invests 1 SEK and is not allowed to short the bonds. $c_k = 0.3$, the probability level p is set to 0.05 and the maximum allowed level of expected shortfall, ES^*, is not chosen here because it will vary in the efficient frontier. The efficient frontiers of the optimizations are shown in Figure 21.

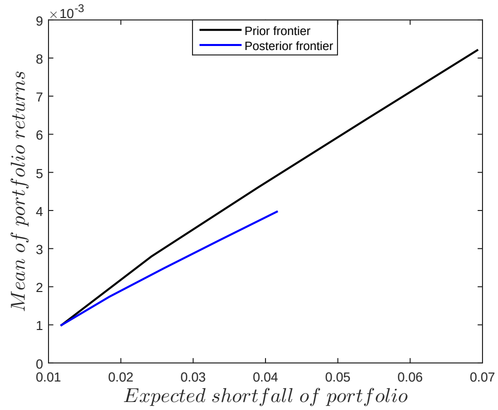

Figure 21: Efficient frontiers.

In this case the posterior frontier is below the prior meaning that the expected mean return is lower for the same level of risk, but this is of course strongly dependent on historical outcomes of the underlying data set. To be able to do a comparison with the classical mean-variance efficient frontier we calculate the standard deviation of the prior and posterior portfolios and build the frontiers shown in Figure 22. The mean-variance frontier has the prior distribution as underlying.

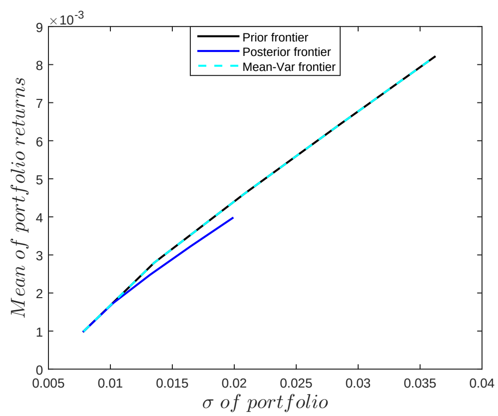

Figure 22: Efficient frontiers.

We see that the mean-variance frontier is almost identical to the prior frontier. To further compare the three frontiers and to understand how COP affects the allocation the amount invested in each bond for five portfolios from each efficient frontier is presented in Table 4.

		1Y	2Y	5Y	10Y
Prior	I	1	0	0	0
	II	0.75	0	0	0.25
	III	0.5	0	0	0.5
	IV	0.25	0	0	0.75
	V	0	0	0	1
Posterior	I	1	0	0	0
	II	0.75	0	0.25	0
	III	0.5	0	0.5	0
	IV	0.25	0	0.75	0
	V	0	0	1	0
Mean-Var	I	1	0	0	0
	II	0.75	0	0	0.25
	III	0.5	0	0	0.5
	IV	0.25	0	0	0.75
	V	0	0	0	1

Table 4: Prior and posterior allocation, each row represents a portfolio, I being the least risky and V the most risky. Increasing risk for each portfolio.

We recall the forecast which says that the yield of the 2 and 10 year bonds should increase and the yield of the 5 year bond should decrease. We remember the bond pricing formula which says that an increase in yield means a lower bond price. The safest portfolio with the lowest risk for the prior case suggests to fully invest in the 1 year bond, the same goes for the posterior case. For the more risky portfolios the prior and posterior differs. The prior suggests to invest in the 1 and 10 year bond but the posterior suggests to invest in the 1 and 5 year bond. The reason for this is that the view was a decreased yield for the 5 year bond, meaning a positive return for a long position, and due to the fact that shorting was not allowed the 2 and 10 year bonds would lead to negative returns. The mean-variance portfolios and the prior portfolios are the same in this particular case. Table 4 is a good example of COP's impact on portfolio allocation.

It is of interest to see what happens when short selling is allowed. The optimization problem becomes

$$\text{maximize} \quad \mathbf{w}^T \boldsymbol{\mu},$$

$$\text{subject to} \quad \mathrm{ES}_{0.05}(\mathbf{w}) \leq \mathrm{ES}^*, \quad \mathbf{w}^T \mathbf{1} \leq 1, \quad w_k \geq -10, \quad k = 1, \ldots, 4\ ,$$

meaning that the short selling in each bond is maximum 10 SEK. $c_k = 0.3$ and it is assumed that short selling can be made without any associated extra costs. The efficient frontiers are shown in Figure 23.

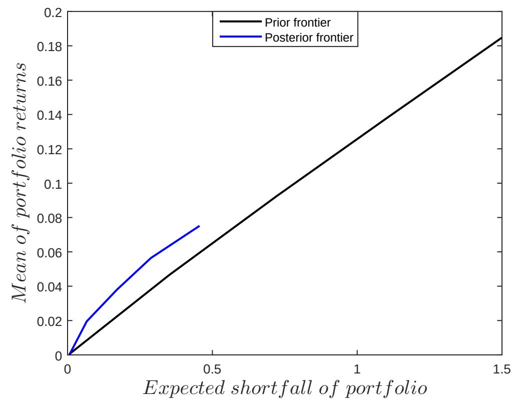

Figure 23: Efficient frontiers. Short selling up to 10 SEK allowed.

Comparison with the mean-variance frontier is shown in Figure 24.

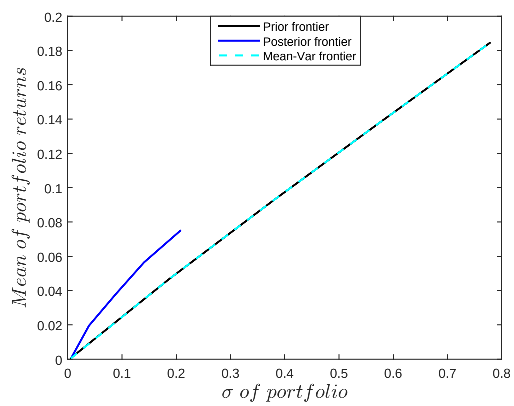

Figure 24: Efficient frontiers.

The allocation is shown in Table 5.

		1Y	2Y	5Y	10Y
Prior	I	1.61	−0.55	−0.14	0.08
	II	−0.04	0.25	−10	10.79
	III	2.73	−10	−10	18.27
	IV	−3.63	−10	−10	24.63
	V	−10	−10	−10	31
Posterior	I	1.57	−0.59	0.05	−0.03
	II	5.85	−10	7.97	−2.82
	III	1.28	−10	16.80	−7.08
	IV	−3.81	−10	24.81	−10
	V	−10	−10	31	−10
Mean-Var	I	1.85	−0.80	−0.11	0.06
	II	1.47	−1.42	−10	10.95
	III	2.75	−10	−10	18.25
	IV	−3.62	−10	−10	24.62
	V	−10	−10	−10	31

Table 5: Prior and posterior allocation, each row represents a portfolio, I being the least risky and V the most risky. Increasing risk for each portfolio.

Now the posterior frontier has higher returns than the prior and the values on the axis are significantly bigger compared to the case when short selling was not allowed. This is expected because now it is possible to cash in on increasing yields as well. The views are clearly shown in the portfolio weights, the investment in the 5 year bond increases with the risk level and the short positions in the rest of the bonds increase with the risk level. As before this is expected because short positions are suitable when yield increase and will lead to a positive return and vice versa. When short selling is allowed the mean-variance and prior frontiers again are almost the same but now the allocation differ for the low risk portfolios.

6.1.5 Further Investigation

Up to now the confidence level c_k for the views has been set to 0.3. Let us look at the case when $c_k = 1$. This case is probably not a realistic investment situation because $c_k = 1$ means that the investor is 100 % confident in the views, but it is of interest to see how this extreme case of COP affects. We start with the probability plots in Figure 25.

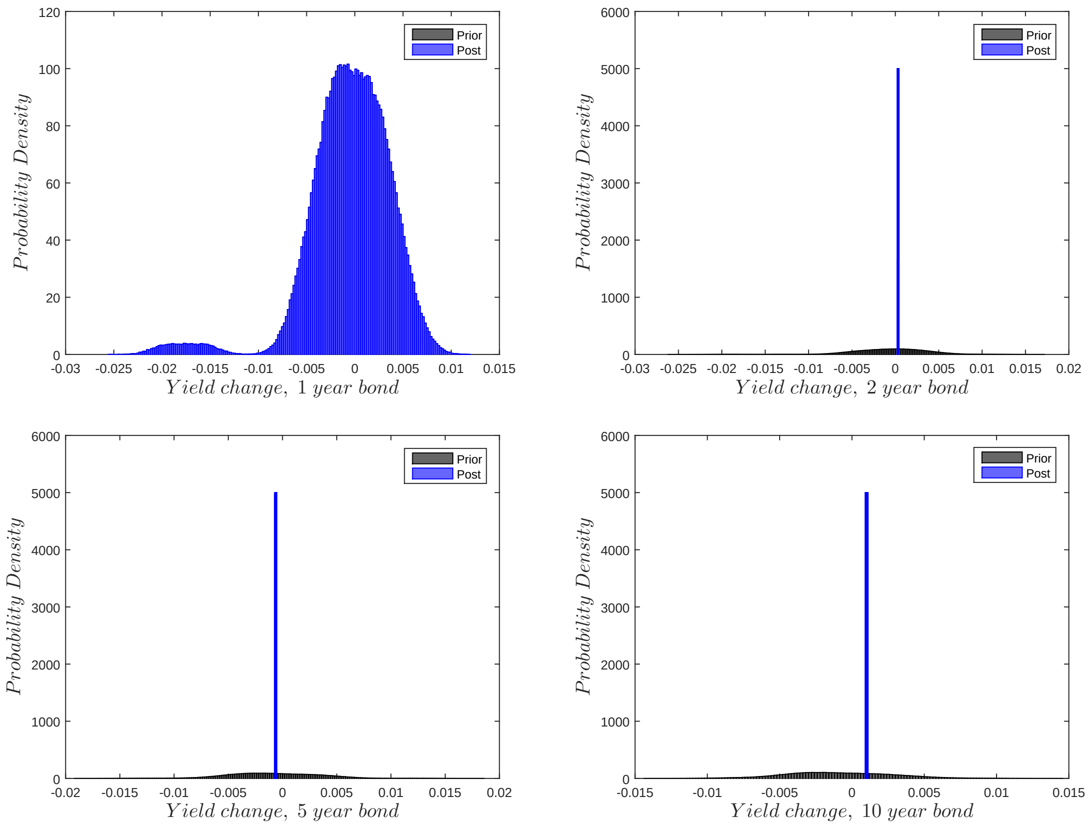

Figure 25: Probability plots of prior and posterior.

As expected, with $c_k = 1$ all probability mass is moved to the small interval given by the views and we get a uniform distribution. The dependence structure is shown in Figure 26 and 27.

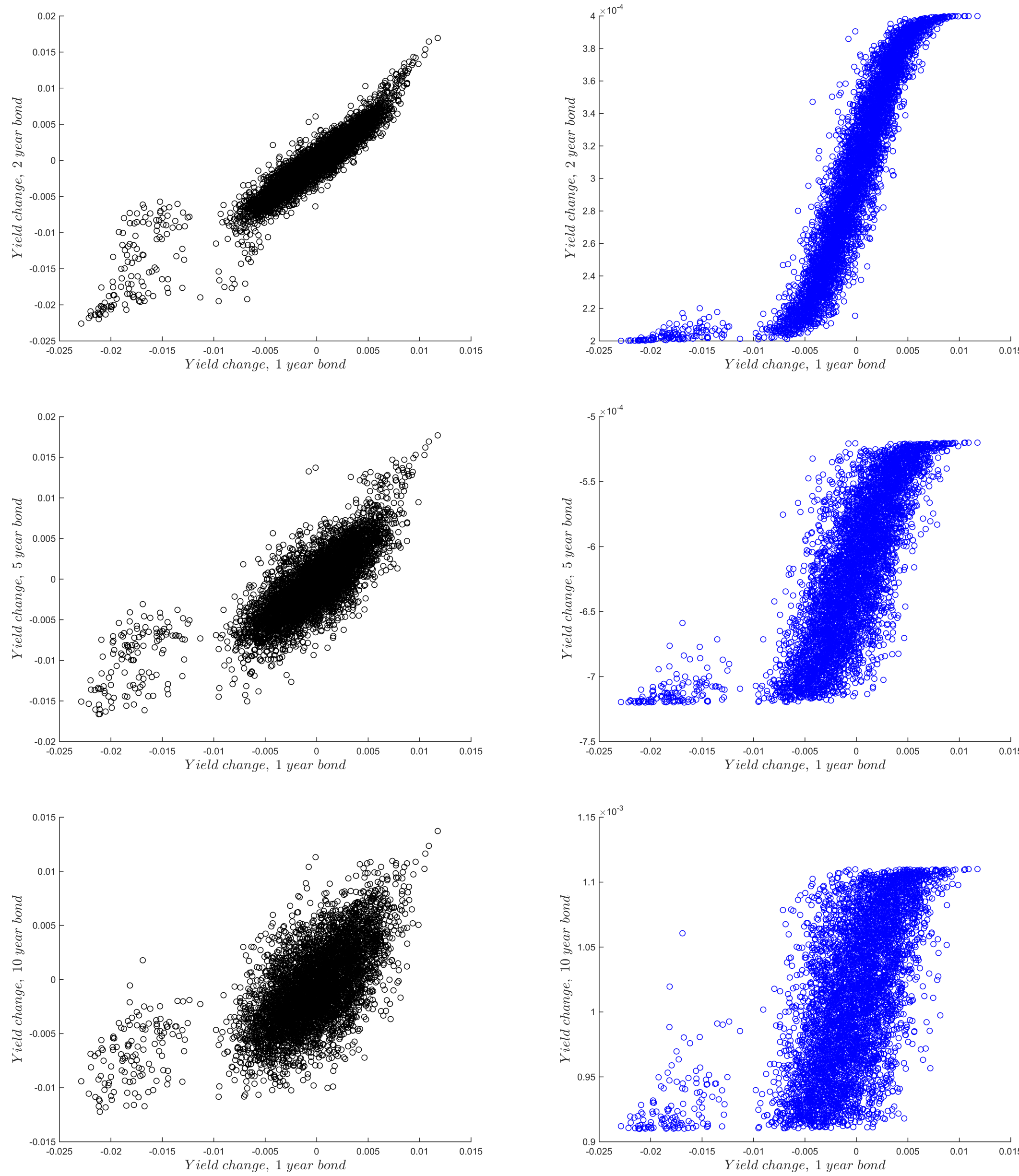

Figure 26: Bond scatter plots. *Left* in black: prior. *Right* in blue: posterior.

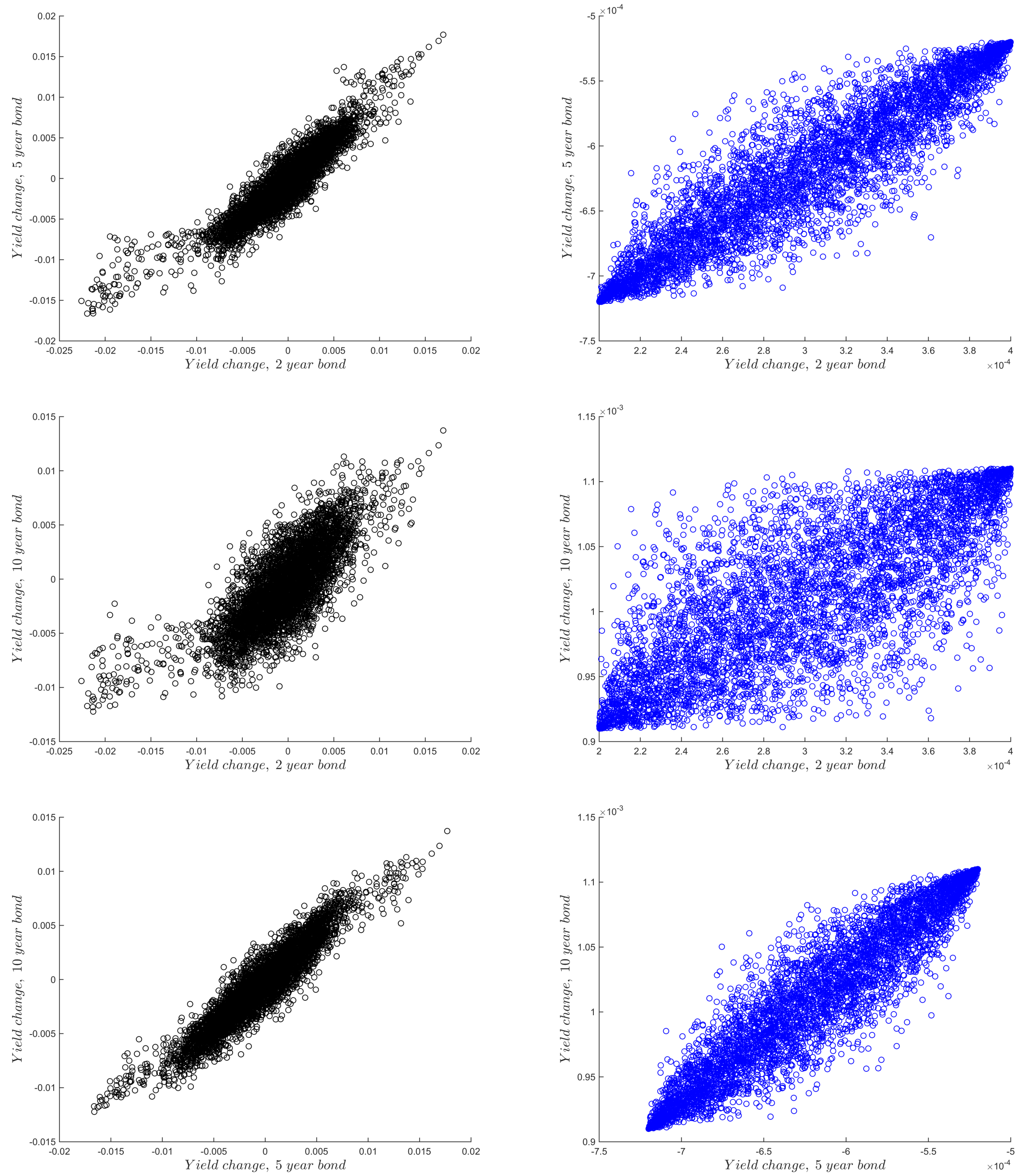

Figure 27: Bond scatter plots. *Left* in black: prior. *Right* in blue: posterior.

The new dependence structure is summarised by the parameters of the fitted t copula as

$$\mathbf{R_{post}} = \begin{pmatrix} 1 & 0.95 & 0.83 & 0.69 \\ 0.95 & 1 & 0.94 & 0.81 \\ 0.83 & 0.94 & 1 & 0.95 \\ 0.69 & 0.81 & 0.95 & 1 \end{pmatrix}, \ \nu_{post} = 10.6, \tag{52}$$

which shows that even though the marginals of the posterior distribution change, the dependence structure is mainly inherited.

When $c_k = 1$ and we do not allow short selling the efficient posterior frontier can be summarised by one point.

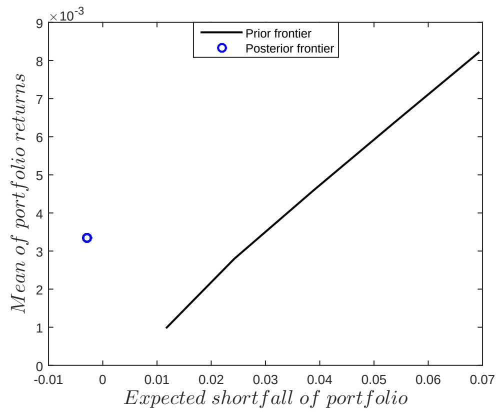

Figure 28: Efficient frontiers.

Figure 28 shows that the posterior frontier has negative expected shortfall which means that if the outcomes of the posterior distribution were the only possible outcomes it would be a risk-less investment.

If we allow short selling the mean of portfolio returns increase significantly for the posterior frontier, shown in Figure 29.

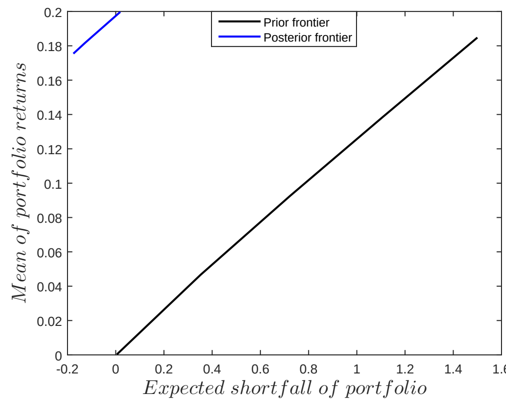

Figure 29: Efficient frontiers. Short selling up to 10 SEK allowed.

COP involves some approximations to get a fast and simple method. We now look at two approximations used and how well they work. Within the framework of COP the prior cdfs of the views are approximated by

$$F_k(W_{j,k}) = \frac{j}{J+1},$$

and the copula of the views are approximated by the normalized ranking. To evaluate the approximations we use the same bond portfolio setup but we set the confidence level in the views to zero. This means that the posterior marginal cdf of each view is modelled as

$$\tilde{F}_{j,k} = \frac{j}{J+1}, \tag{53}$$

and the joint posterior view distribution is computed by

$$\tilde{V}_{j,k} = \mathrm{interp}(C_{j,k}; \tilde{F}_{\cdot,k}, W_{\cdot,k}).$$

This means that the final posterior market distribution

$$\tilde{M} = \tilde{V}\bar{\mathbf{P}}^{\prime -1},$$

will be affected by the approximations but due to $c_k = 0$ the cdfs should be very close to the original ones. Figure 30 shows the prior and approximated cdfs of the 2,5 and 10 year bonds, i.e. the bonds with views.

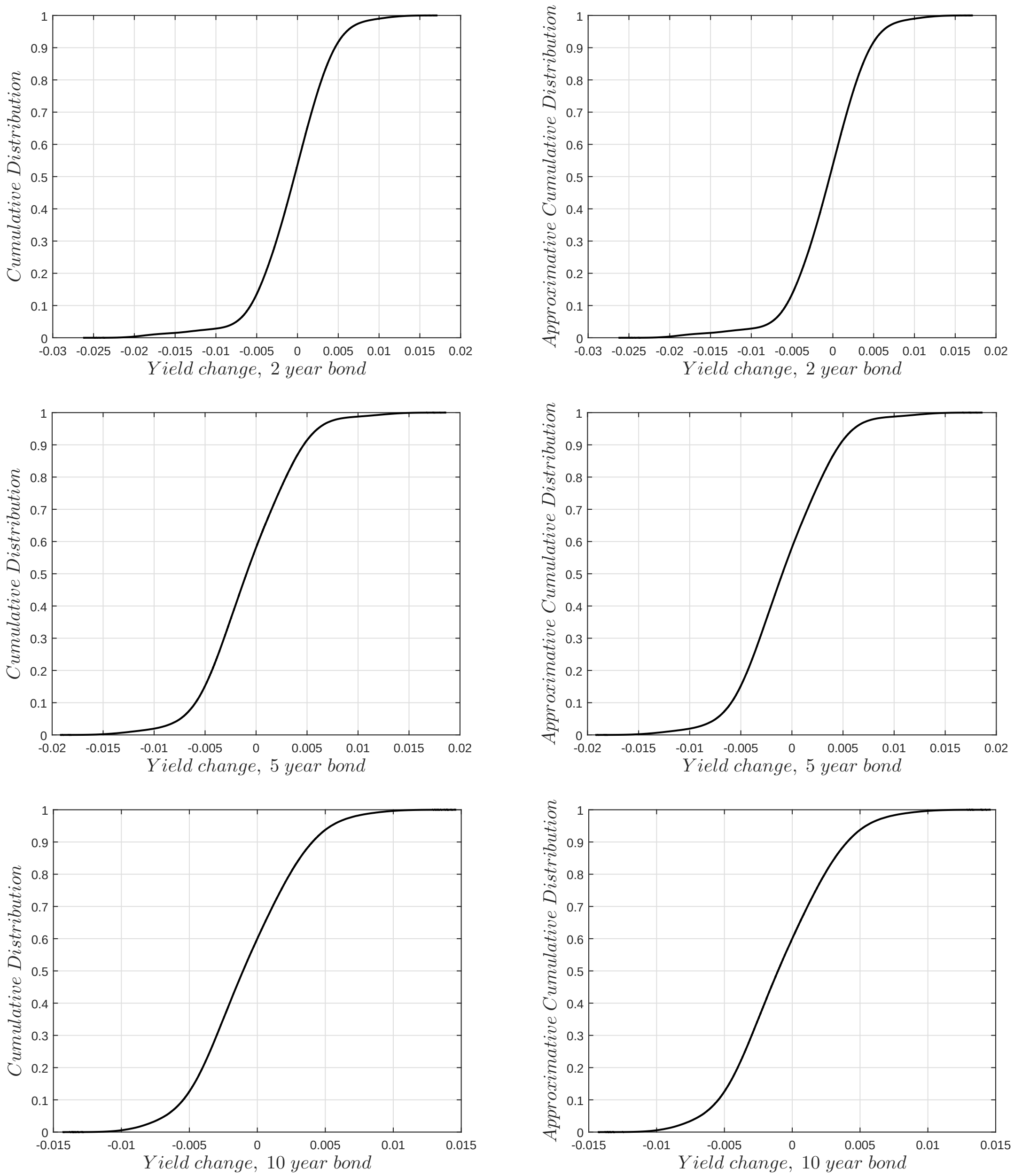

Figure 30: Cumulative distributions. *Left*: prior cdfs. *Right*: approximated cdfs.

It looks like we have plotted the same graphs twice but it is because the approximation works well.

To see how well the ranking approximates the copula we start with generating a sample of size $J = 500000$ from the t copula using the correlation matrix $\mathbf{R}$ and the degrees of freedom $\nu = 11.3$. We then use scatter plots to compare the dependence structure with the one obtained using the *ranking* C. The plots are shown in Figure 31 where we only plot 5000 outcomes to be able to actually see the dependence structure clearly.

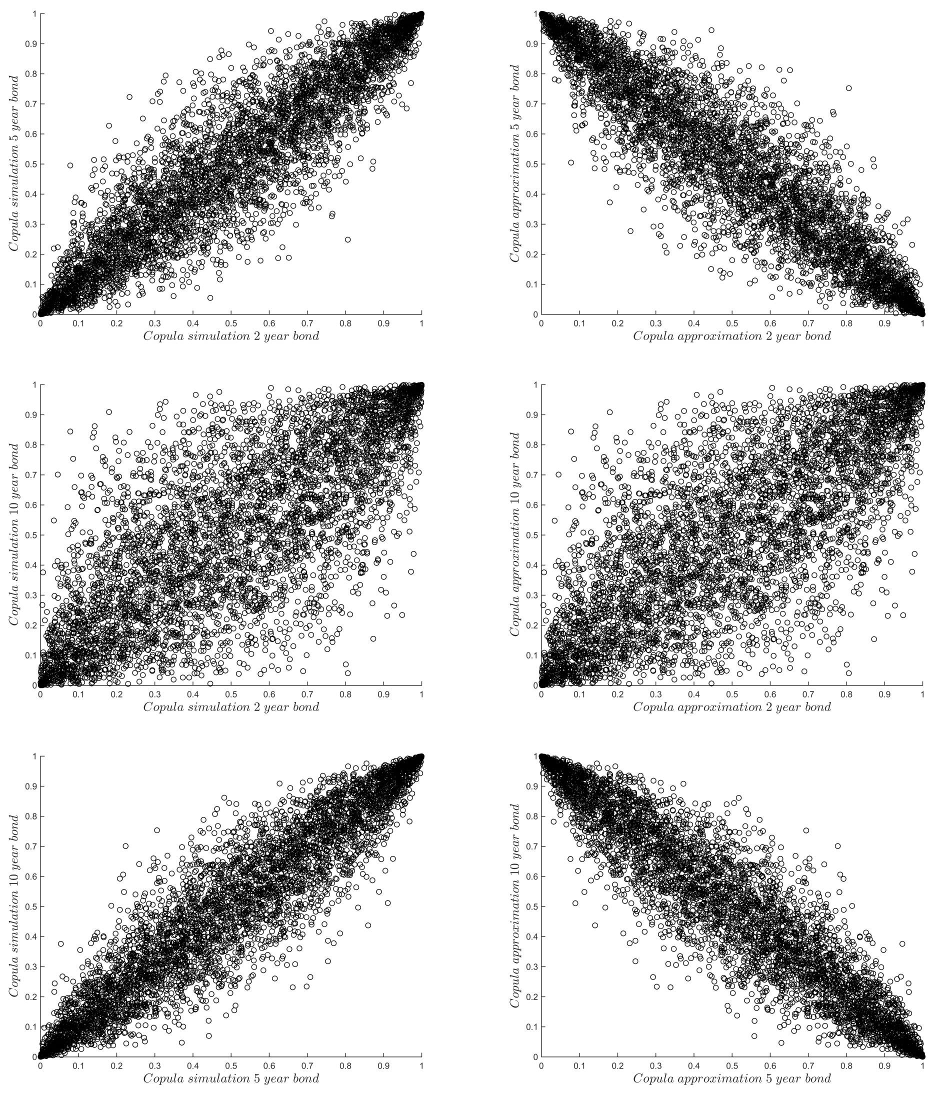

Figure 31: Copula scatter plots. *Left*: simulated copulas. *Right*: approximated copulas.

The approximated copulas are affected by the pick matrix so the negative correlation showing in two of the right plots depends on the fact that the 5 year bond had a negative view. This will change after the mapping to the original distribution which is the next thing to analyze. We now plot the scatter of the prior and posterior distributions, using the smaller sample of 5000, still with $c_k = 0$, see Figure 32.

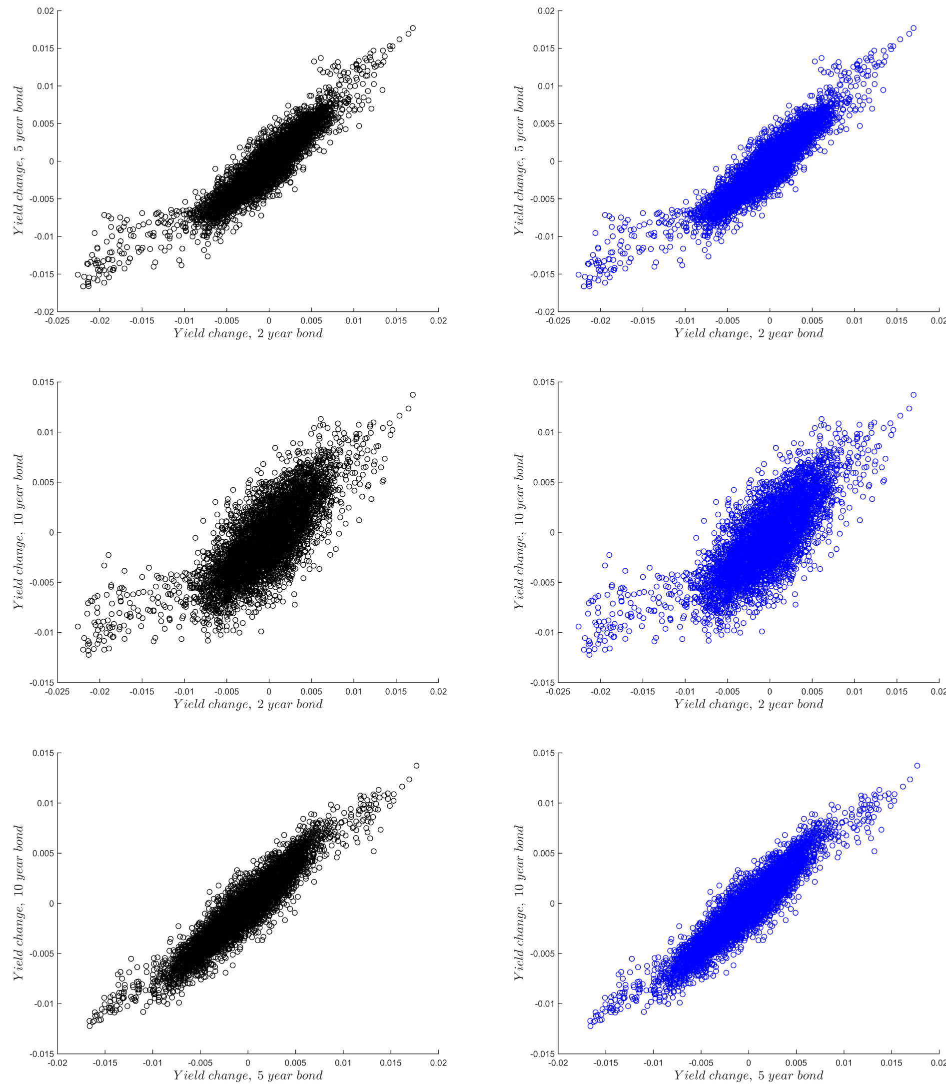

Figure 32: Bond scatter plots. *Left*: prior dependence. *Right*: posterior dependence.

We see that the dependence structure is well approximated by the ranking method. Notice that the posterior market distribution

$$\tilde{M} = \tilde{V}\bar{\mathbf{P}}'^{-1},$$

is rotated back using the inverse of the transpose of the view-adjusted market coordinates so that the negative correlation shown in Figure 31 disappears.

6.2 Equity Portfolio

We will now consider the equity portfolio consisting of the stocks Nordea, Astra Zeneca, Telia Sonera and ABB. We will try arbitrarily chosen views and see what happens if we put more than one forecast in each view. A net return of 1 % is expressed as 0.01.

6.2.1 Views

An investor wants to invest in the four stocks mentioned above and the investment horizon is set to one month. The investor believes that the medical sector which Astra Zeneca operates within will outperform ABB's area of power and automation technologies with up to 10 % units within the next month leading to a relative view between the stock prices of ABB and Astra Zeneca in the intervall 0-10 %. The investor also believes that the stock price of Nordea will outperform the stock price of Telia Sonera with 10-25 % units. The pick matrix becomes

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}, \tag{54}$$

and the uniformly distributed views are expressed as

$$\hat{F}_k(v) = \begin{cases} 0 & v \leq a_k \\ \frac{v-a_k}{b_k-a_k} & v \in [a_k, b_k] \\ 1 & v \geq b_k \end{cases}, \tag{55}$$

for $k = 1, 2$, $\mathbf{a} = 10^{-2}\begin{pmatrix} 0 \\ 15 \end{pmatrix}$ and $\mathbf{b} = 10^{-2}\begin{pmatrix} 10 \\ 25 \end{pmatrix}$. The confidence level will be set to 0.3 to begin with.

The basis for the null space of $\mathbf{P}$ is set to

$$\mathbf{P}^{\perp} = \begin{pmatrix} 2/3 & -1121/4756 & 2/3 & -1121/4756 \\ 1121/4756 & 2/3 & 1121/4756 & 2/3 \end{pmatrix}, \tag{56}$$

and the resulting view-adjusted market coordinates are

$$\bar{\mathbf{P}} = \begin{pmatrix} \mathbf{P} \\ \mathbf{P}^{\perp} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 2/3 & -1121/4756 & 2/3 & -1121/4756 \\ 1121/4756 & 2/3 & 1121/4756 & 2/3 \end{pmatrix}. \tag{57}$$

6.2.2 The Market Prior

The same procedure as for the bond portfolio will be used to generate the market prior. We recall that the purpose of this thesis is to investigate COP and not to test different methods for obtaining distributions from small sets of data. The approach used is good enough for our purpose. Again we fit a t copula to the data set and generate the market prior distribution by $J = 500000$ Monte Carlo simulations which is saved and stored in order to save computation time later on. The correlation matrix of the t copula becomes

$$\mathbf{R} = \begin{pmatrix} 1 & 0.29 & 0.49 & 0.56 \\ 0.29 & 1 & 0.22 & 0.19 \\ 0.49 & 0.22 & 1 & 0.43 \\ 0.56 & 0.19 & 0.43 & 1 \end{pmatrix}, \tag{58}$$

and the number of degrees of freedom is $\nu = 7.5$. The correlations are significantly lower for the stock portfolio compared to the bond portfolio, which is expected. Interest rates of government bonds from one specific country will most often follow each other but the performance of companies from different sectors can differ a lot so their stock prices are not necessarily highly correlated. The resulting prior densities are shown in Figure 33.

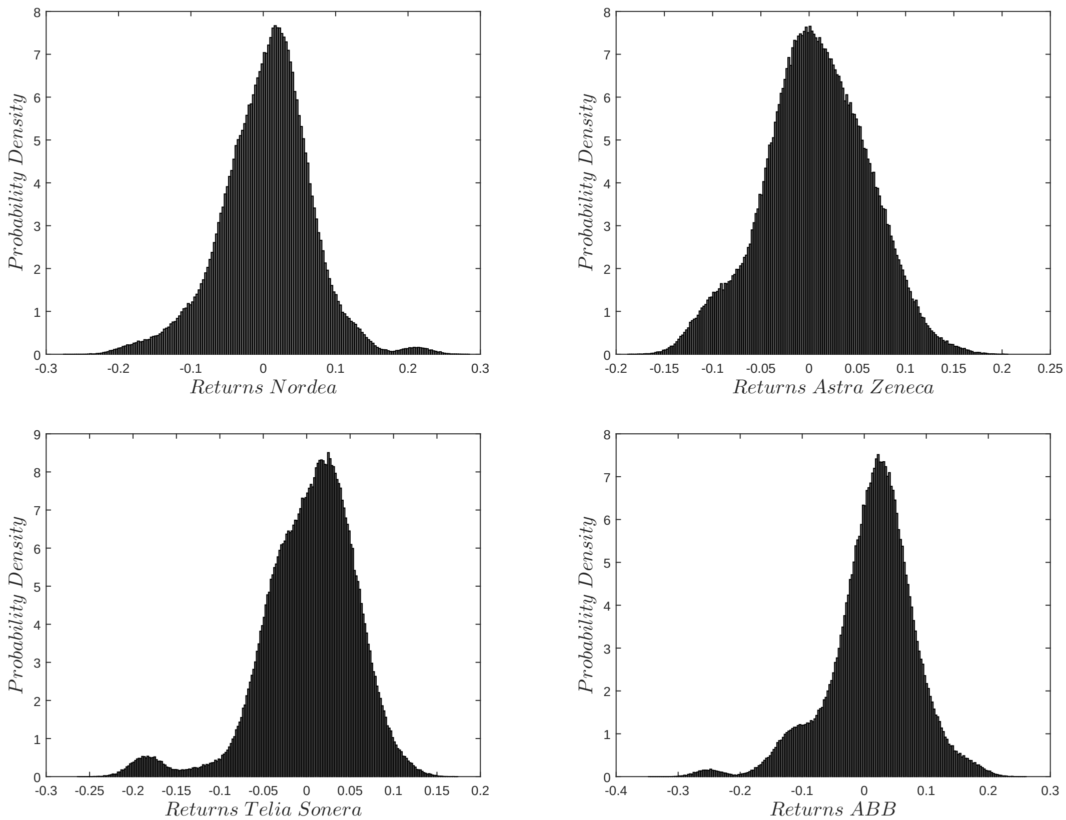

Figure 33: Stock probability plots.

With marginal statistics summarised in Table 6.

	Nordea	Astra Z.	Telia	ABB
μ (%)	0.47	0.68	0.32	1.37
σ (%)	6.22	5.45	5.46	6.91
sk	-0.15	-0.09	-0.97	-0.66
ku	4.13	2.93	5.12	4.55

Table 6: Prior market statistics.

We see that none of the marginal distributions fit to a normal distribution and the probability plots in Figure 33 show some interesting tail behaviour.

6.2.3 The Market Posterior

The prior and posterior probability densities are shown together in Figure 34.

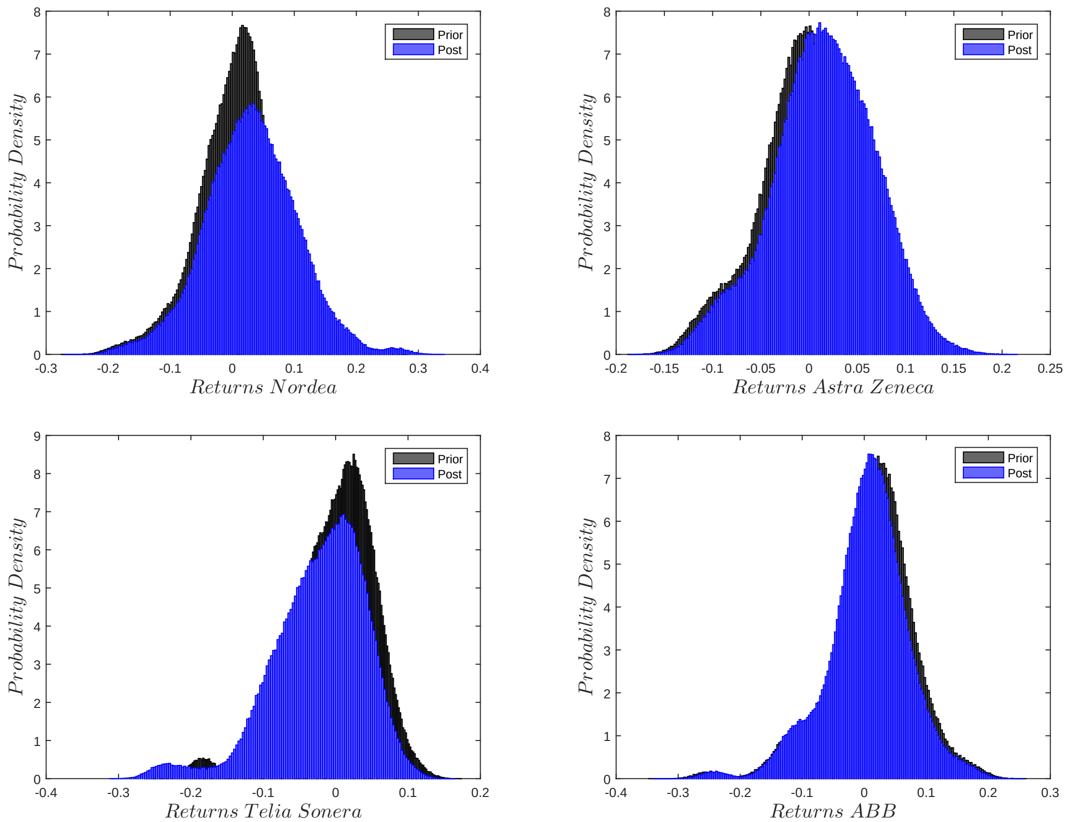

Figure 34: Stock probability plots.

In contrast to the views for the bond portfolio with one bond in each view where the probability mass was moved to the forecasted equally distributed interval we now see a different behaviour. When a view consists of more than one input we see that the density is shifted in the direction of the view but this time the view does not get the uniform distribution. For the densities of Nordea and Telia Sonera mass is moved to the right for Nordea and to the left for Telia Sonera due to the extreme view of 10-25 % returns. The possibility of a positive return increases for Astra Zeneca and a negative return is more likely for ABB. The scatter plots showing the prior and posterior dependence structures are show in Figure 35 and 36. We choose to only plot a smaller sample of 5000 to be able to see the dependence structure.

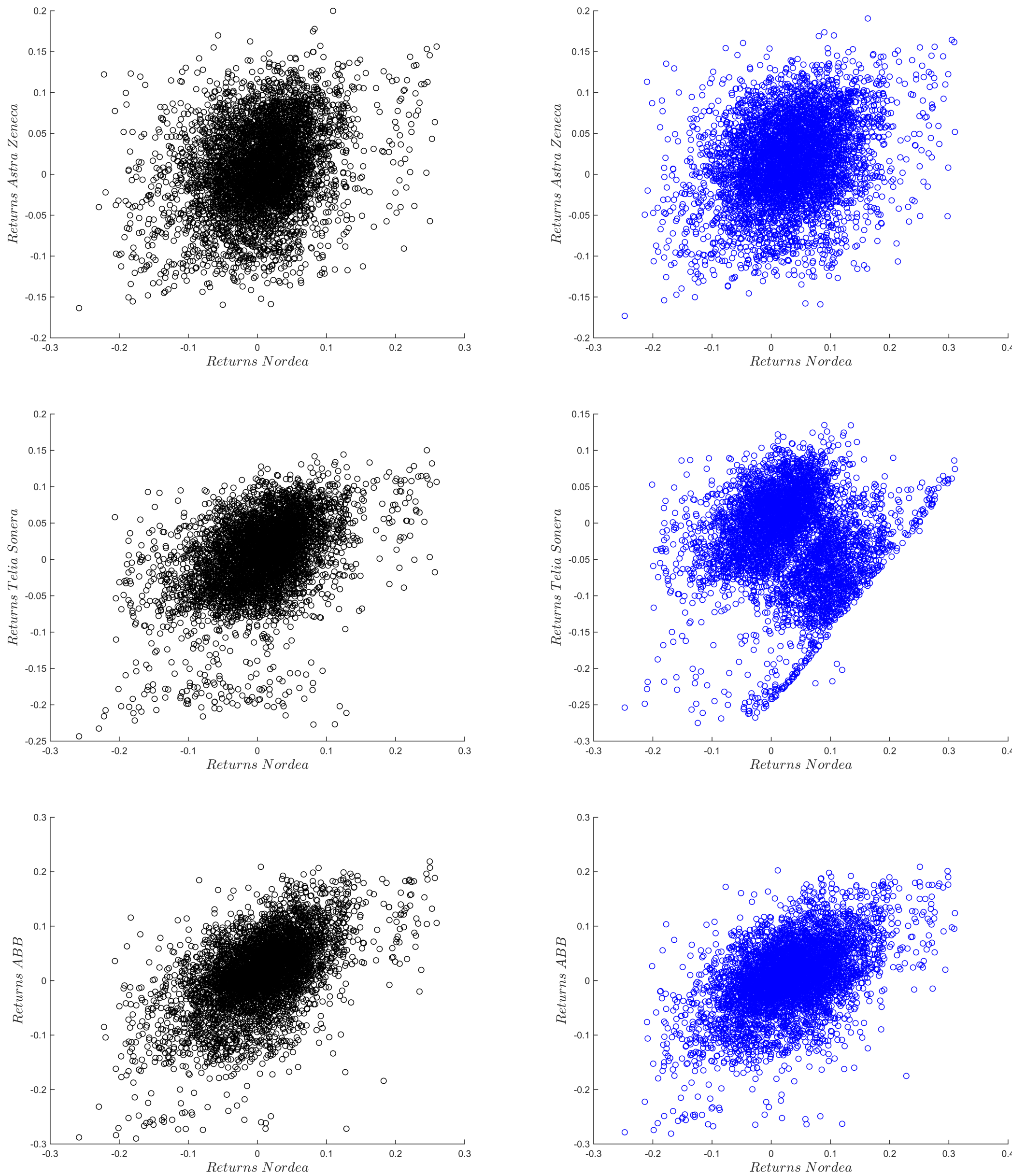

Figure 35: Stock scatter plots. *Left*: prior in black. *Right*: posterior in blue.

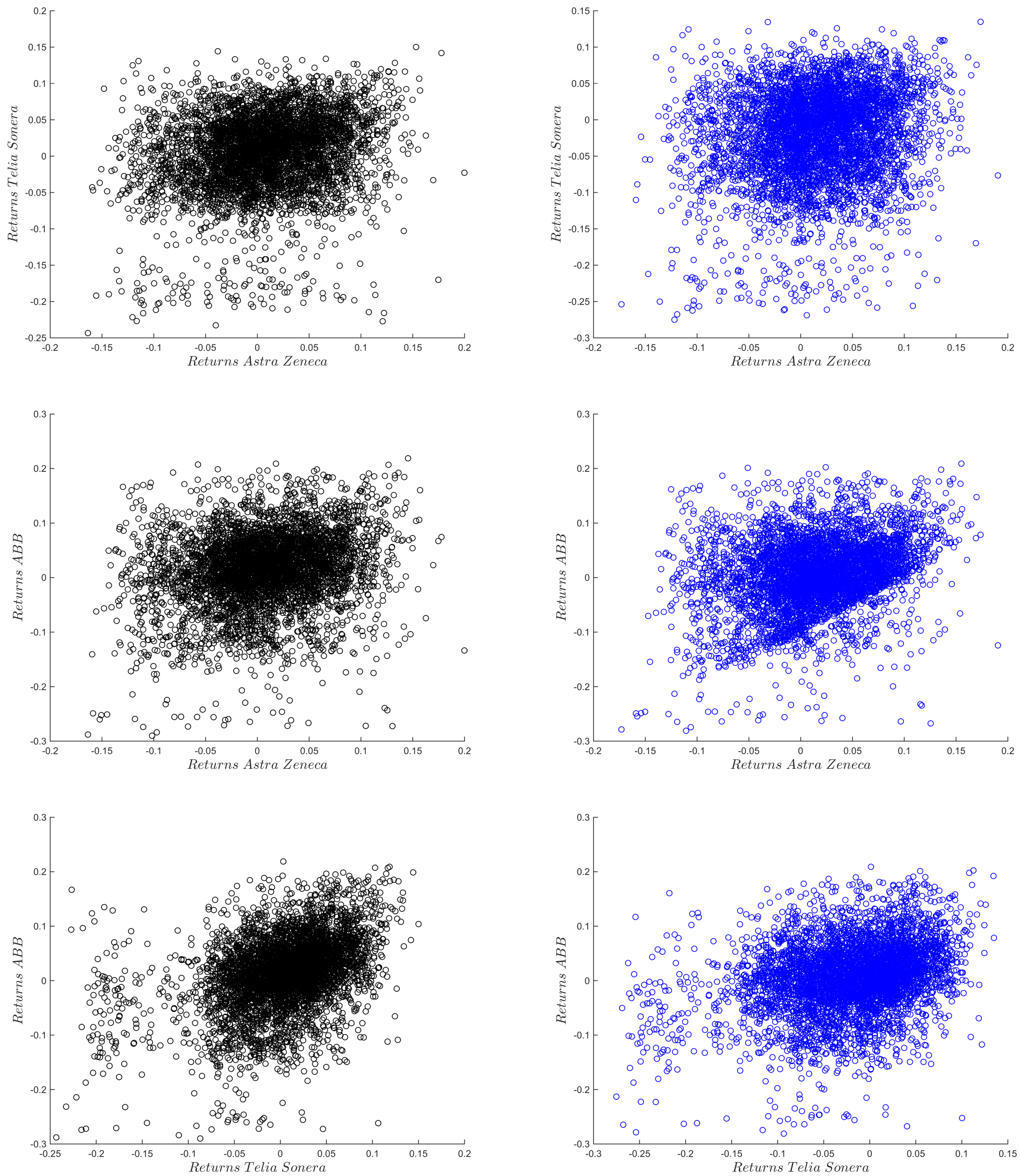

Figure 36: Stock scatter plots. *Left*: prior in black. *Right*: posterior in blue.

The scatter plots confirm low dependency between the stocks but some lower and upper tail dependence due to the t copula. We see some differences between the prior and posterior scatter plots, especially for the scatter plots of the stocks in the same view. The other dependencies do not seem to be affected that much. For the stocks in the same view, i.e. Nordea/Telia Sonera and Astra Zeneca/ABB, we see a new linear pattern. If we fit a t copula to the posterior distribution we get the following parameters

$$\mathbf{R_{post}} = \begin{pmatrix} 1 & 0.28 & -0.03 & 0.50 \\ 0.28 & 1 & 0.16 & 0.27 \\ -0.03 & 0.16 & 1 & 0.29 \\ 0.50 & 0.27 & 0.29 & 1 \end{pmatrix}, \ \nu_{post} = 10.2, \tag{59}$$

which shows that the dependence structure is slightly changed but most for the view-affected stocks. To see what really happens we set $c_k = 1$ and study the margins and scatter plots.

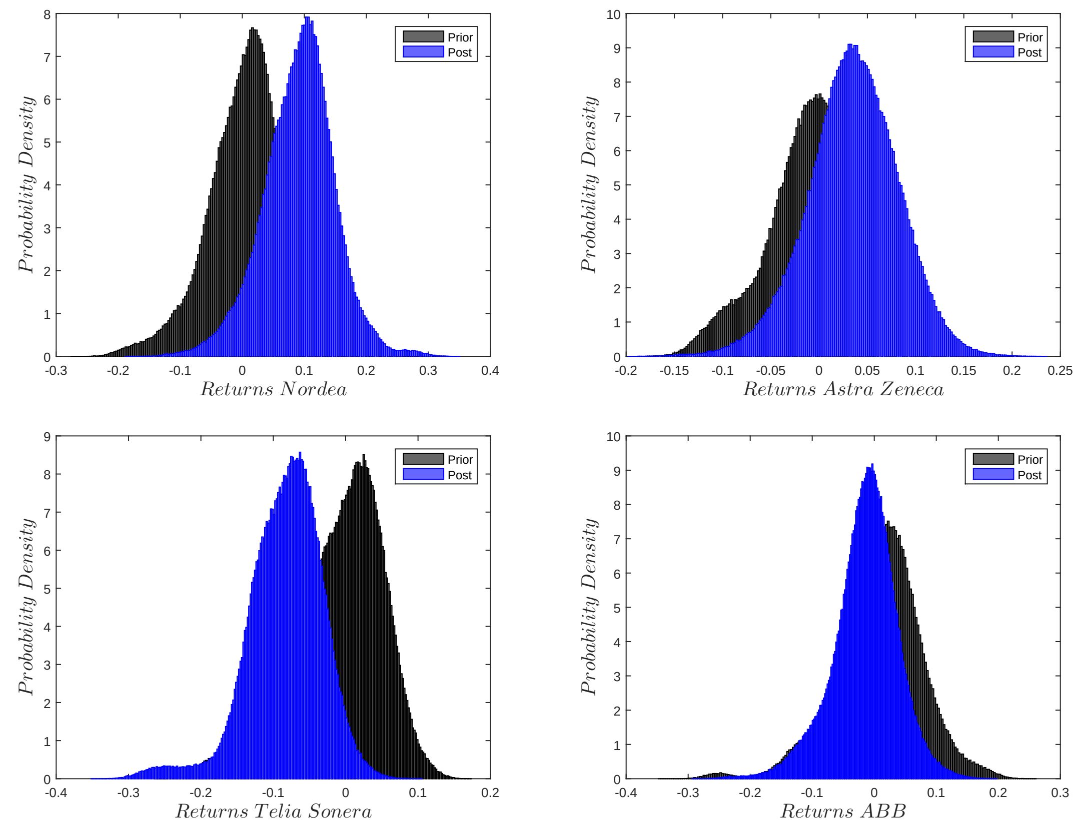

Figure 37: Stock probability plots.

From the pdfs in Figure 37 we conclude that the pdfs of the stocks involved in the same view are moved in the direction chosen in the pick matrix but now when we have a joint view the marginals does not become equally distributed. Scatter plots using $c_k = 1$ are shown in Figure 38 and 39.

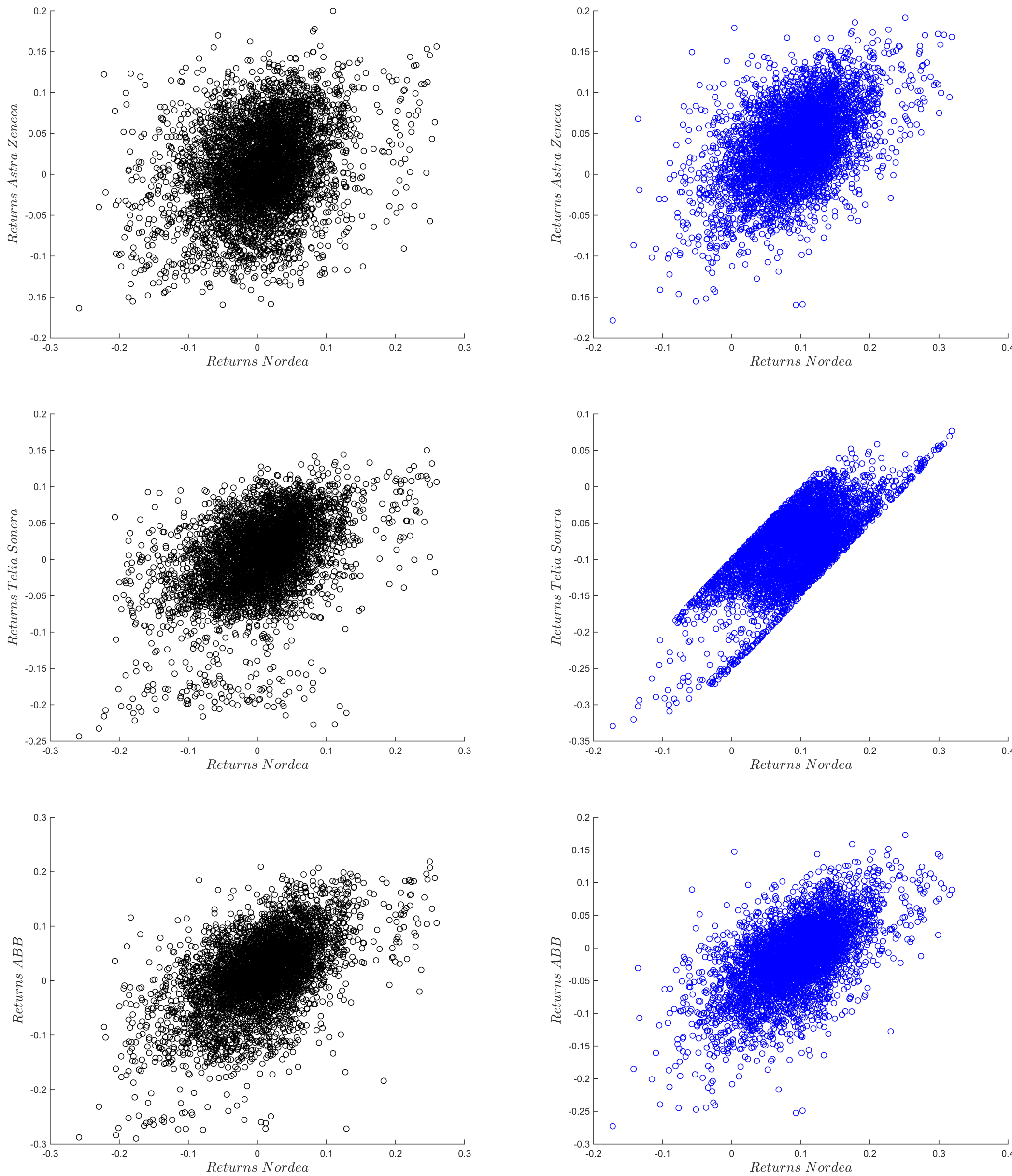

Figure 38: Stock scatter plots. *Left*: prior in black. *Right*: posterior in blue.

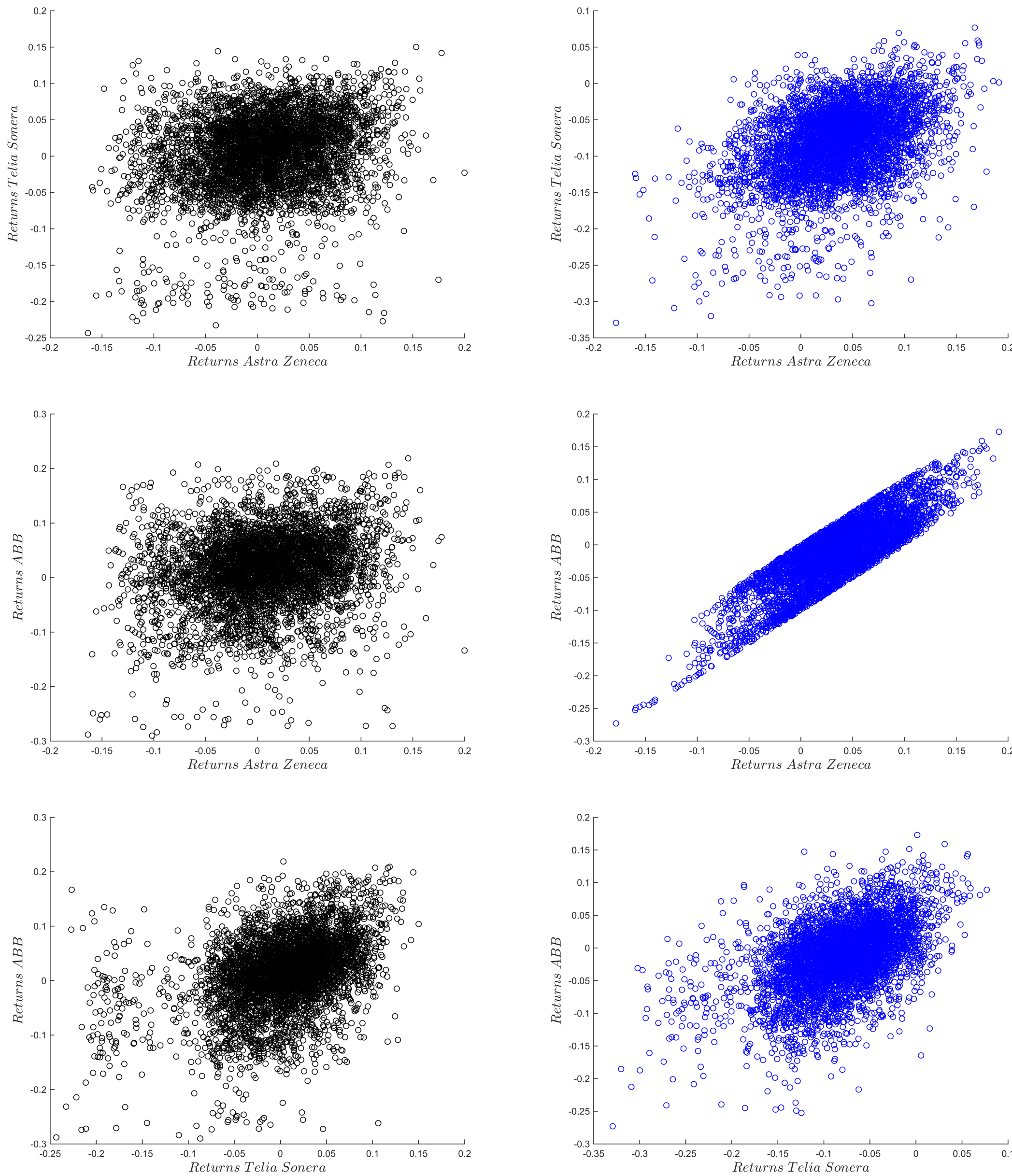

Figure 39: Stock scatter plots. *Left*: prior in black. *Right*: posterior in blue.

Stocks belonging to the same view will get a linear dependence structure and the dependence between other stocks will change but will still be recognizable. The linear dependence shown in Figure 38 and 39 show the relative performance of the stocks in the joint views meaning that the stock return of Nordea always will outperform the return of Telia Sonera with 10-25 % units and the return of Astra Zeneca will be 0-10 % units better than ABB. The posterior dependence structure is given by the parameters of the t copula as

$$\mathbf{R_{post}} = \begin{pmatrix} 1 & 0.49 & 0.68 & 0.60 \\ 0.49 & 1 & 0.42 & 0.83 \\ 0.68 & 0.42 & 1 & 0.50 \\ 0.60 & 0.83 & 0.50 & 1 \end{pmatrix}, \ \nu = 15.4, \tag{60}$$

6.2.4 Portfolio Allocation

To further illustrate COP's impact on portfolio allocation we again use the swapped mean-ES trade-off. The net returns are calculated as

$$R = \frac{S_1}{S_0} - 1,$$

where S_0 is the stock price at the time of the investment and S_1 is the stock price in one month. We set $c_k = 0.3$ and the problem to be solved is

$$\text{maximize} \quad \mathbf{w}^T\boldsymbol{\mu},$$

$$\text{subject to} \quad \text{ES}_{0.05}(\mathbf{w}) \leq \text{ES}^*, \quad \mathbf{w}^T\mathbf{1} \leq 1, \quad w_k \geq 0, \quad k = 1, \ldots, 4.$$

The efficient frontier is shown in Figure 40. We recall that the views changed the pdfs

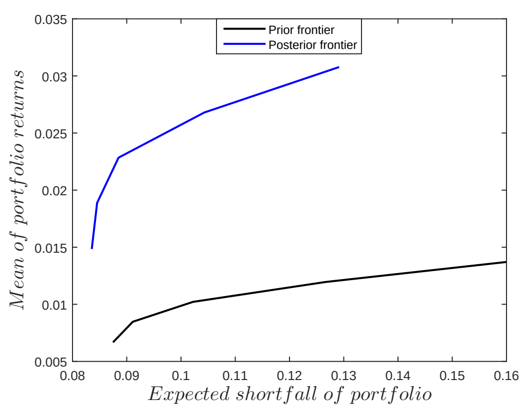

Figure 40: Efficient frontiers.

and will lead to outcomes with higher returns than one could normally expect from some of the stocks and see that the posterior frontier has higher returns than the prior, only long positions allowed. A comparison with the mean-variance frontier is shown in Figure 41.

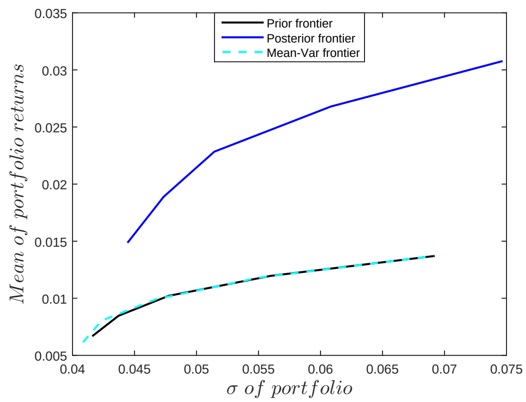

Figure 41: Efficient frontiers.

Again the mean-variance frontier is very close to the prior, the curves differ some in the region where $\sigma \in [0.04, 0.045]$. Five portfolios from each efficient frontier are presented in Table 7.

		Nordea	Astra	Telia	ABB
Prior	I	0.12	0.53	0.22	0.14
	II	0	0.57	0.12	0.31
	III	0	0.50	0	0.50
	IV	0	0.25	0	0.75
	V	0	0	0	1
Posterior	I	0.26	0.57	0.11	0.06
	II	0.33	0.59	0.04	0.04
	III	0.49	0.51	0	0
	IV	0.74	0.26	0	0
	V	1	0	0	0
Mean-Var	I	0.10	0.43	0.35	0.12
	II	0	0.45	0.24	0.31
	III	0	0.45	0.06	0.49
	IV	0	0.27	0	0.73
	V	0	0	0	1

Table 7: Prior and posterior allocation, each row represents a portfolio, I being the least risky and V the most risky. Increasing risk for each portfolio.

For lower risk the posterior allocation suggests more capital invested in Nordea and Astra Zeneca and less in Telia and ABB compared to the prior, which is inline with the views. For portfolio V with the highest risk the posterior means a full investment in Nordea which according to the first view should outperform Telia Sonera with 10-25 % units. The mean-variance and prior allocations are quite similar to each other.

We now allow shorting the stocks up to 10 SEK each when the full investment is 1 SEK. The optimization problem becomes

$$\text{maximize} \quad \mathbf{w}^T \boldsymbol{\mu},$$

$$\text{subject to} \quad \mathrm{ES}_{0.05}(\mathbf{w}) \leq \mathrm{ES}^*, \quad \mathbf{w}^T \mathbf{1} \leq 1, \quad w_k \geq -10, \quad k = 1, \ldots, 4\ ,$$

and the efficient frontier is shown in Figure 42.

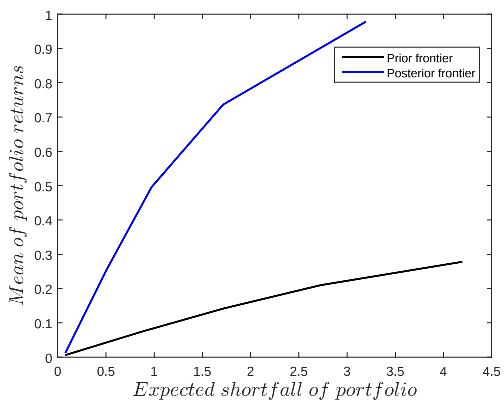

Figure 42: Efficient frontiers.

The return axis now spans between 0 and 1 meaning a possibility of a much higher return which of course goes hand in hand with higher expected shortfall. The comparison with the mean-variance frontier is shown in Figure 43.

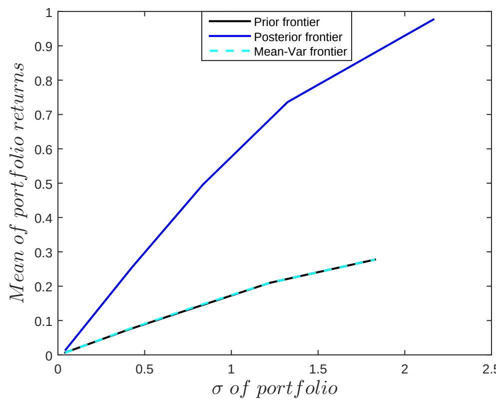

Figure 43: Efficient frontiers.

Again the mean-variance approach gives similar results as the prior distribution together with the swapped mean-ES trade-off. The investment in each stock is presented in Table 8.

		Nordea	Astra	Telia	ABB
Prior	I	0.11	0.54	0.21	0.14
	II	−3.38	2.46	−4.53	6.45
	III	−6.75	4.46	−9.43	12.72
	IV	−10	−0.23	−10	21.32
	V	−10	−10	−10	31
Posterior	I	0.27	0.57	0.10	0.06
	II	3.75	2.24	−4.70	−0.29
	III	7.18	3.97	−9.53	−0.62
	IV	16.41	3.09	−10	−8.50
	V	31	−10	−10	−10
Mean-Var	I	0.10	0.43	0.35	0.12
	II	−3.31	1.34	−3.80	6.77
	III	−6.73	2.25	−7.95	13.43
	IV	−10	−0.21	−10	21.21
	V	−10	−10	−10	31

Table 8: Prior and posterior allocation, each row represents a portfolio, I being the least risky and V the most risky. Increasing risk for each portfolio.

The views again affect the allocation as expected. For the posterior I-IV long positions are suggested in Nordea and Astra together with short positions in Telia and ABB which is inline with the views. The posterior V with a really high risk level puts as much as possible in a long position in Nordea and the rest in short positions. The prior and mean-variance portfolios differ some but are really close to each other.

6.3 Currency Portfolio

At last we will look at a situation where an investor wants to invest in the currency crosses EURSEK, USDSEK, EURUSD, NOKSEK and EURGBP. The investor will use forecasts published at the day of investment by a professional foreign exchange analyst and an investment horizon of three months. This time the views will not be modelled by a uniform distribution, instead we will use the normal and the beta distribution. For both distributions the analyst's forecast will be used as the mean. The forecasts are shown in Table 9.

	EURSEK	USDSEK	EURUSD	NOKSEK	EURGBP
Spot	9.44	7.68	1.23	1.04	0.79
3M	9.20	7.67	1.20	0.99	0.76
2015-06-30	9.10	7.71	1.18	1.01	0.75
Change 3M/Spot	−2.5%	−0.1%	−2.4%	−4.8%	−3.8%

Table 9: Currency forecasts from *Nordea Markets*, published 2014-12-18.

We see that the forecast suggests that all crosses will decrease in value within 3 months but USDSEK and NOKSEK will then rebound. The investor tries to capture these movements in the distributions of the views and decides to model them as follows.

$$\hat{F}_k(v) = \begin{cases} \int_0^v \frac{\Gamma(a_k+b_k)}{\Gamma(a_k)\Gamma(b_k)} v^{a_k-1}(1-v)^{b_k-1}, & k = 1,3,5 \\ \\ \int_{-\infty}^v \frac{1}{\sigma_k\sqrt{2\pi}} e^{-(v-\mu_k)^2/(2\sigma_k^2)}, & k = 2,4 \end{cases}, \tag{61}$$

for $\mathbf{a} = \begin{pmatrix} a_1 \\ a_3 \\ a_5 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \\ b_3 \\ b_5 \end{pmatrix} = \begin{pmatrix} 195 \\ 200 \\ 127 \end{pmatrix}$, $\boldsymbol{\mu} = \begin{pmatrix} \mu_2 \\ \mu_4 \end{pmatrix} = \begin{pmatrix} 0.001 \\ 0.048 \end{pmatrix}$ and $\boldsymbol{\sigma} = \begin{pmatrix} \sigma_2 \\ \sigma_4 \end{pmatrix} = \begin{pmatrix} 0.01 \\ 0.01 \end{pmatrix}$. The chosen parameters a and b lead to means equal to the absolute value of the indicated change 3M/Spot. Due to the negative views the pick matrix becomes

$$\mathbf{P} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \tag{62}$$

and because all currency crosses have a view the null space of $\mathbf{P}$ is empty and $\mathbf{P} = \bar{\mathbf{P}}$.

6.3.1 The Market Prior and Posterior

To generate the market prior distribution we, as in the previous cases, fit a t copula to the data set and simulate $J = 500000$ joint samples. The obtained parameters of the copula becomes

$$\mathbf{R} = \begin{pmatrix} 1 & 0.35 & 0.16 & 0.70 & 0.28 \\ 0.35 & 1 & -0.86 & 0.34 & -0.47 \\ 0.16 & -0.86 & 1 & 0.02 & 0.66 \\ 0.70 & 0.34 & 0.02 & 1 & 0.10 \\ 0.28 & -0.47 & 0.66 & 0.10 & 1 \end{pmatrix}, \ \nu = 4.5. \tag{63}$$

This time we start with setting $c_k = 1$ and plot the prior and posterior together in order to illustrate how the views are modelled, see Figure 44.

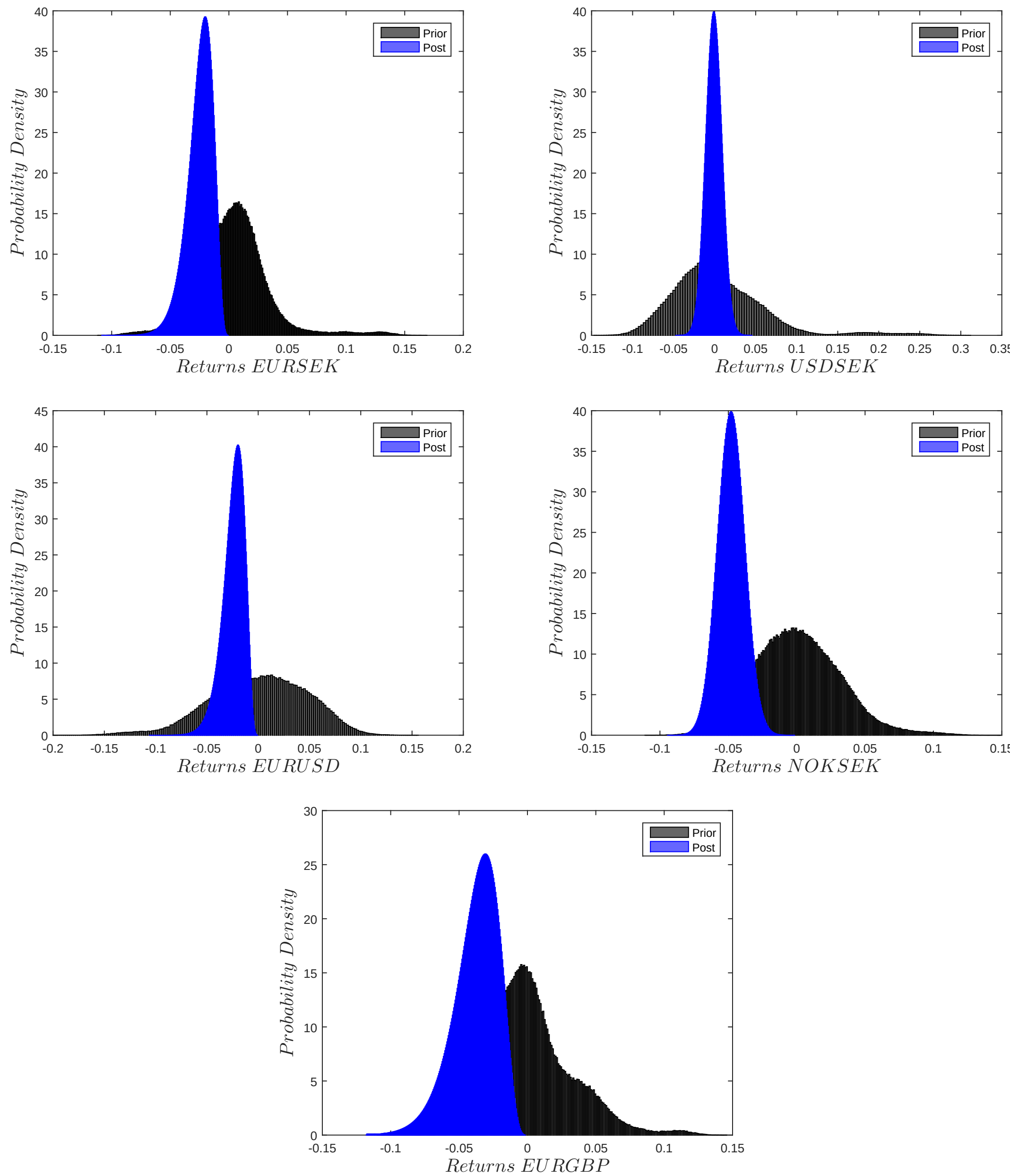

Figure 44: Currency probability plots, $c_k = 1$.

The beta distribution only allows outcomes for values between 0 and 1 but due to the negative view the distributions here have negative outcomes instead. The beta distributed returns put more probability mass near zero but there are still some mass in the left tail. The normally distributed views of USDSEK and NOKSEK both have positive and negative outcomes due to the rebound in the forecasts.

In the investment case the investor sets $c_k = 0.5$ meaning that the posterior distribution will be the mean of the prior and posterior. The distributions are compared in Figure 45.

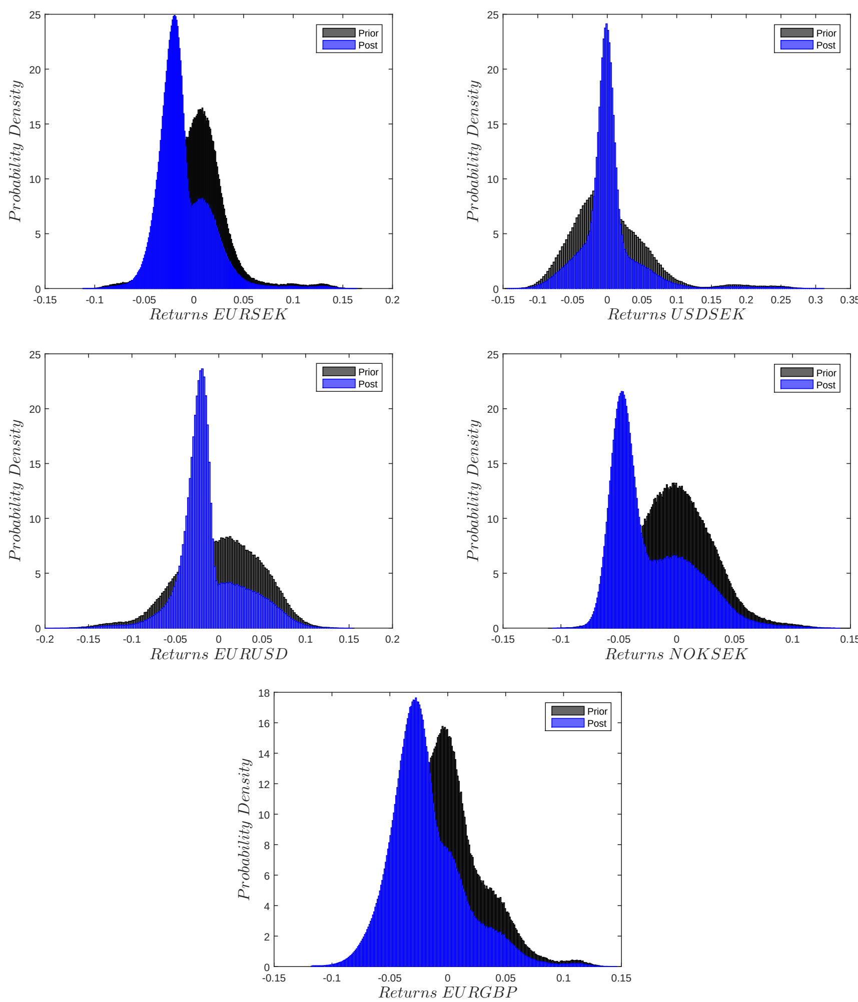

Figure 45: Currency probability plots, $c_k = 0.5$.

If a t copula is fitted to the posterior distribution we get the parameters

$$\mathbf{R_{post}} = \begin{pmatrix} 1 & 0.35 & 0.16 & 0.70 & 0.29 \\ 0.35 & 1 & -0.86 & 0.34 & -0.46 \\ 0.16 & -0.86 & 1 & 0.02 & 0.65 \\ 0.70 & 0.34 & 0.02 & 1 & 0.10 \\ 0.29 & -0.46 & 0.65 & 0.10 & 1 \end{pmatrix}, \ \nu_{post} = 4.9, \tag{64}$$

which shows that the prior dependence structure is inherited.

6.3.2 Portfolio Allocation

The investor is allowed to short sell up to 10 currency units in each asset and wants to solve the following problem

$$\text{maximize} \quad \mathbf{w}^T \boldsymbol{\mu},$$
$$\text{subject to} \quad \mathrm{ES}_{0.05}(\mathbf{w}) \leq \mathrm{ES}^*, \quad \mathbf{w}^T \mathbf{1} \leq 1, \quad w_k \geq -10, \quad k = 1, \dots, 5.$$

The efficient frontier is shown in Figure 46.

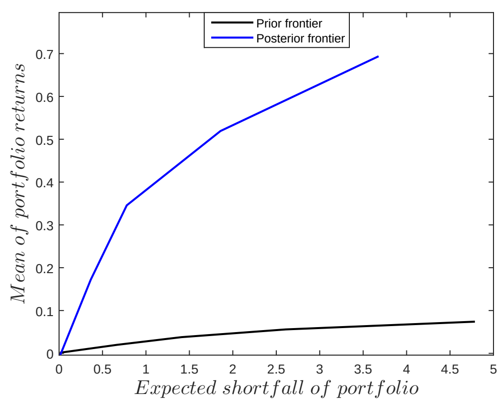

Figure 46: Efficient frontiers.

The posterior distributions are more extreme than the prior allowing for more probability mass on lower returns in this case due to the negative views which is why the frontiers differ. The comparison with the mean-variance frontier is shown in Figure 47.

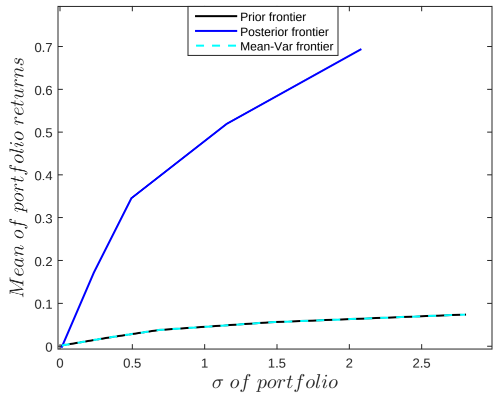

Figure 47: Efficient frontiers.

The investments in each currency cross are presented in Table 10.

		EURSEK	USDSEK	EURUSD	NOKSEK	EURGBP
Prior	I	-0.52	0.71	0.67	0.03	0.11
	II	13.20	1.27	-0.21	8.38	-4.88
	III	28.12	-1.42	-5.70	-10	-10
	IV	16.88	14.12	-10	-10	-10
	V	-10	41	-10	-10	-10
Posterior	I	-0.32	0.74	0.68	-0.07	-0.03
	II	0.91	5.93	5.32	-6.82	-4.34
	III	-4.28	13.87	11.41	-10	-10
	IV	-10	27.56	3.44	-10	-10
	V	-10	41	-10	-10	-10
Mean-Var	I	-0.46	0.61	0.69	0.07	0.09
	II	12.93	1.36	0.94	8.06	-3.17
	III	28.89	-2	-5.89	-10	-10
	IV	16.92	14.08	-10	-10	-10
	V	-10	-41	-10	-10	-10

Table 10: Prior and posterior allocation, each row represents a portfolio, I being the least risky and V the most risky. Increasing risk for each portfolio.

If we look at the posterior allocation we see that USDSEK and EURUSD get long positions and this is because some probability mass is moved from the really bad outcomes, decreasing the risk. NOKSEK and EURGBP are similar because there is already high probability for negative outcomes in the prior distribution. The big difference can be seen in EURSEK where the prior suggests big long positions in portfolio III and IV and the posterior suggests big short position in these portfolios. The prior and mean-variance portfolios are close to each other.

7 Conclusions

We have now seen how to apply COP and we have also made some observations regarding its impact and behaviour. In *Applications and Analysis* the model is applied on two portfolios using the uniform distribution. Two ways of expressing the views are used, outright and relative views. In the bond portfolio implied yields are used to get a kind of market consensus for the views. For the equity portfolio realistic relative views are used. The graphical analysis together with the parameters of the copula shows how these different views affect and it is clear when to use which type. To further illustrate the impact of COP the prior and posterior market distributions are compared using a portfolio optimization approach and here it is also clear that the views have a huge impact on the final allocation, more or less depending on the confidence level. For the third portfolio more realistic distributions are used for the views together with real analyst's forecasts to create a realistic situation. The main conclusions are the following:

- The method used to obtain the market prior distribution seems to be a good and effective method generating realistic results. However, there may be better methods for doing this but for the purpose of this thesis the method is good enough.
- Copula opinion pooling is easy to implement and the calculations are made within a few seconds even though the number of simulations used are huge.
- Three portfolio allocation cases where COP is used in different ways are presented. These cases show the flexibility of the method and generate reasonable results.
- The results rely very much on the types of views used and the distributions of the views. It is up to the practitioner to decide how to use the method but according to the results it could be useful to a portfolio manager using mathematical methods if own views should be included in the allocation process.
- The confidence level is quite hard to decide, i.e. how much the prior distribution should be modified. It is though always possible to study the posterior distribution and then make adjustments if necessary.
- Mean-Variance optimization is a very good approximation yielding almost the same results as the swapped mean-ES approach when the underlying distribution is nearly normally distributed.

The purpose of this thesis was to explain, analyze and apply the copula opinion pooling method introduced by Attilio Meucci. The purpose is considered to be fulfilled.

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