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Investigating usefulness of portfolio optimization with respect to prospect utility in financial advisory

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Abstract

In this paper we derive and analyze the usefulness of a prospect theory based model for selecting optimal portfolios with respect to multiple investment goals. The focus is to determine whether or not the model would be suitable for the advisory process by investigating the result given by the optimal portfolio values and proportion in risky assets in continuous time. The model is based on the framework proposed by Berkeelar et al. [1] and De Giorgi [2] and follows a two step approach. It starts by finding the optimal terminal portfolio value for each investment goal and secondly determines the optimal initial funding for each investment goal based on the optimal terminal portfolio value. We have shown that the initial funding is monotone in the long term investment goal, in other words the investor initially puts all capital in that goal and therefore neglect remaining goals. Moreover we have shown that the model, assuming evenly distributed initial capital among investment goals, results in the investor reaching the short term goal only, for median risk profile but reaching all investment goals for the extreme loss averse profile. Lastly we also point out that the model holds very high leverage in risky assets for the median risk profile and less in risky assets when the investor is considered extreme loss averse. We conclude that this model is not suitable for the financial advisory process mainly because the median risk profile does reach her long term goal.

Sammanfattning

I det här dokumentet tar vi fram och analyserar användbarheten av en prospect theory baserad modell för att välja optimala portföljer, med avseende på flera investeringsmål. Fokus var att avgöra om modellen skulle vara lämplig för en rådgivningsprocess, genom att undersök resultatet från optimala portföljvärden och andelar i risktillgångar, för kontinuerlig tid. Vår modell är baserad på ramverket framtaget av Berkeelar et al. [1] och De Giorgi [2] och följer en tvåstegsmetod. Den börjar med att hitta det optimala terminala portföljvärdet för varje investeringsmål och för det andra bestämmer den optimala finansieringen av varje investeringsmål, baserat på det optimala terminala portföljvärdet. Vi har visat att den initiala finansieringen är monoton i det långsiktiga målet, vilket innebär att investeraren initialt allokerar allt kapital på det långsiktiga målet och därmed försummar resterande mål. Vidare har vi visat att modellen, förutsatt initialt fördelat kapital bland målen, resulterar i att investeraren endast når det kortsiktiga investeringsmålet för en median riskprofil men uppnär alla mål för extrem förlustmotvilja. Slutligen påpekar vi även att investeraren tar väldigt hög leverage när vi antar riskprofilen för en medianinvesterare och investerar mindre i risktillgångar när investeraren anses ha extrem förlusträdsla. Vi drar slutsatsen att denna modell inte är lämplig för den finansiella rådgivningsprocessen på grund av att en median riskprofil inte uppnår det långsiktiga investeringsmålet.

Contents

1 Introduction

Prospect theory was first formulated in 1979 by Kahneman and Tversky [3] as an improved alternative to expected utility theory in the area of decision making under risk. The theory and utility value function developed to describe risk behaviour is based on gains and losses, unlike expected utility theory where utility is determined with respect to the total portfolio value. The research concluded that people value gains and losses differently and in most cases losses has a greater impact than gains. Another key part in prospect theory is the suggestion that people weight probabilities differently. According to Kahneman and Tversky, people are more sensitive to changes in high probabilities than lower probabilities. This is called the certainty effect and leads to people being risk seeking when facing sure losses and vice versa regarding sure gains [4].

Within financial advisory it is import to know what the customer wants. Depending on what risk profile they have, the investment strategies proposed ought to be adapted accordingly. One problem with this is that it will take a considerable amount of time in order to evaluate and determine each customer's risk profile through meetings. Instead, customers can answer questionnaires that will determine their risk profile based on the framework of prospect theory. This data can then be used in order to implement portfolio optimization using the utility value function proposed in prospect theory [4]. The output of this model will suggest a specific portfolio. With this tool the advisor can get an understanding about how to invest in order for the customers to obtain what they want.

The purpose of this thesis is to apply the framework given in prospect theory to portfolio optimization and derive a model that can be used in the advisory process as described earlier. There has in fact already been many applications to prospect theory including two similar models developed by Berkelaar et al. [1] and De Giorgi [2], where the latter is an extension to the former. We focus on evaluating these models in our thesis.

2 Methodology

This section describes why the specific method is chosen and briefly its advantages in comparisons with other similar methods.

People have different perceptions of risk and when it comes to the advisory process it is important to get a grasp of to what extent a customer is willing to be exposed towards risk. Simply maximizing a portfolio according to, for instance modern portfolio theory [5], is a generally acknowledged method to select a portfolio. A disadvantage with this method is that it does not consider what specific risk profile an individual may have but instead what risk level a portfolio must have. By maximizing utility we can customize a portfolio for an investor depending on its specific risk profile. There are of course problematic parts for these kind of models as well, one of the problems lie in the process of quantifying risk profiles. Expected utility theory and prospect theory are two well known models that takes this into account in their respective utility functions.

In expected utility theory the risk profile can be regarded as the concavity of the utility function which is quantified or demonstrated through the Arrow-Pratt's risk aversion coefficient. It is also assumed that people have a rational mind and therefore make rational decisions under uncertainty, this is however not the case. A well known contradiction is the Allai paradox which violates the independence axiom and therefore violates the framework of expected utility maximization [6].

The previously mentioned assumption in expected utility theory has been examined and rejected as an economic behavioural model, by Kahneman and Tversky [3]. They have concluded that people make irrational choices under uncertainty. In their research they propose that people's risk profile depend on changes during the investment horizon, i.e. gains and losses and not to total value of portfolios. Since expected utility theory demand every investor to rationally obey the standard axioms of expected utility one conclude that this maximization approach is naive and has a very small realistic value. Kahneman and Tversky showed that when people face loss they tend to behave different in comparison to gains as mentioned earlier. This assumption is incorporated in prospect theory but not in expected utility theory. Thus by adapting prospect theory to describe investor's risk profile we can possibly improve the portfolio selection based on utility maximization.

3 Theory

In this section the theory behind the applied model is explained and derived. The theory is mainly based on the research regarding prospect theory developed by Tversky and Kahneman [4]. We derive models that are used in research done by Berkelaar et al. [1] and mainly the results derived by De Giorgi [2].

3.1 Prospect Theory

In order to account for the violation of the standard axioms of expected utility theory, Kahneman and Tversky mention five important phenomena. According to them, these cases must be considered in order to be a sufficiently descriptive theory of decision making under risk. First of all they bring up the framing effect. The result of a choice problem may be different depending on how the problem is formulated [7]. The second phenomena regards nonlinear preferences where expected utility theory assumes linear probability. Camerer and Ho [8] discovered nonlinear weighting in probability regarding choices that do not involve sure things. People tend to overweight lower probabilities and underweight higher probabilities. Thirdly they mention source dependence, which implies that people are more willing to bet on events they know more about. Ellsberg [9] found in an experiment that people preferably bet on an urn containing equal numbers of red and green balls over an urn with unknown number of red and green balls. The fourth phenomena regards risk seeking choices where Kahneman and Tversky claims that risk seeking behaviour occurs in two situation. First of all they state that when people evaluate a small probability of winning a large price or the expected value of that prospect, they choose the former. Second, when faced with choosing between a sure loss and an even greater loss with a substantial probability people tend to gamble. The fifth and last phenomena, loss aversion, states that losses have a greater impact than gains for people in general [4].

To clarify the usage of prospect theory we can consider the following experiment performed by Kahneman and Tversky on a group of students. The experiment is based on the Allai paradox which is explained in further detail by Hult et al. [6]. Consider two set of gambles with the following prospects

Gamble 1:

$$\text{A:} \quad 4000, 0.80 \qquad \text{or} \qquad \text{B:} \quad 3000, 1.00$$

and,

Gamble 2:

$$\text{C:} \quad 4000, 0.20 \qquad \text{or} \qquad \text{D:} \quad 3000, 0.25$$

where the first number corresponds to the profit and the second to the probability of that prospect. In this experiment 80% of the students selected prospect B in Gamble 1 even though the expected utility of prospect B is less than prospect A. In Gamble 2 on the other hand, 65% of the students chose the prospect with greater expected utility, i.e. prospect C. This clearly demonstrates the contradiction related to the general assumptions that people are rational when facing decision making under risk, stated in expected utility theory.

3.2 The utility function

In order to account for the violations in expected utility theory, Kahneman and Tversky suggested a new utility function that incorporates the investors uncertainty when facing losses. They found that the value function have three distinguishable properties that are important to point out and that is *(i)* it is evaluated over gains and losses instead of final states. *(ii)* The function is convex for losses and concave for gains. *(iii)* The function derivative is higher on the loss side, i.e. losses have greater impact than gains. The basic perception that individuals value outcomes based on a reference point and not on final states can be exemplified through the notation that changes in temperature depends on the adaption to that temperature. For example Scandinavian individuals may hold a lower reference point of "hot" than Mediterranean individuals. The same notion applies to wealth where poverty for one individual may stand for reference as richness for another individual. Khaneman and Tversky define the prospect theory utility function [4] as

$$u(X) = \begin{cases} X^{\alpha} & , X \geq 0 \\ -\lambda(-X)^{\alpha} & , X < 0 \end{cases} \tag{1}$$

where they found, through non linear regression of the experimental data, that the parameter values of the median investor corresponds to $\alpha = 0.88$ for both gains and losses and the loss aversion $\lambda = 2.25$.

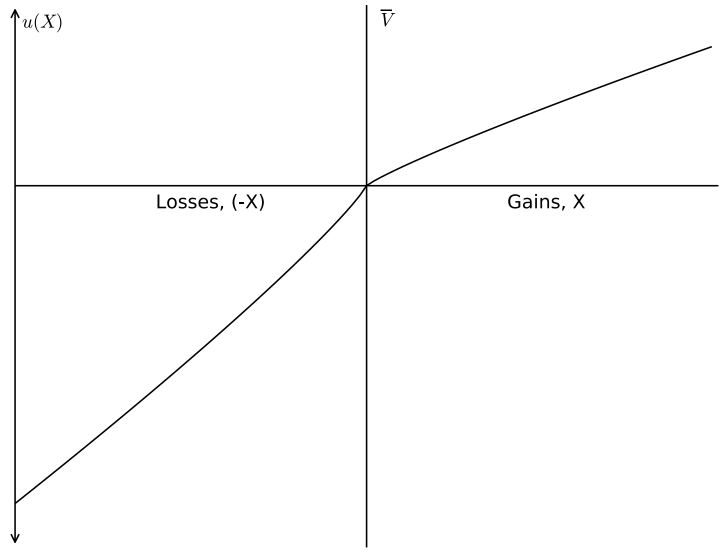

Figure 1: Utility function proposed by Kahneman and Tversky [3] where $\alpha = 0.88$, $\lambda = 2.25$

In Figure 1 we see the behaviour of the utility function described with the properties *(i)*, *(ii)* and *(iii)*. In order to better analyze the change of individual risk profiles the utility function 2 could be reformulated according to [1] in the following way

$$u(X) = \begin{cases} \beta^{+} X^{\alpha} & , X \geq 0 \\ -\beta^{-}(-X)^{\alpha} & , X < 0 \end{cases} \tag{2}$$

where α corresponds to the risk aversion and the loss aversion parameter is redefined as

$$\lambda = \beta = \frac{\beta^{-}}{\beta^{+}}.$$

In this case we set $\beta^- = 2.25$ and $\beta^+ = 1$ in order to make the function correspond to the utility function defined in 1.

Assumptions

Regarding prospect theory and risk preferences we will mainly examine the median investor which are based on results from Kahneman and Tversky [4]. Furthermore we ignore the probability distortion function that describes how an individual weight probabilities. In general a person overweight lower probabilities and underweight medium to high probabilities [4]. For further details on how to incorporate the probability distortion into the framework of portfolio maximization see Jin and Zhou [10].

3.3 Dynamics

First of all we need to establish the dynamics of which the portfolio value, stock and the bank account follows. First we use the well established dynamics of the bank account and the stock, i.e. risky asset. The stocks, under the probability measure P, has the following dynamics as stated by Björk [11]

$$dS_{i,t} = \mu_{i,t} S_{i,t} dt + \sigma_{i,t} S_{i,t} dW_t^P \tag{3}$$

furthermore we have the bank account dynamics

$$dB_t = r_t B_t dt. \tag{4}$$

Now we notice that the portfolio dynamics must depend on the amount invested in the risky assets and the amount invested in the risk-free assets and therefore we get

$$dV_t = \lambda_{i,t} V_t \frac{dS_{i,t}}{S_{i,t}} + V_t(1 - \lambda_{i,t}) \frac{dB_t}{B_t} \tag{5}$$

If we now insert (3) and (4) into (5) we obtain the following portfolio dynamics under probability measure P

$$dV_t = r_t V_t dt + (\mu_{i,t} - r_t)\lambda_{i,t} V_t dt + \sigma_{i,t}\lambda_{i,t} V_t dW_t^P. \tag{6}$$

In order to clarify, we have that V_t is the total portfolio value at time t, $\mu_{i,t}$ is the expected return for risky asset S_i, r_t is the risk free rate at time t, $\sigma_{i,t}$ is the volatility for the risky asset $S_{i,t}$ and $\lambda_{i,t}$ is the fraction invested in risky asset $S_{i,t}$ at time t.

Assumptions

In this report we will assume market completeness. It therefore exists a unique pricing kernel under the probability measure Q. In more detail this assumption means that the drift term, μ_t, under the probability measure P can be eliminated when changing measure from P to Q using the Girsanov kernel. Thus the drift term is replaced by the risk free rate of return, r_t. Furthermore the diffusion term is also deterministic and represent the standard deviation, $\sigma_{i,t}$, (volatility) of the risky asset [11].

3.4 Stochastic discount factor

In this chapter we will use the notation used by Björk [11] and continue to use these throughout the report.

In order to price a contingent T-claim, X, under the probability measure Q, it is well known that it can be priced according to

$$\pi_{t,X} = E^{Q}[e^{-\int_t^T r_s ds} X | F_s]. \tag{7}$$

To make it more general and price it under the probability measure P instead, we can do this by using the measure transformation from P to Q which is defined by

$$\begin{cases} L_t = \frac{dQ}{dP} \\ dQ = L_t dP \end{cases} \tag{8}$$

where L_t is the likelihood process and follows the SDE

$$\begin{cases} dL_t = \varphi_t L_t dW_t^p \\ L_0 = 1 \end{cases}$$

and φ_t is the Girsanov kernel that takes us from the probability measure P to probability measure Q, by eliminating the drift via deterministic values of μ, r and σ, i.e

$$\varphi_t = \frac{\mu_t - r_t}{\sigma_t}.$$

In order to make a more confortable comparison we can express the relation (8) in expected values

$$\int dQ = \int L_t dP$$
$$E^Q[X|F_t] = E^P[L_t X|F_t]$$

by comparing this with (7) we can write

$$E^Q[e^{-\int_t^T r_s ds} X|F_t] = E^P[e^{-\int_t^T r_s ds} L(t)X|F_t] = E^P[\Lambda_t X|F_t]. \tag{9}$$

Now we start by solving the SDE (8) by using the natural logarithm ansatz and applying the Ito's formula and the Girsanov kernel. Then one will arrive at

$$L(t) = e^{-\frac{1}{2}\int_0^t \varphi_s^2 ds + \int_0^t \varphi_s dW_s^P}.$$

Lastly we use the result from (9) and find that the stochastic discount factor Λ_t can be expressed as

$$\begin{aligned}
\Lambda_t &= e^{-\int_0^t r_s ds} L_t \\
&= e^{-\int_0^t r_s ds} e^{-\frac{1}{2}\int_0^t \varphi_s^2 ds + \int_0^t \varphi_s dW_s^P} \\
&= e^{-\int_0^t (r_s + \frac{1}{2}\varphi_s^2) ds + \int_0^t \varphi_s dW_s^P}
\end{aligned}$$

Further by visual inspection of Λ_t we see that its dynamics can be expressed as

$$d\Lambda_t = -r_t \Lambda_t dt - \varphi_t \Lambda_t dW_t^P \tag{10}$$

We have now deduced an explicit expression for the stochastic discount factor and stated the dynamics of the stochastic discount factor. Throughout the report we will use the stochastic discount factor in order to price the claim X which will consist of a portfolio. Lastly we point out that (9) ensures that we price under Q when multiplying an arbitrary claim X with the stochastic discount factor.

3.5 Optimal terminal portfolio value

The principal optimization problem is that we want to maximize the utility for an investor's portfolio based on the utility function formulated in Prospect Theory [4]. The investor can have several investment goals, i.e. payoffs, where she distributes her initial capital between. In other words, we want to determine the optimal terminal portfolio value for each goal, that maximizes the expected value of each investment goal's utility with respect to the difference in terminal portfolio value and investment goal, i.e. gains/losses. Considering the above objective function with the constraint that the expected value of the discounted terminal portfolio has to be funded by the initial capital, i.e., the budget constraint and that the terminal portfolio value is non-negative, we get the following problem

$$\begin{aligned} &\max_{V_j(T_j)} E^P[u(V_{j,T_j} - \overline{V}_j)] \\ &\text{s.t.} \quad E^P[\Lambda_{j,T_j} V_{j,T_j}] \leq \Lambda_0 w_{j,0} V_0 \\ &\qquad V_{j,T_j} \geq 0 \end{aligned} \tag{11}$$

where j corresponds to different investment goals with time horizon T_j, $\overline{V}_j$ is the investment goal or desired payoff for each goal j at given time horizon, which in turn is a constant value. V_0 is the initial capital invested and $w_{j,0}$ is the initial weight of total invested capital, V_0, allocated to each investment goal j. The optimization problem stated above is the same as used by both Berkelaar et al. [1] and De Giorgi [2].

Rewriting optimization problem

From now on we drop investment goal notation j, in order to simplify the notations and to focus on one investment goal. (11) can be simplified if we let $\hat{V}_T$ represent the optimal solution and V_T be any feasible solution that satisfies the budget constraint $E^P[\Lambda_T V_T] \leq \Lambda_0 w_0 V_0$. We consider the difference between the two solutions, which we know has to be greater than, or equal to zero since the objective function of a optimal solution in a maximization problem obviously is greater than or equal to any other feasible solution, i.e.

$$E^P[u(\hat{V}_T - \overline{V})] \geq E^P[u(V_T - \overline{V})]. \tag{12}$$

Now, consider the Lagrangean relaxation of optimization problem (11), corresponding to

$$E^P[u(V_T - \overline{V})] - y(E^P[\Lambda_T V_T] - \Lambda_0 w_0 V_0). \tag{13}$$

Using the properties stated in (12) we know that the objective function is greater than, or equal to any other feasible solution. Thus must the optimal solution to the relaxed problem stated in (13) also be greater than, or equal to any other feasible solution and therefore we get

$$\begin{aligned} &E^P[u(\hat{V}_T - \overline{V})] - y(E^P[\Lambda_T \hat{V}_T] - \Lambda_0 w_0 V_0) \\ &\geq E^P[u(V_T - \overline{V})] - y(E^P[\Lambda_T V_T] - \Lambda_0 w_0 V_0). \end{aligned} \tag{14}$$

By removing the constant $y\Lambda_0 w_0 V_0$ from both sides we get

$$\begin{aligned} &E^P[u(\hat{V}_T - \overline{V})] - yE^P[\Lambda_T \hat{V}_T] \\ &- (E^P[u(V_T - \overline{V})] - yE^P[\Lambda_T V_T]) \quad \geq \quad 0 \end{aligned} \tag{15}$$

which in turn, by rewriting the expectation value, becomes

$$\begin{aligned} &= E^P[u(\hat{V}_T - \overline{V}) - y\Lambda_T \hat{V}_T] - E^P[u(V_T - \overline{V}) - y\Lambda_T V_T] \\ &= E^P[u(\hat{V}_T - \overline{V}) - y\Lambda_T \hat{V}_T - (u(V_T - \overline{V}) - y\Lambda_T V_T)] \quad \geq \quad 0. \end{aligned} \tag{16}$$

Now, we let $\hat{u}(\Lambda_T) = u(\hat{V}_T - \overline{V}) - y\Lambda_T \hat{V}_T$ represent the optimal solution and obtain

$$E^P[\hat{u}(\Lambda_T) - (u(V_T - \overline{V}) - y\Lambda_T V_T)] \quad \geq \quad 0. \tag{17}$$

In order to simplify the expression we only consider the term inside the expectation and get that

$$\hat{u}(\Lambda_T) \ \geq \ u(V_T - \overline{V}) - y\Lambda_T V_T \tag{18}$$

which is obvious since the optimal solution is greater than or equal to all feasible solutions. We want to find the feasible solution that corresponds to $\hat{u}(\Lambda_T)$ and can write this as the following maximization problem

$$\hat{u}(\Lambda_T) \ = \ \max_{V_T \geq 0} \{u(V_T - \overline{V}) - y\Lambda_T V_T\} \tag{19}$$

This can now be divided into two parts for the utility function, $u^P(x)$ for losses and $u^N(x)$ for gains with respect to given goal.

Optimal conditions

In order to find the optimal solution to problem (19) we need to compare local optimal solution for the positive part and the negative part. In our case the function is both convex and concave and therefore we need to find the optimal solution for each part separately and then compare the local optimal solutions. For convex problems every local optimal solution is a global optimal solution so therefore we will find the local optimal solution for the convex part at the boundaries $\hat{V} = 0$ or $\hat{V} = \overline{V}$ by pure inspection of the function. Finding the local optimal solution for the concave part must fulfill the Karush-Kuhn-Tucker (KKT) conditions. Applying the KKT conditions, defined e.g. in Christer Svanberg's optimization book [12], to our maximization problem (19) we get the following equations that needs to be fulfilled

$$\begin{cases} u_P'(\hat{X}) - y\Lambda_T + \lambda = 0 \\ \hat{X} \geq 0 \\ \lambda \geq 0 \\ \lambda\hat{X} = 0 \end{cases} \tag{20}$$

Combining the conditions in (20) we obtain the following local optimal solution

$$\hat{X} = {u_P'}^{-1}(y\Lambda_T)$$

where the derivative with respect to X of the positive utility function is

$$u_P(X) = \alpha\beta^+ X^{\alpha-1}$$

then solving for the inverse function and evaluating the function for Λ_T yields the result

$${u_P'}^{-1}(X) = \frac{X}{\alpha\beta^+}^{\frac{1}{\alpha-1}} \Longrightarrow {u_P'}^{-1}(y\Lambda_T) = \frac{y\Lambda_T}{\alpha\beta^+}^{\frac{1}{\alpha-1}}.$$

Lastly using the notation that X is the change in portfolio value relative the goal, i.e. $\hat{X} = \hat{V} - \overline{V}$, in combination with the earlier results, gives the final solution for the optimal terminal portfolio value

$$\hat{V}_T = \overline{V} + \frac{y\Lambda_T}{\alpha\beta^+}^{\frac{1}{\alpha-1}}. \tag{21}$$

Furthermore we compare the local maximas in order to find the global maximum. Let $\hat{V}_T^N$ represent the optimal solution to the negative part of the utility function, u^N, and let $\hat{V}_T^P$ represent the optimal solution to the positive part, u^P. To find the global optimal solution we examine the difference between $\hat{u}^P$ and $\hat{u}^N$, from (19). From this we determine when $\hat{V}_T^P$ is the global maximum, since we are interested in finding out when gains are optimal. The so called global optimal function [1] could then be defined in the following way

$$\begin{aligned} g(y, \Lambda_T) = u^P(\hat{V}_T^P - y\Lambda_T\hat{V}_T^P) \\ -[u^N(\hat{V}_T^N) - y\Lambda_T\hat{V}_T^N] \geq 0 \end{aligned}$$

and if the function is greater than or equal to zero we know that $\hat{V}_T^P$ is the global optimal solution.

By inserting the local maximums from the negative (convex) part of the utility function we could study the global optimal function g in order to find the global optimal solution when the function changes sign. Here we study the two local maximas from the negative part separately.

$\underline{\hat{V}_T^N = \overline{V}:}$

$$\begin{aligned} g(y, \Lambda_T) &= \beta^+(y\Lambda_T)^{\frac{\alpha}{\alpha-1}}\Big(\frac{1}{\alpha\beta^+}\Big)^{\frac{\alpha}{\alpha-1}} - (y\Lambda_T)^{\frac{\alpha}{\alpha-1}}\Big(\frac{1}{\alpha\beta^+}\Big)^{\frac{1}{\alpha-1}} \\ &= (y\Lambda_T)^{\frac{\alpha}{\alpha-1}}\Big[\beta^+\Big(\frac{1}{\alpha\beta^+}\Big)^{\frac{\alpha}{\alpha-1}} - \Big(\frac{1}{\alpha\beta^+}\Big)^{\frac{1}{\alpha-1}}\Big] \\ &= \frac{1-\alpha}{\alpha}y\Lambda_T\Big(\frac{y\Lambda_T}{\alpha\beta^+}\Big)^{\frac{1}{\alpha-1}}. \end{aligned}$$

At the local maximum $\hat{V}_T^N = \overline{V}$ it is obvious that the function is positive for all values of Λ_T, thus $\hat{V}^P$ is the global optimal condition.

$\underline{\hat{V}_T^N = 0:}$

$$g(y, \Lambda_T) = \frac{1-\alpha}{\alpha}y\Lambda_T\Big(\frac{y\Lambda_T}{\alpha\beta^+}\Big)^{\frac{1}{\alpha-1}} - y\Lambda_T\overline{V} + \beta^-\overline{V}^\alpha \geq 0.$$

We find that this equation is not always positive and therefore want to determine when $g(y, \Lambda_T) = 0$. This is not easily determined and to simplify this problem we notice that function variables y and Λ_T always occur as a product. By letting $a = y\Lambda_T$ as done by De Giorgi [2] we get the following equation

$$g(a) = \frac{1-\alpha}{\alpha} a \Big(\frac{a}{\alpha\beta^+}\Big)^{\frac{1}{\alpha-1}} - a\overline{V} + \beta^- \overline{V}^\alpha \geq 0. \tag{22}$$

Now we can easier examine function $g(a)$ since it is only dependent of variable a, and not y which is unknown at this step. By solving $g(a) = 0$ numerically we get the optimal solution $\hat{a}$. From this we can determine for what interval on Λ_T the global optimal function is positive, making $\hat{V}^P$ the optimal solution. Let the variable Λ_T in the solution $g(\hat{a}) = 0$ be represented by $\hat{\Lambda}_y = \hat{a}/y$, we know that since $g(a)$ is strictly decreasing then if $\Lambda_T \leq \hat{\Lambda}_y$ the global optimal function is positive and as mentioned $\hat{V}^P$ becomes the optimal solution.

We now present the global optimal solution to maximization problem (11), using the knowledge from (21) and (22). The solution include for what conditions we obtain our investment goal at time horizon T, we have

$$\hat{V}_T = \left(\overline{V} + \left(\frac{y\Lambda_T}{\alpha\beta^+}\right)^{\frac{1}{\alpha-1}}\right) \mathbb{1}\{\Lambda_T \leq \hat{\Lambda}_y\} \tag{23}$$

where $\hat{\Lambda}_y = \frac{\hat{a}}{y}$ and $\hat{a}$ is obtained by solving (22) numerically and y is obtained by solving the budget constraint in problem (19). Notice that if the constraint in the indicator function is not fulfilled then the optimal terminal portfolio value is zero.

Budget constraint

In order to find an analytic expression for the budget constraint we can insert the optimal terminal portfolio value given by (23) into the budget constraint. We start by rewriting the budget constraint in the following way

$$w_0 = \frac{E^P[\Lambda_T V_T]}{\Lambda_0 V_0}. \tag{24}$$

Now we insert (23) into the numerator, $E^P[\Lambda_T V_T]$, and simplify the expression in the following way

$$\begin{aligned}
&E^P\left[\Lambda_T\left(\overline{V}+\frac{y\Lambda_T}{\alpha\beta^+}^{\frac{1}{\alpha-1}}\right)\mathbb{1}\{\Lambda_T\leq\hat{\Lambda}_y\}\right]\\
&=E^P\left[\Lambda_T\,\overline{V}\,\mathbb{1}\{\Lambda_T\leq\hat{\Lambda}_y\}+\Lambda_T\left(\frac{y\Lambda_T}{\alpha\beta^+}\right)^{\frac{1}{\alpha-1}}\mathbb{1}\{\Lambda_T\leq\hat{\Lambda}_y\}\right]\\
&=\overline{V}E^P[\Lambda_T\mathbb{1}\{\Lambda_T\leq\hat{\Lambda}_y\}]+\left(\frac{y}{\alpha\beta^+}\right)^{\frac{1}{\alpha-1}}E^P[\Lambda_T^{\frac{\alpha}{\alpha-1}}\,\mathbb{1}\{\Lambda_T\leq\hat{\Lambda}_y\}]
\end{aligned}\tag{25}$$

We know that Λ_T is log-normally distributed with parameters m_T and s_T^2 where $m=-(r+\frac{1}{2}\varphi^2)$ and $s=\varphi$, then let $m_T=mT$ and $s_T=mT$, as denoted by De Giorgi [2].

Let $Z=\log(\Lambda_T)$ which then is normal distributed with the same parameters as Λ_T. We start by considering $E^P[\Lambda_T\mathbb{1}\{\Lambda_T\leq\hat{\Lambda}_y\}]$ and rewrite it in the following way

$$\begin{aligned}
E^P[\Lambda_T\mathbb{1}\{\Lambda_T\leq\hat{\Lambda}_y\}]&=E[e^Z\mathbb{1}\{e^Z\leq\hat{\Lambda}_y\}]\\
&=E^P[e^Z\mathbb{1}\{Z\leq\log(\hat{\Lambda}_y)\}]
\end{aligned}\tag{26}$$

where the indicator function is a function of the random variable Z, thus implying certain limits for the definition of expected value, we obtain

$$\begin{aligned}
&\int_{-\infty}^{\log(\hat{\Lambda}_y)}e^z\frac{1}{s_t\sqrt{2\pi}}e^{\left(-\frac{(z-m_T)^2}{2s_T^2}\right)}dz\\
&=\int_{-\infty}^{\log(\hat{\Lambda}_y)}\frac{1}{s_T\sqrt{2\pi}}e^{\left(-\frac{(z-(m_T+s_T))^2}{2s_T^2}+m_T+\frac{s_T^2}{2}\right)}dz\\
&=e^{\left(m_T+\frac{s_T^2}{2}\right)}\int_{-\infty}^{\log(\hat{\Lambda}_y)}\frac{1}{s_T\sqrt{2\pi}}e^{\left(-\frac{(z-(m_T+s_T))^2}{2s_T^2}\right)}dz
\end{aligned}\tag{27}$$

and now we see that the integral in the last expression in (27) corresponds to the probability density function of a normal distribution with parameters $m_T+s_T^2$ and s_T, i.e., $N(m_T+s_T^2,s_T)$. Thus we input $Z=\log(\Lambda_T)$ and get that

$$E^P[\Lambda_T\mathbb{1}\{\Lambda_T\leq\hat{\Lambda}_y\}]=e^{\left(m_T+\frac{s_T^2}{2}\right)}N\left(\frac{\log(\hat{\Lambda}_y)-m_T-s_T}{s_T}\right)\tag{28}$$

Now we want to rewrite the second expectation in (25), i.e.

$E^P[\Lambda_T^{\frac{\alpha}{\alpha-1}}\mathbb{1}\{\Lambda_T \leq \hat{\Lambda}_y\}]$. Here we have that $\Lambda_T^{\frac{\alpha}{\alpha-1}}$ is log-normally distributed with parameters $\frac{\alpha m_T}{\alpha-1}$ and $\left(\frac{\alpha s_T}{\alpha-1}\right)$ and by applying the same method as earlier we obtain

$$\begin{aligned} &E^P[\Lambda_T^{\frac{\alpha}{\alpha-1}}\ \mathbb{1}\{\Lambda_T \leq \hat{\Lambda}_y\}] \\ &= e^{\left(\frac{\alpha m_T}{\alpha-1}+\frac{1}{2}\frac{\alpha^2 s_T^2}{(\alpha-1)^2}\right)} N\left(\frac{\log(\hat{\Lambda}_y) - m_T - \frac{\alpha}{\alpha-1}s_T^2}{s_T}\right) \end{aligned} \tag{29}$$

Finally we can input the result from (28) and (29) into (25) which then is inserted in the budget constraint, (24). We get that w_0 can be written as

$$\begin{aligned} w_0(y) = AN&\left(\frac{\log(\hat{\Lambda}_y) - m_T - s_T}{s_T}\right) \\ &+ By^{\frac{1}{\alpha-1}}N\left(\frac{\log(\hat{\Lambda}_y) - m_T - \frac{\alpha}{\alpha-1}s_T^2}{s_T}\right) \end{aligned} \tag{30}$$

where

$$\begin{cases} A = \frac{\overline{V}}{\Lambda_0 V_0}e^{\left(m_T+\frac{1}{2}s_T^2\right)} \\ B = \frac{1}{\Lambda_0 V_0}(\beta^+\alpha)^{\frac{1}{\alpha-1}}e^{\left(\frac{\alpha m_T}{\alpha-1}+\frac{1}{2}\frac{\alpha^2 s_T^2}{(\alpha-1)^2}\right)}. \end{cases}$$

(30) now represent the budget constraint depending on y with fixed initial funding for each goal.

The expected optimal portfolio value

The optimal terminal portfolio value can now be explicitly expressed by taking the expected value of (23), which then will result in the following expression

$$
\begin{aligned}
E^P[\hat{V}_T] &= E\left[\overline{V}\mathbb{1}\{\Lambda_T \leq \hat{\Lambda}_y\} + \left(\frac{y\Lambda_T)}{\alpha\beta^+}\right)^{\frac{1}{\alpha-1}}\mathbb{1}\{\Lambda_T \leq \hat{\Lambda}_y\}\right] \\
&= E^P\left[\overline{V}\mathbb{1}\{\Lambda_T \leq \hat{\Lambda}_y\}\right] + E^P\left[\left(\frac{y\Lambda_T)}{\alpha\beta^+}\right)^{\frac{1}{\alpha-1}}\mathbb{1}\{\Lambda_T \leq \hat{\Lambda}_y\}\right]
\end{aligned}
$$

using the result from the deduction of (30) we get the resulting explicit expression for the optimal terminal portfolio value for each investment goal to be

$$
\begin{aligned}
E^P[\hat{V}_T] &= \overline{V}N\left(\frac{\log(\hat{\Lambda}_y) - m_T}{s_T}\right) \\
&+ Cy^{\frac{1}{\alpha-1}}N\left(\frac{\log(\hat{\Lambda}_y) - m_T + \frac{s_T^2}{1-\alpha}}{s_T}\right)
\end{aligned}
\tag{31}
$$

where the constant C corresponds to

$$
C = (\beta^+\alpha)^{\frac{1}{1-\alpha}}e^{\frac{m_T}{\alpha-1}+\frac{1}{2}\frac{s_T^2}{(\alpha-1)^2}}
$$

3.6 The expected optimal portfolio value at any time

By defining the budget constraint from (11) so it holds for any time t, we get

$$\begin{aligned} E^P[\Lambda_T V_T] &= \Lambda_t w_t V_t \\ \hat{V}_t &= \frac{1}{\Lambda_t} E^P[\Lambda_T \hat{V}_T] \end{aligned} \tag{32}$$

now using (30) and multiplying the equation with the product $\Lambda_0 V_0$ yields the expression $E^P[\Lambda_T \hat{V}_T]$ and by inputting this into (32) we get the following result for the optimal portfolio value for any time $t \in [0, T]$

$$\hat{V}_t = \overline{V} e^{-r(T-t)} N(d_1(\hat{\Lambda}_y, t)) + \left(\frac{y\Lambda_t}{\beta^+\alpha}\right)^{\frac{1}{\alpha-1}} e^{\Gamma_t} N(d_2(\hat{\Lambda}_y, t)) \tag{33}$$

where

$$\begin{cases} \Gamma_t = \frac{\alpha}{1-\alpha}\left(r + \frac{1}{2}\varphi^2\right)(T-t) + \frac{1}{2}\left(\frac{\alpha}{1-\alpha}\right)^2 \varphi^2 (T-t) \\ d_1(\Lambda_y, t) = \frac{\log(\hat{\Lambda}(y)) + (r - \frac{1}{2}\varphi^2)(T-t)}{\varphi\sqrt{T-t}} \\ d_2(\Lambda_y, t) = d_1(\Lambda_y, t) + \frac{\varphi\sqrt{T-t}}{1-\alpha}. \end{cases}$$

The result in (33) are derived by both Berkeelar [1] and De Giorgi [2] using exactly the same approach as this article. In short we can explain (33) consisting of two parts where the left part represent the investment goal and the right hand side represent the surplus which is mainly based on individual risk profile and the budget constraint (represented by y).

3.7 Optimal proportion of risky assets

By studying (33) we notice that the function only depends on the time and Λ_t, thus we can express the optimal portfolio value in the form of a function depending on t and Λ_t, i.e.

$$V_t = F(t, \Lambda_t)$$

Applying Ito's formula to this equation we get

$$dF(t, \Lambda_t) = \frac{\partial F(t, \Lambda_t)}{\partial \Lambda_t} \partial \Lambda_t + \frac{1}{2} \frac{\partial F^2(t, \Lambda_t)}{\partial \Lambda_t^2} \partial \Lambda_t^2 \tag{34}$$

and using the dynamics of V_t in (6) we finally arrive at the following expression

$$dV_t = G(t, \Lambda_t)dt - \varphi_t \Lambda_t \frac{\partial F}{\partial \Lambda_t} dW_t \tag{35}$$

Where the function $G(t, \Lambda_t)$ is the function

$$G(t, \Lambda_t) = -r_t \Lambda_t \frac{\partial F}{\partial \Lambda_t} + \frac{1}{2} \frac{\partial^2 F}{\partial \Lambda_t^2} \tag{36}$$

Now we compare diffusion parts in (6) and (35), we arrive at an expression that explicitly explains the proportion invested in risky assets at time t, i.e.

$$\begin{aligned} \lambda_t &= -\frac{\Lambda_t \varphi}{\sigma V_t} \frac{\partial F}{\partial \Lambda_t} \\ &= -\frac{\Lambda_t \varphi}{\sigma V_t} \frac{\partial V_t}{\partial \Lambda_t} \end{aligned} \tag{37}$$

where the partial derivative of F with respect to Λ_t is found by taking the partial derivative of (33) which then yields

$$\begin{aligned} \frac{\partial V_t}{\partial \Lambda_t} = -\frac{\overline{V} e^{-r(T-t)} \varphi(d_1(\hat{\Lambda}_y), t))}{\Lambda_t s_t} &+ \frac{\left(\frac{y\lambda_t}{\alpha\beta^+}\right)^{\frac{1}{\alpha-1}} e^{\Gamma_t} \phi(d_2(\hat{\Lambda}_y), t))}{\Lambda_t s_t} \\ &- \frac{\left(\frac{y}{\alpha\beta^+}\right)\left(\frac{y\lambda_t}{\alpha\beta^+}\right)^{\frac{1}{\alpha-1}-1} e^{\Gamma_t} N(d_2(\hat{\Lambda}_y), t))}{\alpha - 1} \end{aligned}$$

Putting it all together results in the final expression of the proportion in risky assets as

$$\lambda_t = \frac{\phi}{\sigma V_t}\Bigg[\frac{\overline{V}e^{-r(T-t)}\phi(d_1(\hat{\Lambda}_y),t))}{s_t} - \frac{\left(\frac{y\lambda_t}{\alpha\beta^+}\right)^{\frac{1}{\alpha-1}} e^{\Gamma_t}\phi(d_2(\hat{\Lambda}_y),t))}{s_t} + \frac{\left(\frac{y\lambda_t}{\alpha\beta^+}\right)^{\frac{1}{\alpha-1}} e^{\Gamma_t}N(d_2(\hat{\Lambda}_y),t))}{\alpha-1}\Bigg] \tag{38}$$

3.8 Optimal initial funding for multiple investment goals

To solve problem (11) we need to determine the initial funding strategy, i.e. initial weight of total invested capital $w_{j,0}$ allocated to each goal j. In order to find this, we consider the optimization problem formulated by De Giorgi [2].

$$\begin{aligned} &\max_{w_{j,0}} \sum_{j=1}^{n} \delta^{-T_j} E^P[u(\hat{V}_{T_j} - \overline{V}_j)] \\ \text{s.t.} \quad & w_{j,0} \geq 0,\ j = 1, ..., n \\ & \sum_{j=0}^{n} w_{j,0} \leq 1 \end{aligned} \tag{39}$$

The objective function corresponds to the discounted optimal value for problem (11) , summed over all investment goals. The discount factor is denoted as δ^{-T_j}. In other words, problem (39) determines the initial funding allocation, $\hat{w}_{j,0}$, that obtains the maximal total utility where each investment goal is discounted.

Rewriting weight optimization problem

Problem (39) can be reformulated by instead maximizing the expected value of the optimal terminal portfolio value, as stated in (31), similarly to De Giorgi [2].

We now obtain

$$
\begin{aligned}
&\max_{w_{j,0}} \sum_{j=1}^{n} \delta^{-T_j} E^P[\hat{V}_{j,T_j}] \\
&\text{s.t.} \quad w_{j,0} \geq 0,\ j = 1, ..., n \\
&\qquad\ \sum_{j=0}^{n} w_{j,0} \leq 1.
\end{aligned}
$$

furthermore we notice that (31) depends on y so we can write it as a function of y, i.e. $E^P[\hat{V}_{j,T_j}] = S_j(y) = S_j(y(w_{j,0}))$ where y depends on $w_{j,0}$. Now we rewrite the optimization problem in the following way

$$
\begin{aligned}
&\max_{w_{j,0}} \sum_{j=1}^{n} \delta^{-T_j} S_j(y(w_{j,0})) \\
&\text{s.t.} \quad w_{j,0} \geq 0,\ j = 1, ..., n \\
&\qquad\ \sum_{j=0}^{n} w_{j,0} \leq 1
\end{aligned}
\tag{40}
$$

The solution to the initial funding allocation for multiple investment goals, $\hat{w}_{j,0}$, can then be applied to $\hat{V}_t$, through the calculation of y from the budget constraint. This results in the optimal portfolio value for multiple investment goals $\hat{V}_{j,t}$, from which we can derive the optimal proportion in risky assets $\hat{\lambda}_{j,t}$ for each investment goal with the same approach as used in the case for one investment goal.

4 Result

In this section we will present results from the optimal initial funding, defined in the maximization problem (40). Further we compare the relation between $\hat{\Lambda}_T$ and Λ_T for different values on β^+ and lastly we present results that mainly focus on comparing the optimal portfolio value at $t \in [0, T]$ for different type of risk profiles.

Table 1: Table of constants used throughout the result

	r	σ	μ	β^-	V_0	α
Values	0.03	0.2	0.07	2.25	0.75	0.88

In table 1 the constant values are presented that will be used throughout the result section. The parameter β^+ will hold the values $[0, 1, 1.5, 2.25]$ and the discount factor Λ_t will change over time t. Lastly each investment goal in today's value will be, $V_0 = 1$ and the forward discounted investment goal at each time horizon will be $\overline{V}_{T_j} = V_0 e^{rT_j}$.

4.1 Optimal initial funding

Problem (40) maximizes the optimal portfolio value with respect to the initial funding. The problem was solved by defining a grid representing combinations of initial wealth allocation for each investment goal. For a fixed value on fraction in investment goal 1, $w_{1,0}$, and when $w_{3,0} = 1 - w_{1,0} - w_{2,0}$ different combinations of $w_{2,0}$ were tried which resulted in the following figure.

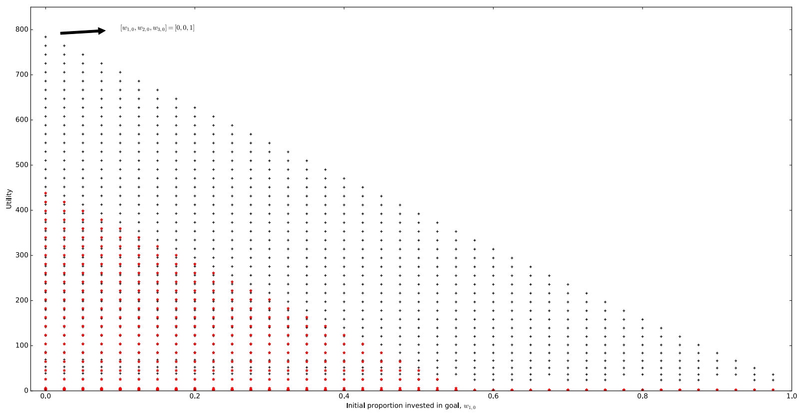

Figure 2: Utility for different combinations of initial funding. Fraction $w_{0,1}$ is iterated over the x-axis and symbols for each step represents different combinations of $w_{0,2}$ and $w_{3,0}$ where $w_{3,0} = 1 - w_{1,0} - w_{2,0}$. The red * corresponds to β^+ approximately zero and the black + corresponds to $\beta^+ = 1$

In Figure 2 we see that the combination of initial capital allocation that corresponds to $[w_{1,0}, w_{2,0}, w_{3,0}] = [0, 0, 1]$ clearly maximizes the utility for the investor given the parameters presented in the introduction of Section 4. Further we see that the monotonocity also holds for the long term goal when $\beta^+ \approx 0$. Because of the result obtained above the remaining results will assume the following initial funding strategy where $w_0 = [1/3, 1/3, 1/3]$. Here the initial capital is evenly distributed between all investment goals.

In order to strengthen the monotonocity argument we computed the derivatives of $E^p[\hat{V}_{j,T_j}]$ corresponding to $S_j(y(w_{j,0}))$ in (40). In order to get a visual presentation of the derivatives the goal horizons were set to T = 1,5,10 due to the derivative of the long term goal being a factor 1000 larger than for the short term goal and the mid term goal, when using T = 20 for the long term goal.

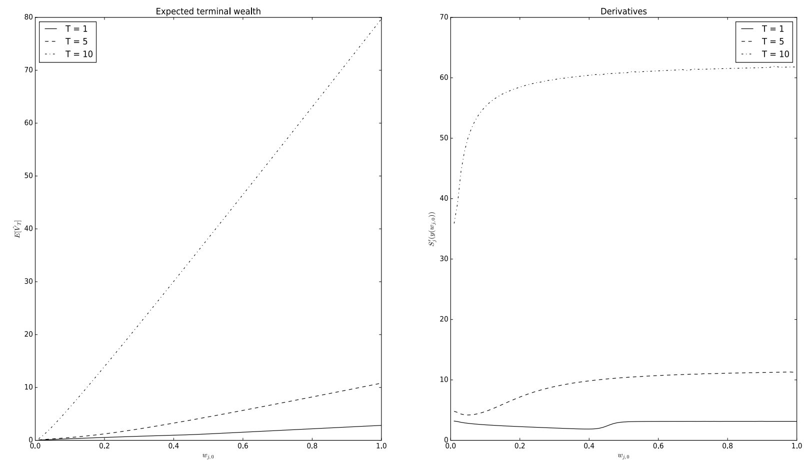

Figure 3: Left figure: Expected optimal portfolio value for the median investor with different goal horizons, T = 1, 5, 10, and different initial portfolio weights. Right figure: Derivatives of expected optimal portfolio value presented in the left figure

In Figure 3 we clearly see that the derivative for the long term goal, T = 10 in this case, is much larger then for the short term goal and the mid term goal. Further we see that this is in line with the expected portfolio value, presented in the left plot. Further more we see that the result in Figure 3 is consistent with the result presented in Figure 2, i.e. the optimization problem (40) is monotone in the long term goal and thus no inner solution exists.

4.2 Global optimal conditions

Here we present the result from evaluating the global optimal condition by comparing $\hat{\Lambda}_y$ and the expected value of Λ_T, that constitutes a constraint in (23). As previously mentioned in the budget constraint section, Λ_T is log-normally distributed with paramters m_T and s_T^2. The constraint were examined by varying the goal horizon for different β^+ where a higher value of β^+ can be viewed as higher risk appetite.

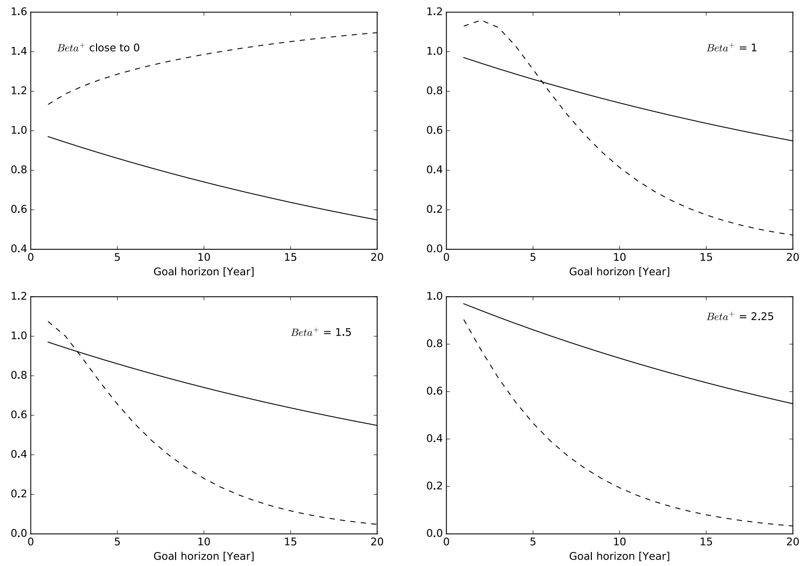

Figure 4: Comparing the expected value of $\Lambda(t)$ *(representing the full line)* and $\hat{\Lambda}(y)$ *(representing the dashed line)* for different time horizons and different values of β^+

As we can see in Figure 4, $\hat{\Lambda}(y) > \Lambda(T)$ appear when the ratio $\beta = \beta^-/\beta^+$ is low. The condtion seems to be fulfilled for all time horizons between $T \in [0, 20]$ when $\beta^+ \approx 0$. This state, $\beta^+ \approx 0$, can be regarded as an investor that is extreme loss averse. The figure also shows that for $\beta^+ = [1, 1.5, 2.25]$ the factor Λ_y^* accelerates faster towards zero than Λ_T.

4.3 Optimal portfolio value and proportion in risky assets

Median loss aversion

In order to get an understanding of the behaviour of optimal portfolio value and the proportion in risky assets for different goals and time horizons, three goals with corresponding time horizons, $T = [1, 5, 20]$, were plotted. The risk profile of the investors corresponds to $\beta^+ = 1$ which can be regarded as a median loss averse investor according to Kahneman and Tversky [3].

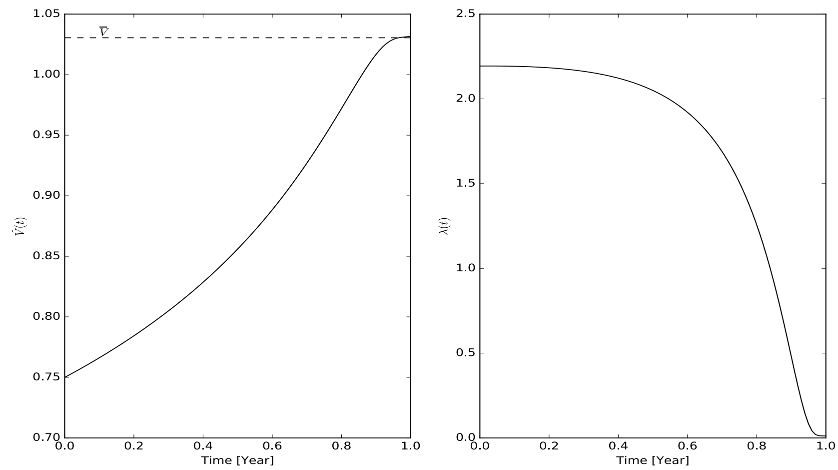

Figure 5: The expected optimal portfolio value and the expected proportion in risky assets for $t \in [0, T]$ when $\beta^+ = 1$ and $T = 1$

Using the risk profile suggested by Kahneman and Tversky [3] representing the median investor we see that in Figure 5, the goal is reached when the time horizon is set to T = 1. For this case the leverage in risky assets is large, furhermore we can see that the proportion in risky assets decline as t approach the time horizon.

In the next case we have time horizon $T = 5$ which results in the following figure.

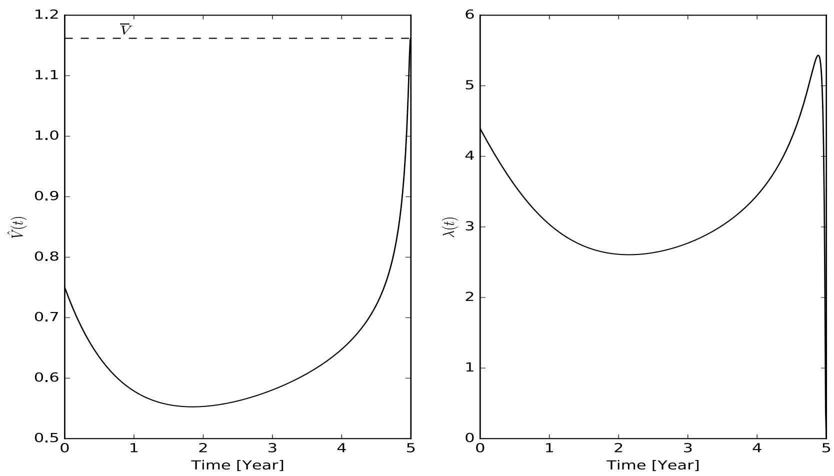

Figure 6: The expected optimal portfolio value and the expected proportion in risky assets for $t \in [0, T]$ when $\beta^{+} = 1$ and $T = 5$

The goal is obtained, similarly to $T = 1$ but in this case with higher leverage. We notice that the optimal portfolio value initially decreases to then start increasing at around $t = 1.5$ and finally reach the investment goal as mentioned. The shape of the risky assets curve is similar to the optimal portfolio value except at the end where it quickly goes to zero.

In the last case we have time horizon $T = 20$ which results in the following figure

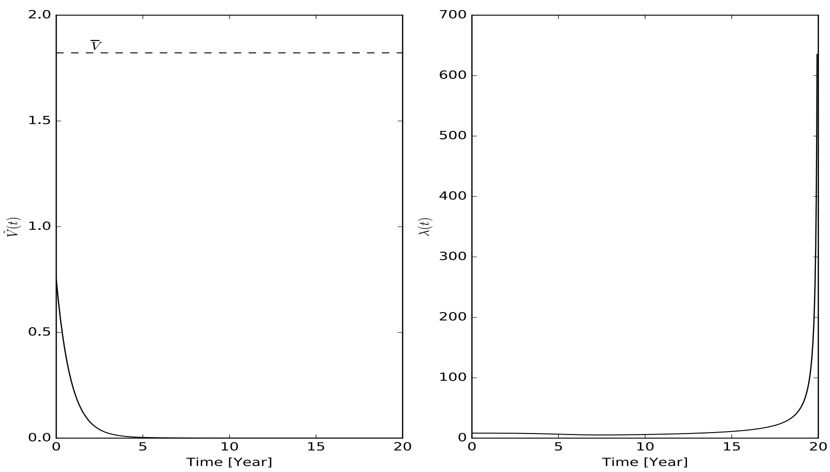

Figure 7: The expected optimal portfolio value and the expected proportion in risky assets for $t \in [0, T]$ when $\beta^+ = 1$ and $T = 20$

The portfolio value goes to zero relatively fast. Moreover we see that the proportion in risky assets drastically increases as t approach the time horizon.

Extreme loss aversion

Now similar comparisons will be made but considering the case of an extreme loss averse investor. This investor is described with preferences according to $\beta^+ \approx 0$. The investor is considered conservative and thus do not want to be exposed to as mush risk as, for instance the median loss averse investor.

We start by examining the investment goal with time horizon $T = 1$ which is similar to Figure 5 except using extreme loss aversion.

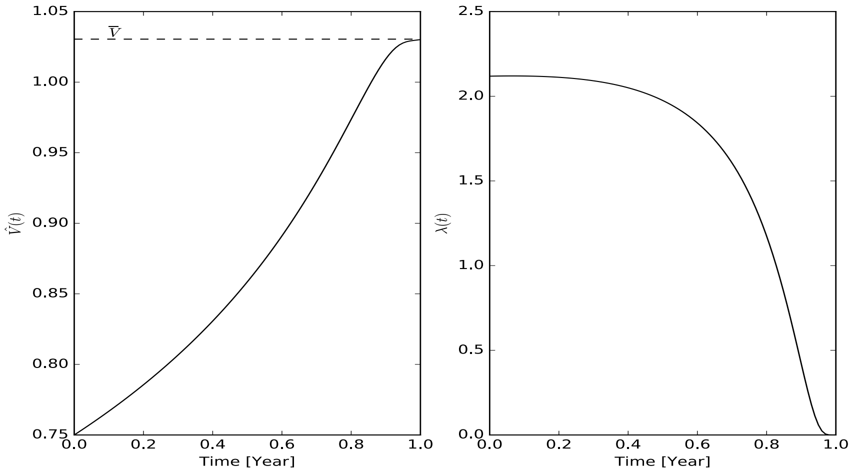

Figure 8: The expected optimal portfolio value and the expected proportion in risky assets for $t \in [0, T]$ when $\beta^+ \approx 0$ and $T = 1$

As we can see in Figure 8 the investment goal is reached and hold relatively high leverage at the beginning but then level out towards the time horizon end. For $T = 5$ the investor also reaches the goal, as we can see in Figure 9.

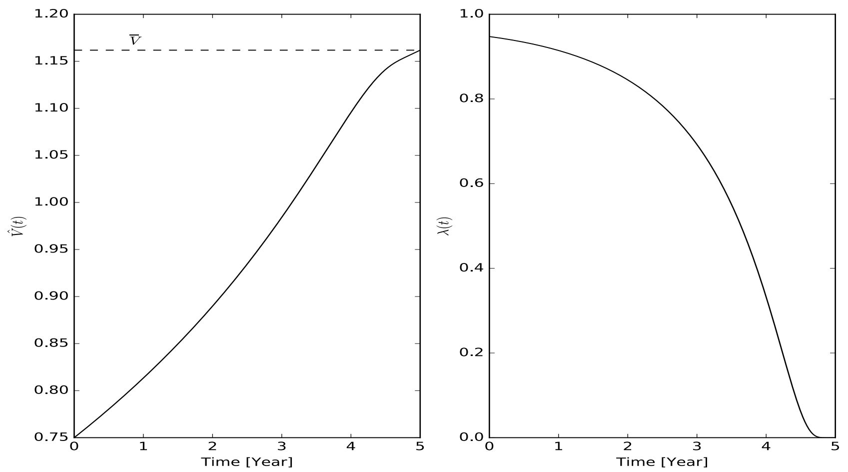

Figure 9: The expected optimal portfolio value and the expected proportion in risky assets for $t \in [0, T]$ when $\beta^+ \approx 0$ and $T = 5$

For this time horizon, presented in Figure 9, the model invest less in risky assets and the proportion invested level out earlier than for the 1 year goal.

Lastly we have the investment horizon $T = 20$ for the case $\beta^+ \approx 0$.

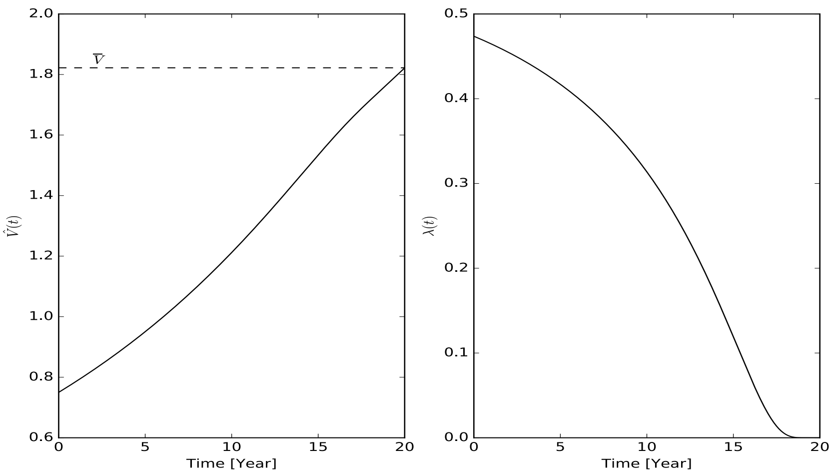

Figure 10: The expected optimal portfolio value and the expected proportion in risky assets for $t \in [0, T]$ when $\beta^+ \approx 0$ and $T = 20$

Similar to the other cases we see that the investment goal is reached with zero surplus and we also notice that the investor holds lower proportion in risky assets than for investment goals $T = 1$ and $T = 5$. The portfolio level out all investment in risky assets towards the time horizon.

5 Discussion

In this section we will discuss and interpret the result presented in Section 4 and possible problems with the model that do not make it suitable for the advisory process.

5.1 Remarks on market conditions

The market conditions is represented by the comparison between $\Lambda(T)$ and $\hat{\Lambda}(y)$. When the market present good conditions for our goals the goal is reachable, that is $\Lambda(T) < \hat{\Lambda}(y)$. By (23) we clearly see that the optimal portfolio value at the time horizon, T, is equal to zero if good market conditions are not fulfilled.

In Figure 4 we clearly see that bad states occur for $\beta^+ \in [1, 1.5, 2.25]$, i.e. we have bad market conditions.

5.2 Remarks on optimal initial funding

After solving the first optimization part (11) we obtain the optimal portfolio value at the time horizon for each investment goal. Secondly we solve (40) in order to find the optimal initial funding for each investment goal. As mentioned in Section 4.1 we found that the initial funding is monotone in the long term goal, i.e. the investor puts all her wealth in the long term goal and ignores the other goals, having preferences according to table 1.

This result indicates that optimizing for initial funding is rather useless in the sense that an investment goal must always be regarded as a goal to reach, not a goal to neglect in favour for, in this case, the long term goal. With respect to that we conclude that maximization problem (40) does not bring any usefull insight into the advisory business. It is meaningful to point out that this scenario changes with respect to risk profile, so for example when not looking at the median investor, proposed by Kahneman and Tversky [3], the maximization problem may change. Therefore we emphasise that the result in Section 4.1 holds for the median investor and not necessarily for other risk preferences.

5.3 Remarks on optimal portfolio value

As derived in (23) the optimal terminal portfolio value is only reached under the condition that $\Lambda_T \leq \hat{\Lambda}_y$, where $\hat{\Lambda}_y = \frac{\hat{a}}{y}$ as mentioned in connection to that equation. This means that for $y \to \infty$ then $\hat{\Lambda}_y \to 0$ and thus the above condition is not fulfilled, at least for a fixed $\Lambda_T > 0$.

In Figure 4 we have showed that in all cases, except extreme loss aversion ($\beta^+ = 0$), both Λ_T and $\hat{\Lambda}_y$ decreases as the investment horizon T increases. We also notice that $\hat{\Lambda}_y$ accelerates faster towards zero than Λ_T and when $\Lambda_T > \hat{\Lambda}_y$ the investment goal will not be obtained. If we now, for example, consider the median risk profile, i.e. $\beta^+ = 1$ [4], that corresponds to the upper right graph in Figure 4 we can see in Figure 5 and 6 that we obtain the investment goal whereas in Figure 7 the optimal terminal portfolio value goes to zero.

Furthermore, the Lagrange multiplier y is determined in the budget constraint presented in (30) and as previously discussed we do not obtain the investment goal for large T, at least not for the median risk profile. For longer investment horizons, the factor C in the budget constraint becomes very large. In order to fulfil the constraint, y also becomes large since the factor $y^{\frac{1}{\alpha-1}}$ decreases for increasing y when we consider the median risk profile. Although, in the case of extreme loss aversion, we stated that the investment goal is obtained. If we examine the budget constraint we notice that since β^+ is approximately zero the factor C is also close to zero and thus y is not calculated to become as large as for other β^+. In other words the second term, corresponding to the surplus, is neglected and the investor mainly focus on reaching the investment goal alone. This makes it more likely to obtain the investment goal, which is in accordance with $\Lambda_T < \hat{\Lambda}_y$ for all T when β is approximately zero.

We mentioned earlier that when the factor C is not neglected, i.e. for $\beta^+ =$

$[1, 1.5, 2.25]$ the investment goal is not always reachable. We also pointed out that C becomes large for increasing T, in fact it increases exponentially. Furthermore the Lagrange multiplier is estimated to be large for more risk taking investors as well, i.e. higher values on β^+. This can be interpreted as when an investor is risk taking and/or want to invest in a long term goal the probability of reaching investment goal and surplus becomes very low as we approach the time horizon. Thus the optimal terminal portfolio value is zero in these cases.

6 Conclusion

The purpose of this thesis have been to examine if the model derived above is useful in the financial advisory process for private investors. We have during the work come to the conclusion that this framework is not suitable in this area.

First of all we discuss in Section 5.2 the initial funding strategy and that the obtained result allocates all wealth in the long term goal, as shown in Figure 2. This is mathematically speaking a reasonable result but completely useless in the sense that an investor can theoretically set up infinitely many goals that all will be ignored in favour of the investment goal with longest time horizon.

Secondly we have noticed, as discussed in Section 5.3, that for the median risk profile [4] we do not obtain the long term investment goal. This is a strong indication that this model might not be suitable for the purpose of advising private investors even though we obtain the investment goal when an investor is extreme loss averse.

Lastly, we also want to point out that the model propose high leverage in risky assets in most cases which we consider not to be a reasonable suggestion for a private investor. The only circumstances when proportion invested in risky asset was below 1 were for the mid- and long term goals for extreme loss aversion. However, since it is irrelevant for the median investor, it should not be used in the advisory process at all.

7 Extensions

We have showed, among many things, that the initial funding optimization is monotone in the long term goal given the preference for a median investor, developed by Kahneman and Tversky. Since this result covers the median investor one may consider alternative values of risk and loss preferences. One example could be investigating whether initial funding would change if we assume different risk preferences, given by α, for the different goals. For example one may assume that the 20 year investment goal could be a pension funding goal and that the 1 and 5 year goals are goals involving material things, i.e. not as important. Therefore loss aversion could be higher for the pension goal and lower for the material goals. Doing these adjustments can change the result for the initial funding optimization and instead the initial capital might be distributed more even between investment goals.

Moreover, one might find that determining risk preferences for private investors is difficult and can be biased if not done correctly. As a counterexample Berkeelar [1] found that the VaR (Value at risk) model presented by Basak and Shapiro [13] genereates similar result as the prospect theory model, present here and by De Giorgi [2], as well. Using the VaR-model, one could instead use lower bounds representing loss limits of investors instead, which in some sense could facilitate the quantification of a private investor's risk profile and make it more accurate.

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