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Asset-Liability Management with in Life Insurance

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Abstract

In recent years, new regulations and stronger competition have further increased the importance of stochastic asset-liability management (ALM) models for life insurance firms. However, the often complex nature of life insurance contracts makes modelling to a challenging task, and insurance firms often struggle with models quickly becoming too complicated and inefficient. There is therefore an interest in investigating if, in fact, certain traits of financial ratios could be exposed through a more efficient model.

In this thesis, a discrete time stochastic model framework, for the simulation of simplified balance sheets of life insurance products, is proposed. The model is based on a two-factor stochastic capital market model, supports the most important product characteristics, and incorporates a reserve-dependent bonus declaration. Furthermore, a first approach to endogenously model customer transitions is proposed, where realized policy returns are used for assigning transition probabilities.

The model's sensitivity to different input parameters, and ability to capture the most important behaviour patterns, are demonstrated by the use of scenario and sensitivity analyses. Furthermore, based on the findings from these analyses, suggestions for improvements and further research are also presented.

Keywords: asset-liability management, participating life insurance policies, bonus policy, surrender

Sammanfattning

Införandet av nya regelverk och ökad konkurrens har medfört att stokastiska ALM-modeller blivit allt viktigare för livförsäkringsbolag. Den ofta komplexa strukturen hos försäkringsprodukter försvårar dock modelleringen, vilket gör att många modeller anses vara för komplicerade samt ineffektiva, av försäkringsbolagen. Det finns därför ett intresse i att utreda om egenskaper hos viktiga finansiella nyckeltal kan studeras utifrån en mer effektiv och mindre komplicerad modell.

I detta arbete föreslås ett ramverk för stokastisk modellering av en förenklad version av balansräkningen hos typiska livförsäkringsbolag. Modellen baseras på en stokastisk kapitalmarknadsmodell, med vilken såväl aktiepriser som räntenivåer simuleras. Vidare så stödjer modellen simulering av de mest väsentliga produktegenskaperna, samt modellerar kundåterbäring som en funktion av den kollektiva konsolideringsgraden.

Modellens förmåga att fånga de viktigaste egenskaperna hos balansräkningens ingående komponenter undersöks med hjälp av scenario- och känslighetsanalyser. Ytterligare undersöks även huruvida modellen är känslig för förändringar i olika indata, där fokus främst tillägnas de parametrar som kräver mer avancerade skattningsmetoder.

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Contents

1 Introduction

In recent years new regulations and stronger competition have further increased the importance of the role stochastic asset-liability management models play within the financial industry. Since the financial crises in 2008, the market has been the target of a turbulent time. With the goal set to turn this trend around, new requirements aiming towards stabilizing the conditions and protecting the investors, have been introduced. One such factor, that was brought forward, meant imposing restrictions on financial firms, forcing them to comply with stricter requirements for how to model and manage their assets and risks. As an effect of such regulations, asset-liability management (ALM) modelling has turned into a mandatory, rather than optional, part of financial firms' everyday business.

A group of financial firms that are largely affected by these changes are the life insurance companies. Furthermore, life insurance companies are generally characterized by the importance of matching large assets with large, and often complex, liabilities. Possessing the right tools for modelling the various dimensions of the balance sheet, in an efficient manner, is therefore critical to these companies. Skandia, as one of Sweden's largest banking- and insurance companies, falls into this category. As a result, Skandia is utilizing an ALM model with the means to capture and simulate the company's main behavior patterns of the balance sheet development, in various conditions.

However, this model is relatively complex. That is one of the reasons why there is a great interest in investigating if, in fact, certain traits of financial ratios could be exposed through a more efficient model, compared to the one currently being used. In order to remove inefficiencies, such a model would only concern the most important aspects of the balance sheet development. The model would incorporate the most important life insurance product characteristics, the surrender of contracts, a reserve-dependent bonus declaration, and a stochastic capital market model.

Another aspect of great interest is to study multiple insurance companies' activities simultaneously. Such an insight could then assist the companies in getting a deeper understanding for the clients' tendency to move, and the implications it has on the balance sheet development, by simply observing the clients' movements between the different competing companies. One can then easier draw conclusions about which offers, guarantees etc, that the clients find the most attractive, and based on those findings investigate the impact of different strategies.

In wide terms, it is precisely this inquiry that describes the main purpose of this thesis. Initially, a basic ALM model will be created, in order to then develop it into a more advanced model taking clients' movements into consideration, which would then enable the analysis and comparison of a number of insurance companies that stands in competition with each other.

2 Theoretical Background

This section provides a description of the theoretical framework lying beneath the creation of an appropriate ALM model. Firstly, a general description of asset-liability management (ALM) is given. Secondly, the theory used for modelling the asset side of the balance sheet is presented, and thirdly, general contract design used to model the liability side is described. Finally, theory used for modelling customer transition is presented, followed by a description of the principal components of Solvency II affecting ALM modelling for insurance firms.

2.1 Asset-Liability Management

Simply expressed, Asset-Liability Management (ALM) refers to managing the asset allocation with respect to the liabilities' cash flows, i.e. handling the risk coming from mismatches between a company's assets and liabilities. ALM can be seen as the practice of managing a business so that there exists a coordination of decisions made as well as actions taken with respect to assets and liabilities [1]. Using the definition given in [1], ALM can be defined as

> *"the ongoing process of formulating, implementing, monitoring and revising strategies related to assets and liabilities to achieve an organization's financial objectives, given the organization's risk tolerances and other constraints."*

On the one hand, capital has to be invested in a profitable way, enabling profits in terms of positive returns (asset management). On the other hand, undertaken liabilities have to be met (liability management), meaning that capital investments should not only provide profits, but also ensure returns that at least cover the firms' obligations [14]. What's more, the lately experienced turbulence on financial markets has induced a shifting focus from the asset side toward the liability side [12]. Company-wide risk management thus requires that both sides of the balance sheet are taken into consideration. Therefore ALM has evolved to be a frequently used term within financial companies' risk departments. A group of companies facing substantial asset management and equally large liabilities are insurance firms. Furthermore, the long-term nature of insurance firms' investments and obligations amplifies the financial rewards and penalties for good and bad decisions [28]. Hence, the insurance business constitutes an area in which ALM has turned out to be particularly important. Also, strengthened competition and new regulations have in recent years further increased the importance of ALM for insurance companies [14].

ALM analyses are usually based on one of two approaches; (1) a computation of particular scenarios (stress tests) that are based on subjective expectations, historical data and guidelines provided by regulatory authorities, or (2) a stochastic modelling and simulation of the market development, customer behaviour and concerned accounts. As the latter, in a more realistic way compared to a small number of deterministic scenarios, takes financial uncertainties into account, it has in the recent years been given more and more attention [14]. Moreover, the standards required by Solvency II have further increased the interest for stochastic simulations of ALM models. As part of pillar I (the quantitative requirements) in Solvency II, the firm is obliged to employ market consistent valuation of each and every account of the balance sheet [3].

Since ALM models are constructed using the components of the firm's balance sheet, the modelling begins by choosing suitable dynamics/models for these parts. Each element of the balance sheet that influences the firm's future states needs to be considered and modelled as accurately as possible. Based on the standards presented in Solvency II, this holds for all parts of the balance sheet, meaning that a relatively high number of sub-models need to be set up before the subsequent analysis may begin [3]. Even though a detailed insurance firm balance sheet is beyond the scope of this paper and a somewhat simplified balance sheet is used, there is still a fairly large number of elements that need to be accounted for. In order to model some of these components, assumptions regarding their dynamics are made, and the underlying theory for these dynamics are presented in the next section.

2.2 Asset Model

The assets of an insurance company significantly affect the outcome of its business. Not only its ability to make profits and high returns to customers is affected, but also (and more importantly) its ability to meet liabilities is largely connected to the assets' development. In order to create a model that as well as possible replicates the every-day business of insurance companies, fairly sophisticated asset modelling is needed. Just to give a clue of the variety of asset types that generally characterizes an insurance company's portfolio, the current asset composition within Skandia is used. The principal asset types being part of Skandia's investment portfolio are

- Stocks
- Real-Estate
- Private Equity
- Commodities
- Swedish Government Bonds
- Swedish Mortgage Bonds
- Inflation-Indexed Bonds

The asset allocation was, as of December 31^{st}, given by the numbers presented in Figure 1.

In the subsequent analysis however, it will be assumed that two types of assets are available for the insurance companies to invest in. The companies can either invest its capital in variable return assets, i.e. a stock or a basket of stocks, or in fixed interest assets, i.e. bonds. Even though insurance companies generally invest in a larger universe than the one given by these two assets, the use of a riskier asset class (stocks in this case) in combination with a less risky asset class (bonds in this case) corresponds well to their investment strategies [24]. Thus, the models needed for simulating the development of the asset side are, one model for the stock prices and one for the interest rates determining the bond prices.

A note to the reader is that the derivation of the Geometric Brownian Motion and of the relation between interest rates and bond prices can be skipped without loss of continuity.

2.2.1 Continuous Stock Return Model

The uncertainty originating from stock price movements plays an essential role in the evolvement of an insurance company's assets and liabilities. In order to model the firm's future balance sheet, we thus need a model reflecting this uncertainty as good as possible. A natural candidate for this task is the Geometric Brownian motion (GBM), which is a widely used model within financial literature and stochastic ALM modelling (see e.g. [4], [6], [7], [8], [11], [14], [19], [22] and [23]). Before introducing the application of a GBM to stock price movements, we start by defining the

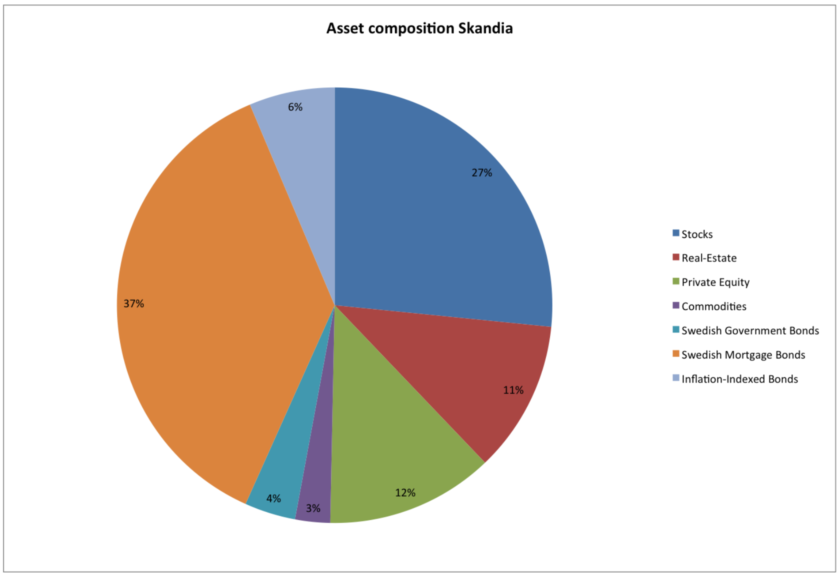


Figure 1: Skandia Investment Portfolio 31^{st} of December 2016

GBM and the included Wiener Process.

Following the definition given in [7], a stochastic process W is a Wiener process if the following conditions are met

- $W_0 = 0$
- *The increments of* W *are independent, i.e.* $W_u - W_t$ *and* $W_s - W_r$ *are independent stochastic variables if* $r < s \leq t < u$.
- *For* $s < t$ *the stochastic variable* $W_t - W_s$ *follows a normal distribution* $N(0, t - s)$.
- W has continuous trajectories.

A computer simulated Wiener trajectory is shown in figure 2.

Given the definition of the Wiener process, we now proceed by defining the GBM. In [7] the GBM is defined as the solution to the following stochastic differential equation (SDE)

$$\begin{cases} dX_t = \alpha X_t dt + \sigma X_t dW_t, \\ X_0 = x_0. \end{cases} \tag{1}$$

Here $\alpha \in \mathbb{R}$ denotes the constant drift rate, $\sigma \in \mathbb{R}$ the constant diffusion rate and W a standard Wiener process as defined above. By taking inspiration from the fact that the solution to the corresponding deterministic linear ODE is an exponential function of time, we can find an explicit solution to equation (1). Let us investigate a process Z_t defined by $Z_t = ln(X_t)$, for which we assume that X_t is strictly positive and solves equation (1). Applying $It\hat{o}$'s formula to Z_t then gives

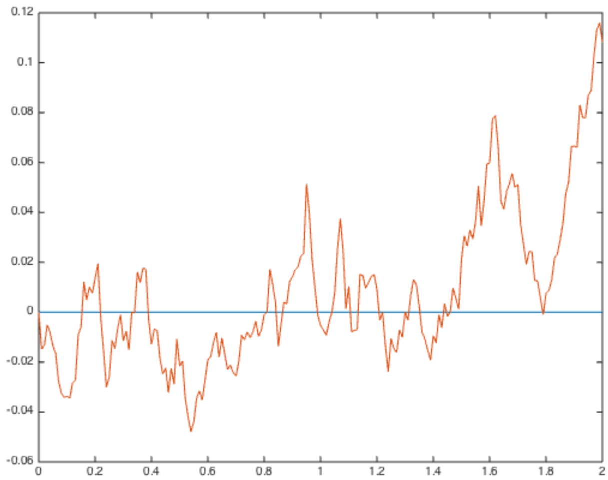


Figure 2: Realization of a Wiener trajectory

$$
\begin{aligned}
dZ &= \frac{1}{X}dX + \frac{1}{2}\Big(-\frac{1}{X^2}\Big)(dX)^2 \\
&= \frac{1}{X}(\alpha X dt + \sigma X dW) + \frac{1}{2}\Big(-\frac{1}{X^2}\Big)\sigma^2 X^2 dt \\
&= (\alpha dt + \sigma dW) - \frac{1}{2}\sigma^2 dt.
\end{aligned}
$$

We thus end up with the equation

$$
\begin{cases} dZ_t = \Big(\alpha - \frac{1}{2}\sigma^2\Big)dt + \sigma dW_t, \\ Z_o = ln(x_0). \end{cases}
$$

Since the right-hand side does not contain Z_t, we can easily integrate both sides and obtain

$$
Z_t = ln(x_0) + \Big(\alpha - \frac{1}{2}\sigma^2\Big)t + \sigma W_t,
$$

giving the following expression for X_t

$$
X_t = x_0 e^{(\alpha - \frac{1}{2}\sigma^2)t + \sigma W_t}. \tag{2}
$$

As is discussed in [7], the correct approach for finding the solution is to define X_t by equation (2) and then show that this definition of X_t satisfies equation (1). As such, one avoids the logical flaw coming from the fact that we had to make two strong assumptions in order for Z to be well defined; (1) we had to assume the existence of a solution X to equation (1), and (2) that the solution is positive. For a more detailed discussion on this matter, the reader can refer to [7].

Finally, by modelling the stock price uncertainty as a Geometric Brownian motion, we thus assume that the stock price S_t at time t evolve according to the following stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{3}$$

where $\mu \in \mathbb{R}$ denotes the constant drift rate, $\sigma \geq 0$ the constant volatility of the stock returns and W a Wiener process. Since the above dynamics are given under the objective probability measure, the parameters μ and σ can be estimated using historical market data.

2.2.2 Continuous Short Interest Rate Model

Another component that constitutes a significant part of an insurance company's asset side is fixed interest assets. Similar to the case of stock price movements, interest rate movements largely affect the development of the firm's balance sheet. In order to model these uncertainties we therefore need a model reflecting the interest movements as well as possible. However, before going into relevant short rate models for this purpose, we will introduce some of the theory behind interest rate models and their relation to bond prices. The steps followed are the same as those in [7], and we therefore begin with introducing a general starting model for the short rate dynamics. Let the short rate dynamics be given by the following SDE

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t \tag{4}$$

It should be mentioned that the dynamics given by equation (4) represent the dynamics of r_t under the objective (real-world) probability measure P. The reason for mentioning this is the upcoming introduction of a risk-neutral measure Q, which will be used when deriving expressions for the bond prices. Based on equation (4), we let the price of zero-coupon bonds with face value 1 at time t, $p(t, T)$, be given by

$$p(t, T) = F(t, r_t; T), \tag{5}$$

where r_t denotes the short rate at time t and T the maturity time of the bond. Hence, we let the bond prices at time t be a function of the short rate at that time and the time to maturity. Since we are dealing with a zero-coupon bond, we also have a simple boundary condition for F given by

$$F(T, r; T) = 1,$$

and that holds for all r. This follows from the fact that the value of a zero-coupon bond at the time of maturity equals its face value, which in this case is 1. In order to derive expressions for the bond prices, we follow the procedures used in [7]. As is stated in [7], bond prices are not uniquely determined by the P-dynamics of the short rate r. However, there exists internal consistency relations between prices of bonds with different maturities in order for bond markets to be free of arbitrage. That is, bond prices are uniquely determined in terms of the r-dynamics and the price of a "benchmark" bond (with maturity greater than the maturity of the priced bonds).

These ideas are implemented by constructing a portfolio that consists of bonds having different maturities. We thus let S and T be two different time of maturities, and let our portfolio consist of zero-coupon bonds with these maturities. Using the Itô formula hence gives the following dynamics for the price of the S-bond and T-bond respectively (index K represents the maturity)

$$dF_t^K = \alpha_K F^K dt + \sigma_K F^K dW_t, \tag{6}$$

where

$$\alpha_K = \frac{\frac{\partial F^K}{\partial t} + \mu \frac{\partial F^K}{\partial r} + \frac{1}{2}\sigma^2 \frac{\partial^2 F^K}{\partial r^2}}{F^K},$$

$$\sigma_K = \frac{\sigma \frac{\partial F^K}{\partial r}}{F^K}.$$

Letting w_T and w_S denote the portfolio weights for each bond, the value dynamics for the whole portfolio become

$$dV_t = w_S \frac{dF_t^S}{F_t^S} V_t + w_T \frac{dF_t^T}{F_t^T} V_t.$$

If we now insert the bond price dynamics given in equation (6) into the value dynamics and perform some restructuring of the terms, we obtain

$$dV_t = (w_S \alpha_S + w_T \alpha_T) V_t dt + (w_S \sigma_S + w_T \sigma_T) V_t dW_t. \tag{7}$$

By choosing the weights w_S and w_T so that they make the dW term vanish, we have obtained a locally riskless portfolio. This in turn implies that the drift term has to equal the risk-free rate of return r in order for the market to be free of arbitrage. Hence, together with the fact that the sum of the weights equals 1, we obtain the following system of equations for w_S and w_T

$$\begin{cases} w_S + w_T = 1, \\ w_S \sigma_S + w_T \sigma_T = 0. \end{cases} \tag{8}$$

The solution to (8) is then given by

$$\begin{cases} w_S = \frac{\sigma_T}{\sigma_T - \sigma_S}, \\ w_T = -\frac{\sigma_S}{\sigma_T - \sigma_S}, \end{cases} \tag{9}$$

and putting this into equation (7) finally yields

$$dV_t = (\frac{\alpha_S \sigma_T - \alpha_T \sigma_S}{\sigma_T - \sigma_S}) V_t dt. \tag{10}$$

As is mentioned above, the absence of arbitrage implies that V's rate of return has to equal the short rate of interest, giving the following relation for all t

$$r_t = \frac{\alpha_S \sigma_T - \alpha_T \sigma_S}{\sigma_T - \sigma_S} \iff \frac{\alpha_S - r_t}{\sigma_S} = \frac{\alpha_T - r_t}{\sigma_T}. \tag{11}$$

Relation (11) holds no matter the choice of S and T, and the common quotient is called the market price of risk $\lambda(t)$

$$\lambda(t) = \frac{\alpha_T(t) - r(t)}{\sigma_T(t)}, \tag{12}$$

and holds for all t and for every choice of maturity T. This means that in a no arbitrage market all bonds will have the same market price of risk, no matter the time of maturity. By inserting the expressions for $\alpha_T(t)$ and $\sigma_T(t)$ in (12) we obtain the so called "term structure equation"

$$
\begin{cases}
\frac{\frac{\partial F^K}{\partial t}+(\mu-\lambda\sigma)\frac{\partial F^K}{\partial r}+\frac{1}{2}\sigma^2\frac{\partial^2 F^K}{\partial r^2}}{F^K} - rF^T = 0, \\
F^T(T,r) = 1.
\end{cases} \tag{13}
$$

A Feynman-Kac representation of F^T then finally gives the price of a zero-coupon bond at time t as

$$
p(t,T) = F(t,r_t;T) = E^Q_{t,r}[e^{-\int_t^T r(s)ds}], \tag{14}
$$

where the dynamics for r under the martingale measure Q are given by

$$
\begin{cases}
dr_s = (\mu - \lambda\sigma)ds + \sigma dW_s, \\
r_t = r.
\end{cases} \tag{15}
$$

Here μ and σ are the drift and diffusion rates given in equation (4), thus the rates under P, and λ as is given in equation (12).

Depending on the model used for the short rates, the bond prices/term structure equation take different forms. Some models are easier to deal with analytically than others by giving rise to a so called affine term structure (ATS). A model is said to possess an ATS if the term structure $p(t,T) = F(t,r_t;T)$ has the form

$$
F(t,r_t;T) = e^{A(t,T)+B(t,T)r_t}, \tag{16}
$$

where $A(t,T)$ and $B(t,T)$ are deterministic functions. Now when the underlying theory for the relation between short interest rate models and bond prices has been covered, we will present the Cox-Ingersoll Ross model together with an expression for its bond prices. The Cox-Ingersoll Ross model has mainly been chosen because of two reasons; (1) it is well-known and frequently used in the literature, and (2) it gives rise to an ATS [11] [14]. The former ensures that they are fairly well tested for the purpose they will serve in this paper, and the latter facilitates the derivation of bond prices once the interest rate has been simulated.

2.2.3 The Cox-Ingersoll Ross model

The Cox-Ingersoll Ross (CIR) model suggests that the instantaneous short rate $r(t)$, under the risk-neutral measure Q, satisfies the following stochastic differential equation

$$
dr(t) = \kappa(\theta - r(t))dt + \sqrt{r(t)}\sigma_r dW_r(t), \tag{17}
$$

where κ, θ and σ_r are positive constants, and $W_r(t)$ is a Wiener process under Q [7]. As is indicated by the drift term $\kappa(\theta - r(t))$, the short rate is mean reverting under the CIR model with mean reversion level θ and reversion rate κ [29]. This means that the short rate is constantly dragged towards its mean level; if the short rate becomes larger than the mean level ($r > \theta$), the drift becomes negative and pulls r back in the direction of θ, and similarly the rate is drawn back towards θ in the event of r being smaller than θ. The speed at which the short rate is corrected when deviating from its long-run mean is given by κ.

Another appealing property of the CIR model is the structure of the diffusion term [14]. The fact that the volatility σ_r is multiplied by $\sqrt{r(t)}$ makes the diffusion term approach zero as the short rate approaches zero, cancelling the model's randomness for low values of r. Hence, the CIR model restricts the short rate r from taking negative values [29]. Furthermore, if parameters fulfil the condition $2\kappa\theta > \sigma_r^2$, the CIR model produces short rates that are always positive [14].

Finally, the CIR model possesses an affine term structure, giving rise to bond prices of the form

$$F(t, r_t; T) = A(T - t)e^{-B(T-t)r_t}, \tag{18}$$

where T is the maturity and r_t satisfies the SDE in equation (17). The deterministic functions $A(T-t)$ and $B(T-t)$ are in this setting given by

$$A(T-t) = \left(\frac{2he^{(\kappa+h)(T-t)/2}}{2h + (\kappa + h)(e^{h(T-t)} - 1)}\right)^{2\kappa\theta/\sigma_r^2},$$

and

$$B(T-t) = \frac{2(e^{h(T-t)} - 1)}{2h + (\kappa + h)(e^{h(T-t)} - 1)},$$

where $h = \sqrt{\kappa^2 + 2\sigma_r^2}$.

2.2.4 Displaced CIR

The low interest rates that markets are currently experiencing make several of the standard short rate models rather unrealistic. For example, the CIR model does not produce negative short rates, and is still used to model rates which can be persistently low and even take negative values at short maturities. In order to compensate for the inability to replicate low-rate environments, displacement of the short rate model can be used. Adding displacement (i.e. shift of the state variable) is a method which is now a relatively common phenomena in the literature for modelling interest rate dynamics [20].

As is outlined in [20], the short rate model becomes displaced by the introduction of a constant displacement δ. The relation between the 'un-displaced' and displaced short rates are given by

$$r'_t = r_t - \delta, \tag{19}$$

where r_t is the 'un-displaced' rate and is modelled according to the dynamics given by the original short rate model used. In the case of a displaced CIR model, the 'un-displaced' rate r_t is modelled according to the dynamics given in equation (17), and the corresponding bond prices $F'(t, r_t; T)$ become

$$F'(t, r_t; T) = A(T - t)e^{-B(T-t)(r_t - \delta)}, \tag{20}$$

where $A(T-t)$ and $B(T-t)$ are as defined in section 2.2.3. Hence, one could say that the effect of introducing a displacement is simply a parallel shift across the entire yield curve [20].

2.2.5 Correlation between Stock Returns and Interest Rates

As is described in [14], stock and bond returns are usually correlated. A simple method for taking this into account is to let the stock price's and the short rate's Wiener processes (W_s & W_r) be correlated with a constant correlation coefficient $\rho \in [-1, 1]$ [14]. Under such an assumption we thus get that the two Wiener processes W_s and W_r satisfy

$$dW_s(t)dW_r(t) = \rho dt, \tag{21}$$

where ρ is a constant in the interval $[-1, 1]$. There are of course far more sophisticated methods for modelling the correlation between stock and bond returns, but the method presented in [14] has been deemed sufficient for the asset model suggested in this paper.

2.3 Liability Model

The liabilities that insurance companies face are closely connected to the characteristics of the insurance products. In order to model and understand the liability side of the balance sheet, a natural starting point is thus to break down the different components of the life insurance contracts. Moreover, in order to avoid an analysis where each and every contract has to be simulated individually, the contracts are grouped together in so called model points. This grouping is made so that each model point contains contracts which share several characteristics. The following two sections will break down traditional life insurance contracts, and based on that, define a division into relevant model points. The theory in this section is primarily based on product information provided by Skandia along with some general assumptions around insurance contracts that can be considered "praxis" in existing literature.

2.3.1 Life Insurance Contracts

A life insurance contract is in the Nordic countries traditionally composed by a guaranteed amount along with a possibility of earning more. The guaranteed amount is usually communicated as a guaranteed rate of return that the policyholder earns on the paid premiums regardless of the economic development. The insurance firm is therefore obliged to at least pay the policyholder the guaranteed amount by the time of maturity. On top of that, there is also a chance for the policyholder to earn more than the guarantee if the market performance is considered to be sufficient. This variable extra amount (from now on referred to as the bonus) depends on the firm's model for bonus declaration, i.e. the model used for determining bonus allocation based on economic circumstances and the firm's reserve rate.

Holding a life insurance contract also comes with an obligation to pay a premium. Normally the premium part is either a single initial premium or a series of premiums that last throughout the contract period. In the case where a series of premiums are paid, the guaranteed rate of return could either be universal (i.e. the same rate holds for all premiums) or be specific for each premium paid. Whatever the contracted terms, each premium thus gives rise to a liability that can be split up into two separate parts; a guaranteed amount, and a bonus part. The two parts grow with the rates of return (the guaranteed rate and the floating rate respectively) and with additional premiums paid. These are therefore the two main parts of the balance sheet's liabilities that are directly connected to the emitted/sold insurance contracts.

In addition to the premiums, return rates and times of maturity, death and surrender characteristics are often included in the contract. These state the terms in the event of death or surrender, in which the policyholder normally is entitled to a benefit that differs from the actual value of his capital. Due to the usually differing obligations in case of death or surrender, these factors affect the development of the liabilities associated with the contract. It is furthermore not only the different terms that has the ability to cause a problem when estimating the future liabilities, but also the unexpectedly shortened time to maturity. As is described in [14], one usually tries to compensate for this uncertainty by letting these liabilities develop as an expected value, where a still active contract, a surrendered contract and a "dead" contract constitute the possible outcomes. This, however, rather belongs to the modelling, and a deeper discussion is therefore given in section 3 (ALM Modelling).

2.3.2 Model Points

For efficiency reasons, the pool of all insurance contracts are often grouped together in a reduced number of so-called model points [14]. The grouping is done so that the contracts in each model point either share, or have similar characteristics. Typical criteria used for this grouping is magnitude of guarantees, surrender and death characteristics, entry and maturity time, and age of the policyholders [14]. The model points are constructed in such a way that the characteristics of the model point as a whole can be represented by a representative of each model point, where the two of them only differ within tolerable margins. When later developing the ALM model, each model point is therefore assumed to have a number of identical contracts and policyholders. In reality this is of course not the case, but the tolerable margins are chosen in such a way that averaged values for the criteria are representative for the whole group. Furthermore, the limited number of concerned contract attributes in this paper makes a successful pooling of the contracts into model points likely even with a customer base taken from a real-world scenario.

2.4 Discretization

The modelling that is to be carried out will be performed using a discrete time framework. This implies that the above defined continuous asset models need to be discretized, thus requiring relevant theory for this task. The following couple of sections therefore present two discretization schemes that have been deemed suitable for the job. These two approaches will later be further investigated as well as elaborated on, in order to locate the one being the most appropriate for this project.

2.4.1 Euler-Maruyama Scheme

The Euler-Maruyama scheme is a natural and simple method for approximating the solutions of various types of stochastic differential equations [21]. Following the descriptions given in [26] and [27], the approximation is constructed in the following way. Let us consider the SDE

$$\begin{cases} dX_t = \alpha(t, X_t)dt + \sigma(t, X_t)dW_t, \\ X_0 = x, \end{cases} \tag{22}$$

where W_t is the Wiener process defined in section 2.2.1. The solution to (22) is given by the process X_t satisfying

$$X_t = x + \int_0^t \alpha(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s. \tag{23}$$

In order to approximate (23) with a discretized solution X_n, we start by dividing a time interval $[0, T]$ into N sub-intervals given by $\delta t = \frac{T}{N}$ and

$$t_n = n \cdot \delta t = n \cdot \frac{T}{N}, \quad n = 0, 1, ..., N. \tag{24}$$

The Euler-Maruyama scheme is then given by

$$X_{n+1} = X_n + \alpha(t_n, X_n)\delta t + \sigma(t_n, X_n)\Delta W_n, \tag{25}$$

where $\Delta W_n = W_{t_{n+1}} - W_{t_n}$. Due to the properties of the Wiener process (see section 2.2.1), ΔW_n are independent random variables following a normal distribution with zero mean and variance δt, i.e. $\Delta W_n \sim N(0, \delta t)$. The Euler-Maruyama scheme is hence constructed using the following two approximations

$$\int_{t_n}^{t_{n+1}} \alpha(s, X_s) ds \approx \alpha(t_n, X_n)\delta t \quad and \quad \int_{t_n}^{t_{n+1}} \sigma(s, X_S) dW_s \approx \sigma(t_n, X_n)\Delta W_n.$$

2.4.2 Milstein Scheme

Another approximation method for stochastic differential equations is the Milstein Scheme. As is described in [27], the Milstein scheme can be seen as an extension of the Euler-Maruyama scheme. The extension is done by including another term of the "stochastic Taylor series". Let us again consider the process X_t satisfying (23) and the partition of $[0, T]$ into N sub-intervals as is presented in (24). The approximation X_n of X_t using the Milstein scheme is then given by

$$X_{n+1} = X_n + \alpha(t_n, X_n)\delta t + \sigma(t_n, X_n)\Delta W_n + \frac{1}{2}\sigma(t_n, X_n)\frac{\partial \sigma}{\partial x}(t_n, X_n)(\Delta W_n^2 - \delta t). \tag{26}$$

By studying (26) one easily sees that, compared to the Euler-Maruyama scheme, both the methods are identical if there is no X-term in the diffusion term $\sigma(t, X_t)$ of equation (22). An investigation of what discretization approach to use is thereby only relevant if the diffusion term contains an X-term.

2.4.3 Convergence

The performance of a numerical scheme is usually defined in terms of weak and strong convergence. A discrete-time approximation is *strongly convergent* if

$$\lim_{\delta t \to 0} E(|X_T - X_T^{\delta t}|) = 0,$$

and *weakly convergent* if

$$\lim_{\delta t \to 0} |E(f(X_T)) - E(f(X_T^{\delta t}))| = 0,$$

for all polynomials $f(x)$, and where X_t is the exact solution and $X_t^{\delta t}$ the approximated solution, computed with constant step size δt. Furthermore, the discrete-time approximation is said to converge strongly with order m if

$$E(|X_T - X_T^{\delta t}|) \sim C(\delta t)^m,$$

where the constant C depends on T and the considered SDE, and converge weakly with order m if

$$|E(f(X_T)) - E(f(X_T^{\delta t}))| \sim C(\delta t)^m,$$

where the constant C depends on T, f and the considered SDE. As is explained in [26] and [27], weak convergence concerns the distribution at time T only, whereas strong convergence concerns the path-wise property. Hence, if the whole path is of interest, strong convergence should be used.

The Euler-Maruyama scheme is strongly convergent with order $\frac{1}{2}$, under appropriate conditions on the functions α and σ in (22), and weakly convergent with order 1. The Milstein scheme, on the other hand, is both strongly and weakly convergent with order 1 [26] [27].

2.5 Customer Transition

In the subsequent modelling and simulation sections, the dimension of customer transition and retention among competing firms will be taken into account. These customer movements will be modelled using a Markov approach, leading us to present some basic Markov theory. The theory used is that of discrete Markov chains, and the following section is therefore limited to only discussing Markov chains in discrete time.

2.5.1 Markov Theory in Discrete Time

Let us consider a stochastic process $\{X_n; \quad n = 0, 1, 2, ...\}$ in discrete time, taking values in a finite state space $\boldsymbol{E} = \{i_k, \quad k = 0, 1, 2, ..., N\}$. The state space could also be a countably infinite set, i.e. $\boldsymbol{E} = \{i_k, \quad k = 0, 1, 2, ...\}$, but relevant to this study is the case where the states are given by a finite set. Following the definition in [15], the stochastic process $\{X_n; \quad n = 0, 1, 2, ...\}$ is said to be a Markov chain if

$$P(X_{n+1} = i_{n+1}|X_0 = i_0, X_1 = i_1, ..., X_n = i_n) = P(X_{n+1} = i_{n+1}|X_n = i_n), \tag{27}$$

for all n and all states $i_0, i_1, ..., i_n, i_{n+1}$. The relation given in (27) implies that the distribution of the next step only depends on the *current* state, and not on any other historical states. A Markov chain is thus memoryless in that sense, and the property of only regarding the current state is called the *Markov property.*

Based on the *Markov property* we can define the transition probabilities p_{ij}, i.e. the probabilities of jumping from one state to another (or remaining in the same). The transition probabilities are defined as

$$p(n)_{ij} = P(X_{n+1} = j|X_n = i), \quad i, j \in \boldsymbol{E}. \tag{28}$$

The transition probability p_{ij}thus represents the probability of moving from i to j, where j could either be the same state ($j = i$) or a different one ($j \neq i$). As opposed to the definition in [15], we let the transition probabilities be dependent on the time step n. This means that we are considering the more general case of a *time inhomogeneous* Markov chain, i.e. Markov chain for which the transition probabilities change across transitions.

The collection of all possible transition probabilities at time step n forms the so-called *transition matrix* $\boldsymbol{P}(n)$. $\boldsymbol{P}(n)$ is hence defined as the matrix $(p(n)_{ij})_{i,j\in\boldsymbol{E}}$, or alternatively put

$$\boldsymbol{P}(n) = \begin{pmatrix} p(n)_{11} & p(n)_{12} & p(n)_{13} & \dots & p(n)_{1N} \\ p(n)_{21} & p(n)_{22} & p(n)_{23} & \dots & p(n)_{2N} \\ p(n)_{31} & p(n)_{32} & p(n)_{33} & \dots & p(n)_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p(n)_{N1} & p(n)_{N2} & p(n)_{N3} & \dots & p(n)_{NN} \end{pmatrix}. \tag{29}$$

Since the aggregated probability of jumping from one state to another (or remaining in the same) is 1 for each time step, the sum of each row equals 1. Alternatively formulated, the ith row represents all possible transitions that can be made from state i and we therefore have that

$$\sum_{j=1}^{N} p(n)_{ij} = 1, \quad \forall i \in \boldsymbol{E}. \tag{30}$$

State i is said *to be leading to* state j, written $i \to j$, if it is possible to move from i to j in zero, one or multiple time steps. Furthermore, if $i \to j$ and $j \to i$ hold, then state i and j are said to be *communicating*, written $i \leftrightarrow j$. Finally, a Markov chain for which the entire state space consists of communicating states, i.e. $i \leftrightarrow j \quad \forall i, j \in \boldsymbol{E}$, is called an *irreducible* chain [15].

2.6 Solvency II

Solvency II is a framework for the supervision and regulation of the insurance and reinsurance industry in the European Union. It serves as a tool for stabilizing and harmonizing the industry, by the introduction of qualitative and quantitative regulations. The content of Solvency II can be categorized into three different pillars [10]:

- Pillar 1: Harmonised valuation and capital requirements
- Pillar 2: Harmonised governance, internal control and risk management requirements
- Pillar 3: Harmonised supervisory reporting and public disclosure

Solvency II is the first framework to introduce economic risk-based solvency requirements across all member states of the European Union, and compared to past requirements the new ones are more risk-sensitive and sophisticated. Furthermore, the requirements are more entity-specific; Solvency II abandons the idea of "one-model-fits-all" and thereby contributes to a better coverage of risks run by any particular insurer [10].

Most relevant to this thesis are the quantitative requirements, and these are all part of *Pillar 1* above. The quantitative requirements of Solvency II are composed by the following components [3]:

- Market consistent valuation of assets and liabilities
- SCR - Solvency Capital Requirement
- MCR - Minimum Capital Requirement

The quantitative focus has primarily been dedicated to the SCR, for which the requirements constitute one of the cores of the directive. The SCR should amount to the one year 99.5% Value-At-Risk (VaR)[1] of the capital base (assets minus liabilities to policyholders) [3]. Alternatively put, this requirement corresponds to a risk of being declared bankrupt, that is lower than $\frac{1}{200} = 0.5\%$.

3 ALM Model

In this chapter we will come up with the model used for the subsequent scenario and sensitivity analysis. Based on the theoretical components presented in the previous section, an ALM model is to be constructed. The fundamental approach is based on the work done in [14], but the construction is then further developed in order to meet the needs of Scandinavian life insurance companies in general, as well as the needs of Skandia in particular. Furthermore, several parts are remodelled, and an additional dimension in terms of competition modelling is introduced.

[1] VaR represents the amount at risk of an investment with a given probability over a certain period of time. The VaR at level $\alpha \in (0, 1)$ of an investment with value X at time 1 is $VaR_\alpha(X) = min\{m : P(mR_0 + X < 0) \leq \alpha\}$, where R_0 is the return of a risk-free asset. It is thus the required amount to be invested in a risk-free asset at time 0, in order to have a probability of a loss at time 1 that is less than or equal to α [18].

3.1 Balance Sheet Projection

The ALM model is put together using a simplified balance sheet, in which the most important balance sheet items for a portfolio of insurance policies are included. Based on the simplified balance sheet, the main focus of the model is to simulate the temporal development of its components. We start by introducing the overall structure of the balance sheet. Once the general structure is in place we move on to determining and modelling each of its components, and their contribution to the balance sheet evolvement as a whole.

We use a discrete time framework, in which we let the time period $[0, T]$ be partitioned into K equal subintervals $[t_{k-1}, t_k]$, $k = 1, 2, ..., K$, with $t_k = k\Delta t$ and $\Delta t = \frac{T}{N}$. The start and end of the simulation are thus, respectively, $t = 0$ and $t = T$, and as in [14] we let t be in years and Δt be equal to the period of one month. The concerned balance sheet items at time t_k, $k = 0, 1, ..., K$, are presented (using the same notation as in [14]) in Table 1.

Assets		Liabilities	
Capital	C_k	Actuarial reserve	D_k
		Allocated bonus	B_k
		Free reserve	F_k
		Equity	Q_k

Table 1: Balance sheet items considered in the model

The asset side consists of the value C_k of the firm's invested capital at time t_{k-1}. It is assumed that the whole asset value is attributable to the firm's investment portfolio, meaning that C_k reflects the portfolio value at time t_{k-1}. Other possible asset items of a typical balance sheet (e.g. cash) are therefore ignored. This is also the case in most of the literature, in which one seldom comes across asset modelling that goes beyond the investment portfolio (see e.g. [14], [17]). Moving on to the liability side, we have the actuarial reserve D_k, allocated bonus B_k, free reserve F_k and equity Q_k. The actuarial reserve reflects the liability coming from the guarantees embedded in the insurance contracts, and thus constitutes an item that has to be covered at the time of maturity (or prior to maturity if it is part of the death and/or surrender benefits). The allocated bonuses represent the returns that have been credited to the policyholders in excess of the guaranteed returns. This liability is governed by the profit participation model, and varies depending on the performance of the underlying investment portfolio. The free reserve is a buffer account consisting of positive returns that have not yet been credited to the policyholders' bonus accounts, enabling a smoothing of the capital market oscillations. By holding a buffer, the insurance firm may save positive returns from better periods in order to avoid bonus retractions during worse periods. It thus contributes to a more stable and low-volatile return participation policy [14]. The equity item reflects the amount that is kept by the shareholders of the company, and in order for the sum of assets to equal the sum of liabilities, we let its value at time t_{k-1} be given by

$$Q_k = C_k - D_k - B_k - F_k.$$

In case the insurance firm is a mutual organization, i.e. the policyholders are also the owners of the company, then F_k and Q_k can be merged into one single account[2] [16]. Similar set-ups of the

[2]Even though the main goal of the model is to replicate Scandinavian mutual insurance companies, the equity account and the free reserve are kept as separate items. This adds clarity to the process of return smoothing and makes the scenarios of default/bankruptcy more tangible/comprehensible

balance sheet are used in existing literature. Apart from [14], one can find similar approaches in [16] and [22].

3.1.1 Asset models in continuous time

As mentioned above, the asset side of the balance sheet is assumed to only include the insurance firm's investment portfolio. We further assume that this portfolio consists of either fixed interests assets, i.e. bonds, or stocks. As in [17], we also assume that there is no borrowing and no short-sales at any time period, and that there constantly exists a liquid market for each and every one of the considered assets. This goes against the bond approach used in [14], where the firm is assumed to hold all bonds to maturity and therefore has to short-sell bonds if liquidity for payments is needed. In this model, via the assumption of liquid markets for each of the traded assets, liquidity problems are instead solved by selling existing assets.

Following the approach used in [14], the development of the capital market is modelled as a coupled system of two continuous stochastic differential equations; one for the stock price movements and one for the short interest rate movements. The coupled system is then discretized using the step size Δt that has been introduced above. The reason for choosing continuous asset models, and not discrete ones, is the fact that they are generally well-known and widely used in this context. Furthermore, the chosen model for the short rate has an affine term structure, and thereby enables explicit derivation of bond prices.

We let the stock price S evolve as a geometric Brownian motion, i.e. S solves the following stochastic differential equation

$$dS(t) = \mu_s S(t)dt + \sigma_s S(t)dW_s(t),$$

where μ_s, σ_s and W_s are as in (3) above. The price $S(t)$ at time t is thus given by

$$S(t) = S(0)e^{(\mu_s - \frac{\sigma_s^2}{2})t + \sigma_s W_s(t)}. \tag{31}$$

For the short interest rates, we assume that they are given by the Cox-Ingersoll-Ross (CIR) model, i.e. that $r(t)$ has the dynamics given by

$$dr(t) = \kappa(\theta - r(t))dt + \sqrt{r(t)}\sigma_r dW_r(t), \tag{32}$$

where κ, θ, σ_r and W_r are as in (17) above. The dynamics of $r(t)$ given in (32) are under the objective probability measure, and the corresponding parameters under the risk-neutral measure are defined below, when stating an expression for the bond prices.

This set-up (GBM for the stock prices and CIR for the short interest rates) is used for example in [11] and [14]. An alternative to the CIR model is the Vasicek model, which is suggested, together with a GBM for the stock prices, in [8] and [23]. Both models share two appealing properties; (1) they are both mean-reverting, and (2) they both possess an affine term structure. The fact that they are mean-reverting means that the the short interest rate is pushed towards its long-run mean θ once it deviates from it. The speed of adjustment is in this case determined by κ. Mean-reversion is partly appealing due to the existence of compelling economic arguments in favor of it. When rates are low, the economy tends to "speed up" and experience an increased demand for funds, implying an increasing interest rate. On the contrary, when rates are high, the economy tends to "slow down" and the demand for funds decreases, implying an eventual decline of the rates [29]. In this context, for which the main purpose of the capital market model is to reflect the economic situation as good as possible, mean-reversion is hence an attractive property. Furthermore, the fact that the models possess an affine term structure enables the bond prices to be derived in closed form, as is shown in equation (16) above. The main difference, on the other hand, between the CIR and Vasicek model is that the Vasicek model has a positive probability of taking negative values. One could argue that the current interest rate environment would be better described by a model that has the possibility of taking negative values. However, following the arguments in [14] and [29], the CIR model is considered more appropriate in this case. Not only as a result of the rates always being non-negative, but also due to the property that makes the rates less volatile as they approach zero, and more volatile at higher levels. What's more, if one wishes to model low rate environments, the CIR model can be used together with a displacement factor, as described in section 2.2.4.

In order to express the bond prices, we use the fact that the CIR model possesses an affine term structure. As is discussed in [14], if we assume a market price $\lambda(t,r)$ of risk having the special form $\lambda(t,r) = \lambda\sqrt{r(t)}$, with $\lambda \in \mathbb{R}$, and the absence of arbitrage, the short interest rate has the same dynamics as in (32) also under the risk-neutral probability measure but with the parameters changed to $\hat{\kappa} = \kappa + \lambda\sigma_r$ and $\hat{\theta} = \frac{\kappa\theta}{\hat{\kappa}}$. Furthermore, the price at time t of a zero-coupon bond with maturity $T = t + \tau\Delta t$, i.e. a duration of τ periods, becomes

$$b(t,\tau) = A(\tau)e^{-B(\tau)r(r)}, \tag{33}$$

where

$$A(\tau) = \left(\frac{2he^{(\hat{\kappa}+h)\tau\Delta t/2}}{2h + (\hat{\kappa}+h)(e^{h\tau\Delta t} - 1)}\right)^{2\kappa\theta/\sigma_r^2}, \quad B(\tau) = \frac{2(e^{h\tau\Delta t} - 1)}{2h + (\hat{\kappa}+h)(e^{h\tau\Delta t} - 1)},$$

and $h = \sqrt{\hat{\kappa}^2 + 2\sigma_r^2}$.

Based on the fact that stock and bond returns usually are correlated, we assume a constant correlation between the Wiener process of the stock price and that of the short interest rates. We thus assume that the two Wiener processes satisfy $dW_s(t)dW_r(t) = \rho dt$, where $\rho \in [-1, 1]$ is a constant correlation coefficient. This is a frequently used assumption for the dependency between stock and bond returns, and can, just to mention a few, be seen in [8], [11], [14], [22] and [23]. Due to the fact that the dynamics are specified under the objective probability measure, the parameters μ, σ_s, κ, θ, σ_r and ρ can be estimated using historical data. Moreover, the parameter λ included in the assumed market price of risk process can be obtained by calibrating the theoretical bond prices given by (33) to observed market prices [7].

3.1.2 Asset models in discrete time

Since the model is set up using a discrete time framework, the continuous capital market models above need to be discretized. We let the stock prices, short interest rates and bond prices be given by $s_k = s(t_k)$, $r_k = r(t_k)$ and $b_k(\tau) = b(t_k, \tau)$. As discussed in section 2.4.3, strong convergence rather concerns the path-wise property, whereas weak convergence rather reflects the distribution at the time horizon T. Since the model's purpose is not only to simulate scenarios focused around the time horizon, but also the behaviour on the way there, it is reasonable to consider both strong and weak convergence. In the case of weak convergence both schemes perform equally well, however the Milstein scheme performs slightly better in terms of strong convergence. A comparison of both methods' performance for the intended step size therefore seems adequate in this case.

In order to decide what scheme to use when approximating the solution to equation (32), we compare the Euler-Maruyama scheme with the Milstein scheme when approximating a geometric Brownian motion (equation (1)) with the step size we intend to use in the simulations. The reason for using a geometric Brownian motion is because it offers an exact solution to compare our approximations with. Figure 3 shows a comparison of the two schemes against the exact solution, using the time step and parameters stated in the caption of the figure. The parameters are taken from [14], and thus belong to the range of values that are to be used in the model.

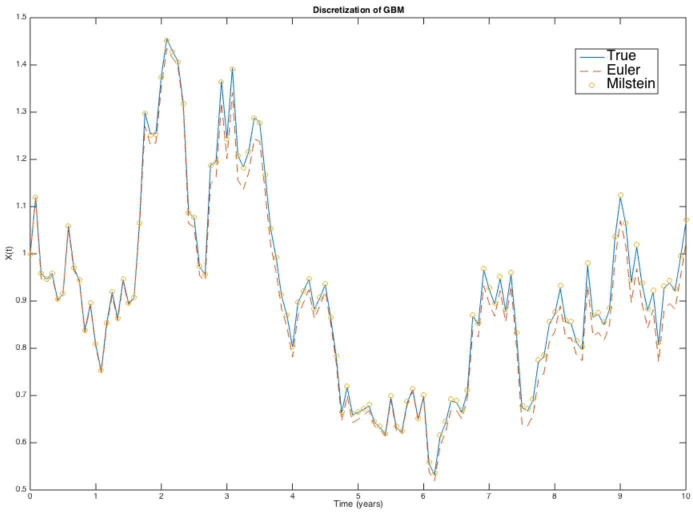


Figure 3: Euler and Milstein approximations along with exact solution to a GBM (equation (1)) with initial value $X(0) = 1$, drift parameter $\alpha = 0.08$, diffusion parameter $\sigma = 0.2$ and step size $\Delta t = \frac{1}{12}$.

As is shown in Figure 3, the Milstein scheme expectedly performs slightly better than the Euler scheme for the chosen step size and set of parameters. Since the dynamics in (32) enable a straightforward expression for the Milstein approximation of the short interest rate, it seems natural to proceed without any further investigation of the two methods' accuracy. Hence, by applying (26) on the model for the short interest rate and using the notation in [14], we obtain the following expression for r_k

$$r_k = r_{k-1} + \kappa(\theta - r_{k-1})\Delta t + \sigma_r \sqrt{|r_{k-1}|}\sqrt{\Delta t}\xi_{r,k} + \frac{1}{4}\sigma_r^2 \Delta t(\xi_{r,k}^2 - 1), \tag{34}$$

where we have used the fact that $\frac{\partial \sigma_r}{\partial x} = \frac{\sigma_r}{2\sqrt{x}}$ and $\xi_{r,k}$ is a $N(0,1)$-distributed random variable. For the stock price, using the approach in [14], one obtains

$$s_k = s_{k-1} e^{(\mu - \sigma_s^2/2)\Delta t + \sigma_s \sqrt{\Delta t}(\rho \xi_{r,k} + \sqrt{1-\rho^2}\xi_{s,k})}, \tag{35}$$

where $\xi_{s,k}$ is a $N(0,1)$-distributed random variable that is independent of $\xi_{r,k}$. The assumed constant correlation ρ between the two Wiener processes $W_s(t)$ and $W_r(t)$ is still respected since

$$Cov(\rho\xi_{r,k} + \sqrt{1-\rho^2}\xi_{s,k}, \xi_{r,k}) = \rho,$$

due to the stated independence between $\xi_{s,k}$ and $\xi_{r,k}$. The discrete bond prices $b_k(\tau)$ are finally given by using the discrete short interest rates as input to equation (33).

3.1.3 Management models

Inspired by the naming conventions used in [14], the management models comprise the allocation of capital between available assets, the bonus declaration mechanism and the shareholder participation, in case the free reserve and equity are modelled separately. For the capital allocation, we assume that all capital is invested in either stocks or zero-coupon bonds. The respective shares invested in each asset are assumed to be given by a fixed trading strategy, implying that the firm has a fixed portion $\beta \in [0,1]$ that it aims to have invested in stocks, and a fixed portion $1-\beta$ invested in bonds. Depending on the development of the two assets, the portfolio may need to be rebalanced, and such rebalances are assumed to take place at the beginning of each period. At the beginning of the kth period, i.e. at time t_{k-1}, we thus have the following capital allocation

$$\begin{cases} C_{s,k} = \beta C_k, \\ C_{b,k} = (1-\beta)C_k, \end{cases}$$

where $C_{s,k}$ and $C_{b,k}$ are the amounts invested at time t_{k-1} in stocks and bonds, respectively. The durations of the held zero-coupon bonds are assumed to be constant, meaning that the firm not only rebalances between assets at the beginning of each period, but also between bonds with different maturities. We hence assume that there exists a constantly liquid market for all traded assets, and that old bonds are sold and new ones bought in order to keep the target distribution in regards to duration. This is similar to the methods used in [4], [16] and [22], where either the portfolio is modelled on an aggregate level or as a fixed combination of two assets. In [14], however, all bonds are assumed to be held until maturity, implying a capital allocation that is restricted to only allocate capital from sold stocks and matured bonds at each period. Furthermore, when capital is needed for financing policy payments, and the amount coming from sold stocks and matured bonds is not sufficient, the firm is assumed to go short in bonds instead of selling existing ones. In the context of Scandinavian insurance firms, the latter is deemed less realistic and the method adopted in this thesis instead follows the suggestions in [4], [16] and [22].

For the bond durations, the firm is assumed to have a fixed set of held durations $\boldsymbol{\tau}$, given by

$$\boldsymbol{\tau} = (\tau_1, \tau_2, ..., \tau_{q-1}, \tau_q),$$

where q denotes the number of held durations and τ_i, $i = 1, 2, ..., q$, the ith duration expressed in number of periods Δt. Furthermore, the distribution $\boldsymbol{\omega}$ of the capital invested in bonds over different maturities is assumed to be constant over time and given by

$$\boldsymbol{\omega} = (\omega_1, \omega_2, ..., \omega_{q-1}, \omega_q),$$

where $\omega_i \in [0,1]$, $i = 1, 2, ..., q$ and $\sum_{i=1}^{q} \omega_i = 1$, denotes the portion of the bond investment put in bonds with duration τ_i. Hence, the total amount invested in bonds at time t_{k-1} can be written

$$C_{b,k} = (1-\beta)C_k = \boldsymbol{\omega} \cdot \boldsymbol{b}_k(\boldsymbol{\tau})^{\boldsymbol{T}},$$

where $\boldsymbol{b}_k(\boldsymbol{\tau}) = (b_k(\tau_1), b_k(\tau_2), ..., b_k(\tau_q))$ is a vector of bond prices at time t_{k-1} for all durations included in $\boldsymbol{\tau}$.

Before defining how the total assets C_k evolve, we need to introduce the premiums received and payments done in each period. Let P_k denote the sum of all premiums received at the beginning of period k, i.e. at time t_{k-1}. We also introduce R_k as the portfolio return gained during period k, i.e. the realized returns over the period $[t_{k-1}, t_k]$. The asset side of the balance sheet then evolves, on an aggregate level, according to

$$C_{k+1} = (1 + R_k)(C_k + P_k) - \phi_{k+1}(D_{k+1}, B_{k+1}), \tag{36}$$

where P_k is assumed to be invested directly in the same portfolio as all other capital is invested in, and $\phi_k(D_k, B_k)$ is a function specifying the payments done at the beginning of period k (more details around ϕ_k are given in the next section). The portfolio return R_k is a weighted average of stock returns $R_{s,k}$, on the one hand, and bond returns $R_{b,k}$, on the other hand. We let the stock returns $R_{s,k}$ be given by

$$R_{s,k} = \frac{s_k}{s_{k-1}} - 1,$$

and the bond returns $R_{b,k}$ by

$$R_{b,k} = \boldsymbol{\omega} \cdot \Delta \boldsymbol{b}_k(\boldsymbol{\tau})^{\boldsymbol{T}},$$

where

$$\Delta \boldsymbol{b}_k(\boldsymbol{\tau}) = \left(\frac{b(t_k, \tau_1 - 1)}{b(t_{k-1}, \tau_1)} - 1, \frac{b(t_k, \tau_2 - 1)}{b(t_{k-1}, \tau_2)} - 1, \ldots, \frac{b(t_k, \tau_q - 1)}{b(t_{k-1}, \tau_q)} - 1 \right)$$

is a vector containing the individual returns for each held duration. The bond returns are thus nothing but a weighted average of the returns realized by the different durations held in the portfolio over the interval $[t_{k-1}, t_k]$. We can now express the portfolio return R_k as

$$R_k = \beta R_{s,k} + (1 - \beta) R_{b,k},$$

where $R_{s,k}$ and $R_{b,k}$ are as defined above.

For the bonus declaration we use a modified version of the model proposed in [14]. A commonly used term within the Scandinavian insurance industry, when it comes to declaring bonus, is "the collective degree of consolidation" (from now on referred to as the KKG[3]). According to [24], the KKG is defined as the relation between the value of the asset portfolio and that of actuarial reserves and allocated bonuses. Using our simplified balance sheet, it thus corresponds to the following quotient

$$KKG_k = \frac{C_k}{D_k + B_k}.$$

Depending on the level of the KKG, the insurance firm decides whether to increase or decrease declared bonuses and by how much [24]. It is therefore desirable to have a mechanism for the bonus declaration that takes this into account, which happens to be the case with the method suggested in [14]. Let us define the reserve rate γ_k at the beginning of period k as

[3] Collective degree of consolidation is Kollektiv Konsolideringsgrad in Swedish, and is usually abbreviated KKG.

$$\gamma_k = KKG_k - 1 = \frac{C_k - D_k - B_k}{D_k + B_k} = \frac{F_k + Q_k}{D_k + B_k}.$$

We further let γ denote the target reserve rate set up by the firm. According to [14], typical values for the target reserve rate are $\gamma \in [0.1, 0.4]$, and the Swedish insurance company Skandia declares additional bonus if their KKG exceeds 115%, yielding a γ equal to 0.15 [24]. Based on the target reserve rate γ and the current reserve rate γ_k, we define the policy return z_k for period k as

$$z_k = \upsilon(\gamma_k - \gamma).$$

Here $\upsilon \in [0, 1]$, typically taking values in $[0.2, 0.3]$, is a constant, working as a smoothing parameter in accordance with the average interest rate principle [14]. Defining the policy return this way yields a smoothened positive return if current reserves are deemed sufficiently large, and a smoothened negative return if not. Compared to the method suggested in [14], there are two minor modifications in this model. Firstly, the reserve rate has been defined by Scandinavian standards, according to which the equity item is part of the numerator [24]. And secondly, the policy return can fall below the guaranteed interest rate g (which will soon be introduced), prohibiting the guaranteed interest from providing positive return on anything else than the actuarial reserve D_k. The latter difference will become much clearer as soon as we have introduced the models for the evolvement of D_k and B_k in the upcoming section.

3.1.4 Liability models

As the asset side of the balance sheet is now defined, we move on to defining the models for the liability items of the balance sheet. Since these are closely connected to the characteristics of the sold insurance contracts, we begin by defining the type of life insurance products considered in this thesis. As is discussed in section 2.3.2, one frequently uses the concept of model points within insurance. For our model we assume a portfolio of insurance contracts that consists of m model points, where each model point shares the following characteristics (super index i represents model point i, $i = 1, 2, ..., m$)

1. Guaranteed rate of return g^i
2. Death, surrender and maturity benefits M_k^i, S_k^i and E_k^i
3. Age and gender distribution
4. Time of maturity $T^i \leq T$ (following from point 3)
5. Probability of death q_k^i (following from point 3)
6. Periodic premiums P_k^i

We thus assume that all policyholders within the same model points have identical values for the characteristics listed above. In regards to the death probabilities, they typically depend on the age and sex of the policyholder, and are extracted from a mortality table possessed by the insurance firm [5], [14]. Furthermore, the systematic development of mortality can, as opposed to the development of capital markets, be predicted more accurately, and by means of portfolio diversification, unsystematic mortality risk can be controlled for [14]. We therefore assume that the death probabilities are given deterministically, and let the probability that a policyholder of model point i dies during the kth period be denoted q_k^i.

According to [14], there exist two distinguished approaches for surrender modelling; either surrender is seen as an exogenously determined event, just like the case of death, or as an event occurring due to rational decisions made by policyholders. In this model the latter approach is adopted. However, the modelling of surrender probabilities is introduced in section 3.3, so for now we assume them to be given and let p_k^i denote the probability that a policyholder of model point i surrenders during period k.

Now when the death and surrender probabilities are introduced, we let n_k^i denote the expected number of active contracts in model point i at the beginning of period k. Using the probabilities of death and surrender, we let the development of n_k^i over time be given by

$$n_k^i = (1 - q_k^i - p_k^i) n_{k-1}^i \mathbb{1}_{k-1}^i, \tag{37}$$

where $\mathbb{1}_k^i$ is an indicator variable defined as

$$\mathbb{1}_k^i = \begin{cases} 1, & if \quad 0 \leq k\Delta t < T^i \\ 0, & if \quad T^i \leq k\Delta t \leq T, \end{cases}$$

where T^i is the time of maturity for model point i. This means that if the contracts of model point i have reached maturity during period $k-1$, then there will be no contracts left in that model point at the beginning of period k, and the expected number of contracts will therefore equal zero for all subsequent periods.

We let the actuarial reserve and allocated bonus evolve on a contract-level in each model point, for which we introduce D_k^i and B_k^i as the actuarial reserve and allocated bonus, respectively, at the beginning of period k for a contract being part of model point i. As mentioned earlier, the actuarial reserve D_k^i should reflect the value of the guaranteed payments being part of model point i's insurance contract. We thus let the actuarial reserves be given by the discounted value of these payments, and let the period's bond prices constitute the discount factor. As the guaranteed payments are determined by the guaranteed rate of return g^i and the paid premiums P_k^i, we obtain the following expression for the actuarial reserves

$$D_{k+1}^i = p(t_k, T_i)(D_{k,T_i}^i + P_{k+1}^i (1 + g^i)^{T_i - t_k}),$$

where T_i is the maturity for model point i, $p(t_k, T_i)$ the price of a ZCB with face value one and maturity T_i, and D_{k,T_i}^i the guaranteed amount at maturity for a policyholder of model point i being at time t_k (i.e. $D_{k,T_i}^i = \sum_{j=1}^{k} P_j^i (1 + g^i)^{T_i - t_{j-1}}$). The somewhat complicated expression above is thus no other than the discounted value of the guaranteed payment that each paid premium gives rise to, where the period's bond prices are used as discount factors. One could argue here that the expression for the actuarial reserve does not take surrender and death probabilities into account, and thereby fails to reflect the expected value of the discounted guarantees. However, as the surrender probabilities are path dependent (they depend on future policy returns), including these factors have been deemed to be beyond the scope of this thesis and will instead later be mentioned as a suggestion for further research.

For the allocated bonus B_k^i, on the other hand, we introduce a variable PK_k^i representing the insurance capital[4]. The insurance capital evolves with the declared policy return z_k and paid

[4]Refers to the Swedish insurance term *pensionskapital*.

premium P_k^i, and can thus be seen as the liability towards a holder of a contract that does not offer any guarantees but only the realized policy return. The variable PK_k^i hence evolves over time according to

$$PK_{k+1}^i = (1 + z_k)(PK_k^i + P_{k+1}^i).$$

We then let the allocated bonus B_k^i be defined as the amount by which PK_k^i potentially exceeds the guarantee D_k^i. That is

$$B_k^i = max(PK_k^i - D_k^i, 0).$$

The totals for these items, D_k and B_k, at the beginning of period k are then obtained by the following sums

$$D_k = \sum_{i=1}^{m} n_k^i D_k^i \quad and \quad B_k = \sum_{i=1}^{m} n_k^i B_k^i,$$

where the use of (37) ensures that potential deaths, surrenders and matured contracts are taken into account.

We further let the death benefits M_k^i, surrender benefits S_k^i and maturity benefits E_k^i be split up into two parts; one part connected to the actuarial reserve D_k^i (denoted with super index G), and one part connected to the allocated bonuses B_k^i (denoted with super index B). We hence have the following relations

$$M_k^i = M_k^{i,G} + M_k^{i,B}, \quad S_k^i = S_k^{i,G} + S_k^{i,B}, \quad and \quad E_k^i = E_k^{i,G} + E_k^{i,B}.$$

We let the characteristics of each benefit be the same as in [14], that is

$$M_k^{i,G} = D_k^i, \quad S_k^{i,G} = \vartheta D_k^i \quad and \quad E_k^{i,G} = \begin{cases} D_k^i, & if \quad (k-1)\Delta t = T^i \\ 0, & if \quad (k-1)\Delta t \neq T^i \end{cases},$$

and

$$M_k^{i,B} = B_k^i, \quad S_k^{i,B} = \vartheta B_k^i \quad and \quad E_k^{i,B} = \begin{cases} B_k^i, & if \quad (k-1)\Delta t = T^i \\ 0, & if \quad (k-1)\Delta t \neq T^i \end{cases},$$

where $\vartheta \in [0, 1]$ is a constant surrender factor reducing the amount paid out to a policyholder surrendering the contract.

Having defined the benefit characteristics for death, surrender and maturity, we can obtain a final expression for the development of the assets C_k and remaining liability items F_k and Q_k. As is shown in equation (36), the recursive relation for C_k is affected by the payments done at the beginning of period k. In order to obtain an explicit expression for the development of C_k we thus have to define $\phi_k(D_k, B_k)$. The payments done at the beginning of period k are determined by the death, surrender and maturity characteristics, giving us the following definition

$$\phi_k(D_k, B_k) = \sum_{i=1}^{m} n_{k-1}^i (q_k^i M_k^i + p_k^i S_k^i) + n_k^i E_k^i = \sum_{i=1}^{m} (D_k^i + B_k^i)(n_{k-1}^i (q_k^i + \vartheta p_k^i) + n_k^i \chi(k-1)), \quad (38)$$

where $\chi(k)$ is an indicator function taking value one if $k\Delta t = T^i$ and zero otherwise. The recursive relation for C_k is thus

$$C_{k+1} = (1 + R_k)(C_k + P_k) - \phi_{k+1}(D_{k+1}, B_{k+1}),$$

where $\phi_{k+1}(D_{k+1}, B_{k+1})$ is defined as in (38). For the reserve and equity items F_k and Q_k, we use a slightly simpler approach than the one employed in [14]. We just let the difference between C_{k+1} and $(D_{k+1} + B_{k+1} + F_k + Q_k)$ be seen as the period's surplus/deficit, and allocate it to F and Q using a shareholder participation coefficient $\alpha \in [0, 1]$. According to [14], a typical value for α is 0.9, which implies that 90% of the surplus goes to the reserve and the remaining 10% to the equity. In the event of deficit, we assume as in [14] that it is taken up by the existing reserves F_k, and if these do not suffice the rest is covered by the equity account Q_k. We let G_k denote the surplus/deficit obtained at the beginning of period k (i.e. the realized surplus/deficit during period $k - 1$), and define the development of F_k in the following way

$$F_{k+1} = max(F_k + min\{G_{k+1}, \alpha G_{k+1}\}, 0),$$

where G_k is defined as

$$G_{k+1} = C_{k+1} - (D_{k+1} + B_{k+1} + F_k + Q_k).$$

Q_{k+1} is finally given so that the totals of the asset side and liability side coincide

$$Q_{k+1} = C_{k+1} - (D_{k+1} + B_{k+1} + F_{k+1}).$$

3.2 Competition model

The inclusion of competition in the model will be based on a simplified Markov set-up in discrete time. The transitions between insurance firms are modelled using time dependent transition matrices, consisting of a probability model that is inspired by the work presented in [9].

3.2.1 Competition characteristics

The competitive environment is assumed to consist of N insurance firms, all competing for an initially fixed number of customers. The total number of customers is fixed in the sense that it is assumed to be no new entrants, meaning that customers gained by one firm have been lost by another. The customer population may of course decrease due to events of death and maturity, but not due to surrendered contracts. It is assumed that a surrendering customer always moves to another firm, hence excluding the aspect of customers completely leaving the insurance industry.

Each firm is an identical copy of its competitors except for a limited number of parameters (typical parameters of interest are the capital allocation and mechanism for policy returns). Furthermore, the balance sheet of each firm is modelled using the ALM model presented in the previous sections, with input values of parameters as the only difference. The main component for attracting new customers will be the firm's allocated period return compared to that of its competitors. Based on the firm's relative performance, its policyholders then decide whether to stay with the firm or to move to one of its competitors, leaving the firm with a potentially new number of clients before entering the next period. The general model framework is illustrated in Figure 4.

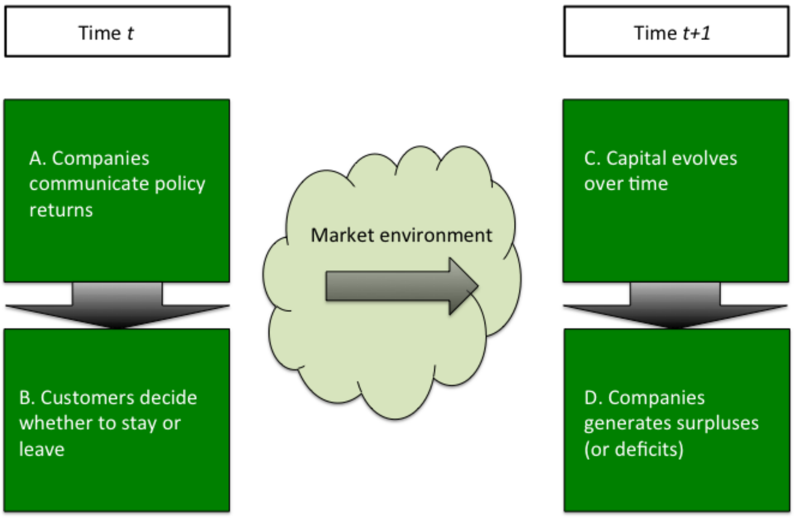


Figure 4: Illustration of competition model during a period $[t, t+1]$

One should mention that in reality not all contract types allow its policyholders to move their capital before maturity. Depending on the contractual terms, the policyholder may be completely prohibited from moving his capital before the end of the contract, or pre-maturity capital movements could be associated with such large fees that it would not be a reasonable thing to do [25]. However, in this setting all model points are assumed to allow pre-maturity capital movements, creating a landscape in which the rivalling firms solely compete based on their performance in terms of allocated returns.

3.2.2 Customer transitions

The customer transitions are modelled as a time inhomogeneous, irreducible Markov chain in discrete time. For the transition probabilities we use a variant of the multinomial logit model presented in [9]. We hence let the unconditional probability of moving to firm j, $j = 1, 2, ..., N$, during period k be given by

$$P_{j,k} = z^{-1} e^{\boldsymbol{\beta}_{trans} \boldsymbol{x}_{j,k}},$$

where $\boldsymbol{x}_{j,k}$ is a vector of variables characterizing firm j during period k, $\boldsymbol{\beta}_{trans}$ a vector of the corresponding response parameters, and z a factor normalizing the sum of probabilities to one. As $\boldsymbol{\beta}_{trans}$ does not depend on the considered period k, we let the response parameters be constant over time.

We further let $P_{ij,k}$, $i, j = 1, 2, ...N$, denote the conditional probability of moving to firm j during period k, given that one currently is a customer to firm i. Using a simplified version of the approach outlined in [9], we define our conditional probabilities as

$$P_{ij,k} = \begin{cases} z_i^{-1}(1+c)e^{\boldsymbol{\beta}_{trans}\boldsymbol{x}_{j,k}}, & if \quad i = j, \\ z_i^{-1}e^{\boldsymbol{\beta}_{trans}\boldsymbol{x}_{j,k}}, & if \quad i \neq j, \end{cases} \tag{39}$$

where $c \geq 0$ is a constant capturing the tendency to stay with the same firm "out of habit", and z_i normalizing factors. Using the Markov set-up presented in section 2.5.1 and the conditional probabilities given in (39), we obtain the following transition matrix for period k

$$\boldsymbol{P}_k = \begin{pmatrix} z_1^{-1}(1+c)e^{\boldsymbol{\beta}_{trans}\boldsymbol{x}_{1,k}} & z_1^{-1}e^{\boldsymbol{\beta}_{trans}\boldsymbol{x}_{2,k}} & \dots & z_1^{-1}e^{\boldsymbol{\beta}_{trans}\boldsymbol{x}_{N,k}} \\ z_2^{-1}e^{\boldsymbol{\beta}_{trans}\boldsymbol{x}_{1,k}} & z_2^{-1}(1+c)e^{\boldsymbol{\beta}_{trans}\boldsymbol{x}_{2,k}} & \dots & z_2^{-1}e^{\boldsymbol{\beta}_{trans}\boldsymbol{x}_{N,k}} \\ \vdots & \vdots & \ddots & \vdots \\ z_N^{-1}e^{\boldsymbol{\beta}_{trans}\boldsymbol{x}_{1,k}} & z_N^{-1}e^{\boldsymbol{\beta}_{trans}\boldsymbol{x}_{2,k}} & \dots & z_N^{-1}(1+c)e^{\boldsymbol{\beta}_{trans}\boldsymbol{x}_{N,k}} \end{pmatrix}$$

The vector $x_{j,k}$ of variables characterizing firm j should ideally contain all factors that one considers important for the customers decision making, e.g. policy returns, management fees, index variables quantifying customer satisfaction etc. However, an extensive analysis of such variables has been deemed beyond the scope of this thesis. We therefore limit $x_{j,k}$ to only concern policy returns, and introduce the following measure for this purpose (for which we let $\boldsymbol{x}_{j,k}$ be dependent on the customer's current state)

$$\boldsymbol{x}_{j,k}(i) = z_k^j - z_k^i,$$

where i is the firm that the customer currently belongs to, j the firm that the customer potentially moves to, z_k^i period k's policy return for firm i and z_k^j period k's policy return for firm j. We thus let $\boldsymbol{x}_{j,k}(i)$ be a scalar representing the difference in period k's policy returns between firm i and firm j. Within this setting, the transition probabilities $P_{ij,k}$ become

$$P_{ij,k} = \begin{cases} z_i^{-1}(1+c), & if \quad i = j, \\ z_i^{-1}e^{\beta_{trans}(z_k^j - z_k^i)}, & if \quad i \neq j. \end{cases}$$

When it comes to determining the constant c, several methods can be argumented for. Without claiming it to be the proper way, we have in this setting chosen c such that the probability of staying when all policy returns are equal amount to a desired value. Alternatively put, when all firms perform equally well we do not want the probability of leaving to be too large. In the event of equally well performance, our transition probabilities become

$$P_{ij,k} = \begin{cases} z_i^{-1}(1+c), & if \quad i = j, \\ z_i^{-1}, & if \quad i \neq j. \end{cases}$$

Hence, the normalizing factors z_i become

$$\sum_{j=1}^{N} P_{ij,k} = 1 \Leftrightarrow z_i = c + \sum_{j=1}^{N} 1 = c + N.$$

The probability of staying $P_{ii,k}$ is thus given by

$$P_{ii,k} = \frac{1+c}{N+c},$$

and if we would like this probability to equal an arbitrary $0 \leq \pi < 1$, we obtain the following expression for c

$$P_{ii,k} = \pi \Leftrightarrow \frac{1+c}{N+c} = \pi \Rightarrow c = \frac{\pi N - 1}{1-\pi}, \tag{40}$$

where we require π to be strictly smaller than one and greater than or equal to zero. Hence, a probility $\pi = 1/N$ would give a constant $c = 0$. Moreover, as π moves to one, c moves towards infinity, i.e. $c \to \infty$ as $\pi \to 1$.

The above introduced transition probabilities are given on a firm-level and not on a model point-level, meaning that their granularity does not provide deeper details than what firms are affected. Assumptions regarding what model points the moving customers belong to are therefore necessary. For this purpose we introduce a vector $\boldsymbol{\varphi}_k^j$ representing firm j's distribution of active contracts over the m different model points at the beginning of period k. Using the notation introduced in section 3.1.4, we define $\boldsymbol{\varphi}_k^j$ as

$$\boldsymbol{\varphi}_k^j = \left(\frac{n_k^{1,j}}{n_k^{tot,j}}, \frac{n_k^{2,j}}{n_k^{tot,j}}, \ldots, \frac{n_k^{m,j}}{n_k^{tot,j}} \right),$$

where $n_k^{i,j}$ is as in (37) but with super index j added for firm j, and $n_k^{tot,j} = \sum_{i=1}^{m} n_k^{i,j}$ denotes firm j's total number of contracts at the beginning of period k. All customers moving from firm j during period k are assumed to be distributed according to $\boldsymbol{\varphi}_k^j$ over the different model points. Moreover, when added to the new firm's total number of contracts, they are allocated using the same distribution. They therefore end up in the corresponding model points within the new firm. Including the transitions described above hence give the following expression for how the number of contracts in model point i evolve for firm j

$$n_k^{i,j} = \left(\sum_{s=1}^{N} P_{sj,k-1} \cdot n_{k-1}^{i,s} - q_k^i \cdot n_{k-1}^{i,j}\right) \cdot \mathbb{1}_{k-1}^i,$$

where $\mathbb{1}_{k-1}^i$ is, just like in section 3.1.4, defined by

$$\mathbb{1}_k^i = \begin{cases} 1, & if \quad 0 \le k\Delta t < T^i \\ 0, & if \quad T^i \le k\Delta t \le T. \end{cases}$$

The balance sheet items are then derived using the formulas presented in section 3.1.4, but with some minor modifications to the recursive relations. Firstly, the new $n_k^{i,j}$ taking transitions into account is used when calculating the total actuarial reserves and total allocated bonuses, i.e. in the following two expressions

$$D_k^j = \sum_{i=1}^{m} n_k^{i,j} D_k^{i,j} \quad and \quad B_k^j = \sum_{i=1}^{m} n_k^{i,j} B_k^{i,j}.$$

Furthermore, we need an expression for the payment function $\phi_k(D_k, B_k)$ explicitly stating the impact of the introduced transitions. By including the transition probabilities introduced above into equation (38), we obtain the following function for the payments made by firm j at the beginning of period k (where super index j has been added to indicate that it concerns firm j)

$$\phi_k^j(D_k^j, B_k^j) = \sum_{i=1}^{m} n_{k-1}^{i,j}(q_k^i M_k^{i,j} + (1 - P_{jj,k-1})S_k^{i,j} + (1 - q_k^i - (1 - P_{jj,k-1})E_k^{i,j}) =$$

$$= \sum_{i=1}^{m} n_{k-1}^{i,j}(D_k^{i,j} + B_k^{i,j})(q_k^i + \vartheta^j(1 - P_{jj,k-1}) + \chi(k-1)(1 - q_k^i - (1 - P_{jj,k-1}))).$$

Having included the lost capital due to transitions into the payments, it only remains to consider the gained capital. In the beginning of period k, firm j receives an expected amount l_k^j given by

$$l_k^j = \sum_{i=1}^{m} \sum_{s \in S}^{N} n_{k-1}^{i,s} S_k^{i,s} P_{sj,k-1}, \quad S = \{1, 2, ..., N | s \neq j\}.$$

We hence let the expected gained capital at the beginning of period k be the expected number of surrendered contracts multiplied by the surrender characteristics, summed up for each competitor and model point. The recursive relation for C_{k+1}^j when accounting for transitions thus becomes

$$C_{k+1}^j = (1 + R_k^j)(C_k^j + P_k^j) - \phi_{k+1}^j (D_{k+1}^j, B_{k+1}^j) + l_k^j.$$

3.3 Simulations

The simulation is based on a sample of N draws from each of the two sequences $\xi_{s,1}, ..., \xi_{s,K}$ and $\xi_{r,1}, ..., \xi_{r,K}$, where K is the number of periods for each simulation. The sample thus consists of $2(N \times K)$ independent draws from a standard normal distribution, i.e. from a $N(0, 1)$. Then, based on these numbers, the development of all involved accounts are derived using the model equations introduced above.

The number of simulated scenarios N has been set to $5,000$ (i.e. $N = 5,000$), and for the set of considered parameters, this gives approximations to all the balance sheet performance figures with relative standard errors about $5 \cdot 10^{-3}$. A relative standard error of that size has been deemed sufficient considering the scope of this thesis. In accordance with Skandia, the period length and time horizon have been set to 1 month and 10 years respectively, giving a total number of periods $K = 120$.

The simulated values are used to perform a scenario analysis as well as a sensitivity analysis. The scenario analysis is based on two scenarios considered either bad or good, and aims to demonstrate whether the model's behaviour appear realistic or not. Since the different scenarios should reflect the varying overall market performance, we use the compound average stock returns as an indicator of whether the scenario should be considered better or worse. In order to define the two scenarios, we use the empirical quantiles of the compound average stock returns and choose these fairly aggressively; the 5%-quantile is used as bad and the 95%-quantile as good. The compound average stock return $\bar{R}$ and its empirical quantiles $\alpha_{\bar{R},x\%}$ are defined in the following way

$$\bar{R}_j = \left[\prod_{i=1}^{K} (1 + R_{s,i}) \right]^{\frac{1}{K}} - 1, \tag{41}$$

where $R_i = \frac{s_i}{s_{i-1}} - 1$ is defined as in section 3.1.3, i.e. defined as the stock return of period i. For the empirical quantiles we use the definition given in [18], and thus start by ordering the sample $\bar{R}_1, \ldots, \bar{R}_N$ such that $\bar{R}_{1,N} \geq \cdots \geq \bar{R}_{N,N}$. The empirical quantiles $\alpha_{\bar{R},x\%}$ are then given by

$$\alpha_{\bar{R},x\%} = \bar{R}_{[N(1-x)]+1,N}, \quad x \in (0, 1), \tag{42}$$

where $[y]$ is the integer part of y, $[y] = sup\{N \in \mathbb{N} : N \leq y\}$ (i.e. the largest integer that is less than or equal to y). The two scenarios are thus chosen by picking the trajectories that correspond to $\alpha_{\bar{R},5\%}$ and $\alpha_{\bar{R},95\%}$, where $\alpha_{\bar{R},x\%}$ is defined as in equation (42).

The sensitivity analysis concerns the investigation of how sensitive a set of performance figures are to different values of the input parameters. Since the model requires several assumptions around input parameters, it is of great importance to understand to what extent obtained results are affected by these assumptions. The purpose of the sensitivity analysis is therefore to give an indication of what input parameters require more accurate estimations in order not to give misleading output figures. Since some parameters (e.g. product parameters and model point parameters) are assumed to be known to the insurance firms, the sensitivity analysis is limited to the parameters considered unknown or uncertain. These parameters are typically those in the capital models and those that require more sophisticated estimation methods. Furthermore, the sensitivity analysis is limited to default probability and expected value of free reserve (definitions of these figures are introduced in section 3.3.2 below), since these are the figures of main interest when performing stochastic ALM analyses [14].

3.3.1 Set of parameters

Since the focus of this paper has been dedicated to the creation of the model, estimation of all its parameters has been left to those interested in using the model. For the upcoming analysis we will therefore use parameter values that have been calibrated by others, and use product parameters that are representative for Scandinavian insurance firms. In particular, the capital market parameters have been taken from [14] (except for the displacement), and the product parameters together with management parameters have been provided by Skandia. For what concerns the displacement factor δ, it has, in consultation with Skandia, been taken from [20]. A summary of the capital market parameters is presented in table 2.

Stock price model	Interest rate model
$\mu = 8\%$	$\kappa = 0.1$
$\sigma_s = 20\%$	$\theta = 4\%$
$\rho = -0.1$	$\sigma_r = 5\%$
	$\lambda = -5\%$
	$r_0 = 3\%$
	$\delta = 2\%$

Table 2: Capital market parameters [14]

Product and management parameters are somewhat simplified and have been determined in cooperation with Skandia as well as collected from [24]. A summary of these parameters is given in table 3.

Management parameters	Product parameters
$\boldsymbol{\omega} = (0.25, 0.25, 0.25, 0.25)$	$nr\ of\ contracts = 1,000,000$
$\boldsymbol{\tau} = (12, 24, 36, 48)$	$P^i \in U(1000, 10000)$
$\gamma = 0.15$	$T^i = 1, 2, ..., 70\ years$
$\upsilon = 0.1$	$g^i = g = 1.25\%$
	$m = 70$
	$\vartheta = 0.9$

Table 3: Management and product parameters [24]

Studying table 3 shows that the only parameters differing between model points are the premiums P_k^i, which are randomly generated from a uniform distribution, and time to maturity T^i.

Furthermore, the management parameters are the same for all five competing firms except for the capital allocation parameter β, leaving the portfolio's riskiness as the single possibility of differentiation. This is indeed a simplification and could naturally be accompanied with additional factors/aspects. However, as the suggested transition model is only affected by the realized policy returns, it was deemed to be a suitable differentiation factor for demonstrating simulated transitions. Finally, the constant c capturing the tendency of staying with the same firm "out of habit" has been determined by using equation (40), where the desired probability π has been set to 95%. Table 4 shows the set of competition parameters.

Competition parameters
nr of firms$= 5$
$\beta_1 = 0.6$
$\beta_2 = 0.5$
$\beta_3 = 0.45$
$\beta_4 = 0.4$
$\beta_5 = 0.35$
$\beta_{trans} = 1$
$c = 75$

Table 4: Competition parameters

The duration of the liabilities has been determined so that it roughly matches Skandia's duration. According to [13], Swedish insurance firms typically have a duration of about 15-20 years. When it comes to Skandia, the duration is about 12-15 years, as the vast majority of the liability mass corresponds to model points with less than 15 years to maturity (model points being closer to maturity has larger claims, as accumulated premiums generally amounts to a larger sum for older policyholders compared to younger). Moreover, the duration of the asset side is chosen to be shorter than that of the liability side, in order to imitate the limited supply of bonds with longer maturities (secured bonds typically have maturities shorter than five years) [13]. Since bond investments are assumed to be periodically rebalanced, the initial distribution over different maturities is maintained throughout the simulations. Also the capital allocation is constant over time, assuming perfect rebalancing every period. This somewhat simplified capital allocation model has been chosen with the purpose to simplify replication of firms' asset portfolios, even though it comes with pretty strong assumptions regarding market perfection.

Mortality rates are assumed to be exogenously given in the model, and have been taken from the Society of Actuaries' *RP-2014 mortality tables* [2]. Ages are calculated as a function of time to maturity, where time to maturity has been assumed to occur the year policy holders of a model point turn 70 years. As no gender distribution for Skandia's client base has been found, mortality rates have been calculated under the assumption that females represent 55% of all policyholders in each model point [14].

Start values for the balance sheet items have been based on Skandia's asset side, which currently amounts to approximately SEK400 billion. The liability items have then been derived using the KKG and solvency communicated in [24]. Since Skandia is a mutual insurance company, the equity part of the balance sheet Q_k has been left out, meaning that the free reserve F_k accounts for all capital that has not been allocated to its policyholders.

The main factors affecting the run time of the model for a single scenario are the number of model points m, the number of competitors and the number of periods K. Running a single scenario with the set of parameters considered in the simulation described above (i.e. $m = 70$, number of competitors= 5 and $K = 120$) takes approximately 0.857 seconds on a laptop with a 1.7 GHz Intel Core i5 processor [5].

3.3.2 Performance figures

The set of performance figures used for the sensitivity analysis are a subset of those presented in [14], and are obtained by averaging over the entire set of generated scenarios. As a risk measure we use the path-dependent probability of default PD_k, which is defined as the estimated likelihood of having negative reserves at some period j prior to or equal to period k. That is

$$PD_k = \mathbb{P}\Big(\min_{j=1,\ldots,k} F_j < 0\Big), \tag{43}$$

where F_j is the free reserve introduced above. The path-dependency of the measure makes it coherent with the assumption that once defaulted the firm remains out of business.

In addition to the probability of default, we also estimate expected values $\mathbb{E}[C_k]$, $\mathbb{E}[D_k]$, $\mathbb{E}[B_k]$ and $\mathbb{E}[F_k]$ for the balance sheet items in each period. The expected number of total contracts in each period is also estimated.

The sensitivity of the performance figures are then determined using the approach outlined in [14]. To determine the sensitivity of measure g to an input parameter x, we look at the relative change in g to a small change in x. The relative change in g is computed as $g\prime(x)/g(x)$, where $g\prime = \frac{\partial g}{\partial x}$ is the partial derivative of g with respect to x computed using a finite difference approximation.

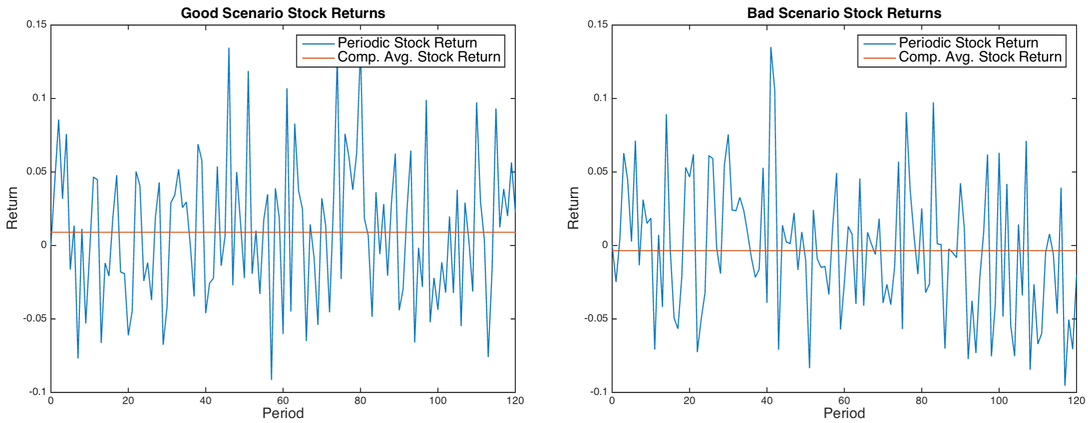
4 Results & Discussion

This section presents and discusses the numerical results originating from the simulation described above. Firstly, the scenario analysis is presented and discussed, mainly focusing on whether the model captures the most important behaviour patterns based on the different scenario characteristics. Secondly, estimation of performance measures and their sensitivity to a set of input parameters are presented. Finally, limitations of the model are discussed based on the numerical results.

4.1 Scenario Analysis

As described in section 3.3, the scenario analysis is based on two different scenarios, where the empirical quantiles of the average compound stock returns have been used as selection criteria. Figure 5 shows the periodic and compound average returns for the two chosen scenarios; good scenario is given in figure 5a, and bad scenario in figure 5b. As is indicated by the plotted stock returns, the good scenario offers a positive compound average return, whereas that of the bad scenario is negative. We will now study the model's behaviour in these two scenarios and limit the analysis to firm 1 and firm 5, as they differ the most in terms of their capital allocation model (see table 4).

[5] Run times were calculated using MATLAB's *Run and Time*-functionality.



(a) Stock returns for $\alpha_{\bar{R},95\%}$ scenario

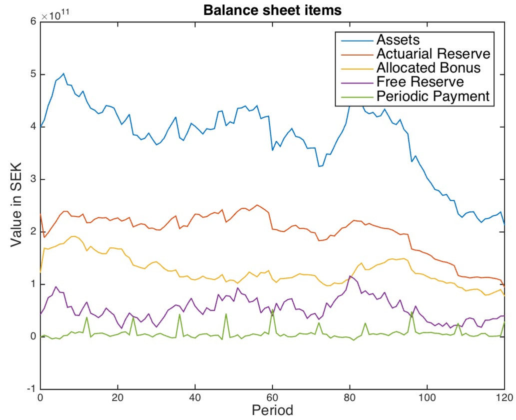
(b) Stock returns for $\alpha_{\bar{R},5\%}$ scenario

Figure 5: Stock returns for the good and bad scenarios

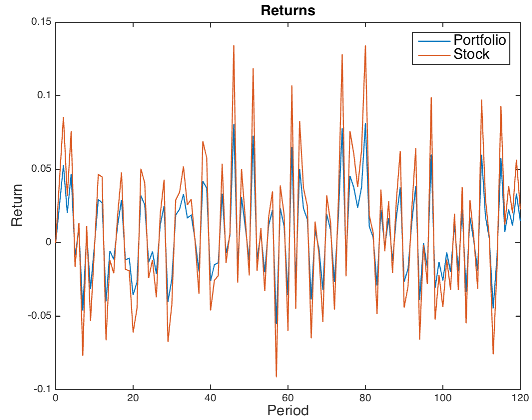
4.1.1 Good scenario

Figures 6 and 7 show balance sheet items and returns in the good scenario for firm 1 and 5 respectively. By looking at figures 6c and 7c, we see the principal difference between the two firms; firm 1 has a greater portion of its portfolio invested in stocks. The higher risk of firm 1's portfolio is also illustrated by figure 6b, which clearly shows, compared to firm 5, that the portfolio returns are more closely connected to the stock returns. The expected effect of this risk difference is an increased volatility of the portfolio value C_k, and thereby also a greater volatility embedded in the free reserve F_k and allocated bonus B_k. The actuarial reserve D_k should not be equally impacted, as its value fluctuates with changing interest rates. However, since the policy returns determine the probabilities for customer transitions, there is still an implicit relation between the size of the actuarial reserve and the free reserve, potentially causing a slightly higher volatility in D_k for firm 1.

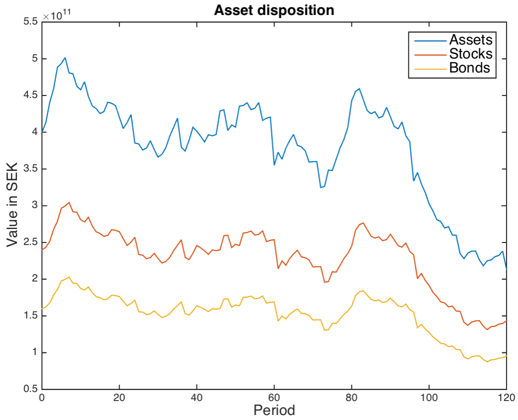
Studying the evolvement of both firms' balance sheet items in figures 6a and 7a demonstrates more significant fluctuations in the values for firm 1, indicating that the model seems to capture these effects. We can also see that the policy returns z_k, plotted in 6d and 7d, show higher peaks and lower dips for firm 1. This is obviously a direct result of the same behaviour being identified in the two firms' value of F_k, as the policy returns are determined using the reserve rate γ. Even though there are signs of varying patterns in the behaviour of the two firms, it can be seen that most behaviour is closely related. Most of the plotted trajectories share similar shapes, and differs only in the order of magnitude of their fluctuations. This is due to the capital allocation parameter β being the only factor differentiating the two firms, whereas all other parameters are identical. It is therefore expected that the two firms share similar behaviour patterns in the plotted variables. However, there are still periods of the simulations for which the firms' trajectories show sign of deviation. For example, if one studies from period 80 and onwards, firm 1 experiences a significant decline that is not reflected in the evolvement of firm 5's balance sheet items. The different risk profiles of the firms' portfolios hence gives rise to a behaviour that is not covered by the reasoning above, leading us to study the effects of customer transitions.



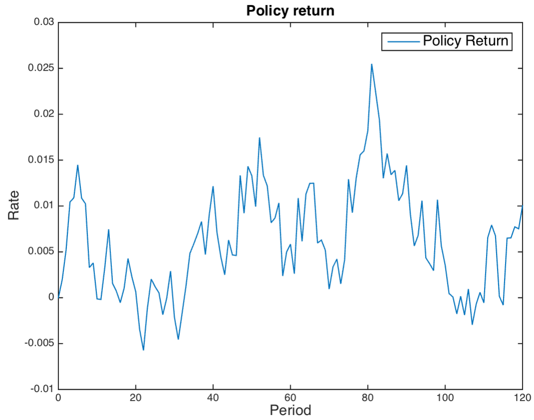
(a) Evolvement of C_k, D_k, B_k, F_k and periodic payments



(b) Periodic stock and portfolio returns

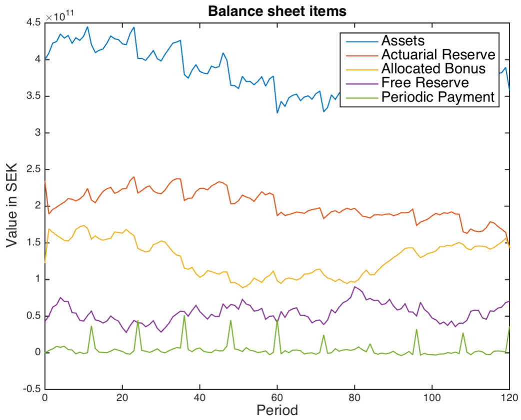


(c) Stock and bond investments' proportions of total assets, i.e. βC_k and $(1 - \beta) C_k$

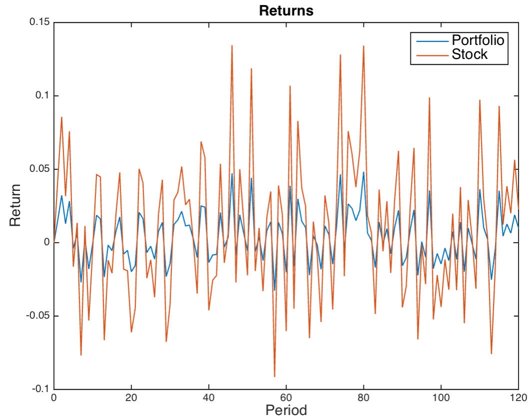


(d) Allocated policy returns z_k

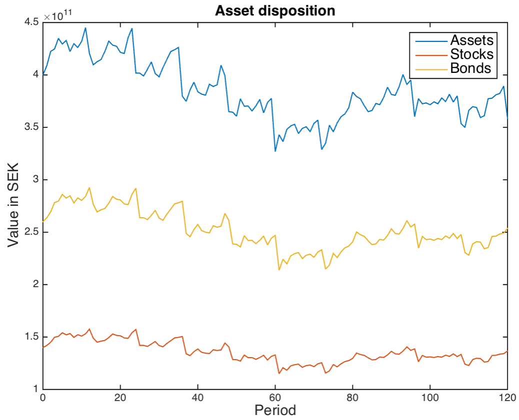
Figure 6: Firm 1 - Balance sheet items and returns in good scenario



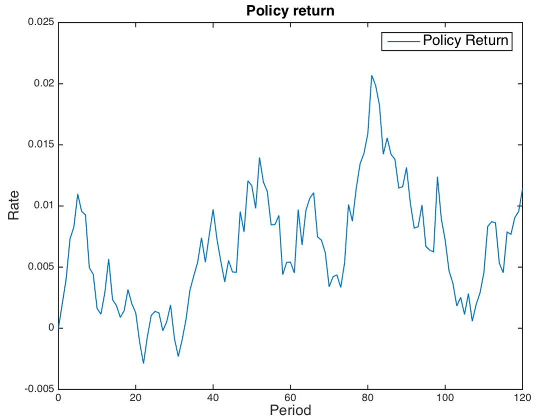
(a) Evolvment of C_k, D_k, B_k, F_k and periodic payments



(b) Periodic stock and portfolio returns



(c) Stock and bond investments' proportions of total assets, i.e. βC_k and $(1-\beta)C_k$



(d) Allocated policy returns z_k

Figure 7: Firm 5 - Balance sheet items and returns in good scenario

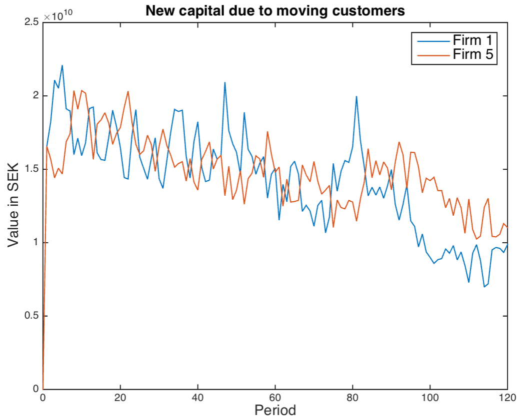
As the transition probabilities modelling customers' movements are directly determined by the firms' allocated policy returns, it is reasonable to expect that more volatile policy returns would yield greater fluctuations in the amount of gained, and lost, capital coming from moving customers. By looking at the two upper plots in figure 8, one can see signs of this behaviour; firm 1's ability to attract new capital changes significantly throughout the simulated period. Obviously, similar fluctuations are also seen in figures 8c and 8d, showing the number of customers that are gained or lost. However, the plots in figure 8 do not only indicate significant variation in customer transitions for firm 1, but firm 5 also experiences a similar situation despite having a less volatile policy return. This is explained by the way transition probabilities are assigned; the greater the difference is between the policy returns, the higher becomes the assigned probability. Thus, the combination of firm 1 having more volatile returns (i.e. lower dips and higher peaks than the other firms) and firm 5 having more stable policy returns (i.e. less sensitive to stock movements than the other firms), makes the policy return difference between these specific firms larger than between the rest. Probabilities for moving between firm 1 and firm 5 are therefore in most cases assigned higher values, implying larger and more fluctuating movements. This may be considered unrealistic, as it does not compensate firm 5 for having a more stable customer return policy, and implies that it experiences fluctuations that are similar to those faced by less stable firms. These shortcomings are further discussed in section 4.3 below.

In regards to the mentioning of deviating trajectories above, one can see in figure 8 that firm 1 persistently faces greater customer losses and lower customer gains than firm 5 during that period. This in turn makes firm 1 suffer from larger capital movements, explaining the decline in its balance sheet items. Furthermore, the reason for customer movements not being as interchanging between the firms during this period, is found in the realized policy returns. By looking at figures 6d and 7d, one sees that firm 5 constantly offers its customer a slightly higher return for this period of time.

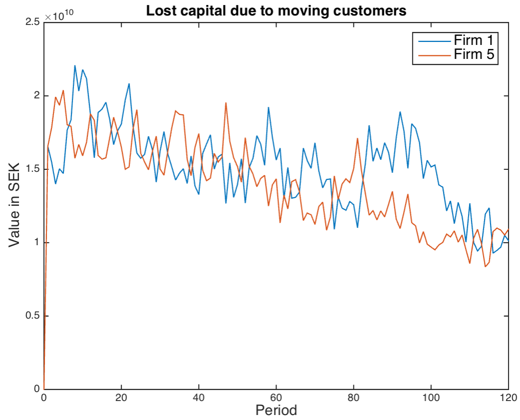
4.1.2 Bad scenario

As is shown in figure 5, the stock movements in the good and bad scenario are not that different. Apart from the compound average returns, both scenarios give rise to similar stock behaviour. It is therefore reasonable to expect that most of the effects identified for the good scenario above, will hold for the bad scenario as well. To begin with, figures 9 and 10 show greater fluctuations for firm 1 than for firm 5, and this can be explained using the same reasoning as for the good scenario. One can also see indications of firm 1's higher portfolio risk being more problematic, now the market performance is weaker. Studying figure 9d shows that there are longer periods of negative policy returns, which in turn makes it harder to keep old and attract new customers, as is indicated in figure 11.

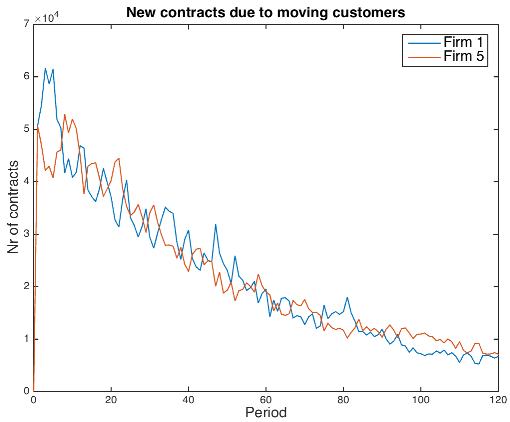
Even though the less attractive market environment is more clearly reflected in firm 1's performance, there is a difference between the scenarios that both firms share; the free reserve F_k eventually becomes low and stays at that level. In the case of firm 1, this even leads to a negative free reserve at the end of the simulated period. Given the more demanding market situation, lower levels (and even negative if portfolio risk is high) is expected, and therefore nicely captured by the model. However, this only holds if the allocated bonus B_k approaches equally low levels, as the relation between B_k and F_k is largely affecting the policy return mechanism. As is described in section 3.1.3, the policy returns z_k are determined as a portion of the difference between reserve rate γ_k, and the target reserve rate γ. The fact that only a portion of the difference is allocated works as a smoothing, making the allocation mechanism comply



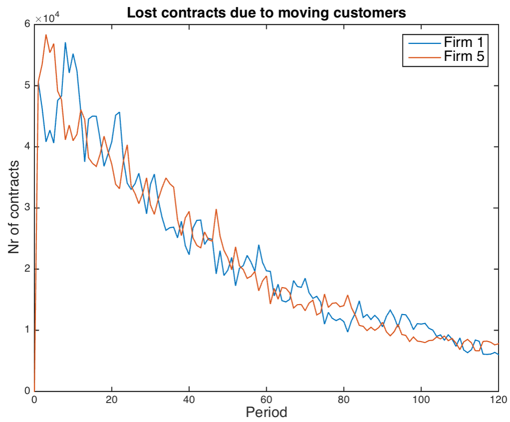
(a) Gained capital due to customers' movements



(b) Lost capital due to customers' movements

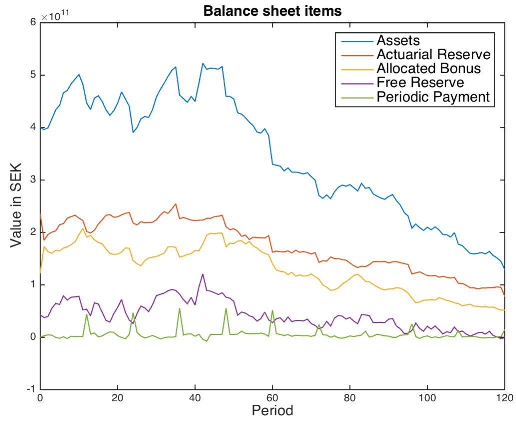


(c) Number of contracts gained due to customers' movements

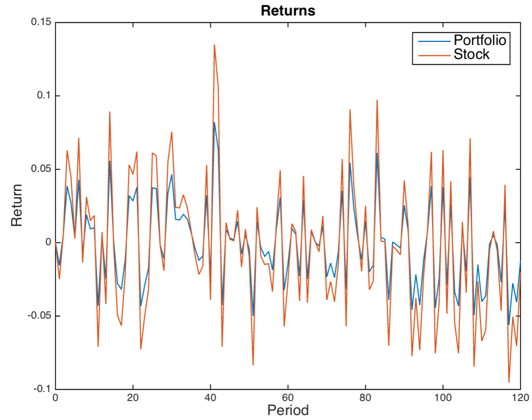


(d) Number of contracts lost due to customers' movements

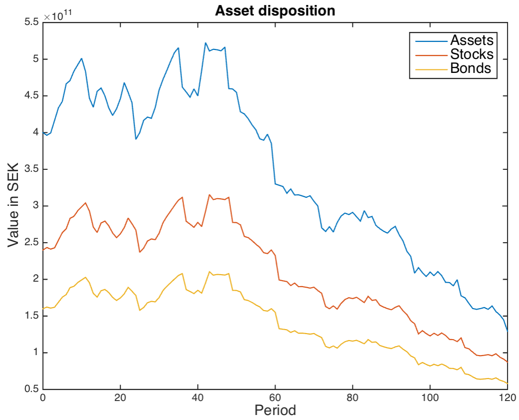
Figure 8: Effect of customer transitions on firms' capital and number of contracts in good scenario



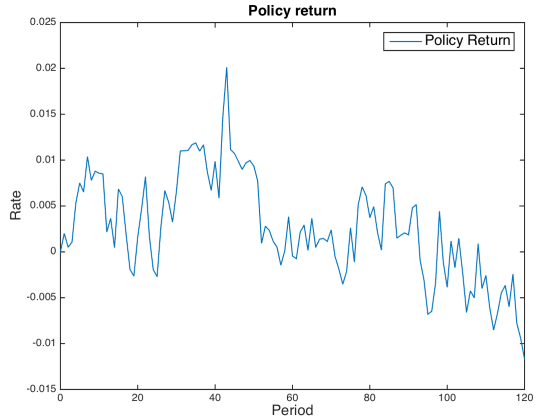
(a) Evolvment of C_k, D_k, B_k, F_k and periodic payments



(b) Periodic stock and portfolio returns



(c) Stock and bond investments' proportions of total assets, i.e. βC_k and $(1-\beta)C_k$



(d) Allocated policy returns z_k

Figure 9: Firm 1 - Balance sheet items and returns in bad scenario

with the average interest rate principle [14]. Based on the simulation carried out in this thesis, one may question if the smoothing parameter υ has been set too low, as it is not desired to produce negative free reserves when the allocated bonus still takes a relatively much higher value.

For what concerns the customer transitions, these are similar to the behaviour identified in the good scenario. The interchanging effect between firm 1 and firm 5 is present also in the bad scenario, and is caused by the mechanism for assigning probabilities as outlined above. However, due to the more extended periods of negative policy returns for firm 1, there are also more extended periods for which firm 1 gets outperformed by firm 5 in terms of gained customers. Also, as both firms' free reserves reached about equally low values towards the end of the simulated period, and thereby realized low policy returns, there are tighter fluctuations in the customer transitions for that period than what was the case in the good scenario.

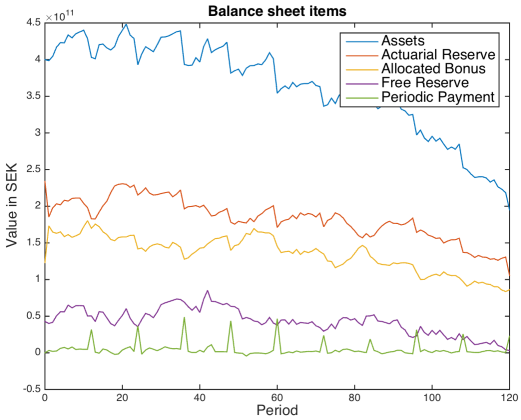
4.2 Sensitivity Analysis

As the scenario analysis only shows the behaviour of the model for two specific trajectories, it is of interest to also study the overall behaviour using the whole generated sample. Figure 12 presents estimations of the performance figures introduced in section 3.3.2, and these are generated based on the sample described in section 3.3. To begin with, in figures 12a and 12c, $\mathbb{E}[C_k]$ and $\mathbb{E}[D_k]$ are plotted for each simulated k. As is shown in the plots, both items face an initial increase due to the period's premium payments, and then show a downward trend caused by mortality and policyholders reaching maturity. The latter gives rise to the clear periodic declines occurring every twelve month. Furthermore, as the distribution of policyholders over the different model points, in this case over the different maturities, are more concentrated around more recent maturities, one can see that the periodic declines eventually becomes smaller. The periodic increases are, as in the case of the initial upturn, caused by the periodic premiums being paid.

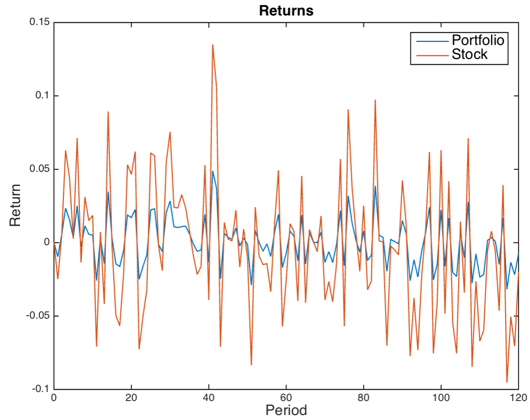
Moving on to figure 12d, one sees that similar patterns exist for the allocated bonus as well. Also in this case, the periodic declines are explained by maturing contracts. However, the periodic increases appear to behave differently. Comparing figure 12d with the plots for C_k and D_k, shows that the periodic increases fluctuate slightly more for B_k than for the two other items. This is due to the bonus allocation not being directly connected to the premiums, but determined by the size of the free reserve F_k. Therefore, as the premiums are deterministic, whereas the free reserve fluctuates, B_k is characterized by more varying periodic increases.

Figure 12b shows the evolvement of the free reserve. As was mentioned in the previous paragraph, $\mathbb{E}[C_k]$ is slightly more fluctuating than the other balance sheet items. This follows from the fact that the bonus allocation mechanism calibrates the free reserve, so that the reserve rate constantly approaches the target reserve rate. However, the downward overall trend is shared with the other balance sheet accounts. This suggests that the bonus allocation mechanism manages, on average, to keep the reserve rate close to the target rate, as the similar trends indicate that a relatively constant relation is maintained between F_k and $(D_k + B_k)$. Maturing contracts lead to a reduction of D_k and of B_k, while not directly affecting the free reserve (the free reserve has not yet been allocated, and thereby still belongs to the firm). Hence, if the bonus allocation mechanism would fail to start distributing greater portions after a large model point matures, the trend of the free reserve would not follow that of the others.

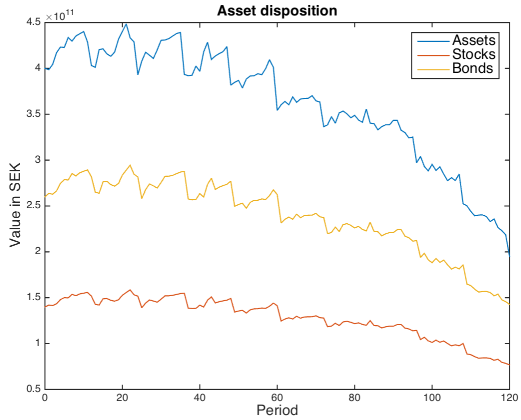
If we look at the spread between the five different firms in figure 12, we see that firm 1, having a riskier portfolio, on average outperforms the other firms during the first 100 periods. For the



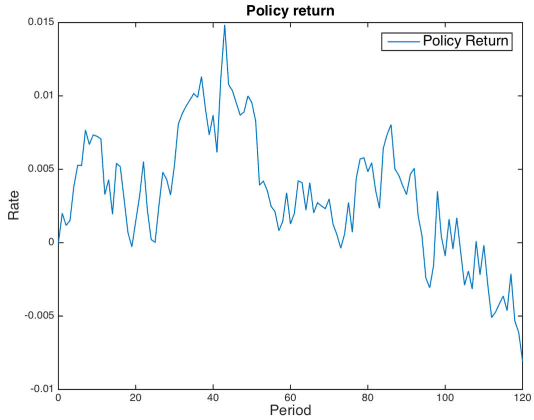
(a) Evolvment of C_k, D_k, B_k, F_k and periodic payments



(b) Periodic stock and portfolio returns

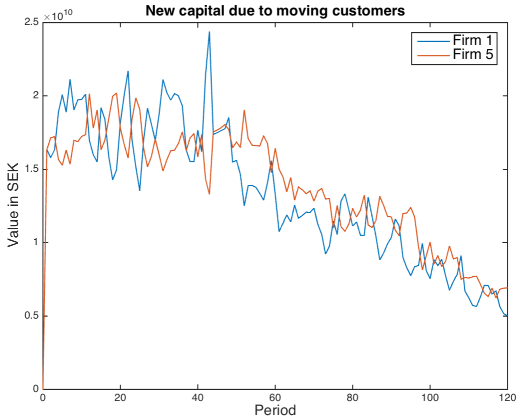


(c) Stock and bond investments' proportions of total assets, i.e. βC_k and $(1-\beta)C_k$

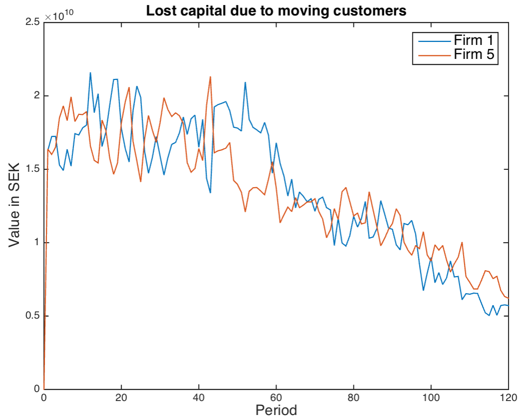


(d) Allocated policy returns z_k

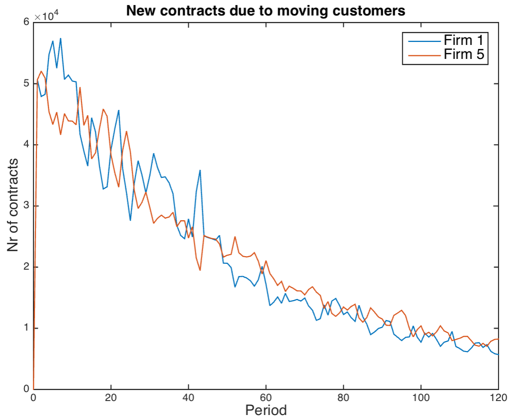
Figure 10: Firm 5 - Balance sheet items and returns in bad scenario



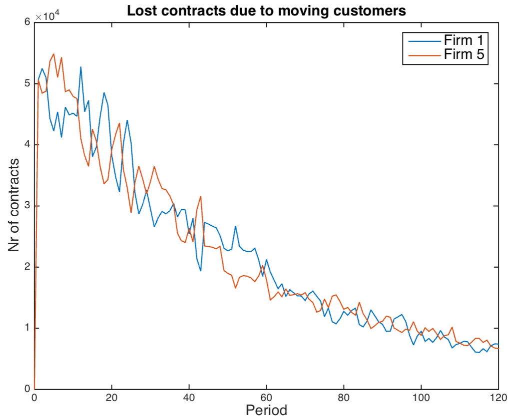
(a) Gained capital due to customers' movements



(b) Lost capital due to customers' movements

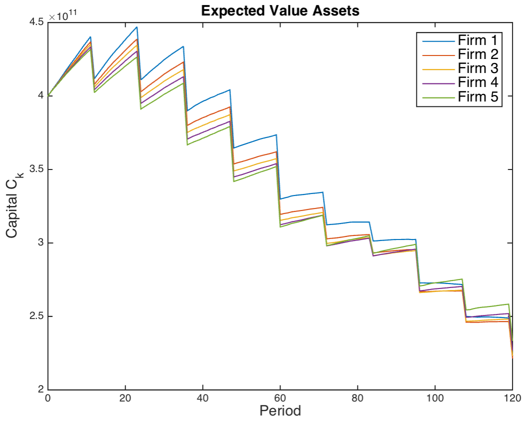


(c) Number of contracts gained due to customers' movements

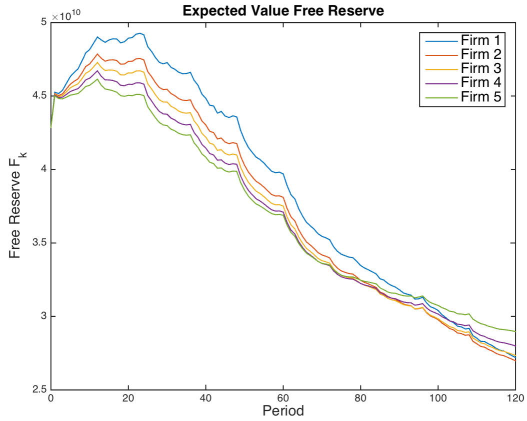


(d) Number of contracts lost due to customers' movements

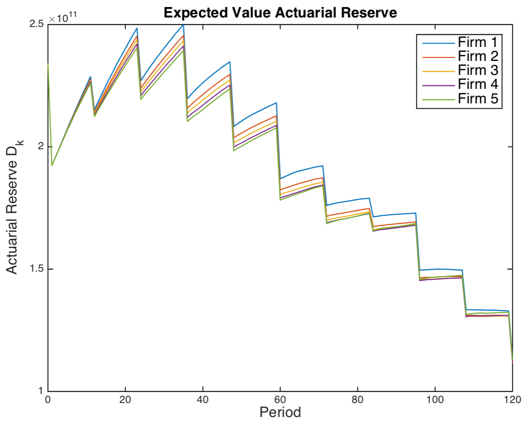
Figure 11: Effect of customer transitions on firms' capital and number of contracts in bad scenario



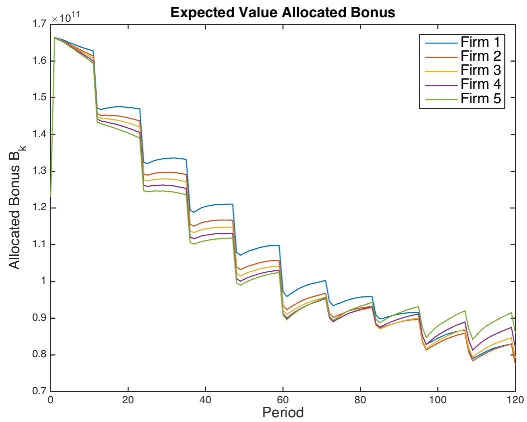
(a) Expected value $\mathbb{E}[C_k]$ of total assets



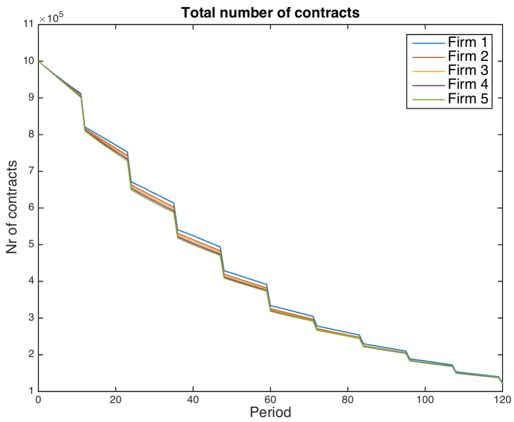
(b) Expected value $\mathbb{E}[F_k]$ of free reserve



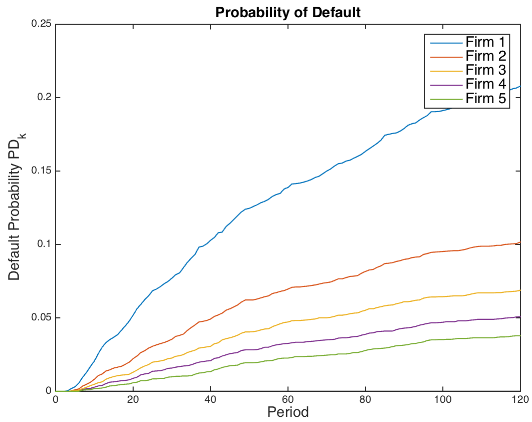
(c) Expected value $\mathbb{E}[D_k]$ of actuarial reserve



(d) Expected value $\mathbb{E}[B_k]$ of allocated bonus



(e) Expected value of total number of contracts



(f) Probability of default PD_k

Figure 12: Performance figure estimates for all firms

	$\mu = 8\%$	$\sigma_s = 20\%$	$\rho = -0.1$
$\mathbb{E}[F_K]$	0.101	-0.002	-0.0002
PD_K	-0.197	0.4376	0.006

Table 5: Stock model parameters x and corresponding quotients $\frac{\partial g}{\partial x}/g$ for $g \in \{\mathbb{E}[F_K], PD_K\}$.

	$\kappa = 0.1$	$\theta = 4\%$	$\sigma_r = 5\%$	$\lambda = -5\%$	$r_0 = 3\%$	$\delta = 3\%$
$\mathbb{E}[F_K]$	0.003	0.0549	-0.0012	-0.0004	0.04	-0.066
PD_K	-0.0133	-0.125	0.0963	0.001	-0.1379	0.1405

Table 6: Interest rate model parameters x and corresponding quotients $\frac{\partial g}{\partial x}/g$ for $g \in \{\mathbb{E}[F_K], PD_K\}$.

subsequent periods, however, firms with less risky portfolios catch up and eventually outperforms firm 1. Alternatively put, the riskier strategy used by firm 1 gives, on average, an advantage in the short run, but at the cost of being less sustainable in the long run. The reason for this is found by looking at the default probabilities in figure 12f. It is shown that the riskier strategy of firm 1 makes default probabilities increase significantly with time. As default probabilities increase, a larger portion of firm 1's scenarios lead to negative free reserves, causing the average number of lost customers to increase. Thanks to this behaviour, the model manages to capture both the upside, and downside, of utilizing a riskier investment strategy.

Finally, figure 12 shows the evolvement of the expected number of contracts in each firm. Just like in the other figures, we can see that firm 1 outperforms the other firms during the first 100 periods, and then gets outrun by the less risky firms. One can also see the steep declines related to maturing contracts, recurring every 12th period. Furthermore, the more flat declines in between the maturing contracts are caused by mortality. As mortality rates are higher for model points that are closer to maturity, one can see that the slope of these declines gradually becomes flatter.

In order to investigate the sensitivity of $\mathbb{E}[B_K]$ and PD_K to changes in input parameters, we use approximations of their partial derivatives as outlined in section 3.3.2. Moreover, in order to allow for direct comparison, the simulations for varying parameters are based on the very same sequence of normally distributed random numbers. The analysis is limited to look at firm 3, as its asset allocation is considered most representative for Skandia, and to only concern capital market parameters, as the other parameters are assumed to be known to the insurance firms. Computed sensitivities are shown in tables 5 and 6.

Studying tables 5 and 6 shows that the default probability PD_K increases in σ_s, ρ, σ_r, λ and δ, and decreases in μ, κ, θ and r_0. Furthermore, the expected free reserve $\mathbb{E}[F_K]$ increases in μ, κ, θ, and r_0, whereas it decreases in σ_s, ρ, σ_r, λ and δ. Hence, the expected behaviour that PD_K and $\mathbb{E}[F_K]$ increase, and decrease, in opposite parameter sets seems to be captured by the model. We can see that PD_K increases in parameters giving higher risk (e.g. σ_s and σ_r), and decreases in parameters yielding higher expected returns (e.g. μ and θ), whereas the opposite holds for $\mathbb{E}[F_K]$.

What concerns the sensitivity to the various input parameters, the values in table 5 suggest that μ and σ_s should be estimated with particular care, as especially default probabilities are sensitive to changes in these parameters. The correlation coefficient ρ is, however, of moderate importance. Moving on to table 6 indicates that θ, r_0, δ, and to some extent σ_r, are of greater importance. As was the case for ρ, λ and κ seem to be of lower importance. It should be mentioned, however, that the presented conclusions are only true for the parameter values used in this simulation. The results may take a different turn if one or several parameters are changed. As an example, it is reasonable to believe that the importance of the stock model parameters may shift towards the short rate parameters, in case the capital allocation is changed.

4.3 Limitations & further research

Based on the results presented in the scenario and sensitivity analysis above, the model appears to capture desired behaviour patterns and behave in accordance with set expectations. Both varying market characteristics, represented by the two simulated scenarios, and varying input parameters were correctly reflected in the model's output. For example, the more unfavorable market conditions characterizing the bad scenario had a larger impact on firm 1, thereby catching the desired effects of a riskier investment strategy. What's more, the different risk profiles of firm 1 and firm 5 were also captured in the sensitivity analysis, where firm 1 on average outperformed the other firms during the first seven years of the simulated period. During the subsequent periods, however, firms having lower risk (in particular firm 5) eventually outperformed firm 1, suggesting that the riskier strategy is correctly modelled as less sustainable in the long run. The sensitivity analysis also demonstrated desired behaviour when it comes to the model's reaction to changes in some of the input parameters. For example, changes in parameters that increased the overall risk led to increased default probabilities, and decreased expected free reserves.

However, as the scenario analysis only consists of two simulated scenarios, it does not provide sufficient output for drawing more detailed conclusions about the model's output. It should rather be seen as an indication of whether the model's behaviour is consistent with the development one could reasonably expect based on the input. Moreover, as mentioned above, the results of the sensitivity analysis only holds for the set of parameters chosen in this paper. Using the model with a different set of parameters therefore requires additional analysis, before any reliable assumptions can be made regarding its performance.

Even though the overall performance of the model is in line with expected behaviour, a number of shortcomings were detected. To begin with, the allocation mechanism for policy returns appeared to be too smoothened. During the bad scenario in the scenario analysis, the firms experienced situations where the free reserve attained very low values (that of firm 1 even turned negative). This happened despite the fact that both firms had relatively high values of their allocated bonuses. If reserves are reaching low values in real-life situations, this would be immediately compensated for. These findings therefore suggest the need of incorporating modelling of capital requirements. Having a bonus allocation mechanism that complies with the average interest rate principle, and no restrictions in form of modelled capital requirements, risk to produce unrealistic outputs.

Another detected limitation in the model is the customer transition model being too volatile. The fact that it only takes the period's policy returns into account when assigning transition probabilities makes customer movements unrealistically volatile. It is not reasonable that policyholders change frequently between different firms, just because the communicated policy return is higher for one period. Furthermore, this way of assigning transition probabilities also punishes firms with more stable returns, which is a characteristic that customers presumably would find attractive. Instead of assuming that customers exclusively look for firms offering the highest momentary returns, the model should benefit from rewarding historically stable returns as well.

The above discussed shortcomings can be summarized in a number of more concrete suggestions for further research. To start with, the customer transition model would benefit from incorporating a historical path dependency. This way, transition probabilities would not only reflect the current period's return, but also historical performance for the different firms. What's more, historical dependence would also enable incorporation of other factors potentially affecting customer transitions. Such a factor could be the tendency of rather choosing a known firm (i.e. a firm that the customer already has a relation with), than a firm that is entirely new to the person.

Another aspect of transition model improvements is to make use of the response parameters β_{trans}, by statistically deriving a more realistic relation between policy returns and customer attraction. This would also add value in case additional factors affecting customer transitions are incorporated into the model, as the response parameter vector $\boldsymbol{\beta}_{trans}$ then serves as statistically determined weights for how responsive customers are to the different factors. Examples of such additional factors would be customer satisfaction, customer awareness, and other attributes certainly playing a role in people's choice of insurance companies. Working out a more advanced customer retention model does not only contribute to more accurate capital requirement modelling, but also opens up for more sophisticated strategy and pricing modelling. See for example [9] for dynamic pricing decisions in oligopolistic markets.

Finally, not only the transition model would benefit from additional research. For example the capital allocation model could be further elaborated on. As is suggested in [14], an interesting extension would be to further optimize the risk-return profile of the modelled firms. This could be done by introducing a more complex capital allocation model, which dynamically matches the duration of the bond investments with that of the liabilities.

5 Summary & Conclusion

Real-life scenarios are complex to model, and modelling the business of insurance firms is no exception. There is a constant trade-off between simplicity and realism, making it hard to combine both in the same model. In this thesis, a discrete time stochastic model framework, for the asset-liability management of insurance products, has been proposed. The model is based on a two-factor stochastic capital market model, and supports several product characteristics that are typical for Scandinavian insurance firms. Moreover, the model also incorporates a bonus allocation that depends on the simulated firm's reserve rate, enabling simulation of a bonus declaration that is similar those used by Scandinavian firms. As an additional dimension, a first approach to endogenously model customer transitions is proposed, where realized policy returns are used for assigning transition probabilities.

Through a number of simulations and examples, it has been demonstrated that the model captures the most relevant behaviour patterns. In order to show the effect of customer transitions, simulations have been performed using five different firms, where each firm differed in terms of capital allocation. As the different capital allocations entailed varying risk profiles for the simulated firms, several desired properties could be detected in a scenario and sensitivity analysis. During less favorable market conditions, the bonus allocation model proved not to be sufficiently responsive, as scenarios of bankruptcy occurred despite the fact reserves could be increased. This demonstrated the need of incorporating modelling of capital requirements into the model, as bonus allocation mechanisms following the average interest rate principle lack the ability to quickly compensate for significant market declines.

Endogenously determining customer transition probabilities, using periodic policy returns, proved to capture the most important effects of firms' performance on customers' movements. However, due to the lack of path-dependency, historical performance is not accounted for. This in turn may lead to volatile customer transitions, and thereby somewhat unrealistic results. An extension of the transition model was therefore suggested, so that historical performance, and how stable returns have been, are accounted for when assigning transition probabilities.

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