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Hedging Foreign Exchange Exposure in Private Equity Using Financial Derivatives

FILIP KWETCZER

CARL ÅKERLIND

KTH ROYAL INSTITUTE OF TECHNOLOGY
SCHOOL OF ENGINEERING SCIENCES

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FILIP KWETCZER
CARL ÅKERLIND

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Supervisor at KTH: Fredrik Viklund
Examiner at KTH: Fredrik Viklund

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Royal Institute of Technology
School of Engineering Sciences
KTH SCI
SE-100 44 Stockholm, Sweden
URL: www.kth.se/sci

Abstract

KTH Royal Institute of Technology
School of Engineering Sciences
Department of Mathematics

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Master of Science in Applied Mathematics
with Specialisation in Financial Mathematics

Hedging Foreign Exchange Exposure in Private Equity Using Financial Derivatives

by Carl Åkerlind and Filip Kwetczer

This thesis sets out to examine if and how private equity funds should hedge foreign exchange exposure. To our knowledge the field of foreign exchange hedging within private equity, from the private equity firms' point of view, is vastly unexplored scientifically. The subject is important since foreign exchange risk has a larger impact on private equity returns now than historically due to increased competition, cross-boarder investments and foreign exchange volatility. In order to answer the research question a simulation model is constructed and implemented under different scenarios. Foreign exchange rates are simulated and theoretical private equity funds are investigated and compared under different performance measures. The underlying mathematical theory originates from the work of Black and Scholes.

The main result of this thesis is that private equity funds cannot achieve a higher internal rate of return on average through hedging of foreign exchange exposure independent of the slope of the foreign exchange forward curve. However, hedging strategies yielding the same mean internal rate of return but performing better in terms of performance measures accounting for volatility of returns have been found. Furthermore, we found that the conclusions are independent of whether the current or forward foreign exchange rate is a better approximation for the future foreign exchange rate.

Keywords: Private Equity, Foreign Exchange Exposure, Hedging, Black-Scholes Model, Financial Derivatives

Sammanfattning

KUNGLIGA TEKNISKA HÖGSKOLAN
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Institutionen för Matematik

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med Specialisering i Finansiell Matematik

Hedging av Valutaexponering inom Private Equity med Finansiella Derivat

av Carl ÅKERLIND och Filip KWETCZER

Uppsatsens syfte är att undersöka om och i sådana fall hur private equity fonder ska hedgea valutaexponering. Ämnet är såvitt vi vet ej tidigare undersökt inom vetenskaplig forskning ur private equity företagens synvinkel. Ämnet är viktigt eftersom valutarisk har fått en större påverkan på private equity företagens avkastning jämfört med hur det har sett ut historiskt på grund av högre konkurrens, mer internationella investeringar samt ökad volatilitet i valutakurser. En simuleringsmodell har konstruerats och implementerats under olika scenarier för att besvara forskningsfrågan. Valutakurser simuleras och teoretiska private equity fonder undersöks samt jämförs utefter olika nyckeltal. Den underliggande matematiska modelleringen härstammar från Black och Scholes forskning.

Uppsatsens viktigaste resultat är att private equity fonder inte kan uppnå en högre avkastning genom att hedgea valutaexponering oavsett lutningen av den förväntade valutautvecklingskurvan. Vi har dock funnit att det existerar hedgingstrategier som ger samma avkastning med lägre volatilitet. Vidare är slutsatserna oberoende av om nuvarande eller förväntad framtida valutakurs är den bästa approximationen av den framtida valutakursen.

Nyckelord: Private Equity, Valutaexponering, Hedging, Black-Scholes Modell, Finansiella Derivat

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Carl Åkerlind
Filip Kwetczer
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List of Notation

Mathematical Notation

E^Q	Expectation taken under the probability measure Q
$\Phi_X(x)$	Cumulative normal distribution function of X at point x. Also referred to as $N(x)$
$\varphi_X(x)$	Normal probability density function of X at point x
$N(\mu, \sigma^2)$	Normal distribution with expected value μ and variance σ^2
$\Pi(t; X)$	Price of a contingent claim X at time t
P	The observable probability measure
Q	The risk neutral probability measure, equivalent martingale measure to P
T	Maturity time
$t_\nu(\mu, \sigma^2)$	Student's t distribution with ν degrees of freedom, location parameter μ and scale parameter σ
t_ν	Standard Student's t_ν distribution, i.e. a Student's t_ν distribution with location parameter 0 and scale parameter 1
W	Q-Wiener process (Q-Brownian motion)
$\overline{W}$	P-Wiener process (P-Brownian motion)
X	An upper case letter, e.g. X, Y or W denotes a random variable
X_t	By a lower case t time dependence is indicated, $X_t = X(t)$

Other Notation

ATS	Affine term structure
ES	Expected shortfall
FX	Foreign exchange
HW	Hull-White
IRR	Internal rate of return
LSE	Least squares estimation
NAV	Net asset value
PC	Portfolio company
PE	Private equity
RW	Random Walk
SDE	Stochastic differential equation
SEK/EUR	Units of EUR per 1 unit of SEK
SR	Sharpe ratio
VaR	Value-at-Risk

Part I

Preliminaries

1 Introduction

Investments in private equity have increased over time since the start of the industry in the 1950s. As the industry is growing and the world is getting more and more globalised an increased amount of cross-border investments are taking place. This has led to private equity firms being more exposed to foreign exchange risk now than before. Moreover, due to increased competition and lower global growth the returns of the private equity firms have decreased making buffers for foreign exchange fluctuations smaller. At the same time foreign exchange volatility has increased as a result of higher political risk and more foreign exchange interventions from central banks. Therefore, hedging foreign exchange exposure within private equity might be more important than ever before. Historically, Nordic based private equity firms have not hedged foreign exchange risk, a strategy that might need to be reconsidered.

This thesis sets out to examine if and how private equity funds should hedge their foreign exchange exposure. To our knowledge the field of foreign exchange hedging within private equity, from the private equity firms' point of view, is vastly unexplored scientifically. In order to answer the research question a simulation model is constructed and implemented under different scenarios. Foreign exchange rates are simulated and theoretical private equity funds are investigated and compared under different performance measures. The underlying mathematical theory originates from the Black-Scholes framework. We hope that this work contributes to the reader's enhanced understanding and other research through defining, structuring and modelling a private equity foreign exchange hedging universe.

The main result of this thesis is that private equity funds cannot achieve a higher mean internal rate of return through hedging of foreign exchange exposure independent of which currency the fund is denominated in. This result is expected since it is in line with no arbitrage theory stating that higher risk should be rewarded with higher mean return. However, hedging strategies yielding the same mean internal rate of return but performing better in terms of performance measures accounting for volatility of returns have been found. Furthermore, we found that the conclusions are independent of whether the current or forward foreign exchange rate is a better approximation for the future foreign exchange rate.

The thesis is divided into five parts. Part I presents the preliminaries and sets the foundation, including previous research, background on private equity and basics of financial derivatives. Part II consists of the base model assuming constant risk free interest rates. First the mathematical framework is presented and applied. Then the methodology is described and finally the results are presented and discussed. Part III extends the model by challenging the assumption of constant risk free interest rates and follows the same structure as Part II. Part IV extends the model further by investigating differences arising from the slope of the foreign exchange forward curve. Finally, Part V concludes the findings and the thesis.

2 Previous Literature

To our knowledge there is no previous scientific research investigating the field of foreign exchange (FX) hedging within private equity (PE), from the PE firms' point of view. This section starts with describing related previous literature. First, previous research from a financial perspective is presented. Second, mathematical research is presented. Finally, related previous research is applied to a PE setting.

2.1 Financial Perspective

There is literature both in favour of and opposed to hedging in general. According to the famous view of Modigliani and Miller, which is part of the foundation of modern finance, hedging is simply a financial transaction and does not affect the value of the operating assets of a company (Modigliani and Miller (1958) [21]). For that reason, hedging does not affect the value of a firm.

Black (1990) [3] reaches the same conclusion as Modigliani and Miller, and argues that investors do not want to fully hedge their FX risk, because Siegel's paradox makes them want a positive amount of FX risk. Furthermore, Black explains that every investor will hold the same mixture of market risk and FX risk, under the assumption that the average risk tolerance is the same across countries.

In line with Modigliani and Miller, as well as Black (1990) [3], Morey and Simpson (2001) [22] is in favour of an unhedged strategy of FX risk. The authors find that an unhedged strategy of FX risk outperforms a hedged strategy, for every sample and time horizon they have investigated. In total, five different hedging strategies were used: (i) to always hedge, (ii) to never hedge, (iii) to hedge when the forward rate is at a premium, (iv) to hedge only when the premium is large, and (v) a strategy based upon relative purchasing power parity. A data set from five different countries was used.

Despite that there is evidence that unhedged strategies are beneficial, Perold and Schulman (1988) [23] argues that investors, when formulating long run investment policies, should hedge the FX risk of their portfolios. The authors find that, on average, FX hedging gives the investors a substantial risk reduction without any loss in expected return. Moreover, the key to their argument is that FX hedging yields zero expected return, in the long run. Glen and Jorion (1993) [14] also find that hedging of FX risk could be beneficial. The authors find that an unconditional strategy including FX hedging could be beneficial, and that conditional hedging strategies, both in sample and out of sample, significantly improve the risk-return trade-off of global portfolios as well as outperform the unconditional hedging strategies.

Worthwhile to note is that both the articles in favour of FX hedging, Perold and Schulman (1988) [23] as well as Glen and Jorion (1993) [14], have assumed no currency risk premium. However, De Santis and Gérard (1998) [7] finds that the premium of bearing FX risk often represents a significant fraction of the total premium.

An article discussing if the investors in the company or the company itself should manage the risk is Froot, Scharfstein and Stein (1994) [10]. The authors conclude that the investors should manage the risk on their own.

Froot, Scharfstein and Stein (1993) [9] explores different strategies to FX hedging involving both linear and nonlinear instruments. They show that if the indexed sensitivity of foreign revenues to the FX rate and the indexed sensitivity of foreign investment costs to the FX rate are equal then futures contracts alone can provide value maximizing hedges. In all other cases, options might be required to obtain the value maximizing hedge. In other words, when there are state dependent financing opportunities nonlinear instruments will be needed in order to obtain the value maximizing hedges.

2.2 Mathematical Perspective and Foundation

Øksendal (1998) [25] covers the introduction to stochastic calculus and its applications, which is needed as a mathematical starting point in this thesis. For example, Itô's famous lemma discovered in the 1940s (Itô (1944) [18]) and Girsanov's theorem which Girsanov proved in 1960 (Girsanov (1960) [13]) are included. For the famous Black-Scholes model for arbitrage pricing in continuous time we take inspiration from Merton (1973) [20] as well as Black and Scholes (1973) [5].

When defining the FX market and deriving a framework for pricing derivatives written on the FX rate as well as generalizing the model to allow for several foreign markets with correlated FX rates we are inspired by Garman and Kohlhagen (1983) [11] as well as Björk (2009) [2].

Inspiration about how to conduct a general historical simulation, the theory concerning copulas as well as the theory about Value-at-Risk and Expected shortfall is gathered from Hult, Lindskog, Hammarlid and Rehn (2012) [17].

The foundation of stochastic short rates is described in Björk (2009) [2]. The martingale model for short rates used in this thesis is the Hull-White (extended Vasiček) model, which is introduced in Hull and White (1990) [16]. Calibration of the Hull-White model is in line with Gurrieri, Nakabayashi and Wong (2009) [15]. Inspiration is also gathered from Brigo and Mercurio (2006) [4] as well as Clark (2011) [6]. Having stochastic short rates has a lot of implications, some of which are problematic. By using the technique of changing the numeraire, described in Geman, El Karoui and Rochet (1995) [12], some of these problems can be tackled.

2.3 Applying Previous Literature

Despite that there is no previous literature regarding FX hedging for PE, from the PE firms' perspective, previous literature can still be applied to this topic. Previous research concludes different strategies regarding the hedging of FX risk. Some are in favour of hedged strategies and other support unhedged strategies. Applying the view of Glen and Jorion (1993) [14] as well as Perold and Schulman (1988) [23], PE firms should hedge FX risk. On the other hand, if accepting the view of Black (1990) [3] as well as Morey and Simpson (2001) [22], PE firms should not hedge their FX risk.

If applying the view of Froot et al. (1994) [10], the choice of hedging the FX risk should be left to the limited partners investing in PE funds. Furthermore, applying the view of Froot et al. (1993) [9] both linear and nonlinear hedging derivatives ought to be considered.

3 Background on Private Equity

Information to this section is gathered from Berk and DeMarzo (2011) [1], Swensen (2009) [24] as well as interviews with private equity professionals and experts. A PE firm is a limited partnership that specializes in raising money to invest in the equity of existing privately held firms. PE firms use debt when investing in companies, e.g. to take publicly traded companies private through so called leveraged buyouts. It is common that PE firms use debt as well as equity to finance their investments which make them risky since the debt leverages the equity yielding higher expected return on equity.

The investors constitutes mainly of limited partners. Among investors PE together with hedge funds and real assets are considered alternative asset classes in comparison to fixed income, public equity and currencies that are considered traditional asset classes or traditional marketable securities. The alternative asset classes allow the investors to create portfolios with higher returns for a given level of risk. Well selected private holdings have the potential to make huge contribution to portfolio returns. The reason for this is that these kinds of holdings have little correlation with domestic marketable securities, so the investors will achieve a diversification effect. Another way investors can achieve diversification to domestic marketable securities involves adding foreign equities to their portfolios.

PE includes both venture capital and leveraged buyout. Leveraged buyout firms will be in focus in this thesis. From now on, when mentioning PE firms throughout the thesis, the actual meaning is leveraged buyout firms.

Investments in alternative asset classes and particularly in PE have increased over recent years. This, in combination with more cross-border investments, increased competition and lower global growth has led to lower returns which in turn makes buffers for FX fluctuations smaller.[1,2] As a result, the topic FX hedging of PE investments is becoming increasingly interesting.

The remained of this section is organised as follows. First, the PE structure is explained in more detail. Second, the investment cycle of a PE fund is described. Third, the portfolio companies within a PE fund are studied. As a fourth part in this section, PE FX exposure is discussed thoroughly. The section ends with the historical view of FX hedging within the PE universe.

3.1 Private Equity Structure

PE firms are organized in a fund structure. Typically the firms constitutes of several funds of different sizes and in different stages of their life cycles. Common types of funds are equity funds, infrastructure funds and credit funds. Hereinafter, main focus will be in equity and infrastructure funds and these will be referred to as PE funds. In turn, the funds consist of different amounts of portfolio companies (PCs),

[1] Preqin, 'The 2016 Preqin Alternative Assets Performance Monitor', 2016
[2] RR Donnelley, Venue Market Spotlight, 'Cross-Border Private Equity Activity', 2015

i.e. companies that are owned by the private equity fund. As a PE fund is set up limited partners commit money to the specific fund.

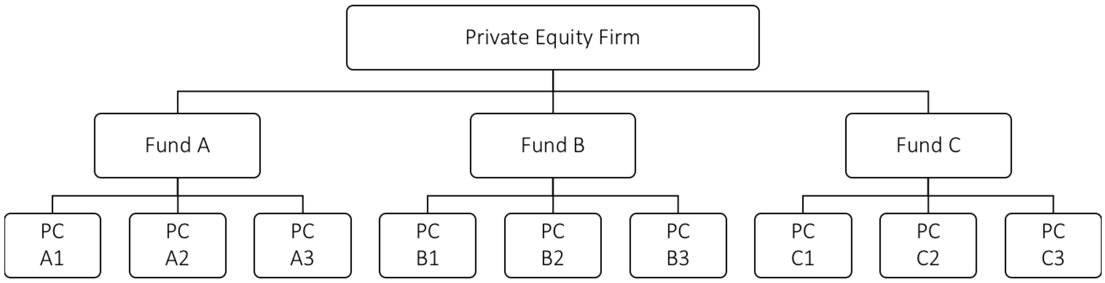

Figure 3.1: Conceptual figure of the structure of a PE firm

3.2 Investment Cycle

The typical life period of a PE fund is ten years. Before the fund starts being active, it is marked in order to attract investors to commit capital to that specific fund. Within the early years of the fund, when the fund invests in portfolio companies, investors pay in their capital commitments to the fund. In the latter years of the fund, exits of the portfolio companies take place. Hence, the PE fund realises the return and the investors receive capital from the exits. If the timing of an exit is unfavourable the fund's life can sometimes be extended by typically two more years, if the investors agree to such a decision.

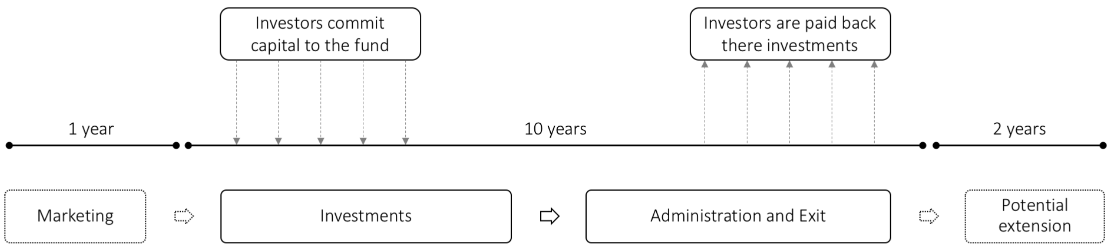

Figure 3.2: Timeline of a typical PE fund

Each investment the fund makes in a portfolio company starts with signing of the deal. After that follows some time to closing of the transaction, typically a few months to an entire year, depending on the size and complexity of the transaction. Within the signing to closing period of the transaction the PE firm will start negotiating with the management of the company regarding their equity roll-over commitment and their incentive option pool. Also, in the period between signing and closing the PE firm needs to assure the debt financing of the deal. Marketing material are presented to proposed debt investors. When all the necessary documentation is completed, the PE firm can close the deal, making sure that all the relevant parties in the transaction receive their capital on time. After closing the deal the company is part of the fund as a portfolio company until it is decided that the company should be exited. This can happen as early as after some months but usually it stays in the fund for several years.

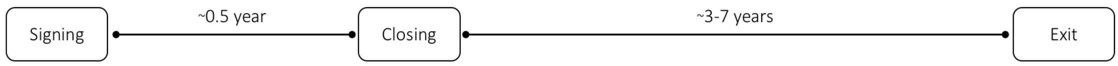

Figure 3.3: Conceptual investment timeline of a PE fund

3.3 Portfolio Companies

The value of a portfolio company is measured by its enterprise value which is defined as the sum of its equity and net debt. Net debt equals debt less cash.

$$Enterprise\ Value = Equity + Net\ Debt \tag{3.1}$$

The enterprise value can be determined through discounting the future expected cash flows of the company. Simplified, it can also be estimated through a multiple of the company's EBITDA[3]. Hence, by observing a portfolio company's net cash flows in different currencies the part of the EV corresponding to each currency can be determined. There are many other ways to determine the enterprise value of a company. However, this is beyond the scope of this thesis.

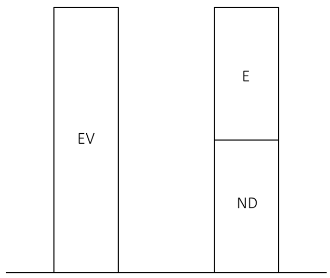

Figure 3.4: Components of the enterprise value of a portfolio company

A commonly used term for the equity value is the net asset value (NAV). The performance of a PE fund is based on the evolvement of the NAVs of each portfolio company. After rearranging the terms in Eq. 3.1, the expression for NAV is found. From now on, the NAV will be in focus.

$$NAV = Enterprise\ Value - Net\ Debt \tag{3.2}$$

3.4 Private Equity Foreign Exchange Exposure

Private equity FX exposure is different from usual corporate FX exposure. This can be understood from studying the PE structure and the PE investment cycle illustrated in Figures 3.1, 3.2 and 3.3.

First, there is FX exposure within each of the portfolio companies of a PE fund. They could e.g. have revenues is several currencies and costs in other currencies.

[3]Earnings before interest, taxes, depreciation and amortisation

If the company is large it might be present in a lot of different countries. Second, on a PE fund level, all the FX exposures of each portfolio company are present. That is, if not any of the portfolio companies fully or to some extent hedge its currency exposure. Third, the PE fund will add new portfolio companies to the fund and also exit some other portfolio companies, implying a substantial change in the aggregated FX exposure of the fund. It is very hard to determine when the exit actually will happen at the time of acquiring a new portfolio company. A fourth FX exposure that is included in the PE setting is the exposure to FX fluctuations in the meantime between signing and closing of each transaction.

Even though the different FX exposures within PE appear complex there is more to take into consideration. There are two possibilities to define the FX exposure of each portfolio company in the fund. Either, the net cash flows should be considered or the NAVs. Both could be measured at an aggregated level for the fund and constitute the fund's total FX exposure.

The rationale for defining the FX exposure based to the fund's net cash flows in different currencies is that the enterprise value and hence the NAV can be calculated from the cash flows in the portfolio companies, either through a multiple or through discounting all expected future free cash flows. Another rationale for defining the FX exposure as the net cash flows in different currencies is that the PE fund that hedges this exposure hedge its indirect profit and loss accounts.

The aggregated FX exposure in the PE fund can also be defined as the net NAV exposures in different currencies. The rationale for this definition is that the performance measures of a PE fund are calculated through the NAV evolvement of the portfolio companies. Since the topic of this thesis concerns hedging of FX risk in PE, and not how to determine the link between cash flows and NAV, the NAVs of the portfolio companies will be considered as given. Thus, the FX exposure of in this thesis is defined from the NAVs of the portfolio companies. By defining the FX exposure as the NAVs in different currencies, and hedging this exposure, the PE fund hedge its balance sheet exposure, i.e. its NAV. Hence, it hedges its performance in terms of IRR. Worth to mention is that the net cash flows, which are part of the profit and loss accounts, will not be hedged. As mentioned earlier, these cash flows will affect both enterprise value as well as net debt of each portfolio company, and hence the NAV of each portfolio company. However, it would be too technical to take all these aspects into account in this thesis.

The NAV of each portfolio company in a PE fund can be regarded as a sum of the NAVs in different currencies which adds up to the total NAV of the portfolio company. This is illustrated in Figure 3.5. Note that the NAV in a specific currency could be negative.

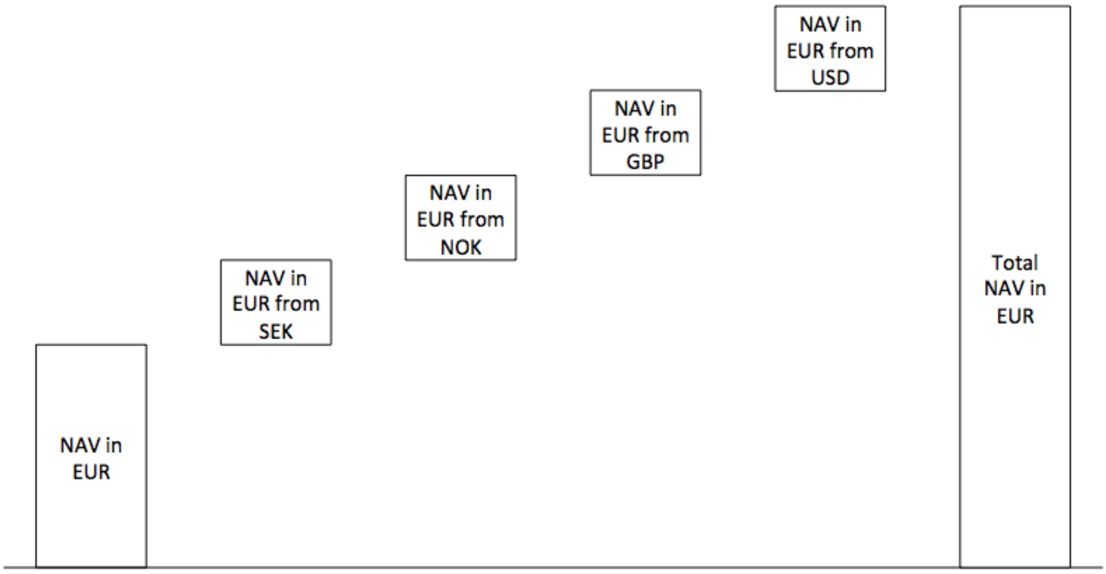

Figure 3.5: Conceptual NAV of a PC regarded as a sum of the parts from different currencies

3.5 Historical Evidence

Historically, Nordic based PE firms have not hedged FX risk in their equity funds. The rationale has been that PE is risky business as it is. Hence, FX risk is only one of several risks each PE fund is facing. Historical returns within PE have been astonishingly high. However, with an increased competition, lower global growth and the usage of lower debt levels in financing transactions, returns have become lower. At the same time, fluctuations in FX rates have increased and cross-border investments have increased.[4] As a consequence, the buffers for FX fluctuations are smaller nowadays for PE firms. For that reason, this topic is interesting and relevant for Nordic based PE firms.

Some PE firms might have more than solely equity funds. For example, they might have credit and infrastructure funds. Credit funds include fixed income instruments with much lower risk compared to equity funds. Therefore, credit funds have hedged their FX exposure historically. Infrastructure funds include more local companies than equity funds and have less volatile cash flows. Furthermore, infrastructure companies usually have contracts with a determined duration. For these reasons, infrastructure funds have a lower NAV volatility and have sometimes hedged their FX risk.

Some American PE funds hedge their FX exposure in their equity funds. Large PE firms have their own Treasury departments that handle the FX risk. Later in the thesis we will analyse why this is the case.

[4]Preqin, 'The 2016 Preqin Alternative Assets Performance Monitor', 2016

4 Foreign Exchange Effect on PE Returns

This section sets out to exemplify the theoretical impact an FX rate can have on the returns of a PE fund. First, some recent examples of large FX movements are discussed. Then a theoretical example of the FX effect is considered.

FX effects can theoretically affect the performance of a PE fund dramatically. The annualised return of the PE fund, the internal rate of return (IRR), is calculated from the NAV in the fund currency, at entry and at exit of a particular investment.

$$IRR(t,T) = \left(\frac{NAV_T}{NAV_t}\right)^{\frac{1}{T-t}} - 1, \tag{4.1}$$

where t is the time of entry, T is the time of exit and $T-t$ is the duration of the investment in years.

Two examples of substantial FX movements are when Brexit happened and when the Swiss National Bank withdrew its management of the value of the Swiss Franc (CHF) against the Euro (EUR), until then it had prevented the value of the CHF from strengthening beyond 1.2 to the EUR. After Brexit was a fact, the UK Sterling (GBP) dropped 12% to the US Dollar (USD) in a single day. After the Swiss National Bank suddenly allowed the CHF to freely appreciate against the EUR, which came as a shock to the market due to Swiss being a conservative country with a predictable central bank, the CHF strengthened by 30% in a matter of minutes.[5]

In order to understand the FX effects on PE returns, let us consider the following simplified example. Take as given a PE fund denominated in EUR. The PE fund makes a fully equity financed investment January 1, 2018, in a company that is denominated in Swedish Krona (SEK). At the investment time the company's value is SEK 100 million, equivalent to EUR 10 million since the SEK/EUR rate is 0.10 at that date, as seen in Figure 4.1. Let us assume that the investment grows 20% during 2018 and that the PE fund exits the company at December 31, 2018. Hence, the IRR in SEK is 20%. Furthermore, let us assume that the PE fund does not hedge its FX exposure.

[5] HiFX, HiFM, 'The three major Foreign Exchange risks faced by Private Equity firms', 2016

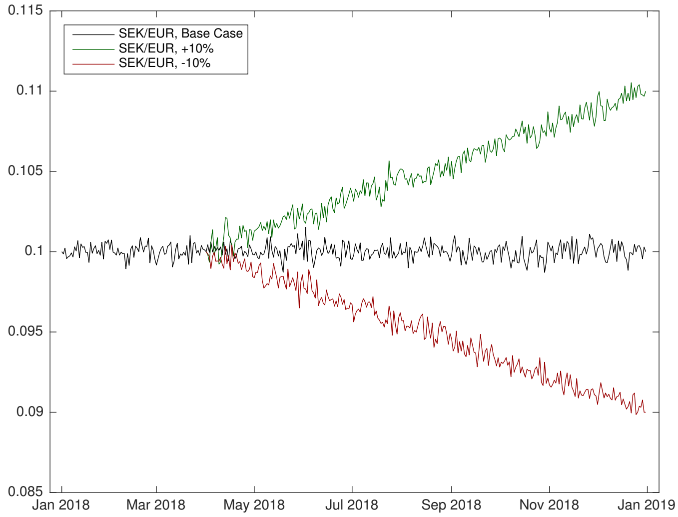

Figure 4.1: Foreign exchange effect on private equity returns, conceptual evolvement of SEK/EUR rate over one year

At the end of 2018, the NAV of the company is SEK 120 million. In the base case, SEK/EUR is unchanged at 0.10 in the end of 2018. Hence, the NAV at exit in EUR is 12.0 million. The resulting IRR, calculated from the entry and exit NAV in EUR, is 20%.

In the case where the SEK/EUR rate ends up in a 10% higher state than the base case, i.e. at 0.11, the NAV at exit in EUR is approximately 13.2 million. Hence, the IRR in this case is 32%.

In the last case, the SEK/EUR rate ends up 10% lower than the base case, i.e. at 0.09. This implies an exiting NAV of EUR 10.8 million and a resulting IRR of 8%.

The results are summarised in Table 4.1. It is evident that, in this simplified example, the FX effects can have a large impact on the performance of a PE fund. In practice, the FX effects on PE returns might be lower, due to diversification effects since PE funds in general have several portfolio companies with different FX exposures, that in turn are invested in at different times. Also, each portfolio company might have several or a lot of different FX exposures. Furthermore, PE investments are rarely fully equity financed. However, the FX effects will still affect the performance of the PE funds. Therefore, this matter is worthwhile to study in greater detail. It might be beneficial for PE funds to hedge their FX exposure.

FX Scenarios	**Base Case**	**+10%**	**-10%**
IRR	20%	32%	8%
SEK/EUR at Exit	0.10	0.11	0.09

Table 4.1: Foreign exchange effect on private equity returns

5 Hypothesis

The research question this thesis sets out to answer is if and how PE funds should hedge their FX exposure. As seen in Section 4, FX effects can dramatically affect the performance of PE funds. Previous scientific research is contradictory regarding hedging of FX risk. Some researchers argue that it could be beneficial to hedge FX risk while others argue that this is not the case. However, there is no specific scientific previous literature regarding the topic of hedging FX risk for PE funds. Applying the research that is opposed to FX hedging, e.g. Black (1990) [3] as well as Morey and Simpson (2001) [22], the null hypothesis of the thesis is

$\mathbf{H_0}$:*PE firms should not hedge FX exposure, regardless of the definition of FX risk, in order to obtain the best performance*

There are several ways to define the FX exposure of a PE fund. As argued in Section 3.4, the FX exposure of a PE fund should be defined as the net FX exposure of the portfolio companies within that fund. Defining the FX exposure in this way, and applying the previous literature claiming that FX hedging could be beneficial, e.g. Perold and Schulman (1988) [23] as well as Glen and Jorion (1993) [14], the alternative hypothesis of this thesis is

$\mathbf{H_1}$:*PE firms should define their FX exposure as the underlying net FX exposure of the portfolio companies, and hedge accordingly, in order to obtain the best performance*

As part of evaluating different hedging strategies for PE funds, several financial derivatives will be considered. In line with Froot et al. (1993) [9] both linear and nonlinear hedging derivatives ought to be considered.

Furthermore, since not all investors necessarily want to hedge the FX risk, it might be wise to let them decide whether to hedge the FX exposure or not. This is in line with the conclusion of Froot et al. (1994) [10]. If not giving them the decision, some PE funds that decide to hedge their FX exposure might lose some potential investors that do not want to hedge FX risk. Also, if giving the investors the decision, they might be able to let FX exposures from different PE funds offset each other. In such way, they might be able to reduce the overall transaction costs related to hedging. The result of this would be a better overall return for the investors.

6 Financial Derivatives

This section sets out to introduce the financial derivatives that will be used to hedge the FX exposure. First we recall the definition of a financial derivative. In the proceeding sections the linear forward contracts and the non-linear option contracts are introduced.

A financial derivative is a financial asset whose value depends on the value of another asset. The underlying asset can for example be a stock, an ounce of gold or an FX rate. In other words, financial derivatives are contracts that give their owners certain rights and obligations depending on what kind of derivative it is. As mentioned above, the two types of derivatives that will be used in this thesis are forwards and options.

6.1 Forward Contracts

Forward contracts are binding contracts both for the buyer and the seller. The two parties agree on a future exchange which neither of the parties can back out from in case it would appear non-beneficial to complete. Below follows the definition.

Definition 6.1 ***Forward contract***
A forward contract between two parties, a seller and a buyer, is a binding agreement in which the seller agrees to sell a predefined asset to the buyer at a future time T to the, at time t, predetermined price K.

The forward price, K, is determined in such a way that the forward contract is worthless at time t when it is being entered. This means that no money transaction takes place until the delivery time, T, i.e. it does not cost anything to enter a forward contract. For the buyer of a forward contract, the payoff of its long position can be expressed as

$$\Pi = S_T - K, \tag{6.1}$$

where S_T is the price of the underlying asset at time T and K is the, at time t, predetermined price. Similarly, the payoff of the seller's short position is given by

$$\Pi = K - S_T. \tag{6.2}$$

An example of the payoff function for a long forward (upper) and short forward (lower) is illustrated in Figure 6.1. Naturally, as seen in the figure, the payoff at time T can be both positive and negative.

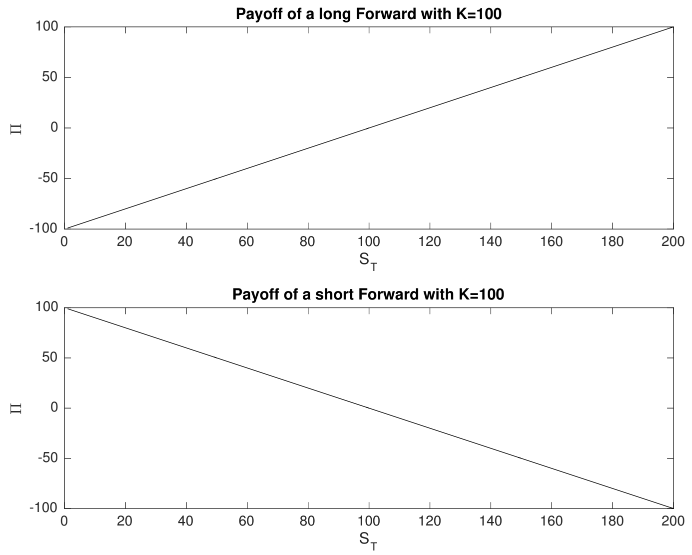

Figure 6.1: Payoff of a forward as a function of the price of the underlying asset

6.1.1 Rolling Forwards

Rolling forwards are not a new financial derivative per se but rather a series of consequent forward contracts over a time period. The concept of rolling forwards is most easily illustrated with an example.

Imagine that you want to buy one dollar in one year from now. If you want to lock in today's price you can buy a forward contract for buying the one dollar in twelve months. In this case one financial derivative will be entered and one payoff will be received in one year, be it positive or negative.

However, rolling forwards could have been used instead of the twelve months forward contract. Instead of fixing the price for twelve months in the future we could have fixed the price for just one month in the future. And in one months time, as the first contract expires, entered a new one month forward contract for the next month and so on. This would have resulted in entering twelve different forward contracts which each would have given some payoff. By this procedure we denote a rolling forward.

An important difference between rolling forwards and forwards is that a rolling forward depends on the price path of the underlying asset whereas a forward only depends on the prices at the entry and maturity times. Another important difference is that the rolling forwards give much more flexibility since the time frame in which the contract is being active is much shorter than in the forward case.

6.2 Option Contracts

In contrast to forward contracts option contracts are not binding for the buyer. The buyer of the contract purchases it at some price at time t. If the payoff of the contract at the exercise time T is non-beneficial the owner of the contract can simply choose not to exercise the option. If however the payoff is beneficial then the holder will of course exercise the option.

Definition 6.2 ***Option contract***
An option contract between two parties, a seller and a buyer, is a right for the buyer of the contract to buy or sell a predefined asset at a specified future time T to the, at time t, predetermined strike price K.

The crucial thing with options is that the holder of the contract gets the right to buy or sell the underlying asset, rather than being obliged to do it. Since the buyer only gets this right and no obligation option contracts make for excellent hedging instruments in times of volatile prices of the underlying assets. If the price of the underlying asset falls significantly and one owns an option giving the right to sell the underlying asset at a predefined price one can exercise the option to avoid a substantial loss. The same argument is applicable for the case that the price of the underlying asset would increase and one owns an option giving the right to buy the asset at a predetermined price.

Potentially there are two transactions involved in an option contract. As the buyer buys the contract the price of the option is paid to the seller. At the maturity time, if the holder of the option chooses to exercise it, an interchange transaction takes place and the asset changes owner for the predefined strike price. The pricing of options requires stochastic calculus and will be investigated in later sections.

Finally, there are two types of options, call options and put options. These are closer investigated in the two below proceeding sections. Furthermore, there are two common types of call options and two common types of put options, namely European call and put options as well as American call and put options. European means that the option only can be exercised at exactly the date of expiration and American means that the option holder has the right to exercise the option at any time before the expiration date as well as on the expiration date. Only European options will be investigated in this thesis.

6.2.1 Calls

An option that gives its holder the right to buy the underlying asset is called a call option. The holder's payoff of a call option is given by

$$\Pi_{C_T} = \max(S_T - K, 0). \tag{6.3}$$

Worth noting from the payoff formula is that the buyer has a potential infinite upside whereas the seller has a potential infinite downside. This makes calls rather risky.

An example of the payoff function for a long call (holder) and short call (seller) is illustrated in Figure 6.2. As seen in the figure, the payoff of the holder at time T will be greater or equal to zero whereas for the seller it will be less or equal to zero. However, since the buyer has paid some amount to the seller at time t the buyer will have lost money if the option is not exercised whereas the seller will have gained money if it is not exercised.

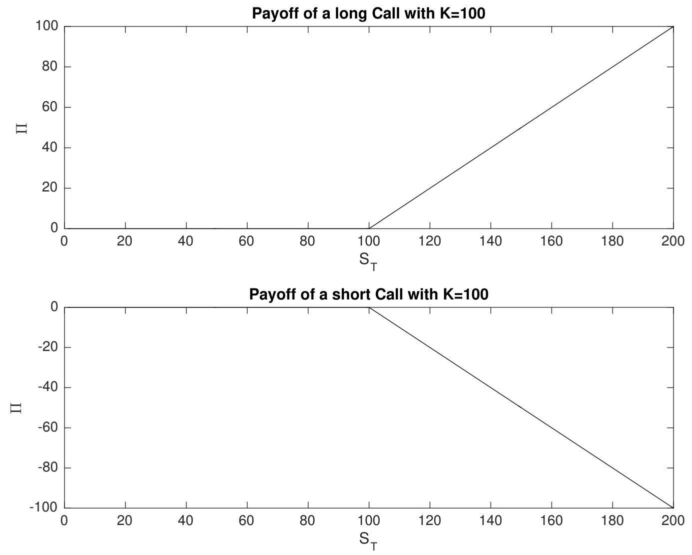

Figure 6.2: Payoff of a call as a function of the price of the underlying asset

6.2.2 Puts

An option that gives its holder the right to sell the underlying asset is called a put option. The holder's payoff of a put option is given by

$$\Pi_{P_T} = \max(K - S_T, 0). \tag{6.4}$$

In contrast to calls the payoff of a put is limited, i.e. the buyer has a capped upside and the seller has a capped downside since the price of the underlying asset can not decrease below zero.

An example of the payoff function for a long put (holder) and short put (seller) is illustrated in Figure 6.3. As seen in the figure, the payoff of the holder at time T will be greater or equal to zero whereas for the seller it will be less or equal to zero. However, as in the call case, since the buyer has paid some amount to the seller at time t the buyer will have lost money if the option is not exercised whereas the seller will have gained money if it is not exercised.

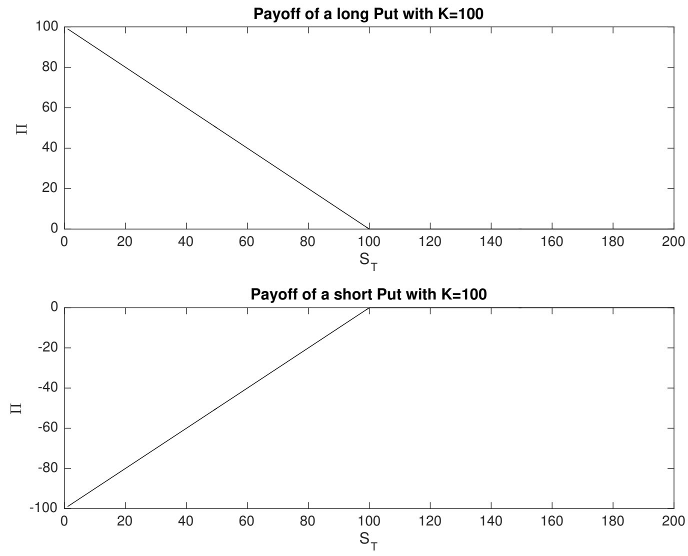

Figure 6.3: Payoff of a put as a function of the price of the underlying asset

6.2.3 Strangles

A strangle is not a financial derivative per se but rather a combination of a call option and a put option with the same expiry date T. The two options do not need to have the same strike prices K. An example of the payoff function for a long strangle (holder) and short strangle (seller), where the call option and the put option have the same strike prices, is illustrated in Figure 6.4.

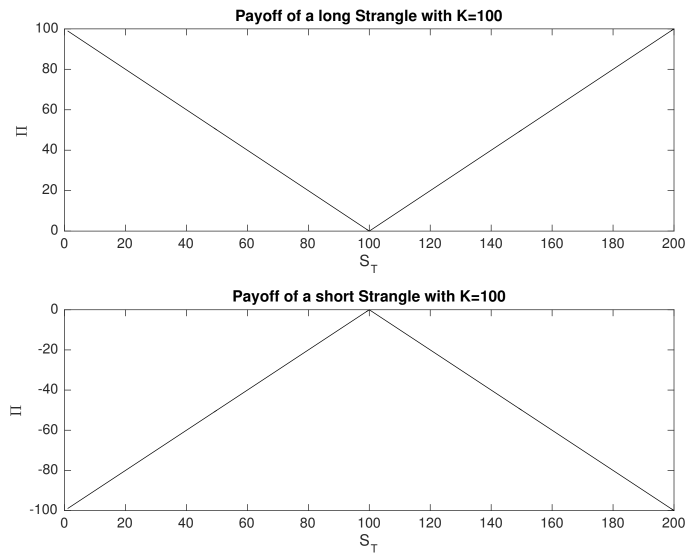

Figure 6.4: Payoff of a strangle as a function of the price of the underlying asset

Part II

Base Model: Constant Interest Rates

7 The Black-Scholes Model

This chapter sets out to present the Black-Scholes model for arbitrage pricing in continuous time. Furthermore, we define a market for the FX rate between the currencies of the regular domestic market and a foreign market and derive a framework for pricing derivatives written on the FX rate. After that we generalize the model to allow for several foreign markets with correlated FX rates. Inspiration is to a large extent gathered from Björk (2009) [2]. In Section 7.1 we follow the arguments of Merton (1973) [20] as well as Black and Scholes (1973) [5]. In Sections 7.2 and 7.3 we follow the arguments of Garman and Kohlhagen (1983) [11]. The underlying mathematical foundation is found in Appendix F.

7.1 Model Definition and Derivative Pricing

In the Black-Scholes model there are two assets, a risk free asset and a risky asset. The risk free asset can be though of as a bond and the risky asset can be though of as a stock. The risk free asset has a deterministic rate of return whereas the risky asset has a stochastic rate of return. The dynamics of the two assets are given in the following definition.

Definition 7.1 ***Black-Scholes model***
The risk free asset and the risky asset have the following dynamics respectively

$$dB_t = rB_t dt, \tag{7.1}$$
$$dS_t = \alpha S_t dt + \sigma S_t d\overline{W}_t, \tag{7.2}$$

where r, α, and σ are deterministic constants.

We begin by noting that the dynamics of the risky asset is modelled as a Geometric Brownian motion. The two constants r and α can be though of as the mean rates of returns of the two assets respectively, and the constant σ can be thought of as the volatility of the risky asset's return. r denotes the short rate of interest which we sometimes will refer to as the risk free interest rate.

As the market has been set up we are now ready to define the claims which are going to be priced. Let us refer to these claims as contingent claims or derivatives since they will depend on the price of the underlying asset S.

Definition 7.2 ***Contingent claim***
The stochastic variable $\mathcal{X}$ is a contingent claim with time to maturity T if $\mathcal{X} \in \mathcal{F}_T^S$. If furthermore $\mathcal{X} = \Phi(S_T)$ then $\mathcal{X}$ is called a simple claim.

Having defined the model and a contingent claim we are ready to price the contingent claim under the given model. The price of a contingent claim $\mathcal{X}$ at time t will be denoted $\Pi(t;\mathcal{X})$ and the relation $\Pi(T;\mathcal{X}) = \mathcal{X}$ holds. Moreover, we recall that under the Black-Scholes model, assuming absence of arbitrage results in contingent claims having unique prices at all times t. To mark the importance of this assumption we formulate it more formally.

Assumption 7.1 ***Absence of arbitrage***
The price process of $\mathcal{X}$, $\Pi(t;\mathcal{X})$, is assumed to be such that there are no arbitrage possibilities on the market. An arbitrage possibility can be though of as a self-financed portfolio h with corresponding value process V_t^h such that

$$\begin{aligned} V_0^h &= 0, \\ P(V_T^h \geq 0) &= 1, \\ P(V_T^h > 0) &> 0. \end{aligned}$$

Assuming that the market is free of arbitrage only two more minor assumptions are required in order to fully be able to derive the pricing equation for a simple claim $\mathcal{X} = \Phi(S_T)$. The first of these is that the contingent claim $\mathcal{X}$ can be bought and sold on the market. The second one is that the price process for contingent claims is on the form

$$\Pi(t;\mathcal{X}) = F(t, S_t). \tag{7.3}$$

Having stated all necessary assumptions, let us now derive the pricing equation.

First we will try to express the dynamics of a self-financed portfolio consisting of the contingent claim and the underlying asset. Applying Itô's formula to Eq. 7.3 and using Eq. 7.2 one obtains

$$\begin{aligned} d\Pi_t &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial s}dS_t + \frac{1}{2}\frac{\partial^2 F}{\partial s^2}d\langle S\rangle_t \\ &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial s}\left(\alpha S_t dt + \sigma S_t d\overline{W}_t\right) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial s^2}dt \\ &= \left(\frac{\partial F}{\partial t} + \alpha S_t \frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial s^2}\right)dt + \sigma S_t \frac{\partial F}{\partial s} d\overline{W}_t. \end{aligned} \tag{7.4}$$

Defining

$$\alpha_\Pi(t) = \frac{\frac{\partial F}{\partial t} + \alpha S_t \frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial s^2}}{F}, \tag{7.5}$$

$$\sigma_\Pi(t) = \frac{\sigma S_t \frac{\partial F}{\partial s}}{F}, \tag{7.6}$$

and using Eq. 7.4 we get

$$d\Pi_t = \alpha_\Pi(t)\Pi_t dt + \sigma_\Pi(t)\Pi_t d\overline{W}_t. \tag{7.7}$$

The value process of a portfolio containing h^S of the underlying asset and h^Π of the derivative is given by

$$V_t^h = h^S S_t + h^\Pi \Pi_t. \tag{7.8}$$

Applying Itô's formula and recalling that for a self-financed portfolio $S_t dh_t + d\langle h, S\rangle_t = 0$ the dynamics of our portfolio is found to be

$$dV_t^h = h^S dS_t + h^\Pi d\Pi_t. \tag{7.9}$$

Defining the relative portfolio weights $\omega_S = \frac{h^S S_t}{V_t^h}$ and $\omega_\Pi = \frac{h^\Pi \Pi_t}{V_t^h}$ the above expression can be written as

$$\begin{aligned} dV_t^h &= V_t^h\Big(\omega_S\big(\alpha dt + \sigma d\overline{W}_t\big) + \omega_\Pi\big(\alpha_\Pi(t)dt + \sigma_\Pi(t)d\overline{W}_t\big)\Big) \\ &= V_t^h\Big(\big(\alpha\omega_S + \alpha_\Pi(t)\omega_\Pi\big)dt + \big(\sigma\omega_s + \sigma_\Pi(t)\omega_\Pi\big)d\overline{W}_t\Big). \end{aligned} \tag{7.10}$$

Choosing the formed portfolio to be locally risk less the second term in the expression is set to zero, $\sigma\omega_s + \sigma_\Pi(t)\omega_\Pi = 0$. This together with the fact that ω_S and ω_Π are weights of a relative portfolio give us two conditions for our portfolio

$$\sigma\omega_s + \sigma_\Pi(t)\omega_\Pi = 0, \tag{7.11}$$
$$\omega_S + \omega_\Pi = 1. \tag{7.12}$$

Solving the system of linear equations the relative portfolio weights are obtained as

$$\omega_S = \frac{\sigma_\Pi(t)}{\sigma_\Pi(t) - \sigma}, \tag{7.13}$$
$$\omega_\Pi = \frac{-\sigma}{\sigma_\Pi(t) - \sigma}. \tag{7.14}$$

Next we recall that if the portfolio is locally risk less then its mean rate of return must equal the return of the risk free asset in order for the model to be free of arbitrage. Thus it must hold that

$$\begin{aligned} r &= \alpha\omega_S + \alpha_\Pi(t)\omega_\Pi \\ &= \frac{\alpha\sigma_\Pi(t) - \sigma\alpha_\Pi(t)}{\sigma_\Pi(t) - \sigma} \end{aligned} \tag{7.15}$$

By inserting the values of $\alpha_\Pi(t)$ and $\sigma_\Pi(t)$ given in Eqs. 7.5 and 7.6 in the absence of arbitrage condition, Eq. 7.4, and rearranging the following equation is obtained

$$\frac{\partial F}{\partial t} + rS_t\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial s^2} - rF = 0. \tag{7.16}$$

Finally, remembering that the price of a contingent claim at time T is given by $\Pi(T) = \Phi(S_T)$ we get the condition

$$F(T, S_T) = \Phi(S_T). \tag{7.17}$$

The final two equations have to hold with probability 1 for each fixed t. This implies that F has to satisfy the following partial differential equation.

$$\frac{\partial F}{\partial t}(t,s) + rs\frac{\partial F}{\partial s}(t,s) + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F}{\partial s^2}(t,s) - rF(t,s) = 0, \tag{7.18}$$

$$F(T,s) = \Phi(s). \tag{7.19}$$

Let us formulate the above findings as a theorem.

Theorem 7.1 ***Black-Scholes equation***
Given the Black-Scholes model and a contingent claim, the price function F that is consistent with absence of arbitrage is the solution to the following boundary value problem on $[0,T] \times R_+$.

$$\frac{\partial F}{\partial t}(t,s) + rs\frac{\partial F}{\partial s}(t,s) + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F}{\partial s^2}(t,s) - rF(t,s) = 0, \tag{7.20}$$

$$F(T,s) = \Phi(s). \tag{7.21}$$

Noting that the Black-Scholes equation is exactly on the form Eqs. F.29-F.30 in Appendix F the analysis of partial differential equations will come in handy as we seek the solution to the system. Using Feynman-Kač's theorem, Proposition F.4 in Appendix F, the solution to the Black-Scholes equation is immediately obtained.

Theorem 7.2 ***Risk neutral valuation***
The arbitrage free price of the simple contingent claim $\mathcal{X} = \Phi(S_T)$ is given by $\Pi(t;\mathcal{X}) = F(t,S_t)$ where F is given by

$$F(t,s) = e^{-r(T-t)} E^Q_{t,s}\big[\Phi(S_T)\big], \tag{7.22}$$

and the Q-dynamics of S are given by

$$dS_u = rS_u du + \sigma S_u dW_u, \tag{7.23}$$

$$S_t = s. \tag{7.24}$$

We close this section by seeking the pricing function in the special case of the contingent claim being a European call option, presented in Section 6. The quite tedious computations will result in the famous Black-Scholes formula. Recalling that the contract function for a European call is given by

$$\mathcal{X} = \max(S_T - K, 0) \tag{7.25}$$

and using the risk neutral valuation formula it must hold that the price at time t is given by

$$\Pi_t(\mathcal{X}) = e^{-r(T-t)} E^Q\big[\max(S_T - K, 0)\big], \tag{7.26}$$

where

$$S_T = S_t \exp\Big(\big(r - \frac{1}{2}\sigma^2\big)(T-t) + \sigma^2\big(W_T - W_t\big)\Big). \tag{7.27}$$

Rewriting S_T as

$$S_T = S_t e^Z,$$

where $Z \sim N\Big(\big(r - \frac{1}{2}\sigma^2\big)(T-t), \sigma^2(T-t)\Big)$, and denoting the density function of Z by φ, $\big(r - \frac{1}{2}\sigma^2\big)(T-t)$ by m and $\sigma^2(T-t)$ by s^2 we have

$$\begin{aligned}
E^Q\big[\max(S_T - K, 0)\big] &= E^Q\big[\max(S_t e^Z - K, 0)\big] \\
&= 0 \cdot Q\big(S_t e^Z \leq K\big) + \int_{\ln\left(\frac{K}{S_t}\right)}^{\infty} \big(S_t e^z - K\big)\varphi(z)dz \\
&= S_t \int_{\ln\left(\frac{K}{S_t}\right)}^{\infty} e^z \frac{1}{\sqrt{2\pi s^2}} e^{-\frac{(z-m)^2}{2s^2}} dz - KQ\Big(Z > \ln\Big(\frac{K}{S_t}\Big)\Big).
\end{aligned}$$

Completing the square in the exponent the first term can be written as

$$\begin{aligned}
&S_t \int_{\ln\left(\frac{K}{S_t}\right)}^{\infty} e^z \frac{1}{\sqrt{2\pi s^2}} e^{-\frac{(z-m)^2}{2s^2}} dz \\
=&S_t e^{\frac{s^2}{2}+m} \int_{\ln\left(\frac{K}{S_t}\right)}^{\infty} \frac{1}{\sqrt{2\pi s^2}} e^{-\frac{(z-(m+s^2))^2}{2s^2}} dz \\
=&S_t e^{\frac{s^2}{2}+m} Q\Big(Y > \ln\Big(\frac{K}{S_t}\Big)\Big),
\end{aligned}$$

where $Y \sim N\big(m + s^2, s^2\big)$. Reinserting this expression back and substituting in the definitions of m and s we have

$$\begin{aligned}
E^Q\big[\max(S_T - K, 0)\big] &= S_t e^{\frac{s^2}{2}+m} Q\Big(Y > \ln\Big(\frac{K}{S_t}\Big)\Big) - KQ\Big(Z > \ln\Big(\frac{K}{S_t}\Big)\Big) \\
&= S_t e^{r(T-t)} Q\Big(Y > \ln\Big(\frac{K}{S_t}\Big)\Big) - KQ\Big(Z > \ln\Big(\frac{K}{S_t}\Big)\Big).
\end{aligned}$$

Finally, using the following characteristics of stochastic random variables and the normal distribution

$$Q(X > x) = 1 - Q(X \leq x), \tag{7.28}$$

$$X \sim N(\mu, \sigma^2) \implies Q(X \leq x) = N\Big(\frac{x-\mu}{\sigma}\Big), \tag{7.29}$$

$$1 - N(x) = N(-x), \tag{7.30}$$

we get

$$
\begin{aligned}
E^Q\big[\max(S_T-K,0)\big] &= S_t e^{r(T-t)}\Bigg(1-Q\Bigg(Y\le \ln\left(\frac{K}{S_t}\right)\Bigg)\Bigg)-K\Bigg(1-Q\Bigg(Z\le \ln\left(\frac{K}{S_t}\right)\Bigg)\Bigg)\\
&= S_t e^{r(T-t)}\Bigg(1-N\Bigg(\frac{\ln\left(\frac{K}{S_t}\right)-\left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\Bigg)\Bigg)\\
&\quad -K\Bigg(1-N\Bigg(\frac{\ln\left(\frac{K}{S_t}\right)-\left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\Bigg)\Bigg)\\
&= S_t e^{r(T-t)} N\Bigg(\frac{\ln\left(\frac{S_t}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\Bigg)\\
&\quad -KN\Bigg(\frac{\ln\left(\frac{S_t}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\Bigg)\\
&= S_t e^{r(T-t)} N[d_1(t,S_t)]-KN[d_2(t,S_t)],
\end{aligned}
$$

where

$$
\begin{aligned}
d_1(t,S_t) &= \frac{\ln\left(\frac{S_t}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}},\\
d_2(t,S_t) &= d_1(t,S_t)-\sigma\sqrt{T-t}.
\end{aligned}
$$

So the price at time t of a European call is given by

$$
\Pi_t(X) = S_t N[d_1(t,S_t)] - e^{-r(T-t)} K N[d_2(t,S_t)]. \tag{7.31}
$$

Above computations can be nicely summarized as a proposition.

Proposition 7.1 ***Black-Scholes formula***
The price at time t of a European call option with strike price K and expiry date T is given by $\Pi_t = F(t,S_t)$ where

$$
F(t,S_t) = S_t N[d_1(t,S_t)] - e^{-r(T-t)} K N[d_2(t,S_t)]. \tag{7.32}
$$

N is the standard cumulative distribution function and d_1 and d_2 are given by

$$
d_1(t,S_t) = \frac{\ln\left(\frac{S_t}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \tag{7.33}
$$

$$
d_2(t,S_t) = d_1(t,S_t)-\sigma\sqrt{T-t}. \tag{7.34}
$$

Drawing the payoff functions it can be seen that a put option is replicated by a portfolio consisting of long positions in a zero coupon T-bond with face value K and a European call option as well as a short position in the underlying asset X.

Proposition 7.2 ***Put-call parity***
Consider a European call option and a European put option, both having strike price K and time of maturity T. Denoting the corresponding pricing functions by $c(t,x)$ and $p(t,x)$, the so called put-call parity holds

$$
p(t,x) = e^{-r(T-t)}K + c(t,x) - x. \tag{7.35}
$$

7.2 The One Dimensional Foreign Exchange Model

In this section the model presented in the previous section is extended to a setting including two different markets, a domestic market and a foreign market. In the domestic market assets are priced in the domestic currency whereas in the foreign market assets are priced in the foreign currency. Since we will only be interested in pure currency derivatives in this thesis we omit the domestic and foreign risky assets. Let us begin by defining the FX rate between the two currencies, state the model assumptions and then state the dynamics of the assets being present on the market.

Definition 7.3 ***Foreign exchange rate***
The FX rate at time t is defined as

$$X_t = \frac{\text{units of domestic currency}}{\text{unit of foreign currency}}. \tag{7.36}$$

The assumptions from the previous section will carry over to this setting as well, for the sake of clarity we restate them together with some additions below.

Assumption 7.2 ***Model assumptions***

(i) *The price process, $\Pi(t;\mathcal{Z})$, of a contingent claim, $\mathcal{Z} = \Phi(X_T)$, is assumed to be such that there are no arbitrage possibilities on the market.*

(ii) *The contingent claim $\mathcal{Z}$ can be bought and sold on a market and the price process for contingent claims is on the form $\Pi(t;\mathcal{Z}) = F(t, X_t)$.*

(iii) *The domestic and foreign risk free interest rates, r_d and r_f, as well as the drift and the volatility of the FX rate, α_X and σ_X, are deterministic constants.*

(iv) *The domestic and foreign markets are frictionless, i.e. there are no transaction costs or taxes.*

Definition 7.4. ***Foreign exchange model***
The domestic risk free asset, the foreign risk free asset and the FX rate have the following dynamics under P

$$dB_t^d = r_d B_t^d dt, \tag{7.37}$$
$$dB_t^f = r_f B_t^f dt, \tag{7.38}$$
$$dX_t = \alpha_X X_t dt + \sigma_X X_t d\overline{W}_t, \tag{7.39}$$

where r_d, r_f, α_X and σ_X are deterministic constants and $\overline{W}$ is a scalar Wiener process.

From this, under the observable measure P, a domestic and a foreign savings account can be expressed as

$$B_t^d = B_0^d e^{r_d t}, \tag{7.40}$$
$$B_t^f = B_0^f e^{r_f t}, \tag{7.41}$$

respectively. The FX rate process is modelled as a Geometric Brownian motion. Thus using Proposition F.3 in Appendix F the FX rate at time t is given by

$$X_t = X_0 \exp\left(\left(\alpha_X - \frac{1}{2}\sigma_X^2\right)t + \sigma_X \overline{W}_t\right). \tag{7.42}$$

Moreover, applying the theory developed in the previous section to the current setting the risk neutral valuation formula will take the form

$$\Pi(t; \mathcal{Z}) = e^{-r_d(T-t)} E_{t,x}^Q\left[\Phi(X_T)\right]. \tag{7.43}$$

In order to be able to use the risk neutral valuation formula the equivalent martingale measure Q has to be defined. The measure Q needs to be a measure such that there are no arbitrage opportunities between the domestic and foreign markets. Since we are interested in prices on the domestic market Q will be a martingale measure for the domestic market, observe that an equivalent martingale measure for the foreign market would have been different.

Recalling that the discounted value of all risky assets are martingales under the domestic martingale measure when the domestic savings account is used as numeraire and noting that $B_t^f X_t$ can be viewed as a risky asset on the domestic market it must hold that

$$\tilde{X}_t := \frac{B_t^f X_t}{B_t^d} = e^{(r_f - r_d)t} X_t \tag{7.44}$$

is a martingale under Q. Inserting the expression for X_t into the above expression $\tilde{X}_t$ is given by

$$\tilde{X}_t = X_0 \exp\left(\left(\alpha_X + (r_f - r_d) - \frac{1}{2}\sigma_X^2\right)t + \sigma_X \overline{W}_t\right), \tag{7.45}$$

or similarly, on differential form

$$d\tilde{X}_t = \big(\alpha_X + (r_f - r_d)\big)\tilde{X}_t dt + \sigma_X \tilde{X}_t d\overline{W}_t. \tag{7.46}$$

Recalling Girsanov's theorem let us now define the measure Q by

$$dQ = L_T dP \text{ on } \mathcal{F}_T, \tag{7.47}$$

where the dynamics of the likelihood process L are given by

$$dL_t = \varphi_t L_t d\overline{W}_t, \tag{7.48}$$

$$L_0 = 1, \tag{7.49}$$

for the so far unknown function φ_t. Assuming that φ_t satisfies the Novikov condition, Eq. F.18 in Appendix F, by Girsanov's theorem

$$dW_t = d\overline{W}_t - \varphi_t dt, \tag{7.50}$$

is a Q-Brownian motion. Inserting Eq. 7.50 into the expression for $d\tilde{X}_t$ yields

$$d\tilde{X}_t = \big(\alpha_X + (r_f - r_d) + \sigma_X \varphi_t\big)\tilde{X}_t dt + \sigma_X \tilde{X}_t dW_t. \tag{7.51}$$

Now, since $\tilde{X}_t$ is supposed to be a martingale under Q its drift term necessarily has to be equal to zero. Hence, we get the following condition for φ_t

$$\alpha_X + (r_f - r_d) + \sigma_X \varphi_t = 0. \tag{7.52}$$

Rearranging, φ_t is obtained as

$$\varphi_t = \frac{(r_d - r_f) - \alpha_X}{\sigma_X}, \tag{7.53}$$

and we see that our assumption that φ_t fulfils the Novikov condition is valid. This marks the completion of fully defining the equivalent martingale measure Q.

Having defined the measure Q we are now ready to state the dynamics of the FX rate under the risk neutral measure. Under Q, X_t has the dynamics

$$\begin{aligned} dX_t &= \alpha_X X_t dt + \sigma_X X_t \big(dW_t + \varphi_t dt\big) \\ &= \left(\alpha_X + \sigma_X \frac{(r_d - r_f) - \alpha_X}{\sigma_X}\right) X_t dt + \sigma_X X_t dW_t \\ &= \big(r_d - r_f\big) X_t dt + \sigma_X X_t dW_t. \end{aligned} \tag{7.54}$$

Moreover,

$$X_t = X_0 \exp\left(\big(r_d - r_f - \frac{1}{2}\sigma_X^2\big)t + \sigma_X W_t\right). \tag{7.55}$$

Let us summarize above exploration as a proposition.

Proposition 7.3 ***Pricing formulas***
In the model consisting of a domestic and foreign risk free asset and an FX rate the arbitrage free price $\Pi(t, \mathcal{Z})$ for the simple T-claim $\mathcal{Z} = \Phi(X_T)$ is given by $\Pi(t; \mathcal{Z}) = F(t, X_t)$, where

$$F(t, x) = e^{-r_d(T-t)} E^Q_{t,x}\big[\Phi(X_T)\big], \tag{7.56}$$

and the Q-dynamics of X are given by

$$dX_t = \big(r_d - r_f\big) X_t dt + \sigma_X X_t dW_t. \tag{7.57}$$

Furthermore, the pricing equation is given by

$$\begin{cases} \dfrac{\partial F}{\partial t}(t,x) + (r_d - r_f)x\dfrac{\partial F}{\partial x}(t,x) + \dfrac{1}{2}\sigma_X^2 x^2 \dfrac{\partial^2 F}{\partial x^2}(t,x) - r_d F(t,x) = 0, \\ \hfill F(T,x) = \Phi(x). \end{cases} \tag{7.58}$$

The option pricing formula derived in the previous section has an immediate counterpart in our model for two markets. Since the derivation is similar we only give the result.

Proposition 7.4 ***Option pricing formula***
The price of a European call with strike price K and expiry date T written on the FX rate, i.e. $\mathcal{Z} = \max(X_t - K, 0)$, is given by the following Black-Scholes formula

$$F(t,x) = xe^{-r_f(T-t)}N[d_1] - e^{-r_d(T-t)}KN[d_2], \tag{7.59}$$

where,

$$d_1(t,x) = \frac{1}{\sigma_X\sqrt{T-t}}\left(\ln\left(\frac{x}{K}\right) + \left(r_d - r_f + \frac{1}{2}\sigma_X^2\right)(T-t)\right) \tag{7.60}$$

$$d_2(t,x) = d_1(t,x) - \sigma_X\sqrt{T-t}. \tag{7.61}$$

7.3 The Multidimensional Foreign Exchange Model

Having derived the Black-Scholes model for a domestic and foreign market and an FX rate we are only one step away from a model satisfying the needs of this thesis. Our model needs to be able to account for several foreign markets and FX rates at the same time. These FX rates will naturally be dependent on each other. We will see that by the small addition of correlated Wiener processes for the different FX rates the theory from the previous section will still be valid.

For a model consisting of a domestic market and n foreign markets there will be n FX rates including the domestic currency. These n FX rates will be correlated according to some correlation matrix ρ. Recalling the concept of correlated Wiener processes and using Eq. F.5 in Appendix F the dynamics of the FX rate between the domestic market and the i^{th} foreign market under the measure Q is given by

$$dX_t^i = (r_d - r_f^i)X_t^i dt + \sigma_{X^i}X_t^i dW_t^i, \tag{7.62}$$

for $i = 1, ..., n$. Here W are correlated Wiener processes according to $W = \delta\tilde{W}$, where $\tilde{W}$ are independent Wiener processes and $\rho = \delta\delta^T$. Moreover,

$$X_t^i = X_0^i \exp\left(\left(r_d - r_f^i - \frac{1}{2}\sigma_{X^i}^2\right)t + \sigma_{X^i}W_t^i\right). \tag{7.63}$$

Updating Definition 7.4 and Proposition 7.3 and 7.4 to general multidimensional versions our final framework takes the following form.

Definition 7.5. ***Multidimensional foreign exchange model***
The domestic risk free asset, the foreign risk free assets and the FX rates have the following dynamics under P respectively

$$dB_t^d = r_d B_t^d dt \tag{7.64}$$
$$dB_t^{f,i} = r_f^i B_t^{f,i} dt \tag{7.65}$$
$$dX_t^i = \alpha_{X^i} X_t^i dt + \sigma_{X^i} X_t^i d\overline{W}_t^i, \tag{7.66}$$

where r_d, r_f^i, α_{X^i} and σ_{X^i} are deterministic constants and $\overline{W}^i$ is a scalar Wiener process, $i = 1, ..., n$.

Proposition 7.5 ***Pricing formulas (multidimensional model)***
In the model consisting of a domestic risk free asset, n foreign risk free assets and n FX rates the arbitrage free price $\Pi(t, \mathcal{Z})$ for the simple T-claim $\mathcal{Z} = \Phi(X_T^i)$ is given by $\Pi(t; \mathcal{Z}) = F(t, X_t^i)$, where

$$F(t, x) = e^{-r_d(T-t)} E_{t,x}^Q \left[\Phi(X_T^i) \right]. \tag{7.67}$$

The Q-dynamics of X^i are given by

$$dX_t^i = \left(r_d - r_f^i \right) X_t^i dt + \sigma_{X^i} X_t^i dW_t^i, \tag{7.68}$$

where W are correlated Wiener processes according to $W = \delta \tilde{W}$ and $\rho = \delta \delta^T$ is the correlation matrix between the different FX rates.

Proposition 7.6 ***Option pricing formula (multidimensional model)***
The price of a European call with strike price K and expiry date T written on the i^{th} FX rate, i.e. $\mathcal{Z} = \max(X_t^i - K, 0)$, is given by the following Black-Scholes formula

$$F(t, x) = x e^{-r_f^i(T-t)} N[d_1^i] - e^{-r_d(T-t)} K N[d_2^i], \tag{7.69}$$

where,

$$d_1^i(t, x) = \frac{1}{\sigma_{X^i}\sqrt{T-t}} \left(\ln\left(\frac{x}{K} \right) + \left(r_d - r_f^i + \frac{1}{2}\sigma_{X^i}^2 \right)(T-t) \right), \tag{7.70}$$
$$d_2^i(t, x) = d_1(t, x) - \sigma_{X^i}\sqrt{T-t}. \tag{7.71}$$

8 Simulation of Foreign Exchange Rates

This section presents the historical dataset and describes the different methodologies that have been used in order to simulate future FX rates. For the sake of brevity the main text includes figures and results concerning the interrelation between the European and the Swedish markets. Figures and results for the other three FX rates are found in Appendix B.

8.1 Historical Dataset

The Thomson Reuters Eikon database has been used to collect weekly historical FX rates. Weekly risk free interest rates have been obtained from the Swedish central bank, Riksbanken.

The dataset consists of interest rates and FX rates quoted weekly from January 7, 2011 to December 29, 2017. The time period was chosen in order to exclude the effects of extremely volatile FX rates that persisted during the financial crisis. Weekly rates were found to give the best trade-off between precision and time consumption in the simulations. The yields of 10 year government bonds have been used as proxies for risk free interest rates.

The currencies used throughout the thesis are Euro (EUR), Swedish Krona (SEK), Norwegian Krone (NOK), Pound Sterling (GBP) and US Dollar (USD). Since the PE funds are denominated in EUR in this thesis the EUR market will be considered the domestic market and the four other currencies will be considered foreign markets. Therefore, the currencies are quoted as EUR per 1 of the other currencies, e.g. SEK/EUR denotes EUR per 1 SEK.

CHF was initially included in the dataset but at a later point discarded since it was found to have been pegged to the EUR until January 15, 2015. As the Swiss National Bank unpegged the CHF it depreciated a lot. Not removing the CHF/EUR rate would have given rise to unnatural correlations. Since all other FX rate data is from the beginning of 2011 to the end of 2017, the CHF was chosen to not be included.

Figure 8.1 shows the historical risk free interest rates. As can be seen in the figure both the Swedish and the EU rate have decreased over time. However, the EU rate has decreased more than the Swedish. Moreover, Figure 8.2 shows the evolution of the SEK/EUR rate and Table 8.1 shows the observable correlation between the FX rates. It does not come as a surprise that the Swedish and Norwegian FX rates are heavily correlated. Similar plots for the other markets are shown in Figures B.1-B.2 in Appendix B.

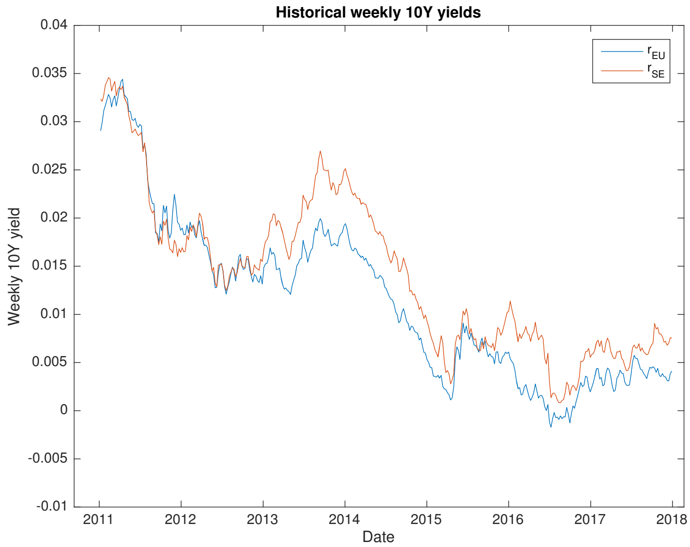

Figure 8.1: Historic development of the European and Swedish risk free interest rates

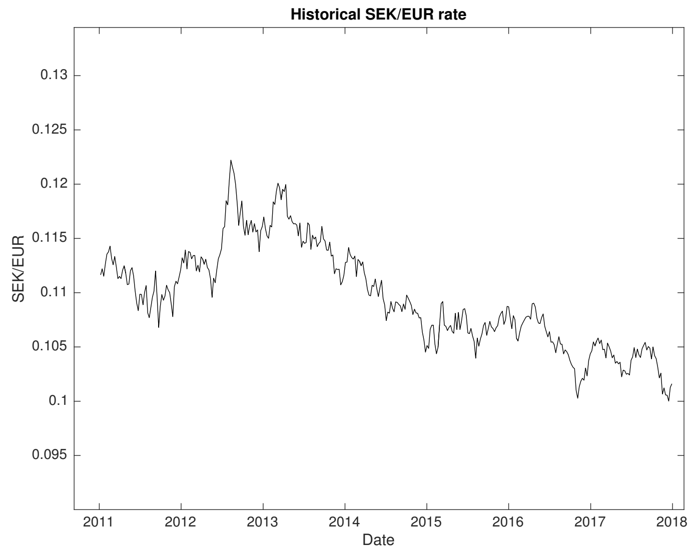

Figure 8.2: Historic development of the SEK/EUR rate

	SEK/EUR	NOK/EUR	GBP/EUR	USD/EUR
SEK/EUR	1	0.86	-0.04	-0.66
NOK/EUR	0.86	1	-0.18	-0.75
GBP/EUR	-0.04	-0.18	1	0.41
USD/EUR	-0.66	-0.75	0.41	1

Table 8.1: Correlation between the different FX rates

8.2 The Random Walk Method

We denote by the random walk method the method for simulating FX rates derived in the multidimensional FX Black-Scholes model. The European market is considered to be the domestic market and the four other markets are considered to be foreign markets 1 to 4. The following formula describes how the FX rates are modelled,

$$X_t^i = X_0^i \exp\left((r_d - r_f^i - \frac{1}{2}\sigma_{X^i}^2)t + \sigma_{X^i} W_t^i \right), \tag{8.1}$$

where $i = 1, ..., 4$ corresponds to the four cases SEK, NOK, GBP and USD respectively. Moreover, the start date, $t = 0$, is the last date of the historical data, i.e. December 29, 2017. The time, t, is measured in weeks, hence the other present constants, r_d, r_f^i and σ_{X^i} must be transformed into effective weekly interest rates and volatilities.

Noting that the most recent values of r_d and r_f^i are the most relevant ones in terms of them being the ones that fiscal policies finds suitable at the moment we chose to set r_d and r_f^i as their last known quoted values, i.e. there values upon December 29, 2017. One could possibly have argued for taking an average over time or making an own choice for their values but we do not find that a relevant focus of this thesis to explore any further. The values that are being used are obtained according to $r_{weekly} = \frac{r}{52}$ and are summarized below,

$$r_d = 0.79 \cdot 10^{-4},$$

$$r_f^1 = 1.45 \cdot 10^{-4}, \qquad r_f^2 = 3.05 \cdot 10^{-4}, \qquad r_f^3 = 2.28 \cdot 10^{-4}, \qquad r_f^4 = 4.66 \cdot 10^{-4}.$$

Having computed the risk free interest rates the only constants left to determine are the volatilities of the FX rates, σ_{X^i}. The following two sections discuss alternative methods for assigning their values.

8.2.1 Historical Volatility

The historical volatility is defined as the volatility of the historical data sample and is obtained through estimating σ_X from historical observations of X. Using maximum likelihood estimation to estimate σ_X the likelihood function is maximized and therefore also the probability of observing the historical data. Note that since observations are used we must work under the observable probability measure P. Below follows the computations.

Since X_t is log-normally distributed, $X_t \sim LN(\alpha_X, \sigma_X^2)$, its log-returns, $Y_{t_i} = \ln\left(\frac{X_{t_i}}{X_{t_{i-1}}}\right)$, are normally distributed according to

$$Y_{t_i} \sim N\left((\alpha_X - \frac{1}{2}\sigma_X)\Delta t, \sigma_X^2 \Delta t \right), \tag{8.2}$$

where $\Delta t = 1$ and therefore can be omitted. Furthermore, the probability density function of Y_{t_i} is thus given by

$$\varphi_{Y(t_i)}(y_i) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left(- \frac{(y_i - (\alpha_X - \frac{1}{2}\sigma_X^2))^2}{2\sigma_X^2} \right). \tag{8.3}$$

The likelihood function can now be written as

$$\begin{aligned} L(y_1, ..., y_n; \alpha_X, \sigma_X^2) &= \prod_{i=1}^{n} \varphi_{Y(t_i)}(y_i) \\ &= \frac{1}{(2\pi\sigma_X^2)^{\frac{n}{2}}} \exp\left(- \frac{\sum_{i=1}^{n}(y_i - (\alpha_X - \frac{1}{2}\sigma_X^2))^2}{2\sigma_X^2} \right), \end{aligned}$$

and consequently, the log-likelihood function is obtained as

$$\begin{aligned} l(y_1, ..., y_n; \alpha_X, \sigma_X^2) &= \ln\left(L(y_1, ..., y_n; \alpha_X, \sigma_X^2)\right) \\ &= \ln\left(\frac{1}{(2\pi\sigma_X^2)^{\frac{n}{2}}} \right) - \frac{\sum_{i=1}^{n}(y_i - (\alpha_X - \frac{1}{2}\sigma_X^2))^2}{2\sigma_X^2} \\ &= -\frac{n}{2} \ln\left(2\pi\sigma_X^2\right) - \frac{\sum_{i=1}^{n}(y_i - (\alpha_X - \frac{1}{2}\sigma_X^2))^2}{2\sigma_X^2}. \end{aligned}$$

Next we seek to find the values of α_X and σ_X^2 that maximize the likelihood function and hence also the log-likelihood function. Differentiating the log-likelihood function with respect to α_X and σ_X^2 and simplifying yields

$$\frac{\partial}{\partial(\alpha_X)} l(y_1, ..., y_n; \alpha_X, \sigma_X^2) = \frac{1}{\sigma_X^2}\left(- n(\alpha_X - \frac{1}{2}\sigma_X^2) + \sum_{i=1}^{n} y_i \right) \tag{8.4}$$

$$\begin{aligned} \frac{\partial}{\partial\sigma_X^2} l(y_1, ..., y_n; \alpha_X, \sigma_X^2) &= \frac{-n\sigma_X^2 - \sum_{i=1}^{n}(y_i - (\alpha_X - \frac{1}{2}\sigma_X^2))\sigma_X^2}{2\sigma_X^4} \\ &\quad + \frac{\sum_{i=1}^{n}(y_i - (\alpha_X - \frac{1}{2}\sigma_X^2))^2}{2\sigma_X^4}. \end{aligned} \tag{8.5}$$

In order to have a maximum the obtained expressions need to satisfy

$$\frac{\partial}{\partial(\alpha_X)} l(y_1, ..., y_n; \alpha_X, \sigma_X^2) = 0 \tag{8.6}$$

$$\frac{\partial}{\partial\sigma_X^2} l(y_1, ..., y_n; \alpha_X, \sigma_X^2) = 0. \tag{8.7}$$

Rearranging these two equations gives an equation system for α_X and σ_X^2 according to

$$\hat{\alpha}_X = \frac{1}{2}\hat{\sigma}_X^2 + \frac{1}{n}\sum_{i=1}^{n} y_i \tag{8.8}$$

$$\hat{\sigma}_X^2 = 2 \cdot \left(-1 \pm \sqrt{1 + \frac{1}{n}\sum_{i=1}^{n}(y_i - (\hat{\alpha}_X))^2} \right). \tag{8.9}$$

Using MATLAB to solve the above system for the $n = 364$ computed log-return values of the SEK/EUR rate, y_i, the maximum likelihood estimates are obtained as

$$\hat{\alpha}_X = -2.59 \cdot 10^{-4} \quad \text{and} \quad \hat{\sigma}_X = 0.86 \cdot 10^{-2}.$$

As motivated in the previous section we will use the last observed values of r_d and r_f when simulating FX rates, hence we will discard $\hat{\alpha}_X$. However, it was still essential for finding a value of $\hat{\sigma}_X$. The historical volatilities for the four FX rates are given by

$$\hat{\sigma}_{X_1} = 0.86 \cdot 10^{-2}, \qquad \hat{\sigma}_{X_2} = 1.04 \cdot 10^{-2}, \qquad \hat{\sigma}_{X_3} = 1.12 \cdot 10^{-2}, \qquad \hat{\sigma}_{X_4} = 1.27 \cdot 10^{-2}.$$

8.2.2 Implied Volatility

The implied volatility is obtained through estimating σ_X from observations of current option prices on the market using Black-Scholes formula. After observing present option prices Black-Scholes formula, Eq. 7.69, can be used to back out the volatility that gives rise to the observed option price.

Implied volatility is a reaction to historical volatility's use of historic data, since in reality volatility is not constant over time historical volatility may prove to be suboptimal. Thus an estimate of the volatility for the upcoming period, rather than the past, is desirable and therefore the market's expectation of the volatility should be used.

Denoting the price of European calls by

$$p = c(x, t, T, r_d, r_f, \sigma_X, K), \tag{8.10}$$

$\hat{\sigma}_X$ is obtained as the value that best fulfils the above equality given the observed values of p. Choosing arbitrary values for T and K option prices p were observed. After that Black-Scholes formula was used to back out the implied volatilities. The obtained implied volatilities for the four FX rates are given by

$$\hat{\sigma}_{X_1} = 1.15 \cdot 10^{-2}, \qquad \hat{\sigma}_{X_2} = 1.25 \cdot 10^{-2}, \qquad \hat{\sigma}_{X_3} = 1.28 \cdot 10^{-2}, \qquad \hat{\sigma}_{X_4} = 1.28 \cdot 10^{-2}.$$

Comparing the obtained values for the implied volatilities we find them to be larger than the historic volatilities for the SEK/EUR and NOK/EUR cases and similar in size for the GBP/EUR and USD/EUR cases. Overall, the implied volatilities are similar to the historic volatilities in size which is satisfactory. Thus, following the above arguments and the fact that the discrepancies between the results of two methods are minor, from hereinafter the implied volatilities will be used as estimates for the true volatilities.

8.2.3 Simulated Foreign Exchange Rates

Using the random walk method with the implied volatilities the development of the FX rates can now be simulated. We then recalculate the obtained weekly values to quarterly values assuming that each quarter is exactly 13 weeks. MATLAB is used to simulate correlated Wiener processes as discussed in Section 7.3. Note that even though the historical correlations given in Table 8.1 are observed under P they are invariant under the Girsanov transformation from P to Q and therefore the same correlations are valid under Q. The general motivation follows below and leads to the same result that can be found in Karatzas and Shreve (1998) [19].

Under P, using the same notation as in Definition F.3 in Appendix F, it holds that

$$d\overline{W}_t^i = \sum_{k=1}^{d} \delta_{ik} d\overline{\tilde{W}}_t^k. \tag{8.11}$$

Under Q, according to Girsanov's theorem

$$d\overline{\tilde{W}}_t^k = \varphi_k(t)dt + d\tilde{W}_t^k, \tag{8.12}$$

and therefore it also holds that

$$dW_t^i = \sum_{k=1}^{d} \delta_{ik} d\tilde{W}_t^k = \sum_{k=1}^{d} \delta_{ik} \left(d\overline{\tilde{W}}_t^k - \varphi_k(t)dt \right). \tag{8.13}$$

Thus it can be deduced that

$$d\langle W_t^i, W_t^j \rangle = \sum_{k=1}^{d} \delta_{ik}\delta_{jk} dt = \rho_{ij} dt, \tag{8.14}$$

and hence the correlation coefficients are invariant under Girsanov transformations.

The upper plot of Figure 8.3 shows the expected development of the SEK/EUR rate together with three simulated example paths over a 40 quarter period. Since the expectation of the log-normally distributed FX rate X_t is given by $E[X_t] = X_0 e^{(r_d - r_f)t}$ and $r_d - r_f < 0$ it comes as no surprise that the SEK/EUR rate is expected to depreciate over time.

The lower plot of Figure 8.3 shows the probability distribution function of $n = 10,000$ simulations of the SEK/EUR rate, X^1, at the end of the fund period, i.e. after 40 quarters. We observe that the majority of the simulations end up close to the expected value but that there however are cases in which the FX rate both halved and doubled compared to today's value. These are however very rare.

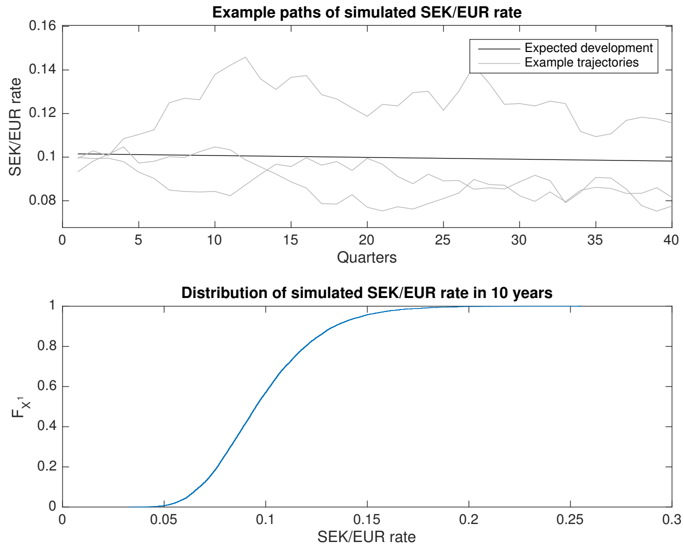

Figure 8.3: Expected future development (upper) and empirical distribution function of the 10y SEK/EUR rate (lower)

Similar results are obtained for the three other FX rates as well. Since $r_d - r_f^i < 0$ holds for $i = 1, ..., 4$ all FX rates are expected to depreciate over time. Expected development and example paths for all FX rates are found in Figure B.3 in Appendix B. Probability distribution functions of $n = 10,000$ simulations after 40 quarters for all FX rates are found in Figure B.4 in Appendix B.

8.2.4 The Zero Drift Case

In this section we discuss our first alternative way of simulating FX rates. Namely a random walk model with zero drift, i.e. we set $r_f^i = r_d$ for $i = 1, ..., 4$. Hence $r_d - r_f^i = 0$ which implies that $E[X_t] = X_0 e^{(r_d - r_f)t} = X_0$. Put in words, we force the future expected values of the FX rates to be equal to today's values.

A motivation for this alternative model is that the Black-Scholes model's assumption of interest rates being constant is quite unreasonable. It is not probable that the observed interest rates today will remain constant for a 10 year period to come, on the contrary, it is utterly unlikely. Therefore, one can motivate that by letting $r_f^i = r_d$ we do not take today's view of what is going to happen with the FX rates in the future, but rather choose to not include any drift and instead let the diffusion decide the entire path.

Expected development and example paths for all FX rates are found in Figure B.5 in Appendix B. Probability distribution functions of $n = 10,000$ simulations after 40 quarters for all FX rates are found in Figure B.6 in Appendix B.

8.3 The Historical Simulation Method

The historical simulation method, described in Hult et al. (2012) [17], is an alternative way of simulating FX rates. The method models future returns from observed historical returns. Having observed n historical values, $X_{-n+1}, ..., X_0$, $n-1$ historical returns can be computed according to

$$y_k = \frac{X_k}{X_{k-1}}, \quad k = -n+2, ..., 0. \tag{8.15}$$

Assuming that the historical returns are independent the future returns can be simulated by drawing uniformly with replacement from the set of historical returns $y = \{y_{-n+2}, ..., y_0\}$.

Moreover, the correlation between the four different FX rates is taken into account by drawing all four FX rates from the same date, i.e. we draw a random number from the index set of y and then we chose all four y^i from the same date.

Expected development and example paths for all FX rates are found in Figure B.7 in Appendix B. Probability distribution functions of $n = 10,000$ simulations after 40 quarters for all FX rates are found in Figure B.8 in Appendix B.

Comparing the results of the current simulation method to the random walk simulation method it is possible to make some interesting observations. The returns of the FX rates have on average been negative historically except for the USD/EUR rate. This results in the expected value of the SEK/EUR and GBP/EUR rates having approximately the same development now as under the random walk method. Though, the expected development of the NOK/EUR rate depreciates faster than before and the USD/EUR rate actually appreciates. This implies that the overall results of the fund simulations might depend on which simulation method is used.

8.4 The Copula Simulation Method

Inspiration about the theory of copulas is to a large extent gathered from Hult et al. (2012) [17]. Copulas are used to construct a multivariate distribution for a random vector $\boldsymbol{X} = (X_1, ..., X_d)$ with specified univariate marginal distribution functions $F_1, ..., F_d$ but unspecified multivariate distribution. Using the quantile transform the desired dependence can be introduced by setting $\boldsymbol{X} = (F_1^{-1}(U_1), ..., F_d^{-1}(U_d))$ where $\boldsymbol{U}$ has the desired multivariate distribution and its components satisfy $U_k \sim U(0,1)$. The vector $\boldsymbol{X}$ then inherits the dependence from $\boldsymbol{U}$ among its components. The distribution function C of the random vector $\boldsymbol{U}$, i.e. $C(u_1, ..., u_d) = P(U_1 \leq u_1, ..., U_d \leq u_d)$, is called the copula of $\boldsymbol{X}$.

If $X_1, ..., X_d$ have the multivariate distribution function $F(x_1, ..., x_d)$ and the univariate marginal distributions, $F_k(x)$, are continuous then the fallowing relation holds

$$C(F_1(x_1), ..., F_d(x_d)) = P(U_1 \leq F_1(x_1), ..., U_d \leq F_d(x_d)) =$$
$$= P(F_1^{-1}(U_1) \leq x_1, ..., F_d^{-1}(U_d) \leq x_d) = F(x_1, ..., x_d),$$

showing that the joint distribution function F can be expressed entirely by the marginal distributions, $F_1, ..., F_d$, and the copula, C.

Our goal is to model the four FX rates' comovements using a copula. In order to do this we need to know their univariate marginal distributions as well as deciding on the form of their multivariate distribution. Two alternative copulas will be used to model the multivariate distribution, a Gaussian copula and a Student's t copula. First we begin by assigning suitable marginal distributions.

8.4.1 Fitting of Marginal Distributions

In our particular case the univariate distributions of the four FX rates are not known. However, by the use of Quantile-Quantile plots (Q-Q plots) and least square estimation (LSE) we can find what distribution that best suits the observed data and hence approximate the unknown distributions by these best fitting distributions.

Given a set of observations, we recall that a Q-Q plot is a plot of the observations' empirical quantiles against the quantiles of a reference distribution F. Q-Q plots can thus be used to graphically test whether the observations from a sample come from a specified reference distribution F. If the observations come from a probability distribution similar to the reference distribution the Q-Q plot will be approximately linear with intercept 0 and slope 1. If the empirical distribution of the observations has a heavier right tail than F the Q-Q plot will curve up for large values on the x-axis. If the empirical distribution of the observations has a heavier left tail than F the Q-Q plot will curve down for small values on the x-axis. Finally, if the empirical tails of the observations are lighter than the tails of F the Q-Q plot will show the opposite behavior as previously described.

The least square estimates, are defined as the parameter values that minimize the sum of the squared deviations between the empirical quantiles of the observations and the quantiles of the parametric reference distribution. Given n observations ordered from the largest to the smallest, $z_{1,n} \geq z_{2,n} \geq ... \geq z_{n,n}$, and a reference distribution F the LSE parameters are given by

$$\hat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \sum_{i=1}^{n} \left(z_{i,n} - F_{\theta}^{-1}\left(\frac{n-i+1}{n+1}\right)\right)^2. \qquad (8.16)$$

In particular, the least square estimate for a Student's t distribution is the parameter triplet $(\hat{\nu}, \hat{\mu}, \hat{\sigma})$ that minimizes the sum of the squared deviations between the empirical and Student's t quantiles

$$(\hat{\nu}, \hat{\mu}, \hat{\sigma}) = \arg\min_{\nu,\mu,\sigma} \sum_{i=1}^{n} \left(z_{i,n} - \mu - \sigma \cdot t_{\nu}^{-1}\left(\frac{n-i+1}{n+1}\right)\right)^2, \qquad (8.17)$$

where $\sigma > 0$ and $\nu > 0$ by definition.

Figure 8.4 shows the observed quantiles of the log-returns of the SEK/EUR rate against standard normal tails (upper) and the least square fitted Student's t tails (lower.) Similar plots for the other FX rates are to be found in Figure B.9 in Appendix B. For all FX rates it is found that a Student's t distribution is the best fit. The lower the degrees of freedom, ν, the heavier the tails are. As seen in Table 8.2 the log-returns of the NOK/EUR rate is found to have the heaviest tails whereas the tails of the log-returns of the SEK/EUR rate are the lightest and therefore the SEK/EUR rate resembles a normal distribution the most.

For the other fitted parameters both $\hat{\mu}$ and $\hat{\sigma}$ are of similar sizes as found in the maximum likelihood estimation of Section 8.2. However, as previously motivated, currently prevailing values of r_d and r_f^i as well as implied volatility will be used to price derivatives.

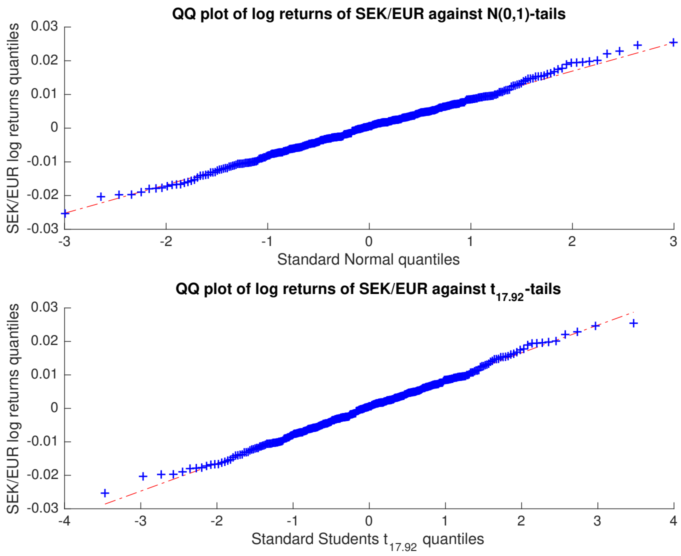

Figure 8.4: Q-Q plots of the log-returns of the SEK/EUR rate against normal and Student's t tails

	$\hat{\nu}$	$\hat{\mu}$	$\hat{\sigma}$
SEK/EUR	19.92	$-0.68 \cdot 10^{-4}$	$0.83 \cdot 10^{-2}$
NOK/EUR	5.95	$-4.09 \cdot 10^{-4}$	$0.90 \cdot 10^{-2}$
GBP/EUR	10.28	$2.79 \cdot 10^{-4}$	$1.04 \cdot 10^{-2}$
USD/EUR	9.97	$-0.32 \cdot 10^{-4}$	$1.15 \cdot 10^{-2}$

Table 8.2: Least square fitted parameters for a Student's t location-scale family

8.4.2 Gaussian Copula

The third alternative way of simulating FX rates is through the use of a Gaussian copula. The FX rates are assumed to have the univariate marginal distributions found in the previous section and a Gaussian multivariate distribution. Below follows the algorithm for simulating future FX rates with a Gaussian copula.

Random variate generation from the Gaussian copula is done as follows. First the Cholesky decomposition of the correlation matrix ρ is computed according to

$\rho = \delta\delta^T$. Then d independent random variates $Z_1, ..., Z_d$ are simulated from a standard normal distribution. Setting $\boldsymbol{X} = \delta\boldsymbol{Z}$ and $U_k = \Psi(X_k)$ for $k = 1, ..., d$ the vector $\boldsymbol{U} = (U_1, ..., U_d)$ has the distribution function C_ρ^{Ga}.

Expected development and example paths for all FX rates are found in Figure B.10 in Appendix B. Cumulative distribution functions of $n = 10,000$ simulations after 40 quarters for all FX rates are found in Figure B.11 in Appendix B. Comparing the results of the current simulation method to the random walk simulation method it is possible to make some interesting observations. The fitted marginal distributions assign negative expected values for the log-returns of all FX rates except for the GBP/EUR rate. From the expected development plot it is found that the SEK/EUR rate is expected to have the same development as in the random walk simulation, the NOK/EUR rate depreciates more whereas the USD/EUR rate depreciates less. Finally, the GBP/EUR rate is found to appreciate. This implies that the overall results of the PE fund simulations might depend on which simulation method is used.

8.4.3 Student's t Copula

The fourth alternative way of simulating FX rates is through the use of a Student's t_4 copula. As before the FX rates are assumed to have the univariate marginal distributions found in Section 8.4.1 and a Student's t_4 multivariate distribution. The difference between a Gaussian copula and a Student's t copula is that the Student's t copula will have heavier tails. This implies that extreme values in one FX rate to a larger degree will infer extreme values in the other FX rates. The lower the degrees of freedom the heavier the tails. Having $\nu = 4$ degrees of freedom is considered relatively heavy tails. Below follows the algorithm for simulating future FX rates with a Student's t copula.

Random variates from a t_ν distribution is generated by computing the Cholesky decomposition of the correlation matrix ρ according to $\rho = \delta\delta^T$ and simulating d independent random variates $Z_1, ..., Z_d$ from a standard $N(0, 1)$ distribution. Then a random variate, S, is simulated from a χ^2_ν distribution independent of $Z_1, ..., Z_d$. Proceeding by setting $\boldsymbol{X} = \frac{\sqrt{\nu}}{\sqrt{S}}\delta\boldsymbol{Z}$ and $U_k = t_\nu(X_k)$ for $k = 1, ..., d$ the vector $\boldsymbol{U} = (U_1, ..., U_d)$ has the distribution function $C_\rho^{t_\nu}$.

Expected development and example paths for all FX rates are found in Figure B.12 in Appendix B. Cumulative distribution functions of $n = 10,000$ simulations after 40 quarters for all FX rates are found in Figure B.13 in Appendix B. Comparing the results of the current simulation method to the random walk simulation method it is possible to do some interesting observations. The fitted marginal distributions assign negative expected values for the log-returns of all FX rates except for the GBP/EUR rate. From the expected development plot it is found that the SEK/EUR rate is expected to have the same development as in the random walk simulation, the NOK/EUR rate depreciates more whereas the USD/EUR rate depreciates less. Finally, the GBP/EUR rate is found to appreciate. However, these are the exact same behaviours shown in the Gaussian copula case. Though, the tails of the Student's t copula simulation are heavier as seen when comparing Figure B.13 to Figure B.11 in Appendix B.

9 Performance Measures

This section sets out to present a handful of performance measures that will be taken into consideration when analysing whether it would be beneficial or not for a PE fund to hedge its FX exposure. First the main measure, the IRR, is formalized. After that the concepts Value-at-Risk, Expected shortfall and Sharpe ratio are defined. The theory of IRR and Sharpe ratio is covered in Berk and DeMarzo (2011) [1] whereas the theory regarding Value-at-Risk and Expected shortfall is covered in Hult et al. (2012) [17].

9.1 Internal Rate of Return

The IRR is the discount rate that makes the net present value (NPV) of all cash flows from a specific project equal to zero. That means that the IRR is obtained through solving the following equation

$$NPV = \sum_{t=0}^{T} \frac{C_t}{(1+r)^t} = 0, \tag{9.1}$$

where C_t is the cash flow at time t. Due to the form of the equation it has to be solved numerically.

In the setting of this thesis the IRR will be calculated for an entire PE fund. This means that all acquisitions of portfolio companies in the beginning of the fund life cycle, divestment of portfolio companies in the end of the fund life cycle as well as cash flows from hedging derivatives will be included as cash flows in the above formula and therefore affect the IRR. Moreover, the IRR will depend on the growth rates of the portfolio companies as well as the developments of the FX rates and what hedging instruments are used since all these parameters affect the cash flows.

9.2 Value-at-Risk

We begin by stating the classic definition of Value-at-Risk (VaR). However, in this thesis' setting an alternative definition is preferable and is introduced afterwards.

The VaR at level $p \in (0, 1)$ of a position with value X at time $t = 1$ is the smallest amount of money that if added to the position by investing it in the risk free interest rate at time $t = 0$ ensures that the probability of a strictly negative value of the position at time $t = 1$ is less than or equal to p. Thus, VaR_p is given by

$$VaR_p(X) = \min(c : P(cr_0 + X < 0) \leq p), \tag{9.2}$$

where r_0 denotes the risk free interest rate. The value X can be thought of as $X = V_1 - V_0 r_0$, where V_1 is the value of the position at time $t = 1$ and V_0 is the value of the position at time $t = 0$, i.e. the original money invested was borrowed.

$VaR_p(X)$ may be interpreted as the smallest value c such that the probability of the discounted portfolio loss, $L = -\frac{X}{r_0}$, being at most c is at least $1-p$. This can be seen through

$$\begin{aligned}
VaR_p(X) &= \min(c : P(cr_0 + X < 0) \leq p) \\
&= \min(c : P\big(-\frac{X}{r_0} > c\big) \leq p) \\
&= \min(c : 1 - P\big(-\frac{X}{r_0} \leq c\big) \leq p) \\
&= \min(c : P\big(-\frac{X}{r_0} \leq c\big) \geq 1-p) \\
&= \min(c : P(L \leq c) \geq 1-p) \\
&= \min(c : F_L(c) \geq 1-p).
\end{aligned}$$

Recalling that the u-quantile of a random variable L with distribution function F_L is given by

$$F_L^{-1}(u) = \min(c : F_L(c) \geq u), \tag{9.3}$$

we reach the conclusion that $VaR_p(X)$ is nothing else than the $(1-p)$-quantile of L. Thus it can be expressed as

$$VaR_p(X) = F_L^{-1}(1-p). \tag{9.4}$$

In the setting of this thesis using the above definition the VaR will always be smaller than zero, since the returns of the PE funds are extremely high. Thus the measure becomes somewhat pointless. Therefore, instead of looking at monetary values we chose to look directly at the quantile values of the IRR instead. Hence, by VaR_p we denote the IRR value corresponding to the p-quantile. So for example, if we are interested in the VaR at the $p = 5\%$ level and $n = 100$ simulations are performed the $VaR_{0.05}$ is the fifth lowest IRR value obtained since $np = 5$. We are aware of that denoting this measure VaR is somewhat misguiding since it is not a monetary value per se. But through the obvious resemblance to the classic definition of VaR we think it is the neatest notation to use.

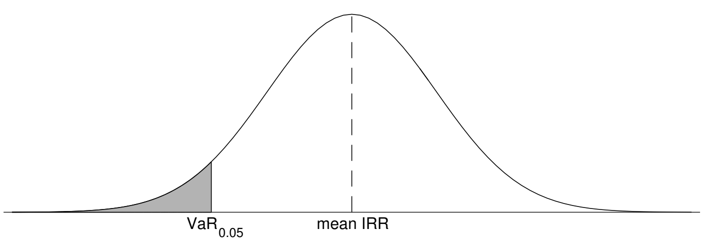

Figure 9.1: Illustration of VaR, note that the average of the shaded area is $ES_{0.05}$

9.3 Expected Shortfall

Classically the expected shortfall (ES) is defined as the average VaR value below the level p, i.e.

$$ES_p(X) = \frac{1}{p}\int_0^p VaR_u(X)du. \tag{9.5}$$

The advantage of ES over VaR is that instead of only considering a quantile value it takes all the most extreme cases beyond level p into consideration. This means that highly unlikely scenarios but with catastrophic effects in the case they were to come through also are taken into consideration.

Since we use a modified VaR the ES will also indirectly become modified. In the same way as VaR will be expressed as a percentage value ES is also going to be expressed as a percentage value, namely the average IRR of the np worst outcomes in terms of IRR, as seen in Figure 9.1.

9.4 Sharpe Ratio

The Sharpe ratio (SR) is a measure of risk-adjusted return. It is defined as the average return in excess of the risk free rate per unit of volatility,

$$SR = \frac{r_{fund} - r_d}{\sigma_{fund}}, \tag{9.6}$$

where r_d is the domestic risk free interest rate. The measure is thus high if the returns of an asset are stable and low if they vary a lot.

The relevance of the Sharpe ratio measure for a PE fund can be discussed since the returns in their nature are very risky. However, in terms of hedging, it would come as no surprise if the Sharpe ratio of the hedged funds were larger than the Sharpe ratio of the unhedged funds.

10 Methodology

This section sets out to describe the approach used to obtain the results of the thesis. First, PE FX exposure is defined. Second, we describe how the PE funds under consideration are constructed. Third, the structure of the simulation model is described. Fourth, we highlight the assumptions and limitations of our model.

Our study of the topic began by the conduction of four interviews. Two of the interviews were with experienced private equity professionals. Another interview was with a banking professional specialised in hedging within PE. The last interview was with a Stockholm School of Economics PE professor. These interviews partly laid the foundation of our work.

10.1 Private Equity Foreign Exchange Exposure

As argued and concluded in Section 3.4, the FX exposure of each portfolio company in a PE fund is regarded to be its NAV exposures in each different currency. The most beneficial way of defining PE FX exposure is as the total net NAV exposure of all the portfolio companies in each currency within a fund. This definition yields a fully hedged position while minimizing transaction costs since the least amount of positions is taken.

Alternatively, the PE FX exposure could have been defined as the net exposure of each portfolio company within a specific PE fund separately. This definition also yields fully hedged positions, though more positions need to be taken. Hereinafter the PE fund's FX exposure in each currency will be defined as the total net NAV exposure of all the fund's portfolio companies in that currency.

10.2 Fund Construction

In order to make the characteristics of the funds as realistic as possible, the data of the PE funds has been composed by the authors together with a private equity professional. However, note that the PE funds are hypothetical and not examples of real funds.

In total twelve different hypothetical PE funds of size EUR 3 billion are considered. There are three different dimensions in which the funds differ. A fund can be either *local*, meaning that each portfolio company have all its business in one currency, or *global*, meaning that each portfolio company may have business in several currencies. Moreover, both *equity* and *infrastructure* funds are considered. Equity funds are modelled to have an expected IRR of 20% whereas infrastructure funds are modelled to have an expected IRR of 12%. Finally, three different FX exposure scenarios, A,B and C, are considered. In all scenarios 50% of the equity is in EUR. In scenario A the FX exposure is equal between the foreign currencies, in scenario B there is an over exposure in SEK and in scenario C there is an over exposure in USD. Table 10.1 summarizes the FX exposures of the funds. Taking all possible different combinations of the three dimensions a total of twelve funds are obtained.

In turn, all funds contain twelve portfolio companies. The investments in the portfolio companies take place in the beginning of the first, the second and the third year of the fund's lifetime, which is assumed to be ten years. For all funds, the expected duration of each investment is assumed to be five years, i.e. the expected holding time of each portfolio company is five years. Moreover, a quarterly cumulative average growth rate is assigned to each portfolio company's NAV. The assigned growth rates are semi-randomized in the sense that they are set by the authors to realistic scenarios under the constraint of the given expected IRR of the funds. The expected growth rate of each portfolio company is set to the expected growth rate of the fund, i.e. 20% for equity funds and 12% for infrastructure funds. For a complete summary of the twelve funds, see Figures A.1-A.6 and Tables A.1-A.6 in Appendix A. Note that in the Tables A.1-A.6 there is a high and a low quarterly growth rate, corresponding to equity funds and infrastructure funds respectively.

Fraction of Exposure	**EUR**	**SEK**	**NOK**	**GBP**	**USD**
Global PE Fund A	50%	12.5%	12.5%	12.5%	12.5%
Global PE Fund B	50%	35%	5%	5%	5%
Global PE Fund C	50%	5%	5%	5%	35%
Local PE Fund A	50%	12.5%	12.5%	12.5%	12.5%
Local PE Fund B	50%	35%	5%	5%	5%
Local PE Fund C	50%	5%	5%	5%	35%

Table 10.1: Foreign exchange exposure of the PE funds upon entering the portfolio companies

The above FX exposures are realistic for large Swedish Euro denominated PE funds. The three different FX exposure setups, A, B and C, are chosen in order to investigate if there are any differences between balanced funds, SEK heavy funds and USD heavy funds in terms the efficiency of hedging. Since SEK and USD might have different relations with EUR having a larger fraction of SEK might imply a greater or lower FX risk than having the same fraction in USD. Other overweights are not investigated since SEK and USD are the most likely overweights a Swedish EUR denominated PE fund would have.

10.3 Simulation Model

This section describes the construction of the simulation model in MATLAB. The simulation model builds on the theory developed in the previous sections and is based on quarterly data. It can of course be built for any time steps. However, quarterly data was found to be a good trade-off between speed of iteration and quality of iteration since the main simulation can be run over a night.

10.3.1 Private Equity Funds

The simulation model begins by reading in the information of the specific PE fund under consideration. Information of the PE fund includes all the portfolio companies, their NAVs in different currencies and when the entry in each portfolio company takes place. Furthermore, information regarding quarterly growth as well as expected quarterly growth and expected holding time is included for each portfolio company. For full information of the PE funds used see Section 10.2 as well as Appendix A.

The expected quarterly growth rates for the portfolio companies are set to 4.66% for the equity funds and 2.87% for the infrastructure funds, which means annualised growth rates of 20% and 12%, respectively. The expected growth rates and the expected holding times are needed later in the model when applying the hedging strategies.

10.3.2 Lifetime of the PE Fund and Adding Uncertainty

Next, the lifetime of the PE funds is considered. As described in Section 3, a normal lifetime of a PE fund is ten years, this is also chosen. In order to make the model realistic, stochastic holding times for each portfolio company are included in the model. The rational for this is that fund managers do not know when the portfolio companies will be spun off when investing in them. In more detail, random normally distributed numbers, with zero mean are drawn. These numbers, which represents quarters, are added to the expected holding times to obtain the actual holding times. This procedure is done for each portfolio company separately.

We have determined that the minimum holding time cannot be less than eight quarters and the maximum holding time cannot exceed 32 quarters. This is because symmetry around the expected holding time of 20 quarters is wanted and because a portfolio company should be exited within the lifetime of the PE fund.

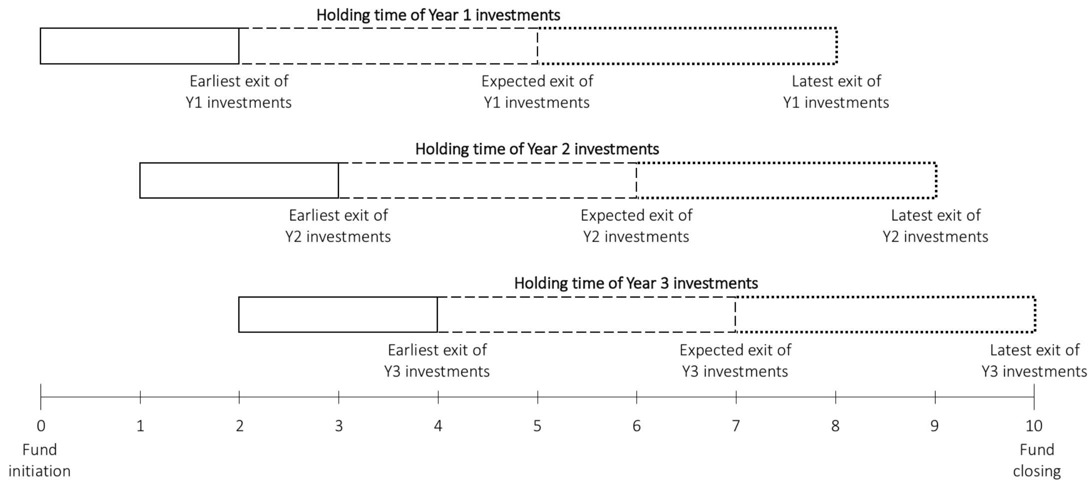

Figure 10.1: Illustration of the stochastic holding times, note that each portfolio company has an individual stochastic holding time

10.3.3 Foreign Exchange Rates

The second input argument to the model is simulated quarterly FX rates. FX rates are simulated separately, as described in Section 8, and then stored in order for the exact same rates to be used across all different fund simulations. These FX rates are used to calculate the NAVs of the portfolio companies in different currencies at different times and value the hedging derivatives.

10.3.4 Cash Flows from the Portfolio Companies

At this stage, all required information to perform the fund simulation and compute the cash flows of the portfolio companies exist. Next a cash flow matrix which keeps track of the cash flows of the PE fund and at which quarter they occur is created. Investments are represented by negative cash flows and divestments by positive cash flows.

10.3.5 FX Exposure and Hedging Strategies

Next the different hedging strategies are incorporated. The hedging strategies used in this thesis are

(i) An unhedged strategy

(ii) A hedging strategy using forward contracts

(iii) A hedging strategy using rolling forward contracts on a quarterly basis

(iv) Three hedging strategies using call options, each with a different strike prices

(v) Three hedging strategies using put options, each with a different strike prices

(vi) Three hedging strategies using strangles, each with a different strike prices

Note that the unhedged strategy and the call strategies are not hedging strategies per se since they do not decrease volatility in returns. The unhedged strategy is simply not to hedge the FX exposure and serves as a benchmark. The call strategies serve to enhance overall understanding. It is essential to keep track of the FX exposure at every time period when a hedging transaction is initiated when applying the different hedging strategies. Therefore, an exposure vector is created. The FX exposure is defined as the expected value of the portfolio companies at expected exit. It is this value that is hedged. Hence, the expected growth rate and expected holding time are taken into account for this calculation.

For the forward and rolling forward hedging strategies, a payoff vector is created which keeps track of the cash flows from the contracts and when they occur. For the hedging strategies including options, both a price vector and a payoff vector is created. Since these hedging strategies include solely long positions in options, the price vector will include negative cash flows. It will also keep track of when the long positions in the options are initiated. The payoff vector will include the payoffs of the options as well as at which quarter these occurs.

10.3.6 Aggregating the Cash Flows

After having applied different hedging strategies, the cash flows of the PE funds have changed. A new cash flow vector is created which includes the net cash flows of the portfolio companies and the hedging strategies and their respective timing. From this cash flow vector, the IRR of of the fund under each hedging strategy can be calculated, according to Section 9.1.

10.3.7 Performing the Simulations

After having completed the steps in Sections 10.3.1-10.3.6, the model is implemented and ready to be used. Using the model $n = 10,000$ simulations for each of the twelve PE funds, for each of the five different FX simulation methods and for each of the twelve different hedging strategies are performed. That means that a total of $n = 10,000$ simulations are made for 720 different scenarios. Hence, 7.2 million simulations are made in total.

10.3.8 Performance of the Hedging Strategies

After each $n = 10,000$ simulations, the result is 10,000 simulated IRR values. These IRRs are further analysed according to Section 9.2-9.4, mean IRR, VaR, ES as well as Sharpe ratio are computed, stored and then presented as output in this thesis.

10.4 Model Assumptions and Limitations

In order to build a model replicating an actual fund setting some assumptions have been required. It is important to remember that the results of this thesis are obtained under these assumptions. Therefore, let us restate them again. First, we have assumed some fundamental characteristics of the funds and their portfolio companies; (i) each fund consists of twelve portfolio companies, (ii) each fund has an expected IRR of 20%, given no FX movements, (iii) each fund has a specific initial FX exposure, (iv) the expected holding time of each portfolio company is five years, (v) the lifetime of a fund is ten years and (vi) each fund has an initial NAV of EUR 3 billion.

Second, some additional assumptions have been made in order to create a realistic fund setting and fulfil the first assumptions; (i) the fund invests in four portfolio companies in the beginning of the first three years, (ii) each portfolio company is given a constant quarterly growth rate, (iii) each portfolio company has a specific currency mix and (iv) the holding times are stochastic.

We have assumed that the hedging derivatives can be bought on a market and used by the funds. This might not be the case. Forward contracts might be practically infeasible for funds to implement since regulation requires the fund to provide collateral for the hedge. Moreover, rolling forwards give rise to periodic cash flows which are undesired by the fund. However, these facts have been disregarded in the thesis.

11 Results and Discussion

This section sets out to present and discuss the results obtained. The performance of the hedging strategies are measured in terms of mean IRR, VaR, ES and Sharpe ratio for $n = 10,000$ simulations. For the sake of brevity solely results for the best, in terms of mean IRR, performing option strategies have been included in this section. Although three different strike prices, $K_1 = 5\%$, $K_2 = 10\%$ and $K_3 = 20\%$, have been investigated it makes no sense to discuss the suboptimal ones in the main text. The optimal strike prices were found to be the highest ones investigated namely 20% out of the money options, i.e. the strike price for the call option, K_3^C, is 20% higher than the FX rate at initiation and the strike price of the put option, K_3^P, is 20% lower than the FX rate at initiation. A more profound analysis of the performance of all hedging strategies including options is included in Section 11.4.

The remainder of the section is structured as follows. First, we discuss the simulated FX rates. Second, the performances of the hedging strategies under the random walk method are considered. Third, the hedging strategies' performances under the random walk method with zero drift are considered. Last, a comparison of the hedging strategies including options is presented.

11.1 Foreign Exchange Rate Results

Section 8 described and presented the results of five different methods to simulate FX rates. As the FX simulation methods produced largely similar results two of the methods are chosen to be included in the main analysis of the results of the hedging strategies, the results of the other three methods are available in Appendix C. Only minor marginal value would stem from including them all in this section whereas it would hurt the readability of the text severely. The chosen FX simulation methods are the random walk method of FX rates with and without drift.

The random walk simulation method with drift is chosen to be the main simulation method since it is the method aligned with the Black-Scholes theory. The random walk simulation method thus represents a theoretically correct scenario of FX rate evolvement.

Since there on average is a negative drift in the interest rates, a downward trend is noticed for the simulated FX rates, as can be seen in Figure B.3 in Appendix B. Thus, there will be a systematic FX effect on the values of the portfolio companies. Since the US risk free rate is larger than the Swedish risk free rate, the downward drift will be more negative for the USD/EUR FX rate. For that reason the portfolio companies will lose more value on average in Fund C, which has a higher exposure to USD than the other funds. The effect is illustrated by the following example.

Consider a PE fund that invests in a portfolio company with NAV EUR 10 million in SEK, and makes an exit of this portfolio company in one year. The transaction is all-equity financed. Also, assume that the SEK/EUR FX rate is 0.1 now and 0.09 in one year, i.e. the FX rate has decreased with 10% over the year. Hence, the portfolio company is bought for SEK 100 ($= \frac{10}{0.1}$) million. After one year, at exit, the

NAV of the company is still worth SEK 100 million but only EUR 9 (= $100 \cdot 0.09$) million. Now, instead assume the portfolio company is denoted in USD, and that the USD/EUR FX rate today is 0.8 and in one year is 0.64, i.e. it has decreased with 20% since the drift in FX rate is more negative. Then the company value at entry, and at exit is USD, is USD 12.5 (= $\frac{10}{0.8}$) million and the company value at exit is EUR 8 (= $12.5 \cdot 0.64$) million. Thus, in the first case the fund lost EUR 1 million whereas it lost EUR 2 million in the second case. The conclusion is that the more negative the drift in the FX rate is, the more money is lost.

There are different views on whether the forward FX rate or the current FX rate is the best approximation for the future FX rate. Therefore, a random walk simulation method with zero drift is an interesting scenario to analyse further as well.

11.2 Performance of Foreign Exchange Hedging Strategies under the Random Walk Simulation Method

This section presents the results obtained for the funds under the random walk simulation method for FX rates. First the results for the global equity funds are presented and discussed. Then results are shown and analysed for the local equity funds, global infrastructure funds and local infrastructure funds. Finally, comparisons between the global and local equity and infrastructure funds are done.

11.2.1 Global Equity Funds

The performances of all hedging strategies for the three different global equity funds are found in Tables C.1-C.4 in Appendix C. First, performance in terms of IRR is presented and discussed. Second, VaR and ES are considered. Third, the Sharpe ratios are shown and analysed. Finally, a comparison of the different hedging strategies is included.

We begin by noting that the systematic FX effect affects the value of the portfolio companies negatively, hence the mean IRRs of the unhedged case are lower than the expected IRR of 20%. Furthermore, in the unhedged case the SEK heavy fund, Fund B, outperforms the equally weighted fund, Fund A, which in turn outperforms the USD heavy fund, Fund C, in terms of mean IRR. This is due to the USD/EUR rate having the most negative drift whereas the SEK/EUR rate has the least negative drift as explained in Section 11.1. Moreover, in terms of VaR, ES and Sharpe ratio, i.e. measures accounting for volatility in returns, the most diversified fund, Fund A, outperforms the other two. Thus the effect of diversification has a larger impact than the FX drift effect on these measures.

11.2.1.1 Internal Rate of Return

The mean IRRs of the different hedging strategies are shown in Table 11.1. Comparing the hedging strategies' performances within each fund, the mean IRRs of the unhedged strategy, the forward hedging strategy and the rolling forward hedging strategy are the highest and almost of exact same size. The option strategies are found to give significantly lower mean IRRs, the put and call option strategies performs better than the strangle strategy.

Comparing the hedging strategies' performances between the different funds minor differences are observed. For example, in Fund C the forward and rolling forward strategies may appear to perform relatively better than they do in the other funds. This is however not the case but rather a consequence of the outcome of the $n = 10,000$ FX rate simulations. Since it is not possible to make an infinite number of simulations the obtained average FX evolution of our simulations will ever so slightly deviate from the theoretical expected value, i.e. the forward curve. Therefore these kind of small deviations will occur. Moreover, due to the fact that the USD/EUR rate has the highest volatility, the call option strategy performs relatively less bad in Fund C than it does in the other funds.

Fund	**Un-hedged**	**Forward**	**Rolling forward**	**Call**$_{K_3^C}$	**Put**$_{K_3^P}$	**Strangle**$_{K_3^S}$
GE A	19.43%	19.45%	19.45%	19.17%	19.24%	18.99%
GE B	19.69%	19.72%	19.71%	19.41%	19.54%	19.27%
GE C	18.99%	19.05%	19.05%	18.79%	18.78%	18.59%

Table 11.1: IRR for random walk simulation of FX rates for the equity funds as a function of hedging derivative. Note that the hedging strategies including the call option, the put option and the strangle include 20% out of the money options

11.2.1.2 Value-at-Risk and Expected Shortfall

The VaR and ES of the different hedging strategies are shown in Tables 11.2-11.3. Comparing the hedging strategies' performances within each fund, the VaR of the rolling forward strategy is the highest followed by the forward strategy. The put option strategy outperforms the unhedged strategy except for in Fund A. Finally, the call and strangle strategies are the worst except for in Fund C in which the strangle strategy outperforms the unhedged strategy. Largely similar results applies to the ES, as can be seen in Table 11.3.

Comparing the hedging strategies' performances between the different funds, in terms of VaR and ES, an important difference is observed. For the hedging strategies protecting against the downside of the FX effect, i.e. forwards, rolling forwards and put options, a relatively better performance is observed for Fund B and Fund C than Fund A. This is explained by the diversification effect. Since Fund B and C are overweighted in a particular currency bad outcomes in these respective currencies will affect the over all result of Fund B and C more than Fund A. Therefore, from a VaR and ES perspective, hedging will offer greater benefits in the less diversified funds.

Fund	Un-hedged	Forward	Rolling forward	Call$_{K_3^C}$	Put$_{K_3^P}$	Strangle$_{K_3^S}$
GE A	17.37%	18.08%	18.21%	17.02%	17.35%	16.99%
GE B	17.02%	18.35%	18.47%	16.65%	17.28%	16.91%
GE C	16.44%	17.70%	17.86%	16.21%	16.89%	16.63%

Table 11.2: VaR for random walk simulation of FX rates for the equity funds. Note that the hedging strategies including the call option, the put option and the strangle include 20% out of the money options

Fund	Un-hedged	Forward	Rolling forward	Call$_{K_3^C}$	Put$_{K_3^P}$	Strangle$_{K_3^S}$
GE A	16.90%	17.72%	17.90%	16.55%	16.90%	16.54%
GE B	16.45%	17.95%	18.16%	16.09%	16.87%	16.49%
GE C	15.90%	17.33%	17.58%	15.69%	16.48%	16.21%

Table 11.3: ES for random walk simulation of FX rates for the equity funds. Note that the hedging strategies including the call option, the put option and the strangle include 20% out of the money options

11.2.1.3 Sharpe Ratio

The Sharpe ratios of the different hedging strategies are shown in Table 11.4. Comparing the hedging strategies' performances within each fund, the Sharpe ratio is highest for the rolling forward strategy followed by the forward strategy and then the put option strategy. The differences between the strategies including forward contracts and the other strategies are large. Furthermore, the differences between the unhedged strategy and the strangle strategy are minor. The call option strategy is always the worst performer.

Comparing the hedging strategies' performances between the different funds the same difference as in the VaR and ES case is observed. For the hedging strategies protecting against the downside of the FX effect, i.e. forwards, rolling forwards and put options, a relatively better performance is observed for Fund B and Fund C than Fund A. As discussed in the previous section, this is a result of the diversification effect taking place in Fund A.

Fund	Un-hedged	Forward	Rolling forward	$\text{Call}_{K_3^C}$	$\text{Put}_{K_3^P}$	$\text{Strangle}_{K_3^S}$
GE A	14.97	22.86	25.76	13.78	15.51	14.06
GE B	11.39	22.65	25.93	10.40	12.36	11.12
GE C	11.51	22.54	26.65	10.87	13.68	12.56

Table 11.4: SR for random walk simulation of FX rates for the Equity funds. Note that the hedging strategies including the call option, the put option and the strangle include 20% out of the money options

11.2.1.4 Comparison of the Hedging Strategies

The performances of the different hedging strategies for the global equity funds are summarised in Table 11.5 where ranking 1 represents the best performing strategy. If only minor differences are present several hedging strategies are awarded the same rank. The first rank is connected to Fund A, the second to Fund B and the third to Fund C.

Metric	Un-hedged	Forward	Rolling forward	$\text{Call}_{K_3^C}$	$\text{Put}_{K_3^P}$	$\text{Strangle}_{K_3^S}$
IRR	1,1,1	1,1,1	1,1,1	4,4,4	4,4,4	6,6,6
VaR	3,4,5	2,2,2	1,1,1	5,6,6	3,3,3	5,5,4
ES	3,4,5	2,2,2	1,1,1	5,6,6	3,3,3	5,5,4
SR	4,4,5	2,2,2	1,1,1	6,6,6	3,3,3	5,5,4

Table 11.5: Ranking of the hedging strategies' performances for the global equity PE funds (A,B,C) under the random walk simulation of FX rates

The differences between the unhedged strategy, the forward hedging strategy and the rolling forward hedging strategy, in terms of IRR, are insignificant since the forward strategies lock in the forward rate which also is the average of the simulated FX rates. Therefore, all of these hedging strategies receive the highest rank. The performances of the options are similar due to the fact that they have similar price-payoff trade-off. The reason as to why the options perform worse than the unhedged and forward strategies is that the option strategies cost more to implement. Since the strangle strategy consists of two long positions its implementing cost is the highest implying the most negative impact on the mean IRR and consequently the worst hedging strategy in terms of mean IRR.

In terms of the measures accounting for volatility in returns, VaR, ES and Sharpe ratio, the rolling forward strategy performs best followed by the forward strategy. Having the same mean IRRs as the unhedged strategy but lower volatilities imply better performing strategies. The reason for the rolling forward hedging strategy performing better than the forward strategy is that it is updated each quarter without cost. Since the strategy is updated quarterly, it does not have to make assumptions about when the portfolio companies in the PE fund will be exited. The forward hedging strategy have to make that assumption, which implies a mismatch of the cash flows of the portfolio companies and the cash flows from the forward contracts, both in terms of absolute amount and FX effect.

The put option strategy performs better than the call option strategy, in terms of VaR, ES and Sharpe ratio, since the two have similar mean IRR but the put option strategy decreases volatility whereas the call option strategy increases volatility. The strangle strategy performs in between the put and call strategies. This is explained by the fact that it has a lower mean IRR than the other option strategies and reduces volatility more than the call option strategy but less than the put option strategy. Moreover, whether the unhedged strategy or the option strategies perform better depends on the degree of diversification of the fund. In a diversified fund the unhedged strategy is equally good as the option strategies, except for Sharpe ratio, whereas in single currency overweighted funds the put option strategy outperforms the unhedged strategy.

11.2.2 Local Equity Funds

The hedging strategies' performances, in terms of mean IRR, VaR, ES and Sharpe ratio, for the local equity funds are shown in Tables C.1-C.4 in Appendix C. A summary of the results is shown in Table 11.6. Largely similar results as obtained for the global equity funds are obtained for the local equity funds as well. This is somewhat expected since the only difference between the global and local funds is how the different currencies are distributed between the different portfolio companies. This fact decreases the diversification but there are still other parameters, e.g. stochastic holding time of the portfolio companies, mitigating the total effect. Though, by construction of the simulation model, which currency ending up in portfolio companies with higher assigned growth will affect the results slightly.

To conclude, in terms of mean IRR the same results as in the global equity funds are obtained, both in terms of size and relative order of the strategies. In terms of VaR, ES and Sharpe ratio the same results as in the global equity fund are obtained except for the unhedged strategy now being comparable to the put option strategy in terms of VaR and ES. Having a lower degree of diversification the opposite results were expected. The results must therefore be accountable to the particular construction of the portfolio companies and is thus not a general result for local equity funds.

Metric	Un-hedged	Forward	Rolling forward	Call$_{K_3^C}$	Put$_{K_3^P}$	Strangle$_{K_3^S}$
IRR	1,1,1	1,1,1	1,1,1	4,4,4	4,4,4	6,6,6
VaR	3,3,3	2,2,2	1,1,1	5,5,5	3,3,3	5,5,5
ES	3,4,3	2,2,2	1,1,1	5,5,5	3,3,3	5,5,5
SR	4,4,4	2,2,2	1,1,1	5,6,6	3,3,3	5,5,5

Table 11.6: Ranking of the hedging strategies' performances for the local equity PE funds (A,B,C) under the random walk simulation of FX rates

11.2.3 Global Infrastructure Funds

The hedging strategies' performances, in terms of IRR, VaR, ES and Sharpe ratio, for the global infrastructure funds are shown in Tables C.5-C.8 in Appendix C. A summary of the results is shown in Table 11.7. Again, largely similar results as for the global equity funds are obtained, except for the fact that the overall IRR levels are lower due to the lower growth of the infrastructure funds. This is expected since the only difference between the global equity and infrastructure funds is the assigned growth rate of the funds.

To conclude, disregarding the size of the results, in terms of mean IRR the same results as in the global equity funds are obtained. In terms of VaR, ES and Sharpe ratio, disregarding the size of the results, the same results as in the global equity funds are obtained. It should however be noted that the put option performs slightly better relative the unhedged case boosting the performance of the strangle option strategy. This is probably due to the construction of the model since there are no logical arguments in favour of put options performing better for lower exposure in a currency.

Metric	Un-hedged	Forward	Rolling forward	Call$_{K_3^C}$	Put$_{K_3^P}$	Strangle$_{K_3^S}$
IRR	1,1,1	1,1,1	1,1,1	4,4,4	4,4,4	6,6,6
VaR	3,5,5	2,2,2	1,1,1	6,6,6	3,3,3	4,4,4
ES	3,5,5	2,2,2	1,1,1	6,6,6	3,3,3	4,4,4
SR	4,4,5	2,2,2	1,1,1	5,6,6	3,3,3	5,5,4

Table 11.7: Ranking of the hedging strategies' performances for the global infrastructure PE funds (A,B,C) under the random walk simulation of FX rates

11.2.4 Local Infrastructure Funds

The hedging strategies' performances, in terms of IRR, VaR, ES and Sharpe ratio, for the local infrastructure funds are shown in Tables C.5-C.8 in Appendix C. A summary of the results is shown in Table 11.8. Largely similar results as obtained for the global infrastructure funds are obtained.

To conclude, in terms of mean IRR the same results as in the global infrastructure funds are obtained. In terms of VaR, ES and Sharpe ratio very similar results as in the global infrastructure funds are obtained. The only small difference is that the unhedged strategy and the strangle strategy have similar performances in terms of VaR and ES whereas there was a difference attributable to the construction of the funds in the global case. The reason why this is not observable in the local infrastructure fund is probably explained by the choices of other parameters, e.g. entry time and growth of the local portfolio companies.

Metric	**Un-hedged**	**Forward**	**Rolling forward**	**Call**$_{K_3^C}$	**Put**$_{K_3^P}$	**Strangle**$_{K_3^S}$
IRR	1,1,1	1,1,1	1,1,1	4,4,4	4,4,4	6,6,6
VaR	3,4,4	2,2,2	1,1,1	6,6,6	3,3,3	5,4,4
ES	3,4,4	2,2,2	1,1,1	6,6,6	3,3,3	5,4,4
SR	4,4,5	2,2,2	1,1,1	5,6,6	3,3,3	5,5,4

Table 11.8: Ranking of the hedging strategies' performances for the local infrastructure PE funds (A,B,C) under the random walk simulation of FX rates

11.2.5 Comparison of All Funds

Similar results have been found for the four different types of funds, global equity funds, local equity funds, global infrastructure funds and local infrastructure funds. Comparing the equity funds to the infrastructure funds no major differences are observed. Comparing the global funds to the local funds we expected the diversification effect to be smaller in the local funds. This was however not observed in the results which is believed to be attributable to construction differences.

In terms of mean IRR, the hedging strategies receive the same rank independent of overall fund type. When comparing the measures accounting for volatility in returns the rolling forward strategy is optimal followed by the forward strategy. The put option strategy is always number three and the call option strategy is always the worst. The unhedged strategy and the strangle strategy vary in rank depending on overall fund type as well as fund, due to diversification effects and construction differences.

11.3 Performance of FX Hedging Strategies under the Random Walk Simulation Method with Zero Drift

Recalling that there are different views on whether the forward FX rate or the current FX rate is the best approximation for the future FX rate a random walk simulation method with zero drift is analysed below.

A summary of the results in Tables C.9-C.16 is shown in Tables 11.9 and 11.10. We begin by noting that since no drift in FX rates is present no systematic FX effect affects the value of the portfolio companies and hence all performance measures are in general higher than in the drift case. The mean IRRs of the unhedged case are close to 20% for the equity funds and 12% for the infrastructure funds, which are the expected IRRs of the funds. The performance of the different funds, the SEK heavy fund, Fund B, the equally weighted fund, Fund A, and the USD heavy fund, Fund C, is very similar in terms of mean IRR. Moreover, in terms of VaR, ES and Sharpe ratio, the most diversified fund, Fund A, outperforms the other two since the effect of diversification is still present whereas the FX drift effect is now removed.

Comparing the hedging strategies' performances within each fund and between the funds we observe very similar results leading to the same conclusions as in the random walk simulation with drift. Hence decisions regarding if and how to hedge ought to be independent on one's view of whether the current FX rate or the forward FX rate is the better approximation for the future FX rate.

Metric	**Un-hedged**	**Forward**	**Rolling forward**	**Call**$_{K_3^C}$	**Put**$_{K_3^P}$	**Strangle**$_{K_3^S}$
IRR	1,1,1	1,1,1	1,1,1	5,5,5	4,4,4	6,6,6
VaR	3,4,4	2,2,2	1,1,1	5,6,6	3,3,3	5,5,5
ES	3,4,4	2,2,2	1,1,1	5,6,6	3,3,3	5,5,5
SR	4,4,4	2,2,2	1,1,1	5,6,6	3,3,3	5,5,5

Table 11.9: Ranking of the hedging strategies' performances for the equity funds (A,B,C) under the random walk simulation of FX rates with zero drift

Metric	**Un-hedged**	**Forward**	**Rolling forward**	**Call**$_{K_3^C}$	**Put**$_{K_3^P}$	**Strangle**$_{K_3^S}$
IRR	1,1,1	1,1,1	1,1,1	5,5,5	4,4,4	6,6,6
VaR	4,4,4	2,2,2	1,1,1	6,6,6	3,3,3	5,5,5
ES	4,4,4	2,2,2	1,1,1	6,6,6	3,3,3	5,5,5
SR	4,4,4	2,2,2	1,1,1	6,6,6	3,3,3	5,5,5

Table 11.10: Ranking of the hedging strategies' performances for the infrastructure funds (A,B,C) under the random walk simulation of FX rates with zero drift

11.4 Comparison of the Option Hedging Strategies

In Sections 11.2 and 11.3, the hedging strategies that performed best in terms of mean IRR, among the option strategies with various strike prices, were presented. These strategies were 20% out of the money options. However, other strategies including 5% and 10% out of the money options were also applied but proved to be suboptimal for mean IRR.

In this section we extend the simulation model and investigate a wider spectrum of strike prices in order to find the optimal strike price for the hedging strategies. The call option strategy and the put option strategy will be compared and analysed separately. Since the strangle is a combination of the call option and put option strategies it will not be separately analysed. 0% to 100% out of the money options are considered. Note that the more far out of the money the options are the less they cost and the less payoff they give. Therefore, in the limit they tend to the unhedged strategy. For all option strategies, performance in terms of mean IRR, VaR, ES and Sharpe ratio is calculated from the $n = 10,000$ simulations. However, since the results point in the same direction for the different PE funds, only global equity Fund A under the random walk simulation of FX rates is considered.

The results can be found in Figures 11.1-11.2. For the call option strategy the best performance in terms of all performance measures is as far out of the money call options as possible, as seen in Figure 11.1. However, for around 40% out of the money call options and more, the mean IRR, VaR and ES seem to almost have stagnated. The results of the put option strategy are shown in Figure 11.2. In terms of mean IRR, the performance seems to almost have stagnated for around 45% out of the money put options and more. However, the results are different for VaR, ES and Sharpe ratio. VaR and ES are maximised for 29% out of the money put options. The at the money put option strategy is the best performer in terms of Sharpe ratio.

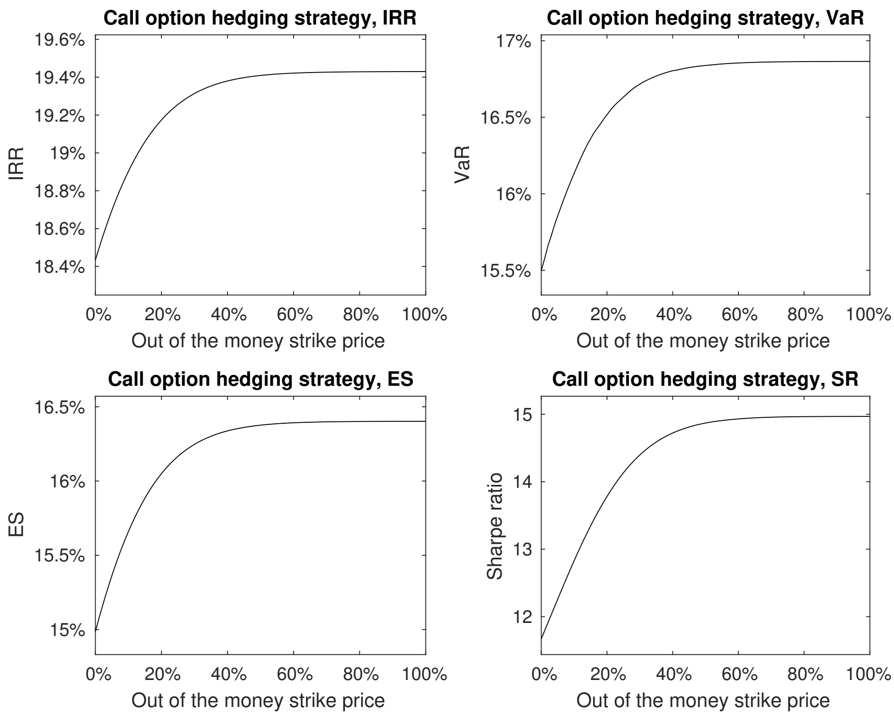

Figure 11.1: Call option hedging strategy performance as a function of the out of the money strike price

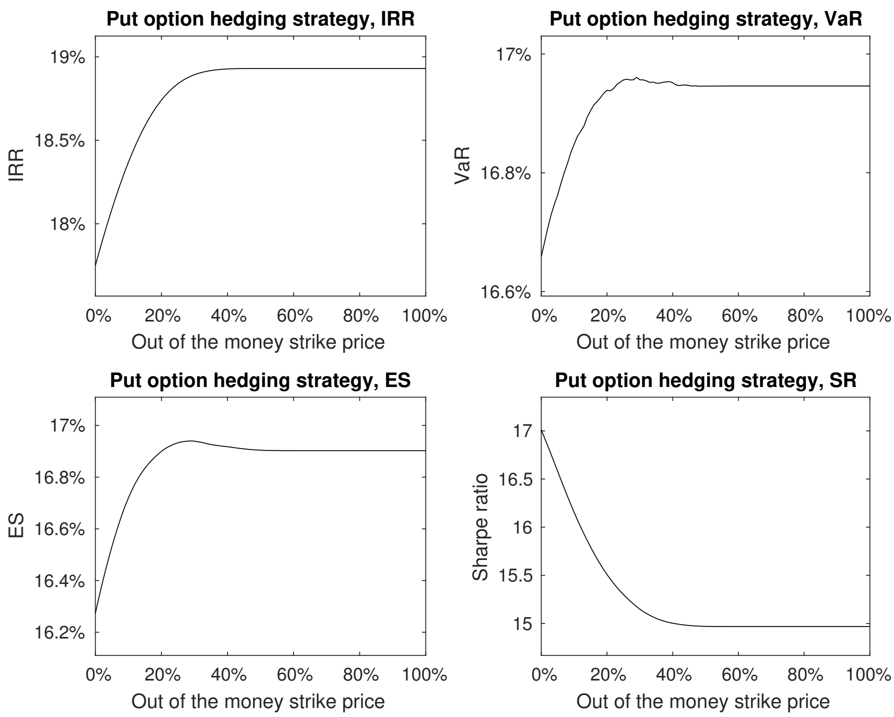

Figure 11.2: Put option hedging strategy performance as a function of the out of the money strike price

From the mean IRR results in Figures 11.1-11.2 the same conclusion can be drawn for both the call and put option strategies. The cost of buying the options exceeds the present value of their five year later payoffs. For that reason in order to perform optimally in terms of mean IRR as far out of the money options as possible are preferred, hence options should not be used.

In terms of VaR and ES the results of the two option strategies differ. For the call option strategy the result is again to not use these options since they do not protect from the downside FX effect whilst costing money to buy. For the put option strategy the results are interesting. There is actually an optimal strike price for the put option strategy, namely 29%. Put options with this strike price gives the best price-payoff trade-off in terms of VaR and ES.

For our given range of IRRs and their volatilities the volatility has a greater impact on the Sharpe ratio than the IRR by its definition. The Sharpe ratio increase with more out of the money call options, and decrease with more out of the money put options. The reason for this is that the payoff of the call option increases if the portfolio companies increase in value due to FX effects whilst the opposite is true for the put option strategy. Hence, the call option strategy will have a high volatility for options closer to at the money, which is reflected in a low Sharpe ratio as seen in Figure 11.1. The payoff of the put options increase as the portfolio companies decrease in value. Thus, the put option strategy will reduce the volatility of returns, especially for out of the money options having strike prices close to at the money.

To conclude, the call option strategy should never be applied, no matter which performance measure is focused on. The put option hedging strategy should not be used if focusing on mean IRR. However, if focusing on VaR and ES, the put option hedging strategy with the optimal strike price, i.e. 29% out of the money options, perform slightly better than the unhedged strategy. Also, a high Sharpe ratio can be achieved with put options, even higher than an unhedged strategy, comparing the results to Tables C.1-C.4 in Appendix C.

Part III

Extended Model I: Stochastic Interest Rates

12 Short Rate Models

So far constant interest rates have been assumed. In reality, this is utterly unlikely. Monetary policy, political risk and many other factors affect FX rate evolvements. In order to test the impact of the constant interest rate assumption a model including stochastic interest rates will be developed. This chapter lays out the general theory of interest rates, introduces the Hull-White model and presents formulas for pricing derivatives under stochastic short rates. Inspiration is to a large extent gathered from Björk (2009) [2].

12.1 Dynamics of the Short Rate of Interest

Our objective is to model an arbitrage free family of zero coupon bond price processes $\{p(\cdot, T); T \geq 0\}$. The price $p(t,T)$ depends on the short rate of interest over the interval $[t,T]$. Therefore, as a starting point the dynamics of the short rate of interest will be introduced. The short rate can, under the objective probability measure P, be modelled as the solution of an SDE of the form

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))d\overline{W}_t. \tag{12.1}$$

Since we solely focus on the short rate, the only exogenously given asset is a money account with price process B which has the dynamics

$$dB_t = r(t)B_t dt. \tag{12.2}$$

This model can be interpreted as a model of a bank account with stochastic short rate of interest r. The bank is considered as a risk free asset. Hence, the price of the risk free asset is given by B.

We also assume that there exits a market for zero coupon T-bonds for each maturity T. Hence, we assume a market containing infinite many assets but only the risk free asset is exogenously given. Formulating this differently, the risk free asset is considered as the underlying asset whilst all bonds are regarded as derivatives of the underlying short rate of interest r. Note that the price of a particular bond will not be completely determined by the specification Eq. 12.1 of the r-dynamics and the requirement that the bond market is free of arbitrage. The reason for this is that arbitrage pricing is always a case of pricing a derivative in terms of the price of some underlying asset, and in our market we do not have sufficiently many underlying assets.

Intuitively, in order to avoid arbitrage on the bond market, prices of bonds with different maturities will have to satisfy certain internal consistency relations. Taking the price of a particular bond, a benchmark bond, as given then the prices of all other bonds having maturities prior to the benchmark bond will be uniquely determined in terms of the price of the benchmark bond and the r-dynamics.

12.2 The Term Structure Equation

This section formalizes the intuition in the end of the previous section and expands the theory further. The ultimate goal is to find a pricing equation for arbitrary interest rate derivatives. Let us begin with stating the main assumption.

Assumption 12.1
Assume that there is an arbitrage free market for T-bonds for every choice of maturity T. Furthermore, assume that the price of a T-bond is given by

$$p(t,T) = F(t, r(t); T), \tag{12.3}$$

for every T, where F is a smooth function of the real variables r, t and T.

One can think of F as a function of the two variables r and t, whereas T can be considered as a parameter. For that reason, we will sometimes write $F^T(t,r)$ rather than $F(t,r;T)$. We want to find out what F^T looks like on a market which is absent of arbitrage.

The boundary condition is very simple. At the time of maturity a T-bond is worth exactly 1 unit of the denominating currency, i.e. its face value equals 1. Thus, the following relation for the boundary condition holds

$$F(T, r; T) = 1, \quad \forall r \in \mathbb{R}. \tag{12.4}$$

Note that the letter r denotes both a real variable as well as the name of the stochastic process for the short rate. However, the meaning will be clear from the context.

We are now ready to find out what F^T looks like under the given assumptions. In order to do this we start by forming a portfolio consisting of two bonds that have different times to maturity, namely S and T. Using Assumption 12.1 and Itô's formula we get the following price dynamics for the T-bond

$$dF^T = F^T \alpha_T dt + F^T \sigma_T d\overline{W}, \tag{12.5}$$

where

$$\alpha_T = \frac{\frac{\partial F^T}{\partial t} + \mu \frac{\partial F^T}{\partial r} + \frac{1}{2}\sigma^2 \frac{\partial^2 F^T}{\partial r^2}}{F^T}, \tag{12.6}$$

$$\sigma_T = \frac{\sigma \frac{\partial F^T}{\partial r}}{F^T}, \tag{12.7}$$

and similarly for the S-bond. Moreover, the value dynamics of the relative portfolio

(u_S,u_T) are given by

$$dV = V\left(u_T\frac{dF^T}{F^T} + u_S\frac{dF^S}{F^S}\right). \tag{12.8}$$

Inserting the differential found in Eq. 12.5, as well as the corresponding one for the S-bond, and rearranging terms gives

$$dV = V(u_T\alpha_T + u_S\alpha_S)dt + V(u_T\sigma_T + u_S\sigma_S)d\overline{W}. \tag{12.9}$$

Choosing the formed portfolio to be locally risk less it must hold that $u_T\sigma_T + u_S\sigma_S = 0$. Furthermore, since (u_T, u_S) is a relative portfolio it must hold that the weights u_T and u_S sums to one. Thus we have the two following conditions for our portfolio

$$u_T + u_S = 1, \tag{12.10}$$
$$u_T\sigma_T + u_S\sigma_S = 0. \tag{12.11}$$

Inserting the second condition the diffusion term in Eq. 12.9 will vanish, and the value dynamics are reduced to

$$dV = V(u_T\alpha_T + u_S\alpha_S)dt. \tag{12.12}$$

The two conditions, Eqs. 12.10-12.11, has the solution

$$u_T = -\frac{\sigma_S}{\sigma_T - \sigma_S}, \tag{12.13}$$
$$u_S = -\frac{\sigma_T}{\sigma_T - \sigma_S}. \tag{12.14}$$

Substituting this into Eq. 12.12 gives us the final expression for the value dynamics of our portfolio

$$dV = V\left(\frac{\alpha_S\sigma_T - \alpha_T\sigma_S}{\sigma_T - \sigma_S}\right)dt. \tag{12.15}$$

Next we recall that if the portfolio is locally risk less then its rate of return must equal the rate of return of the risk free asset, i.e. the short rate of interest, in order for the model to be free of arbitrage. Thus it must hold that

$$\frac{\alpha_S\sigma_T - \alpha_T\sigma_S}{\sigma_T - \sigma_S} = r(t), \quad \forall t, \text{ with probability 1.} \tag{12.16}$$

Rearranging gives

$$\frac{\alpha_S(t) - r(t)}{\sigma_S(t)} = \frac{\alpha_T(t) - r(t)}{\sigma_T(t)}. \tag{12.17}$$

Noting that the left-hand side of Eq. 12.17 is a stochastic process independent of T and the right-hand side is a stochastic process independent of S it must hold that the common quotient will be independent of the choice of either T and S. Let us state this important result as a proposition.

Proposition 12.1
Assume that the bond market is arbitrage free. Then there exists a process λ such that the relation

$$\frac{\alpha_T(t) - r(t)}{\sigma_T(t)} = \lambda(t) \tag{12.18}$$

holds for all t and for every choice of maturity time T.

Let us take a closer look at the process λ. In the numerator of Eq. 12.18 we have the term $\alpha_T(t) - r(t)$, i.e. the risk premium of the T-bond. In other words, $\alpha_T(t) - r(t)$ measures the risky T-bond's excess rate of return over the risk less rate of return that is required by the market to avoid arbitrage opportunities. Since the denominator of Eq. 12.18 is the local volatility, $\sigma_T(t)$, of the T-bond the dimension of λ is "risk premium per unit of volatility". This unit is known as the market price of risk. Thus, Proposition 12.1 tells us that all bonds will, regardless of maturity time, have the same market price of risk, in a market absent of arbitrage.

Finally, by inserting the expressions for α_T and σ_T given by Eqs. 12.6-12.7 into Eq. 12.18, and rearranging we obtain the important term structure equation.

Proposition 12.2 ***Term structure equation***
In an arbitrage free bond market, F^T satisfies the term structure equation

$$\begin{cases} \dfrac{\partial F^T}{\partial t} + (\mu - \lambda\sigma)\dfrac{\partial F^T}{\partial r} + \dfrac{1}{2}\sigma^2\dfrac{\partial^2 F^T}{\partial r^2} - rF^T = 0, \\ F^T(T, r) = 1. \end{cases} \tag{12.19}$$

As seen from Eqs. 12.6, 12.7 and 12.18 λ is on the form $\lambda = \lambda(t, r)$, which means that the term structure is a standard partial differential equation. The problem is that λ is not determined within the model. Hence, in order to solve the term structure equation λ must be specified exogenously just as we have to specify μ and σ. Despite this problem it is not hard to obtain a Feynman-Kač representation of F^T. We do this by fixing (t, r) and then using the process

$$\exp\left(- \int_t^s r(u)du \right) F^T(s, r(s)). \tag{12.20}$$

Applying Itô's formula to Eq. 12.20 and using the fact that F^T satisfies the term structure equation the following stochastic representation formula is obtained.

Proposition 12.3 ***Risk neutral valuation***
Bond prices are given by the formula $p(t, T) = F(t, r(t); T)$ where

$$F(t, r; T) = E^Q_{t,r}\left[e^{-\int_t^T r(s)ds} \right]. \tag{12.21}$$

Furthermore, the Q-dynamics of the short rate is given by

$$dr(s) = (\mu - \lambda\sigma)ds + \sigma dW_s, \tag{12.22}$$

$$r(t) = r. \tag{12.23}$$

Writing Eq. 12.21 as

$$F(t, r; T) = E_{t,r}^{Q}\left[e^{-\int_t^T r(s)ds} \cdot 1\right]. \tag{12.24}$$

we observe that the value of a T-bond is the expected value of the final payoff of 1 unit of currency discounted to present value. We also note that since the model is not complete we have different martingale measures Q for different choices of λ. This is the main difference to the Black-Scholes model, where the martingale measure is uniquely determined.

We end this section by studying more general contingent claims than just T-bonds. Let us introduce a general contingent T-claim on the form

$$\mathcal{X} = \Phi(r(T)), \tag{12.25}$$

where Φ is some real valued function. Using the same arguments as earlier in this section the general term structure equation is obtained.

Proposition 12.4 ***General term structure equation***
Let $\mathcal{X}$ be a contingent T-claim of the form $\mathcal{X} = \Phi(r(T))$. In an arbitrage free market the price $\Pi(t; \Phi)$ will be given as

$$\Pi(t; \Phi) = F(t, r(t)), \tag{12.26}$$

where F solves the boundary value problem

$$\begin{cases} \dfrac{\partial F}{\partial t} + (\mu - \lambda\sigma)\dfrac{\partial F}{\partial r} + \dfrac{1}{2}\sigma^2\dfrac{\partial^2 F}{\partial r^2} - rF = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad F(T, r) = \Phi(r), \end{cases} \tag{12.27}$$

and has the stochastic representation

$$F(t, r; T) = E_{t,r}^{Q}\left[\exp\left(-\int_t^T r(s)ds \times \Phi(r(T))\right)\right]. \tag{12.28}$$

Furthermore, the Q-dynamics of the short rate is given by

$$dr(s) = (\mu - \lambda\sigma)ds + \sigma dW_s, \tag{12.29}$$

$$r(t) = r. \tag{12.30}$$

12.3 Martingale Models for the Short Rate

In the two following subsections we tackle the problem of solving the general term structure equation obtained in the previous section. First we introduce the technique of martingale modelling. Then we explore a certain class of martingale models, possessing a so called affine term structure, suitable for calculations.

12.3.1 Fundamentals of Martingale Modelling

In the previous section we arrived at the general term structure equation which completely determines the price of all interest rate derivatives after the drift term, μ, the diffusion term, σ, and the market price of risk, λ, have been specified,

$$\begin{cases} \dfrac{\partial F}{\partial t} + (\mu - \lambda\sigma)\dfrac{\partial F}{\partial r} + \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 F}{\partial r^2} - rF = 0, \\ F(T, r) = \Phi(r). \end{cases} \tag{12.31}$$

In order to solve this equation the procedure of martingale modelling, i.e. modelling the short rate r directly under the martingale measure Q instead of under the objective probability measure P, will be used. Thus, r will have the following dynamics under Q

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW_t, \tag{12.32}$$

where μ and σ are given functions. Hereinafter in this section μ will always denote the drift term of the short rate of interest under the martingale measure Q.

There are a lot of models proposing different ways of how to specify the Q-dynamics of r. As we model directly under Q, and not under the observable measure P, we cannot calibrate the model to market data since the data we observe on the market is under P. However, this is only true for the drift term and not the diffusion term since a Girsanov transformation only affects the drift term and not the diffusion term.

Next we proceed by determining the parameters of a martingale model. The typical approach to determine the parameters in a martingale model is to invert the yield curve. After having selected a martingale model for the short rate r, we introduce the parameter vector α and express the r-dynamics under Q as

$$dr(t) = \mu(t, r(t); \alpha)dt + \sigma(t, r(t); \alpha)dW_t, \tag{12.33}$$

and solve the term structure equation, for every conceivable time of maturity T

$$\begin{cases} \dfrac{\partial F^T}{\partial t} + \mu\dfrac{\partial F^T}{\partial r} + \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 F^T}{\partial r^2} - rF^T = 0, \\ F^T(T, r) = 1. \end{cases} \tag{12.34}$$

Thus, we have calculated a theoretical term structure according to

$$p(t, T; \alpha) = F^T(t, r; \alpha). \tag{12.35}$$

Note that the parameter vector α yet has not been chosen. In order to do this bond price data needs to be collected. For all values of T, it is possible to observe $p(0, T)$. Denoting the observed empirical term structure by $\{p^\star(0, T); T \geq 0\}$ the next step is to choose the parameter vector α in such a way that the theoretical curve $\{p(0, T; \alpha); T \geq 0\}$ fits the empirical curve $\{p^\star(0, T); T \geq 0\}$ as good as possible. We denote the chosen parameter vector $\alpha^\star$ and insert it into μ and σ to obtain $\mu^\star$ and $\sigma^\star$. Following this procedure we have now determined which martingale measure Q we are working with.

After having specified Q we are now ready to price interest rate derivatives. Denoting an arbitrary interest derivative by $\mathcal{X} = \Gamma(r(T))$ the price process of $\mathcal{X}$ is given by $\Pi(t;\Gamma) = G(t, r(t))$, where G solves the term structure equation

$$\begin{cases} \dfrac{\partial G}{\partial t} + \mu^\star \dfrac{\partial G}{\partial r} + \dfrac{1}{2}(\sigma^\star)^2 \dfrac{\partial^2 G}{\partial r^2} - rG = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad G(T, r) = \Gamma(r). \end{cases} \tag{12.36}$$

The difficulty of solving the above partial differential equation depends on the choice of martingale model. This leads us to the subject affine term structures which will be investigated in the following section.

12.3.2 Affine Term Structures

Let us begin by giving the definition of a martingale model possessing an affine term structure.

Definition 12.1
If the term structure $\{p(t,T); 0 \leq t \leq T,\ T > 0\}$ *has the form*

$$p(t,T) = F(t, r(t); T), \tag{12.37}$$

where F *has the form*

$$F(t, r; T) = e^{A(t,T) - B(t,T)r}, \tag{12.38}$$

and where A *and* B *are deterministic functions, then the model is said to possess an affine term structure (ATS).*

As seen in Eq. 12.38, A and B are both functions of the real variables t and T. We choose to regard A and B as functions of t, while T serves as a parameter. From both an analytical and computational point of view, the model having an ATS is beneficial. Therefore, we now seek to understand what conditions must be fulfilled in order for the model to have an ATS.

As before, we assume that r has the following Q-dynamics

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW_t, \tag{12.39}$$

and furthermore that this model possesses an ATS. Next, we compute the partial derivatives of F using Eq. 12.38. Since F solves the term structure equation, Eq. 12.31, we obtain

$$\frac{\partial A}{\partial t}(t,T) - \left(1 + \frac{\partial B}{\partial t}(t,T)\right) r - \mu(t,r)B(t,T) + \frac{1}{2}\sigma^2(t,r)B^2(t,T) = 0. \tag{12.40}$$

Furthermore, the boundary value $F(T, r; T) \equiv 1$ implies that

$$\begin{cases} A(T,T) = 0, \\ B(T,T) = 0. \end{cases} \tag{12.41}$$

In order for an ATS to exist, the relations between A, B, μ and σ in Eq. 12.40 must hold. It turns out that if μ and σ are affine functions of r, i.e. linear plus a constant, with possibly time dependent coefficients, then Eq. 12.40 becomes a separable differential equation for the functions A and B. Thus, assuming that μ and σ are on the form

$$\begin{cases} \mu(t,r) = \alpha(t)r + \beta(t), \\ \sigma(t,r) = \sqrt{\gamma(t)r + \delta(t)}, \end{cases} \tag{12.42}$$

Eq. 12.40 becomes

$$\begin{aligned} &\frac{\partial A}{\partial t}(t,T) - \beta(t)B(t,T) + \frac{1}{2}\delta(t)B^2(t,T) \\ -&\left(1 + \frac{\partial B}{\partial t}(t,T) + \alpha(t)B(t,T) - \frac{1}{2}\gamma(t)B^2(t,T)\right)r = 0 \quad \forall t,T,r \in \mathbb{R}. \end{aligned} \tag{12.43}$$

Let us consider the above equation for a fixed t and T. Since the equation must hold for all values of r the coefficient of r must equal zero,

$$1 + \frac{\partial B}{\partial t}(t,T) + \alpha(t)B(t,T) - \frac{1}{2}\gamma(t)B^2(t,T) = 0 \tag{12.44}$$

Thus, the equation reduces to

$$\frac{\partial A}{\partial t}(t,T) - \beta(t)B(t,T) + \frac{1}{2}\delta(t)B^2(t,T) = 0. \tag{12.45}$$

At this stage, we formulate the result in a proposition.

Definition 12.2 ***Affine term structure***
Assume that μ and σ are of the form

$$\begin{cases} \mu(t,r) = \alpha(t)r + \beta(t), \\ \sigma(t,r) = \sqrt{\gamma(t)r + \delta(t)}. \end{cases} \tag{12.46}$$

Then the model admits an ATS of the form

$$F(t,r;T) = e^{A(t,T) - B(t,T)r}, \tag{12.47}$$

where A and B satisfy the system

$$\begin{cases} 1 + \dfrac{\partial B}{\partial t}(t,T) + \alpha(t)B(t,T) - \dfrac{1}{2}\gamma(t)B^2(t,T) = 0, \\ B(T,T) = 0. \end{cases} \tag{12.48}$$

$$\begin{cases} \dfrac{\partial A}{\partial t}(t,T) = \beta(t)B(t,T) - \dfrac{1}{2}\delta(t)B^2(t,T), \\ A(T,T) = 0. \end{cases} \tag{12.49}$$

We end this section by noting that Eq. 12.48 does not involve A. Hence, having solved Eq. 12.48 we can insert the solution B into Eq. 12.49 and integrate in order to obtain the solution A. For the reader interested in extensions and notes on the affine term structure theory we suggest Duffie and Kan (1996) [8].

12.4 The Hull-White Model

This section sets out to present the Hull-White model for the short rate dynamics under Q. Furthermore, the parameters of the model are determined and the term structure equation under the Hull-White model is presented.

The martingale model for short rates chosen in this thesis is the Hull-White (extended Vasiček) model, hereinafter referred to as the Hull-White or HW model, which is introduced in Hull and White (1990) [16]. Under the Hull-White model the Q-dynamics of the short rate are given by

$$dr = (\Theta(t) - a(t)r)dt + \sigma(t)dW_t, \quad (a(t) > 0). \tag{12.50}$$

The Hull-White model is chosen due to its accuracy and simplicity. The model describes the short rate r using a linear SDE. A linear SDE is easy to solve and the corresponding short rate process is normally distributed. Furthermore, another reason for choosing the Hull-White model is because of its time dependent coefficients, which make it possible for the model to fit data much better than a model solely containing time independent coefficients. Moreover, the Hull-White model is a mean reverting model under Q, which means that the model tends to revert to a mean level.

We set $r(0) = r_0$, i.e. the present short rate on the market. Furthermore, we choose the coefficient a in the Hull-White model to be time independent, $a(t) = a$. We also choose the coefficient $\sigma(t)$ to be piecewise constant. The piecewise constant function $\Theta(t)$ will do the job of fitting the model to market data. The remainder of this section is devoted to finding a suitable $\Theta(t)$ given values of a and $\sigma(t)$. The process is commonly known as inverting the yield curve.

For the zero coupon yield $y(t,T)$ it holds that

$$y(t,T) = -\frac{\ln p(t,T)}{T-t}. \tag{12.51}$$

Choosing $t = 0$ we obtain the following equation for the yield curve

$$y(0,T) = -\frac{\ln p(0,T)}{T}. \tag{12.52}$$

Next, we will convert the yield curve into a piecewise constant forward rate curve $f(0,T)$ for $0 \leq T \leq T^\star$, where $T^\star < \infty$ is the longest maturity. We proceed with the definition of the forward curve $f(0,T)$

$$f(0,T) = -\frac{\partial}{\partial T}\ln p(0,T). \tag{12.53}$$

Rearranging Eq. 12.52 and inserting into the above expression for the forward rate gives the theoretical forward curve

$$f(0,T) = -\frac{\partial}{\partial T}\ln p(0,T) = \frac{\partial}{\partial T}(Ty(0,T)) = y(0,T) + T\frac{\partial}{\partial T}y(0,T). \tag{12.54}$$

Similarly, the observed forward curve is given by

$$f^\star(0,T) = -\frac{\partial}{\partial T}\ln p^\star(0,T) = \frac{\partial}{\partial T}(Ty^\star(0,T)) = y^\star(0,T) + T\frac{\partial}{\partial T}y^\star(0,T). \quad (12.55)$$

As stated in Definition 12.2, the model has an ATS if μ and σ are of the form

$$\begin{cases} \mu(t,r) = \alpha(t)r + \beta(t), \\ \sigma(t,r) = \sqrt{\gamma(t)r + \delta(t)}. \end{cases} \quad (12.56)$$

Comparing Eq. 12.50 to Eq. 12.56 and identifying coefficients we find that $\alpha(t) = -a$, $\beta(t) = -\Theta(t)$, $\gamma(t) = 0$ and $\delta(t) = \sigma^2(t)$. Having an ATS, bond prices are given by

$$p(t,T) = e^{A(t,T)-B(t,T)r}, \quad (12.57)$$

where the functions A and B, after inserting the obtained values of $\alpha(t)$, $\beta(t)$, $\gamma(t)$ and $\delta(t)$ into Eqs. 12.48-12.49, solve

$$\begin{cases} \frac{\partial B}{\partial t}(t,T) = aB(t,T) - 1, \\ B(T,T) = 0, \end{cases} \quad (12.58)$$

$$\begin{cases} \frac{\partial A}{\partial t}(t,T) = \Theta(t)B(t,T) - \frac{1}{2}\sigma^2 B^2(t,T), \\ A(T,T) = 0. \end{cases} \quad (12.59)$$

The solutions to these equations are given by

$$B(t,T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right), \quad (12.60)$$

$$A(t,T) = \int_t^T \left(\frac{1}{2}\sigma^2(t)B^2(s,T) - \Theta(s)B(s,T)\right)ds. \quad (12.61)$$

By using Eqs. 12.54 and 12.57 we find the following expression for $f(0,T)$

$$\begin{aligned} f(0,T) &= \frac{\partial}{\partial T}(Ty(0,T)) && (12.62)\\ &= -\frac{\partial}{\partial T}\ln(p(0,T)) && (12.63)\\ &= \frac{\partial B}{\partial T}(0,T)r(0) - \frac{\partial A}{\partial T}(0,T). && (12.64) \end{aligned}$$

Using Eqs. 12.58-12.59 and approximating $\sigma(t)$ as constant, $\frac{\partial B}{\partial T}(0,T)$ and $\frac{\partial A}{\partial T}(0,T)$ are given by

$$\frac{\partial B}{\partial T}(0,T) = e^{-aT}, \quad (12.65)$$

and

$$\frac{\partial A}{\partial T}(0,T) = \int_0^T \left(\frac{\sigma^2(t)}{a}(1-e^{-a(T-s)})e^{-a(T-s)} - \Theta(s)e^{-a(T-s)}\right)ds \tag{12.66}$$

$$= -\int_0^T \Theta(s)e^{-a(T-s)}ds + \frac{\sigma^2(t)}{2a^2}\left(1-e^{-aT}\right)^2. \tag{12.67}$$

Finally we obtain the anticipated expression for $f(0,T)$ as

$$f(0,T) = e^{-aT}r(0) + \int_0^T \Theta(s)e^{-a(T-s)}ds - \frac{\sigma^2(t)}{2a^2}\left(1-e^{-aT}\right)^2. \tag{12.68}$$

Similarly, for the observed values we have

$$f^\star(0,T) = e^{-aT}r(0) + \int_0^T \Theta(s)e^{-a(T-s)}ds - \frac{\sigma^2(t)}{2a^2}\left(1-e^{-aT}\right)^2 \tag{12.69}$$

We solve Eq. 12.69 by rewriting $f^\star(0,T)$ as

$$f^\star(0,T) = x(T) - g(T), \tag{12.70}$$

where x and g are defined by

$$\begin{cases} \dfrac{\partial x}{\partial t} = -ax(t) + \Theta(t), \\ x(0) = r(0), \end{cases} \tag{12.71}$$

$$g(t) = \frac{\sigma^2(t)}{2a^2}(1-e^{-at})^2 = \frac{\sigma^2(t)}{2}B^2(0,t). \tag{12.72}$$

Finally, $\Theta(T)$ can be expressed as

$$\Theta(T) = \frac{\partial x}{\partial t}(T) + ax(T) \tag{12.73}$$

$$= \frac{\partial f^\star}{\partial T}(0,T) + \frac{\partial g}{\partial T}(T) + ax(t) \tag{12.74}$$

$$= \frac{\partial f^\star}{\partial T}(0,T) + \frac{\partial g}{\partial T}(T) + a(f^\star(0,T) + g(T)), \tag{12.75}$$

and hence we have determined a martingale measure Q, for a given choice of a and $\sigma(t)$. Thus, if we can specify a and $\sigma(t)$, we are ready to implement the Hull-White model, given observed market data of the yield curve. More on how a and $\sigma(t)$ are specified is investigated in Section 12.6. The reason for not doing it directly is that some of the results of Section 12.5 are required.

We finish off this section by presenting the Hull-White term structure using the results above. Due to its importance, we state it as a proposition.

Proposition 12.5 ***Hull-White term structure***
Consider the Hull-White model with given a and $\sigma(t)$ (approximated as constant). Choosing Θ as

$$\Theta(T) = \frac{\partial f^\star}{\partial T}(0,T) + \frac{\partial g}{\partial T}(T) + a(f^\star(0,T) + g(T)), \tag{12.76}$$

where $g(t)$ is given by

$$g(t) = \frac{\sigma^2(t)}{2a^2}(1-e^{-at})^2 = \frac{\sigma^2(t)}{2}B^2(0,t), \tag{12.77}$$

gives the bond prices as

$$p(t,T) = \frac{p^\star(0,T)}{p^\star(0,t)} \exp\left(B(t,T)f^\star(0,t) - \frac{\sigma^2(t)}{4a}B^2(t,T)(1-e^{-2at}) - B(t,T)r(t) \right), \tag{12.78}$$

where B is given by

$$B(t,T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right). \tag{12.79}$$

12.5 Pricing under Stochastic Interest Rates

Having time dependent stochastic short rates instead of constant short rates will imply consequences on most parts of the Black-Scholes model. Let us explore the immediate consequences on pricing of derivatives further in the following section.

As an introduction to the topic consider pricing of a contingent claim $\mathcal{X}$ under a model with a stochastic short rate r. We want to price the claim using the standard procedure with the risk neutral valuation formula

$$\Pi(0;\mathcal{X}) = E^Q\left[e^{-\int_0^T r(s)ds} \cdot \mathcal{X}\right]. \tag{12.80}$$

Since there are two stochastic variables under the expectation, $\int_0^T r(s)ds$ and $\mathcal{X}$, the formula will be problematic to evaluate. In order to compute the expectation one would have to find the joint distribution of the two stochastic variables under Q and then perform the integration with respect to that distribution. Since this is complicated an alternative method that tackles the problem more effectively will be used, namely the technique of changing the numeraire, developed in Geman et al. (1995) [12].

Let us begin by recalling what a numeraire actually is. According to the First Fundamental Theorem the market model is free of arbitrage if and only if there exists a martingale measure, i.e. a measure $Q^0 \sim P$ such that the normalized price processes

$$\frac{S_0(t)}{S_0(t)}, \frac{S_1(t)}{S_0(t)}, \ldots, \frac{S_N(t)}{S_0(t)} \tag{12.81}$$

are martingales under Q^0. The processes $S_0(t)$ is referred to as numeraire process and is assumed to be strictly positive. Usually the natural choice of numeraire is the money account B but sometimes computations can be facilitated by choosing another asset as numeraire. Thus it is useful to be able to go from one numeraire-martingale measure pair (S_0, Q^0) to another numeraire-martingale measure pair (S_1, Q^1). This procedure is referred to as change of numeraire and has previously been used indirectly when going from P to Q.

Suppose that the corresponding martingale measure Q^0 for a specific numeraire S_0 is known and we want to change the numeraire from S_0 to S_1. Then the appropriate Girsanov transformation taking us from Q^0 to Q^1, where Q^1 is the martingale measure corresponding to the numeraire S_1, needs to be found.

Next, let us find the relevant Girsanov transformation. We begin by recalling that in order to have no arbitrage possibilities $X(T)$ must be priced according to

$$\Pi(t; X) = S_0(t) E^0\left[\frac{X}{S_0(T)}\middle|\mathcal{F}_t\right]. \tag{12.82}$$

Using the pricing formula above two conditions can be established,

$$\Pi(0; X) = S_0(0) E^0\left[\frac{X}{S_0(T)}\right], \tag{12.83}$$

and

$$\Pi(0; X) = S_1(0) E^1\left[\frac{X}{S_1(T)}\right]. \tag{12.84}$$

Denoting the Radon-Nikodym derivative by $L_0^1(T)$ we define

$$L_0^1(T) = \frac{dQ^1}{dQ^0}, \quad \text{on } \mathcal{F}_T. \tag{12.85}$$

Then Eq. 12.83 can be expressed as

$$\Pi(0; X) = S_1(0) E^0\left[\frac{X}{S_1(T)} \cdot L_0^1(T)\right], \tag{12.86}$$

and thus it holds that

$$S_0(0) E^0\left[\frac{X}{S_0(T)}\right] = S_1(0) E^0\left[\frac{X}{S_1(T)} \cdot L_0^1(T)\right] \tag{12.87}$$

for a sufficiently integrable arbitrary $X(T)$. Thus it must hold that

$$\frac{S_0(0)}{S_0(T)} = \frac{S_1(0)}{S_1(T)} \cdot L_0^1(T). \tag{12.88}$$

Reorganizing the obtained expression the induced likelihood process is found to be

$$L_0^1(t) = \frac{S_0(0)}{S_1(0)} \cdot \frac{S_1(t)}{S_0(t)}, \ 0 \leq t \leq T. \tag{12.89}$$

Let us state our exploration as a proposition.

Proposition 12.6 ***Change of numeraire***
Assume that Q^0 is a martingale measure for the numeraire S_0 on $\mathcal{F}_T$ and assume that S_1 is another asset price process that is positive such that $\frac{S_1(t)}{S_0(t)}$ is a Q^0-martingale. Define Q^1 on $\mathcal{F}_T$ by the likelihood process

$$L_0^1(t) = \frac{S_0(0)}{S_1(0)} \cdot \frac{S_1(t)}{S_0(t)}, \ 0 \leq t \leq T. \tag{12.90}$$

Then Q^1 is a martingale measure for S_1.

Now returning to our original problem, pricing a contingent claim when the short rate is time dependent and stochastic, we need to find the appropriate change of numeraire. Since the foundation of the section on short rates builds on zero coupon bonds, a natural choice of numeraire to try is the T-bond. The corresponding martingale measure will be denoted Q^T and referred to as the T-forward measure as specified in the following definition.

Definition 12.3 T-*forward measure*
For a fixed T, the T-forward measure Q^T is defined as the martingale measure for the numeraire process $p(t,T)$.

In a model using the money account B as numeraire with the corresponding risk neutral martingale measure Q an explicit description for Q^T can be obtained as presented in the following proposition.

Proposition 12.7 ***Change of martingale measure from* $\mathbf{Q}$ *to* $\mathbf{Q^T}$**
Let Q denote the risk neutral martingale measure. The likelihood process

$$L^T(t) = \frac{dQ^T}{dQ}, \quad on\ \mathcal{F}_t,\ 0 \leq t \leq T, \tag{12.91}$$

is then given by

$$L^T(t) = \frac{p(t,T)}{B(t)p(0,T)}. \tag{12.92}$$

Moreover, if the Q-dynamics of the T-bond are Wiener driven, i.e. on the form

$$dp(t,T) = r(t)p(t,T)dt + v(t,T)p(t,T)dW_t, \tag{12.93}$$

then the dynamics of L^T are given by

$$dL^T(t) = L^T(t)v(t,T)dW_t. \tag{12.94}$$

In other words, the Girsanov kernel for the transition from Q to Q^T is given by the T-bond volatility $v(t,T)$.

Using the change of numeraire of Proposition 12.7 and observing that $p(T,T) = 1$ the pricing formula for a contingent claim $X(T)$, Eq. 12.82, takes the following form under the T-forward measure.

Proposition 12.8 ***Pricing under the*** $\mathbf{Q^T}$ ***measure***
The price of an arbitrary T*-claim* X *at time* t *is given by*

$$\Pi(t;X) = p(t,T)E^T[X \mid \mathcal{F}_t], \tag{12.95}$$

where E^T *denotes integration with respect to* Q^T*.*

Moreover, the following lemma will be useful later on. Here we give it without a proof.

Lemma 12.1
It holds that $Q = Q^T$ *if and only if* r *is deterministic.*

Having arrived at a general pricing formula under Q^T we are now ready to price options. Before we proceed with an option pricing formula we would like to quickly investigate the so called *Expectation Hypothesis*. Recalling that the forward rate process $f(t,T)$ can be viewed as an estimate of the future short rate $r(T)$ we would like to investigate whether the market expects the short rate at T to be high implies that the forward rate $f(t,T)$ is also high. For the sake of brevity we give the result directly.

Proposition 12.9 ***The Expectation Hypothesis***
Assume that, for all $T>0$*, we have* $\frac{r(T)}{B(T)} \in L^1(Q)$*. Then, for every fixed* T*, the process* $f(t,T)$ *is a* Q^T*-martingale for* $0 \le t \le T$*, and in particular*

$$f(t,T) = E^T[r(T) \mid \mathcal{F}_t]. \tag{12.96}$$

Proof. In order to establish Proposition 12.9 the pricing equation of Proposition 12.8 with $X = r(T)$ is used

$$\Pi(t;X) = E^Q\big[r(T)e^{-\int_t^T r(s)ds}\big|\mathcal{F}_t\big] = p(t,T)E^T\big[r(T)\big|\mathcal{F}_t\big].$$

Reorganizing we get

$$\begin{aligned}
E^T\big[r(T)\big|\mathcal{F}_t\big] &= \frac{1}{p(t,T)}E^Q\big[r(T)e^{-\int_t^T r(s)ds}\big|\mathcal{F}_t\big] \\
&= -\frac{1}{p(t,T)}E^Q\big[\frac{\partial}{\partial T}e^{-\int_t^T r(s)ds}\big|\mathcal{F}_t\big] \\
&= -\frac{1}{p(t,T)}\frac{\partial}{\partial T}E^Q\big[e^{-\int_t^T r(s)ds}\big|\mathcal{F}_t\big] \\
&= -\frac{p_T(t,T)}{p(t,T)} \\
&= f(t,T).
\end{aligned}$$

□

We can thus deduce that the Expectation Hypothesis is true under the Q^T-measure but not generally under neither the risk neutral Q-measure nor the observable P-measure.

We now turn to the delicate problem of finding an option pricing formula in a financial market with a stochastic short rate and a strictly positive asset price process $S(t)$. The problem is considered in Geman et al. (1995) [12]. The option we want to price is a European call on S with strike price K and date of maturity T,

$$\mathcal{X} = \max(S(T) - K, 0). \tag{12.97}$$

Using an indicator function,

$$I\{S(T) \geq K\} = \begin{cases} 1 \text{ if } S(T) \geq K \\ 0 \text{ if } S(T) < K, \end{cases} \tag{12.98}$$

the option can be written as

$$\mathcal{X} = (S(T) - K) \cdot I\{S(T) \geq K\}. \tag{12.99}$$

The price of $\mathcal{X}$ at time $t = 0$ can thus be expressed as

$$\Pi(0; \mathcal{X}) = E^Q\left[\frac{S(T) - K}{B(T)} \cdot I\{S(T) \geq K\}\right] \tag{12.100}$$

$$= E^Q\left[e^{-\int_0^T r(s)ds} S(T) \cdot I\{S(T) \geq K\}\right] \tag{12.101}$$

$$- KE^Q\left[e^{-\int_0^T r(s)ds} \cdot I\{S(T) \geq K\}\right]. \tag{12.102}$$

We can now use the change of numeraire technique to compute the two terms separately. For the first term we use S as numeraire and therefore the measure Q^S and for the second term we use the T-forward measure Q^T. Applying the two pricing formulas Eqs. 12.82 and 12.95 we arrive at the general option pricing formula

$$\Pi(0; \mathcal{X}) = S(0)Q^S(S(T) \geq K) - Kp(0, T)Q^T(S(T) \geq K). \tag{12.103}$$

Next we would of course like to compute the probabilities in the general option pricing formula explicitly. In order to be able to compute the probabilities the volatilities have to be deterministic. More precisely, we define

$$Z_{S,T}(t) = \frac{S(t)}{p(t, T)} \tag{12.104}$$

and assume that $Z_{S,T}$ has a stochastic differential on the form

$$dZ_{S,T}(t) = Z_{S,T}(t)m_{S,T}(t)dt + Z_{S,T}(t)\sigma_{S,T}(t)dW_t, \tag{12.105}$$

where the volatility process $\sigma_{S,T}(t)$ is deterministic. We are now ready to compute the probabilities explicitly. The probability of the second term can be written as

$$Q^T(S(T) \geq K) = Q^T\left(\frac{S(T)}{p(T, T)} \geq K\right) = Q^T(Z_{S,T}(T) \geq K). \tag{12.106}$$

Noting that $Z_{S,T}$ is Q^T martingale, since it is an asset price normalized by a T-bond, its Q^T-dynamics are given by

$$dZ_{S,T}(t) = Z_{S,T}(t)\sigma_{S,T}(t)dW_t^T. \tag{12.107}$$

Recalling the solution of a Geometric Brownian motion the solution to the above equation is given by

$$Z_{S,T}(T) = \frac{S(0)}{p(0,T)} \exp\left(-\frac{1}{2}\int_0^T ||\sigma_{S,T}||^2(t)dt + \int_0^T \sigma_{S,T}(t)dW_t^T \right). \tag{12.108}$$

Next we observe that the exponent consists of a deterministic time integral and a stochastic integral. Furthermore the integrand of the stochastic integral is deterministic, thus we can deduce that the stochastic integral will have a Gaussian distribution with zero mean and variance

$$\Sigma_{S,T}^2(T) = \int_0^T ||\sigma_{S,T}(t)||^2 dt. \tag{12.109}$$

Thus the entire exponent is normally distributed and therefore it holds that

$$Q^T(S(T) \geq K) = N[d_2], \tag{12.110}$$

where

$$d_2 = \frac{\ln \frac{S(0)}{Kp(0,T)} - \frac{1}{2}\Sigma_{S,T}^2(T)}{\sqrt{\Sigma_{S,T}^2(T)}}. \tag{12.111}$$

The probability of the first term of the general option pricing formula is a Q^S-probability, we can thus write

$$Q^S(S(T) \geq K) = Q^S\left(\frac{p(T,T)}{S(T)} \leq \frac{1}{K} \right) = Q^S\left(Y_{S,T}(T) \leq \frac{1}{K} \right), \tag{12.112}$$

where $Y_{S,T}(t) = \frac{p(t,T)}{S(t)} = \frac{1}{Z_{S,T}(t)}$. The Q^S-dynamics of $Y_{S,T}$ are on the form

$$dY_{S,T}(t) = Y_{S,T}(t)\delta_{S,T}(t)dW_t^S. \tag{12.113}$$

Using the fact that $Y_{S,T} = Z_{S,T}^{-1}$ and applying Itô's formula we find that $\delta_{S,T}(t) = -\sigma_{S,T}(t)$. The solution of the above equation is thus given by

$$Y_{S,T}(T) = \frac{p(0,T)}{S(0)} \exp\left(-\frac{1}{2}\int_0^T ||\sigma_{S,T}||^2(t)dt - \int_0^T \sigma_{S,T}(t)dW_t^S \right). \tag{12.114}$$

Similar to the first case we again have a normally distributed exponent and it therefore holds after simplifications that

$$Q^S(S(T) \geq K) = N[d_1], \tag{12.115}$$

where

$$d_1 = d_2 + \sqrt{\Sigma_{S,T}^2(T)}. \tag{12.116}$$

We can now summarize our exploration as a proposition.

Proposition 12.10 ***Geman-El Karoui-Rochet***
Assuming that volatilities are deterministic, the price of the call option defined in Eq. 12.97 is given by

$$\Pi(0, \mathcal{X}) = S(0)N[d_1] - p(0,T)KN[d_2], \tag{12.117}$$

where d_2 and d_1 are given by

$$d_2 = \frac{\ln \frac{S(0)}{Kp(0,T)} - \frac{1}{2}\Sigma^2_{S,T}(T)}{\sqrt{\Sigma^2_{S,T}(T)}}, \tag{12.118}$$

$$d_1 = d_2 + \sqrt{\Sigma^2_{S,T}(T)}, \tag{12.119}$$

respectively and $\Sigma^2_{S,T}(T)$ is given by

$$\Sigma^2_{S,T}(T) = \int_0^T ||\sigma_{S,T}(t)||^2 dt. \tag{12.120}$$

Applying the put-call parity we have the following pricing formula for a put option, $\mathcal{Y} = \max(K - S, 0)$,

$$\Pi(0; \mathcal{Y}) = p(0,T)KN[-d_2] - S(0)N[-d_1]. \tag{12.121}$$

12.6 Calibration of the Hull-White Model

In order to be able to use Proposition 12.5 to simulate short rates we need to determine a and $\sigma(t)$. To do this, the values of a and $\sigma(t)$ that best matches theoretical prices of some interest rate derivatives to observed prices need to be found. Common choices of such derivatives are caps and caplets as well as swaptions. We have chosen to use caps and caplets and will in the following subsection go through what a cap and a caplet is and how they are priced. After that the calibration steps of the Hull-White model are derived. We mainly follow Gurrieri et al. (2009) [15], inspiration is also gathered from Brigo and Mercurio (2006) [4] as well as Clark (2011) [6].

12.6.1 Caps and Caplets

Caps are among the most traded interest rate derivatives. A cap protects its owner from having to pay more than a specific rate that is prespecified, the cap rate, even though the owner has a floating rate loan. Hence, the cap can be seen as a financial insurance contract. A cap is the sum of a number of caplets, which are defined below.

Definition 12.4 ***Caplet***
Let the time interval for the cap be $[0,T]$. *Divide the time interval into equidistant points,* $0 = T_0, T_1, ..., T_n = T$, *and denote the length of each elementary interval by* δ. *Furthermore, denote the cap rate by* R *and denote the principal amount of money for the cap by* K. *Finally, denote the underlying rate of interest of the cap by* L. *Hence,* $L(T_{i-1}, T_i)$ *is the* LIBOR *spot rate over the interval* $[T_{i-1}, T_i]$. *Then the caplet* i, *paid at time* T_i, *is defined as the following contingent claim*

$$\mathcal{X}_i = K\delta \max(L(T_{i-1}, T_i) - R, 0) \tag{12.122}$$

The next step is to price the caplet. Using the shorthand notation $L = L(T_{i-1}, T_i)$ and $p = p(T_{i-1}, T_i)$, L can be expressed as

$$L = \frac{1-p}{p\delta}. \tag{12.123}$$

Hence, by using the notation $x^+ = \max(x, 0)$ as well as without loss of generality setting $K = 1$, the caplet can be priced as

$$\mathcal{X} = \delta(L-R)^+ = \delta\left(\frac{1-p}{p\delta} - R\right)^+ = \left(\frac{1}{p} - (1+\delta R)\right)^+ = \frac{1+\delta R}{p}\left(\frac{1}{1+\delta R} - p\right)^+. \tag{12.124}$$

Since a payment of $\frac{1+\delta R}{p}(\frac{1}{1+\delta R} - p)$ at time T_i is equivalent to a payment of $(1+\delta R)(\frac{1}{1+\delta R} - p)$ at time T_{i-1}, a caplet is equivalent to $1 + \delta R$ put options on an underlying T_i-bond, where the exercise date of the option is T_{i-1} and the exercise price is $\frac{1}{1+\delta R}$. Thus, a cap can be viewed as a portfolio of put options.

12.6.2 Determining the Coefficients a and σ(t) in the Hull-White Model

In the previous section, we showed that the price of a caplet with cap rate R is equal to the price of $1 + \delta R$ put option on an underlying T_i zero coupon bond, also called zero-bond put options (ZBP), with strike price $\frac{1}{1+\delta R}$, where δ is the length of the time interval on which the caplet is active. Hence, we have the following formula for pricing a caplet, with fixing time T_F and paying time T_P

$$\Pi_t^{Caplet}(R, T_F, T_P) = (1+\delta R)\Pi_t^{ZBP}\left(T_F, T_P, \frac{1}{1+\delta R}\right). \tag{12.125}$$

In order to price the caplet, we need to price the ZBP. We recall that the Q-dynamics of r in the Hull-White model are given by

$$dr = (\Theta(t) - ar)dt + \sigma(t)dW_t. \tag{12.126}$$

We also recall that the HW model has an affine term structure which means that

$$p(t,T) = e^{A(t,T) - B(t,T)r(t)}, \tag{12.127}$$

where A and B are deterministic functions and where B is given by

$$B(t,T) = \frac{1}{a}\Big(1 - e^{-a(T-t)}\Big). \tag{12.128}$$

We want to price a European put option with exercise date T_F and strike price K, on an underlying T_P-bond, where $T_P > T_F$. Observing the process

$$Z(t) = \frac{p(t,T_P)}{p(t,T_F)}, \tag{12.129}$$

we want to check if the volatility, σ_Z, of this process is deterministic, in line with the assumption of Proposition 12.10. Inserting Eq. 12.127 into Eq. 12.129 gives

$$Z(t) = \exp\big(A(t,T_P) - A(t,T_F) - (B(t,T_P) - B(t,T_F))r(t)\big). \tag{12.130}$$

Next, applying Itô's formula and using Eq. 12.126 gives the following Q-dynamics

$$dZ_t = Z(t)(\cdots)dt + Z(t)\sigma_Z(t)dW_t, \tag{12.131}$$

where σ_Z is given by

$$\sigma_Z(t) = -\sigma(t)(B(t,T_P) - B(t,T_F)) = \frac{\sigma(t)}{a}e^{at}(e^{-aT_P} - e^{-aT_F}), \tag{12.132}$$

and the drift coefficient only is indicated by $(\cdots)$ since it is not of interest. Hence, we have shown that σ_Z is deterministic and can therefore apply Proposition 12.10 in order to price a ZBP under the Hull-White model.

Proposition 12.11 ***Hull-White bond option***
The price of a European put option with strike price K and time of maturity T_F, in the Hull-White model at $t = 0$, on a T_P-bond, is given by

$$\Pi_0^{ZBP}(T_F,T_P,K) = p(0,T_F)KN[-d_2] - p(0,T_P)N[-d_1], \tag{12.133}$$

where d_1 and d_2 are given by

$$d_2 = \frac{\ln\left(\frac{p(0,T_P)}{Kp(0,T_F)} - \frac{1}{2}\Sigma^2\right)}{\sqrt{\Sigma^2}}, \tag{12.134}$$

$$d_1 = d_2 + \sqrt{\Sigma^2}, \tag{12.135}$$

and where Σ^2 is given by

$$\Sigma^2 = \frac{\sigma^2(t)}{2a^3}(1 - e^{-2aT_F})\Big(1 - e^{-a(T_P - T_F)}\Big)^2. \tag{12.136}$$

Having determined a pricing formula for a ZBP the theoretical price of a caplet can now be computed according to Eq. 12.125. Thus by using for example the LSE approach a and $\sigma(t)$ can be determined such that the theoretical prices best match the observed market prices.

13 Simulation of FX Rates and Pricing under the Hull-White Model

The aim of this section is to describe how the theory of Section 12 has been implemented in our particular simulation model and PE fund setting. First, we show how the Hull-White model was calibrated and present the calibration results. Second, we describe how short rates were simulated and the assumptions that were made. Third, we describe how FX rates were simulated and the assumptions that were made. Last, some aspects regarding pricing of the hedging derivatives under an FX model with stochastic short rates are presented and our assumptions are presented and motivated.

13.1 Calibration of the Hull-White Model to Market Data

This section sets out to present how the Hull-White model was calibrated to market data. We begin by determining the coefficients a and $\sigma(t)$ by pricing caplets according to Eq. 12.125 and seeking the values of a and $\sigma(t)$ that fit the theoretical caplet prices to the caplet prices observed on the market best. After that, using the obtained values of a and $\sigma(t)$ and yield market data, we determine the piecewise constant function Θ according to Proposition 12.5.

The model is calibrated for each country separately using the market price of ten different caplets, $\Pi_i^*,\ i = 1, ..., 10$, for each country. Caplet prices have been collected for two different maturities each year for five years. All caplets used in the calibration have a three months tenor. Using the LSE approach we seek estimates of a and $\sigma(t)$ such that

$$(\hat{a}, \hat{\sigma}(t)) = \arg \min_{a,\sigma(t)} \sum_{i=1}^{10} \left(\frac{\Pi_i}{\Pi_i^*} - 1 \right)^2. \tag{13.1}$$

Performing the above minimization we sometimes found that global minimums were obtained for $a < 0$ or a very large. Since by definition a has to be bigger than zero the first finding is problematic. Also, when comparing to the results of Gurrieri et al. (2009) [15] we came to the conclusion that a should be of size 0%-12%. Therefore, instead of seeking global minimums we sought a local minimum in the given range. Moreover, we found that the least square errors became the smallest when we allowed $\sigma(t)$ to be piecewise constant. In more detail, we allow for sigma to change value for each caplet maturity, i.e. each half year. The results obtained are presented in Table 13.1 below and in Figure B.14 in the Appendix. Note that $\hat{\sigma}_{low}$ and $\hat{\sigma}_{high}$ are the smallest respectively largest value of the piecewise constant $\hat{\sigma}(t)$.

	EU	SE	NO	GB	US
LS error	0.078	0.026	0.006	0.003	0.016
$\hat{a}$	0.030	0.086	0.039	0.037	0.106
$\hat{\sigma}_{low} - \hat{\sigma}_{high}$	0.006-0.020	0.007-0.015	0.014-0.020	0.008-0.017	0.005-0.008

Table 13.1: Least square fitted parameters for the Hull-White model

After having determined the values of a and $\sigma(t)$ that best fits the theoretical caplet prices to the caplet prices observed on the market we proceed by determining the piecewise constant function Θ according to Proposition 12.5. To exemplify how the curves look we show the Swedish case below in Figure 13.1. The curves of all markets are found in Figure B.15 in Appendix B.

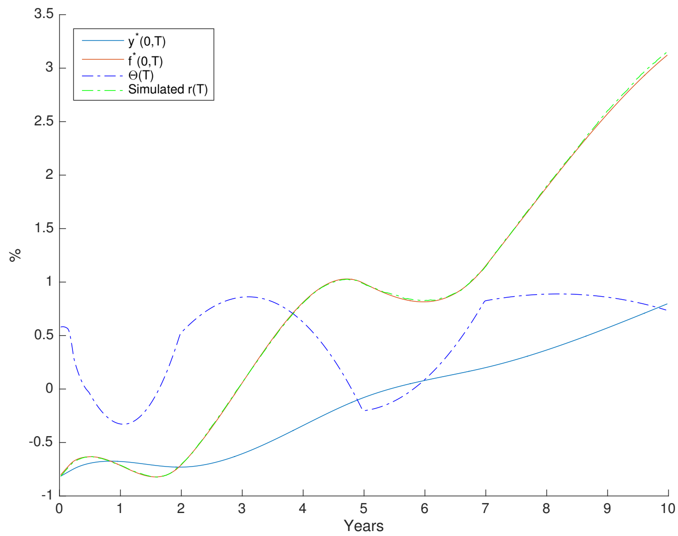

Figure 13.1: Observed and fitted Hull-White curves for Sweden

Having obtained piecewise constant functions $\Theta(T)$ for all five countries the calibration of the Hull-White model is complete. Moreover, since the least square errors obtained when comparing the theoretical caplet prices to the actual ones are fairly small the calibrated model is believed to be accurate.

13.2 Simulation of Short Rates

In order to simulate future short rates we use our calibrated Hull-White model. Since interest rates are highly correlated to one another the correlation between the historical yields needs to be taken into account when simulating future short rates. The historical correlations are given in Table 13.2.

	EU	SE	NO	GB	US
EU	1	0.984	0.976	0.992	0.948
SE	0.984	1	0.974	0.989	0.890
NO	0.976	0.974	1	0.980	0.944
GB	0.992	0.989	0.980	1	0.930
US	0.948	0.890	0.944	0.930	1

Table 13.2: Correlation between the historical yields

Weekly interest rates are recursively simulated according to

$$r_t^i = r_{t-1}^i + \big(\Theta(t) - ar_{t-1}^i\big) + \sigma_{t-1}^i\big(W_t^i - W_{t-1}^i\big),\ t = 1, ..., 520 \tag{13.2}$$
$$r_0^i = y_0^{*,i}, \tag{13.3}$$

where $i = 1, ..., 5$ corresponds to the five cases EU, SE, NO, GB and US respectively and correlated Wiener processes are simulated as discussed in Section F.1. Moreover, the start date, $t = 0$, is the last date of the historical data, i.e. December 29, 2017. The time, t, is measured in weeks, hence the other present parameters, a, $\sigma(t)$ and y_0^*, must be transformed into effective weekly values.

Figure 13.2 shows the first $1,000$ trajectories of the full simulated sample of $n = 10,000$ simulated interest rates. The results are expected, since $\sigma(t)$ is fairly small and a mean reversion is present the simulated rates do not depart significantly from their respective markets' forward rate curves. Also, by the Expectation Hypothesis stated in Proposition 12.9, this should be the case since $f(t,T) = E^T[r(T)|\mathcal{F}_t]$.

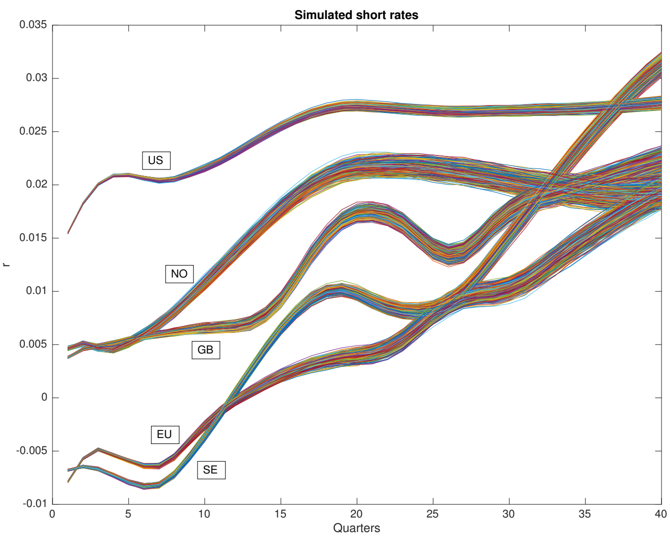

Figure 13.2: Simulated short rates for the five different markets

13.3 Simulation of FX rates

The simulation of FX rates using the random walk method is very similar to the one described in Section 8.2 with some important differences. Instead of repeating the description of the full procedure we direct the reader to that section and instead focus on pointing out the differences.

As before the European market is considered to be the domestic market and the four other markets are considered to be foreign markets 1 to 4. The following formula describes how the FX rates are modelled,

$$X_t^i = X_0^i \exp\left(\sum_{s=1}^{t}\left(r_d(s) - r_f^i(s)\right) - \frac{1}{2}\sigma_{X^i}^2 t + \sigma_{X^i} W_t^i\right), \tag{13.4}$$

where $i = 1, ..., 4$ corresponds to the four cases SEK, NOK, GBP and USD respectively. In contrast to before we now use our simulated short rates instead of constant ones, thus the drift term has to be summed over instead of just amplified by the time t. The same implied volatilities as before are used.

It is important to point out that we have not inferred a correlation between the Wiener processes of the short rates and the Wiener processes of the FX rates. Existing literature, e.g. Clark (2011) [6], does not seem to cover this topic when considering a model with several FX rates and not just one. However, there is of course a correlation between the Wiener processes of the FX rates and the Wiener processes of the interest rates. Though, to account for the dependence between the FX rates and interest rates further than just through Eq. 13.4 is to be out of the scope of this work.

Using the random walk method with implied FX volatilities and simulated short rates the development of the FX rates can now be simulated. MATLAB is used to simulate correlated Wiener processes driving the FX rates as discussed in Section 7.3. The results obtained are very similar to the ones obtained in the constant interest case. Probability distribution functions of $n = 10,000$ simulations after 40 quarters for all FX rates are found in Figure B.17 in Appendix B.

13.4 Derivative Pricing

In the two following subsections we show how we price the hedging derivatives and clarify the assumptions that have been made in order to price them under the FX model with stochastic short rates.

13.4.1 Forwards

Valuing forwards and rolling forwards requires the computation of a forward rate. This is fairly easily done by using the Hull-White term structure given in Proposition 12.5 and then using the relation

$$f(t,T) = -\frac{\partial}{\partial T}\ln\big(p(t,T)\big). \tag{13.5}$$

No further assumptions are required.

13.4.2 Options

In order to be able to use the pricing formula of Proposition 12.10 some assumptions have to be made. Below we present the assumptions and their rationale.

The general option pricing formula, Eq. 12.103, for a call option on the FX rate reads

$$\Pi(0,\mathcal{X}) = X(0)e^{-\int_0^T r_f(s)ds}Q^d(X_T \geq K) - p(0,T)KQ^T(X_T \geq K). \tag{13.6}$$

In order to be able to evaluate this formula we want to transform it to the form of Geman-El Karoui-Rochet's formula. Noting that the simulated short rates, shown in Figure 13.2, only slightly deviates from their mean we assume that the simulated short rates are deterministic. Now we can use Lemma 12.1 to deduce that under our assumption $Q^d = Q^T$ and $Q^f = Q^T$. This fact allows us to write Eq. 13.6 as

$$\begin{aligned}\Pi(0,\mathcal{X}) &= X(0)e^{-\int_0^T r_f(s)ds}N[d_1] - p_d(0,T)KN[d_2] &(13.7)\\ &= X(0)p_f(0,T)N[d_1] - p_d(0,T)KN[d_2], &(13.8)\end{aligned}$$

where d_2 and d_1 are given by

$$\begin{aligned}d_2 &= \frac{\ln\frac{X(0)}{Kp_d(0,T)} - \frac{1}{2}\Sigma^2_{S,T}(T)}{\sqrt{\Sigma^2_{S,T}(T)}}, &(13.9)\\ d_1 &= d_2 + \sqrt{\Sigma^2_{S,T}(T)}. &(13.10)\end{aligned}$$

Furthermore, we also assume that $\sigma_{S,T}(t) = \sigma_{implied}$, i.e. the same implied volatilities that were used in the model with constant interest rates. This implies that $\sigma_{S,T}(t)$ not only is deterministic but also constant and hence we can write

$$\begin{aligned}\Sigma^2_{S,T}(T) &= \int_0^T ||\sigma_{S,T}(t)||^2 dt &(13.11)\\ &= \int_0^T ||\sigma_{implied}||^2 dt = \sigma^2_{implied}\cdot T. &(13.12)\end{aligned}$$

The bond prices used as discount factors, $p_f(0,T)$ and $p_d(0,T)$ are observable on the market at time $t = 0$. However, as the model time evolves and new investments are made we will be required to compute these prices at new times, then the Hull-White terms structure equation given in Eq. 12.78 is used.

14 Results and Discussion

This section presents the results of the hedging strategies under the random walk simulation method with stochastic interest rates. The difference to the base model is that short rates now also are simulated, which have many consequences as described in Sections 12 and 13.

14.1 Foreign Exchange Rate Results

Section 13.3 described the methodology and presented the results of the random walk simulation method of FX rates with stochastic interest rates. Since there on average is a negative drift in the FX rates a systematic negative FX effect on the portfolio companies is implied as in the base model.

14.2 Performance of FX Hedging Strategies under the RW Simulation Method with Stochastic Interest Rates

A summary of the results in Tables D.1-D.8 in Appendix D is shown in Tables 14.1 and 14.2. Even though short rates are simulated the obtained results are similar compared to the case of constant risk free interest rates, comparing the results to Tables C.1-C.8 in Appendix C. Hence, the assumption of constant risk free interest rates does not have a crucial impact on the performances of the hedging strategies.

Comparing the hedging strategies' performances within each fund and between the funds we observe very similar results leading to the same conclusions as in the random walk simulation with drift. Hence the previous assumption of constant risk free interest rates does not affect one's decisions regarding if and how to hedge.

Metric	**Un-hedged**	**Forward**	**Rolling forward**	**Call**$_{K_3^C}$	**Put**$_{K_3^P}$	**Strangle**$_{K_3^S}$
IRR	1,1,1	1,1,1	1,1,1	4,4,4	4,4,4	6,6,4
VaR	3,4,4	2,2,2	1,1,1	5,6,6	3,3,3	5,5,4
ES	3,4,4	2,2,2	1,1,1	5,6,6	3,3,3	5,5,4
SR	4,4,4	2,2,2	1,1,1	5,6,6	3,3,3	5,5,4

Table 14.1: Ranking of the hedging strategies' performances for the equity funds (A,B,C) under the RW simulation of FX rates with stochastic interest rates

Metric	**Un-hedged**	**Forward**	**Rolling forward**	**Call**$_{K_3^C}$	**Put**$_{K_3^P}$	**Strangle**$_{K_3^S}$
IRR	1,1,1	1,1,1	1,1,1	4,4,4	4,4,4	6,6,4
VaR	4,4,5	2,2,2	1,1,1	6,6,6	3,3,3	5,5,4
ES	4,4,5	2,2,2	1,1,1	6,6,6	3,3,3	5,5,4
SR	4,4,5	2,2,2	1,1,1	6,6,6	3,3,3	5,5,4

Table 14.2: Ranking of the hedging strategies' performances for the infrastructure funds (A,B,C) under the RW simulation of FX rates with stochastic interest rates

Part IV

Extended Model II: Dollar Denominated Funds

15 Methodology, Results and Discussion

So far EUR denominated funds have been considered. Since the European risk free interest rate is the lowest among all risk free interest rates considered in this thesis, having EUR denominated funds will imply a systematic negative FX effect on the non-EUR denominated portfolio companies under the random walk simulation method, as described in detail in Section 11.1. Hence, the IRR of the unhedged strategy for the equity funds and infrastructure funds is lower than the expected IRR of 20% and 12% respectively, as seen in Tables C.1 and C.5 in Appendix C.

Since the American risk free interest rate is the highest among all risk free interest rates considered in this thesis, it is of great interest to analyse the performance of USD denominated funds under the random walk simulation method with constant risk free interest rates. The remainder of this section is organised as follows. First the methodology is described. Then the FX rate results and the results of the hedging strategies are presented.

15.1 Methodology

Theoretically, there will be a positive drift in FX rates quoted to the USD because of the interest rate difference, as seen in Eq. 8.1 which we recall as

$$X_t^i = X_0^i \exp\left(\left(r_d - r_f^i - \frac{1}{2}\sigma_{X^i}^2\right)t + \sigma_{X^i} W_t^i\right), \tag{15.1}$$

where $i = 1, ..., 4$ corresponds to the four cases SEK, NOK, GBP and EUR respectively, and where the American market is considered to be the domestic market and the four other markets are considered to be foreign markets 1 to 4. There will be a positive drift in FX rates because $r_d > r_f^i$ for all i. We use the same simulation technique for the random walk simulation method and the same historical data as previously.

The USD denominated funds are built in exactly the same way as the EUR denominated funds were built, i.e. as in Appendix A with the only difference that USD and EUR have switched places. For example, global PE fund A now consists of 50% USD and 12.5% EUR, focusing on the initial values of the portfolio companies. After these changes, the rest of the methodology is exactly the same as earlier in the thesis.

15.2 Foreign Exchange Rate Results

The results of the simulated FX rates quoted to the USD are presented in Figures B.18 and B.19 in Appendix B. The simulated FX rates quoted to the USD in Figure B.18 can be compared to the simulated FX rates quoted to the EUR in Figure B.3 in Appendix B. In Figure B.18 we see a positive drift whilst in Figure B.3 we see a negative drift, as expected. The positive drift will have a systematic positive FX effect on the non-USD denominated portfolio companies, which easily can be understood by the following simplified example.

Consider a USD denominated fund that invests in a portfolio company with a NAV of USD 10 million in SEK, and makes an exit of this portfolio company in one year. The transaction is all-equity financed, and the value of the portfolio company in its denominated currency does not change. Also, assume that the SEK/USD FX rate is 0.1 now and 0.11 in one year, i.e. the FX rate has increased with 10% over the year. Hence, the portfolio company is bought for SEK 100 ($= \frac{10}{0.1}$) million. After one year, at exit, the NAV of the company is still SEK 100 million but USD 11 ($= 100 \cdot 0.11$) million. Hence, the FX effect for the PE fund has been positive, as will be the case on average for all non-USD denominated portfolio companies, due to the positive drift.

15.3 Performance of Foreign Exchange Hedging Strategies for US Dollar Denominated Funds under the Random Walk Simulation Method

The results of the performances of the different hedging strategies are presented in Tables E.5-E.8 in Appendix E and summarised in Tables 15.1 and 15.2. There are several differences compared to the results of the EUR denominated funds.

We begin by noting that the systematic FX effect affects the value of the portfolio companies positively, hence the mean IRRs of the unhedged case are higher than the expected IRR of 20% for the equity funds and 12% for the infrastructure funds. Furthermore, in the unhedged case the EUR heavy fund, Fund C, outperforms the SEK heavy fund, Fund B, which in turn outperforms the equally weighted fund, Fund A, in terms of mean IRR. This is due to the EUR/USD rate having the most positive drift whereas the SEK/USD rate has the second most positive drift. Moreover, in terms of VaR, ES and Sharpe ratio, i.e. measures accounting for volatility in returns, the most diversified fund, Fund A, outperforms the other two when observing the local funds. For the global funds, the funds' performances are equal. Thus the effect of diversification still has a larger impact than the FX drift effect on these measures, but the relative difference has diminished compared to the EUR denominated funds.

Comparing the hedging strategies' performances within each fund and between the funds we mainly observe similar results leading to the same conclusion as in the base model part's random walk simulation method with drift. The put option strategy performs better in a USD denominated fund relative to a EUR denominated fund. This is explained by the pricing of the option which accounts for the drift in FX rates and therefore the put option is relatively cheaper in the current setting. The rationale for the option being cheaper is that the strike price is a fraction of the current FX rate which is below the expected future FX rate and hence the option will be exercised less frequently implying a lower price. Moreover, the opposite is true for the call options, they are relatively more expensive in the current setting implying that the call option strategy performs worse.

Metric	**Un-hedged**	**Forward**	**Rolling forward**	**Call**$_{K_3^C}$	**Put**$_{K_3^P}$	**Strangle**$_{K_3^S}$
IRR	1,1,1	1,1,1	1,1,1	5,5,5	1,1,1	5,5,5
VaR	4,4,4	2,2,2	1,1,1	6,6,6	3,3,3	5,5,5
ES	4,4,4	2,2,2	1,1,1	6,6,6	3,3,3	5,5,5
SR	4,4,4	2,2,2	1,1,1	6,6,6	3,3,3	5,5,5

Table 15.1: Ranking of the hedging strategies' performances for the equity funds (A,B,C) under the random walk simulation of FX rates for US dollar denominated funds

Metric	**Un-hedged**	**Forward**	**Rolling forward**	**Call**$_{K_3^C}$	**Put**$_{K_3^P}$	**Strangle**$_{K_3^S}$
IRR	1,1,1	1,1,1	1,1,1	5,5,5	1,1,1	5,5,5
VaR	4,4,4	2,2,2	1,1,1	6,6,6	3,3,3	5,5,5
ES	5,5,4	2,2,2	1,1,1	6,6,6	3,3,3	4,4,5
SR	4,4,4	2,2,2	1,1,1	6,6,6	3,3,3	5,5,5

Table 15.2: Ranking of the hedging strategies' performances for the infrastructure funds (A,B,C) under the random walk simulation of FX rates for US dollar denominated funds

Even though the forward strategies are not performing relatively better for USD denominated funds than for EUR denominated funds they might still be an attractive alternative for the fund to implement. The reason is that the fund can use them to secure a higher mean IRR than the expected NAV growth of the portfolio companies. The argument follows below.

The forward FX rate F_{T_0,T_1}, i.e. standing in time $t = T_0$, to time $t = T_1$, $T_1 > T_0$, quoted to the USD is calculated as

$$F^i_{T_0,T_1} = X^i_{T_0} \exp\left((r_d - r^i_f)(T_1 - T_0)\right), \tag{15.2}$$

where $i = 1, ..., 4$ corresponds to the four cases SEK, NOK, GBP and EUR respectively, $X^i_{T_0}$ is the FX rate quoted to the USD at time T_0. The forward rate is higher than the FX rate at time T_0 since $r_d > r^i_f$ for all i. In other words, the slope of the forward curve is positive, which is equivalent to having a positive drift in the FX rate. Since the forward rate is the predetermined forward price when entering forward contracts, the owner of the forward contract will make money on the forward contract in the case of a movement of the FX rate that ends below the forward rate. However, in this case the fund will loose money on the portfolio companies. In a case where the FX rate becomes larger than the forward rate, the opposite is true, the owner of the forward contract will loose money on the contract and gain on the portfolio companies. Hence, there is no net gain arising from hedging but the advantage rather comes from the positive slope of the forward curve, which the fund can lock in. It is however important to understand that by doing this the fund locks in the expected FX effect and not a higher than expected FX effect. This is indicated by the mean IRR results since the forward strategies and the unhedged strategy are tied.

To conclude, everything else equal it is better to denominate a fund in a currency whose market has a higher risk free interest rate than those of the other portfolio companies' markets due to the favourable effects from the positive slope of the FX forward curve. More specifically, using the data in this thesis, it is better to have a USD denominated PE fund if investing in portfolio companies denoted in SEK, NOK, GBP and EUR since the American risk free interest rate is higher than the other risk free interest rates.

Part V

Conclusion

16 Conclusion, Limitations and Future Research

16.1 Conclusion

This thesis has set out to examine if and how private equity funds should hedge their FX exposure. In order to answer the research question at hand a simulation model has been constructed and implemented under different scenarios. FX rates have been simulated in five different ways and twelve theoretical funds have been investigated. Furthermore, the underlying mathematical theory for the main FX simulation method and pricing of the derivatives originates from the Black-Scholes framework.

The main result of this thesis is that private equity funds cannot achieve a higher mean IRR through hedging of FX exposure independent of which currency the fund is denominated in. This result is expected since it is in line with no arbitrage theory stating that higher risk should be rewarded with higher mean return. However, hedging strategies yielding the same mean IRR but performing better in terms of performance measures accounting for volatility of returns have been found.

The hedging strategies using forward contracts, the forward strategy and the rolling forward strategy, have the same mean IRR as the unhedged fund but perform better in terms of Value-at-Risk, Expected shortfall and Sharpe ratio. Having the same mean IRR but lower volatility implies a better performing hedging strategy. Thus, we conclude that the rolling forward strategy is the best hedging strategy followed by the forward strategy. Therefore, the null hypothesis of this thesis can be rejected and the alternative hypothesis can be accepted. The hedging strategies using options all have a lower mean IRR than the unhedged fund. Though, the put option strategy with optimal strike prices has a better Value-at-Risk, Expected shortfall and Sharpe ratio performance than the unhedged fund.

No differences affecting the conclusion have been observed between neither the local and global nor the equity and infrastructure funds investigated. Though, hedging have been found to be relatively more efficient in less diversified funds, which is an expected result. Furthermore, different FX simulation methods indicate the same results. We found that the conclusions are independent of whether the current or forward FX rate is a better approximation for the future FX rate. Moreover, the extended models including stochastic interest rates and dollar denominated funds imply the same conclusion as previously. Hence the results obtained seem to be robust and independent of the assumptions of constant risk free interest rates as well as underlying fund currency. Finally, due to the slope of the forward curve we observe a positive FX effect for funds denominated in a currency with higher risk free interest rate than those of the portfolio companies' currencies. This FX effect can be captured using forward contracts but is not an improvement over the expected outcome.

16.2 Limitations

As described throughout the thesis several assumptions have been made. Although we expect their impact to be minor they might still affect the results to some extent. The most important simplifications compared to reality are the assigned constant growth rates of the portfolio companies as well as their assigned currency mixes. These assumptions imply that each fund has a systematic NAV evolvement in each currency due to how currencies and growth rates are paired. Thus, by construction of the model, the FX exposure will grow more in some currencies than in other. Ideally, both growth rate and currency assigning had been randomised. Moreover, investment time of each portfolio company should ideally have been randomised as well rather than assumed.

We have assumed that the hedging derivatives can be bought on a market and used by the funds which might not be the case. Forward contracts might be practically infeasible for funds to implement since regulation requires the fund to provide collateral for the hedge. Moreover, rolling forwards give rise to periodic cash flows which are undesired by the fund. In view of these facts the conclusion of the thesis might not be implementable in practice. Moreover, no particular notice has been given to transaction costs which might be problematic from a practical perspective.

Finally, a data related issue might be that historical data has been collected from a certain time interval. During this time interval, the market might have had certain characteristics that are not representative for other time intervals.

16.3 Contribution and Future Research

This thesis contributes to the field of research concerning FX hedging within private equity, from the private equity firms' point of view, since there to our knowledge does not exist any scientific previous research within this field. We hope that this work can contribute to the reader's enhanced understanding and other research through defining, structuring and modelling a private equity FX hedging universe.

There are numerous ways of extending our setting to perform a more general and broader analysis. First, to address the issues listed as abovementioned limitations, i.e. stochastic growth rates and randomised currency assigning of the portfolio companies as well as randomised investment time of each portfolio company. Second, other interesting aspects to investigate further include to (i) do the same analysis for currencies within emerging markets, (ii) hedge entry NAV instead of expected exit NAV, (iii) model FX rates with other stochastic processes than Brownian motions, e.g. Cauchy processes and (iv) update hedges during the life of the fund depending on holding times of the portfolio companies.

Two other studies to investigate the same research question as in this thesis could be to (i) evaluate other FX risks within private equity such as the one arising between signing and closing an acquisition of a portfolio company and (ii) divide portfolio companies in different industries and include correlations between industries and FX rates, which could affect hedging decisions.

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Interviews

[a] Bygge, Johan, EQT AB [interviewed by Carl Åkerlind], October 11, Hong Kong.

[b] Lindberg, Magnus, EQT AB [interviewed by Carl Åkerlind and Filip Kwetczer], February 6, 2018, Stockholm, Sweden.

[c] Strömberg, Per, Stockholm School of Economics [interviewed by Carl Åkerlind and Filip Kwetczer], February 8, 2018, Stockholm, Sweden.

[d] Sylvan, Fredrik, Nordea Bank AB [interviewed by Carl Åkerlind and Filip Kwetczer], January 31, 2018, Stockholm, Sweden.

Appendices

A Fund Data

A.1 Global Fund A

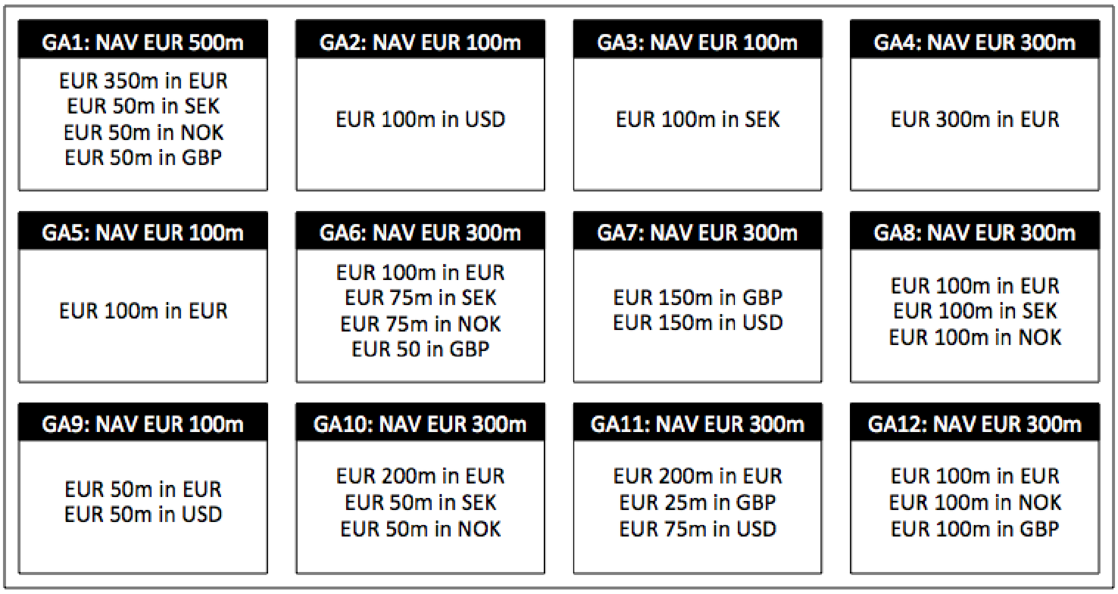


Figure A.1: Global fund A, the fund invests in global companies, i.e. companies that have FX exposures in several currencies

Portfolio Company	Quarterly Growth (Equity)	Quarterly Growth (Infrastructure)	Investment Quarter
GA1	3.56%	2.18%	1
GA2	4.66%	2.87%	1
GA3	8.78%	5.53%	1
GA4	0.00%	0.00%	1
GA5	4.66%	2.87%	5
GA6	4.66%	2.87%	5
GA7	5.74%	3.56%	5
GA8	3.56%	2.18%	5
GA9	5.74%	3.89%	9
GA10	6.78%	4.22%	9
GA11	5.74%	3.56%	9
GA12	4.28%	2.87%	9

Table A.1: Characteristics of global fund A, investments take place in the beginning of each quarter

A.2 Global Fund B

Figure A.2: Global fund B, the fund invests in global companies, i.e. companies that have FX exposures in several currencies

Portfolio Company	Quarterly Growth (Equity)	Quarterly Growth (Infrastructure)	Investment Quarter
GB1	3.56%	2.18%	1
GB2	4.66%	2.87%	1
GB3	8.78%	5.53%	1
GB4	0.00%	0.00%	1
GB5	4.66%	2.87%	5
GB6	4.66%	2.87%	5
GB7	5.74%	3.56%	5
GB8	3.56%	2.18%	5
GB9	5.74%	3.89%	9
GB10	6.78%	4.22%	9
GB11	5.74%	3.56%	9
GB12	4.28%	2.87%	9

Table A.2: Characteristics of global fund B, investments take place in the beginning of each quarter

A.3 Global Fund C

Figure A.3: Global fund C, the fund invests in global companies, i.e. companies that have FX exposures in several currencies

Portfolio Company	Quarterly Growth (Equity)	Quarterly Growth (Infrastructure)	Investment Quarter
GC1	3.56%	2.18%	1
GC2	4.66%	2.87%	1
GC3	8.78%	5.53%	1
GC4	0.00%	0.00%	1
GC5	4.66%	2.87%	5
GC6	4.66%	2.87%	5
GC7	5.74%	3.56%	5
GC8	3.56%	2.18%	5
GC9	5.74%	3.89%	9
GC10	6.78%	4.22%	9
GC11	5.74%	3.56%	9
GC12	4.28%	2.87%	9

Table A.3: Characteristics of global fund C, investments take place in the beginning of each quarter

A.4 Local Fund A

Figure A.4: Local fund A, the fund only invests in local companies, i.e. companies that only have FX exposures in one currency

Portfolio Company	Quarterly Growth (Equity)	Quarterly Growth (Infrastructure)	Investment Quarter
LA1	3.56%	2.18%	1
LA2	4.66%	2.87%	1
LA3	8.78%	5.53%	1
LA4	0.00%	0.00%	1
LA5	4.66%	2.87%	5
LA6	4.66%	2.87%	5
LA7	5.74%	3.56%	5
LA8	3.56%	2.18%	5
LA9	5.74%	3.89%	9
LA10	6.78%	4.22%	9
LA11	5.74%	3.56%	9
LA12	3.51%	2.55%	9

Table A.4: Characteristics of local fund A, investments take place in the beginning of each quarter

A.5 Local Fund B

Figure A.5: Local fund B, the fund only invests in local companies, i.e. companies that only have FX exposures in one currency

Portfolio Company	Quarterly Growth (Equity)	Quarterly Growth (Infrastructure)	Investment Quarter
LB1	3.56%	2.18%	1
LB2	4.66%	2.87%	1
LB3	8.78%	5.53%	1
LB4	0.00%	0.00%	1
LB5	3.56%	2.18%	5
LB6	3.56%	2.18%	5
LB7	5.74%	4.00%	5
LB8	6.78%	4.22%	5
LB9	3.56%	2.18%	9
LB10	5.74%	4.22%	9
LB11	5.74%	3.56%	9
LB12	3.47%	2.18%	9

Table A.5: Characteristics of local fund B, investments take place in the beginning of each quarter

A.6 Local Fund C

Figure A.6: Local fund C, the fund only invests in local companies, i.e. companies that only have FX exposures in one currency

Portfolio Company	Quarterly Growth (Equity)	Quarterly Growth (Infrastructure)	Investment Quarter
LC1	3.56%	2.18%	1
LC2	4.66%	2.87%	1
LC3	8.78%	5.53%	1
LC4	0.00%	0.00%	1
LC5	3.56%	2.18%	5
LC6	3.56%	2.18%	5
LC7	5.74%	4.00%	5
LC8	6.78%	4.22%	5
LC9	3.56%	2.18%	9
LC10	5.74%	4.22%	9
LC11	5.74%	3.56%	9
LC12	3.47%	2.18%	9

Table A.6: Characteristics of local fund C, investments take place in the beginning of each quarter

B Foreign Exchange Rates Simulations

B.1 Historical Rates

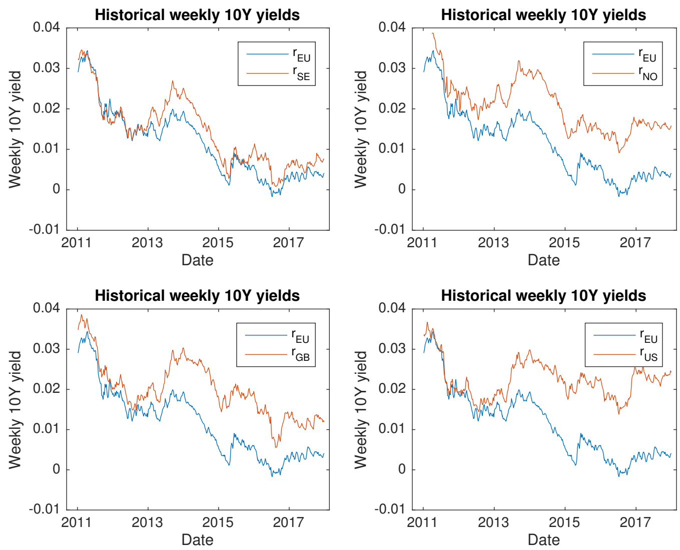


Figure B.1: Historic development of risk free interest rates

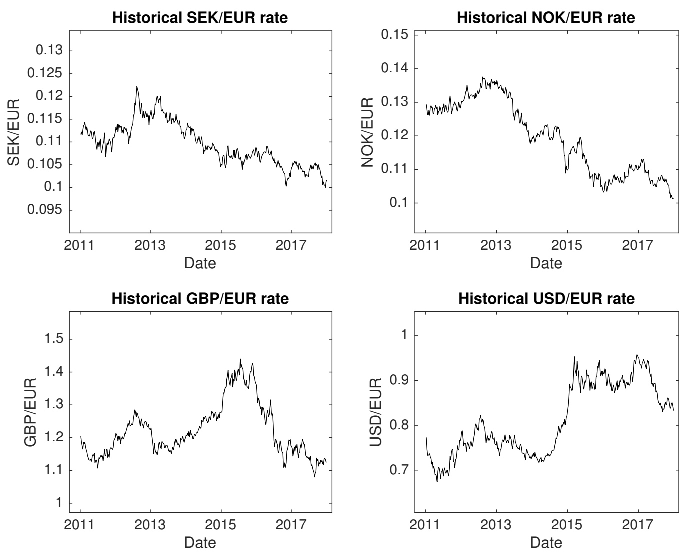


Figure B.2: Historic development of the SEK/EUR, NOK/EUR, GBP/EUR and USD/EUR rates

B.2 Random Walk Simulation Method

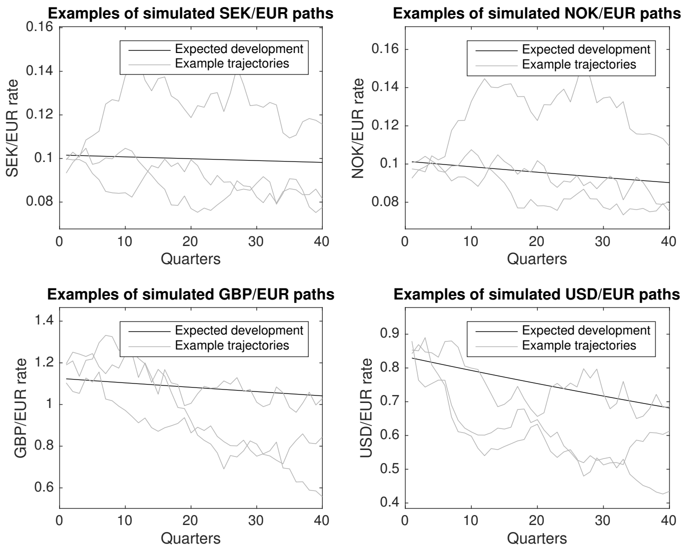


Figure B.3: Expected development and example paths of the FX rates under the random walk simulation method

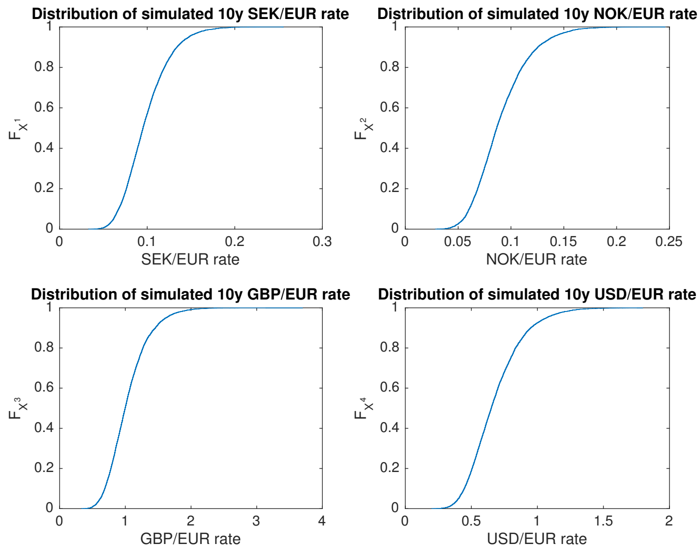


Figure B.4: Empirical distribution of $n = 10,000$ simulated FX rates in 10 years under the random walk simulation method

B.3 Random Walk Simulation Method with Zero Drift

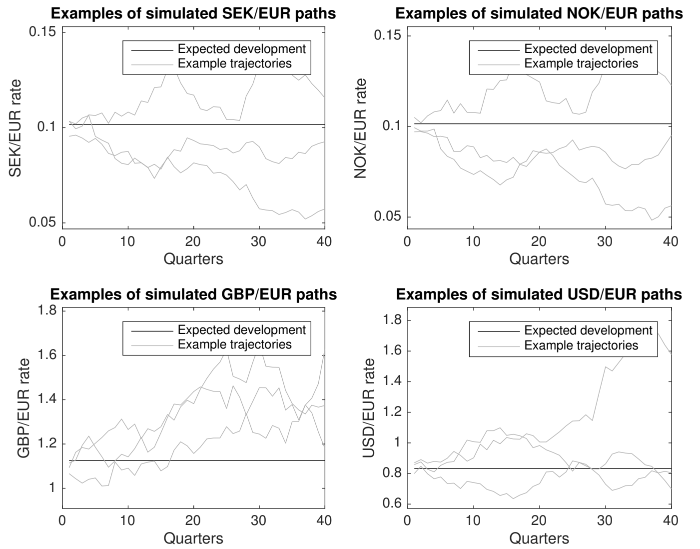


Figure B.5: Expected development and example paths of the FX rates under the random walk simulation method with zero drift

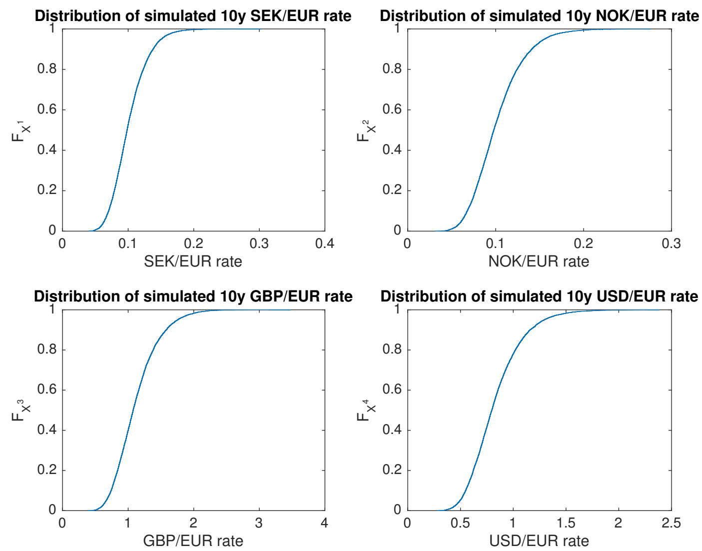


Figure B.6: Empirical distribution of $n = 10,000$ simulated FX rates in 10 years under the random walk simulation method with zero drift

B.4 Historical Simulation Method

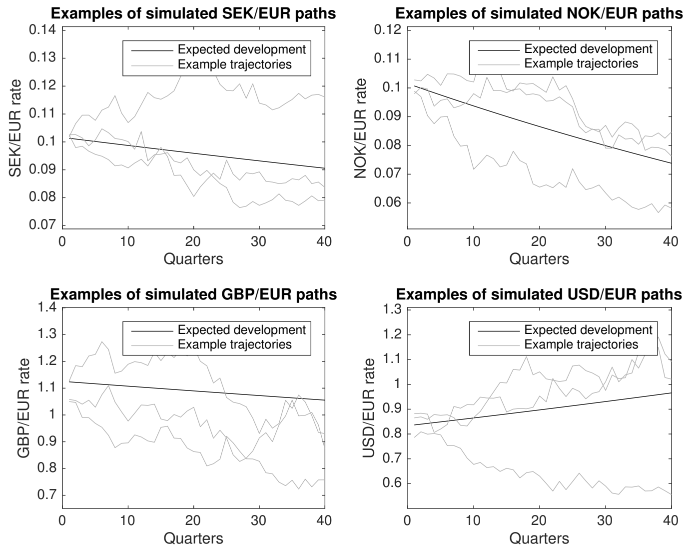


Figure B.7: Expected development and example paths of the FX rates under the historical simulation method

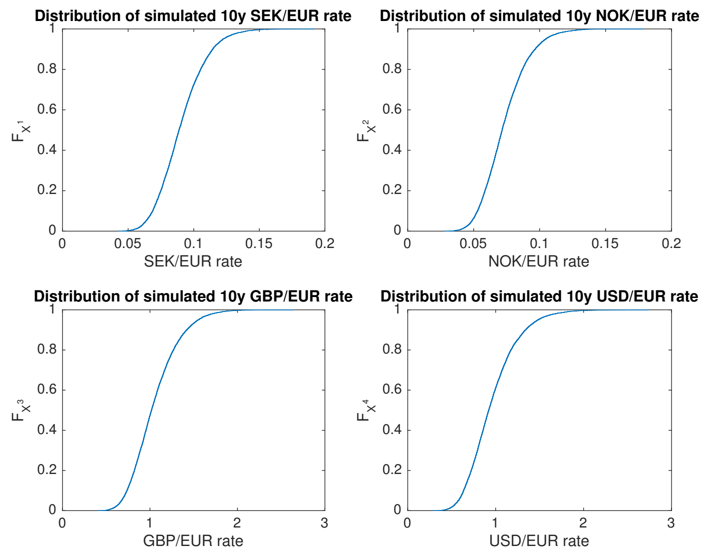
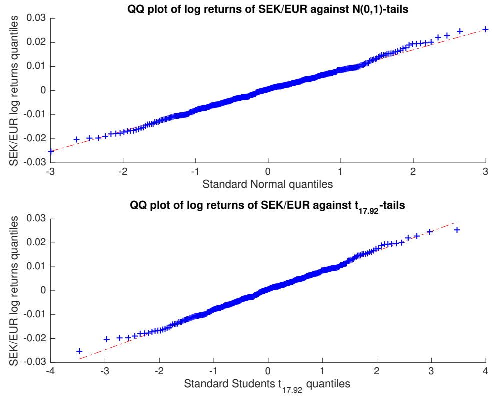
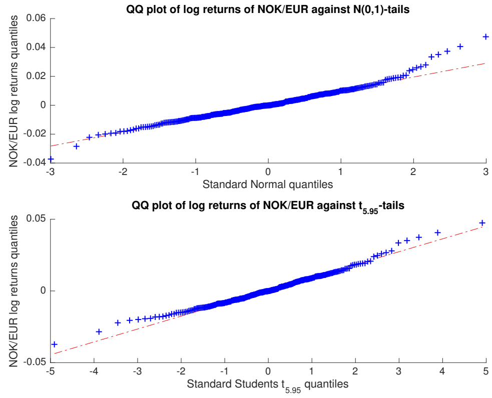


Figure B.8: Empirical distribution of $n = 10,000$ simulated FX rates in 10 years under the historical simulation method

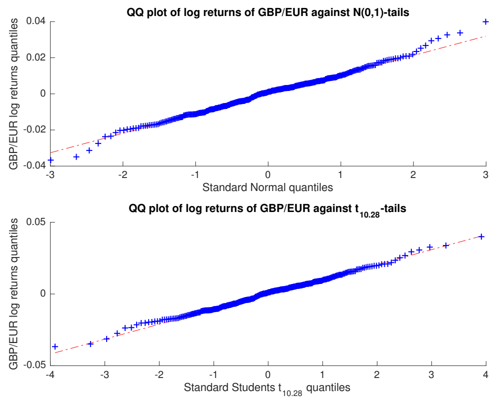
B.5 Q-Q Plots of Marginal Distributions for Copula Methods



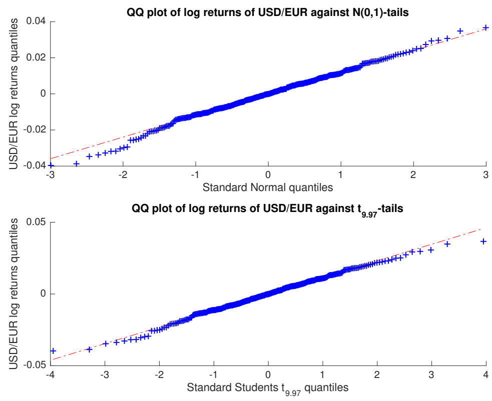
(a) SEK/EUR



(b) NOK/EUR



(c) GBP/EUR



(d) USD/EUR

Figure B.9: Q-Q plots of the log-returns of the FX rates against normal and Student's t tails

B.6 Gaussian Copula Simulation Method

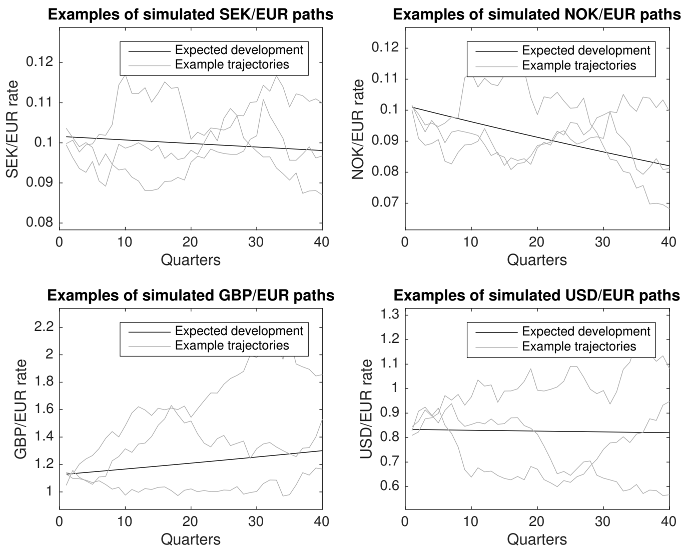


Figure B.10: Expected development and example paths of the FX rates under the Gaussian copula simulation method

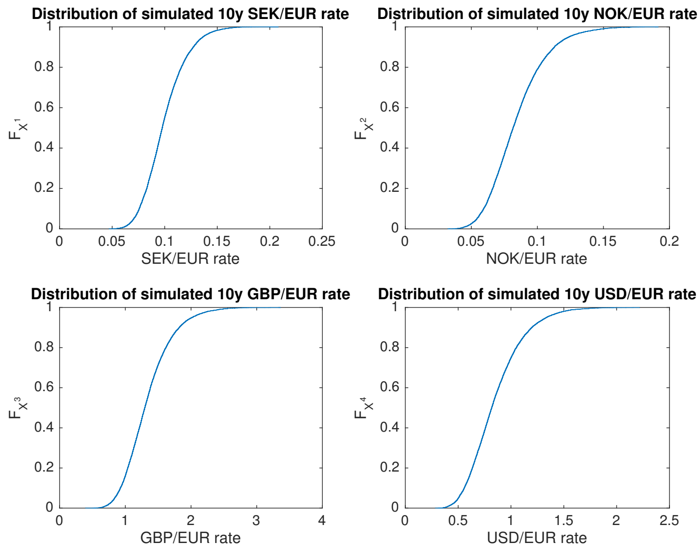


Figure B.11: Empirical distribution of $n = 10,000$ simulated FX rates in 10 years under the Gaussian copula simulation method

B.7 Student's *t* Copula Simulation Method

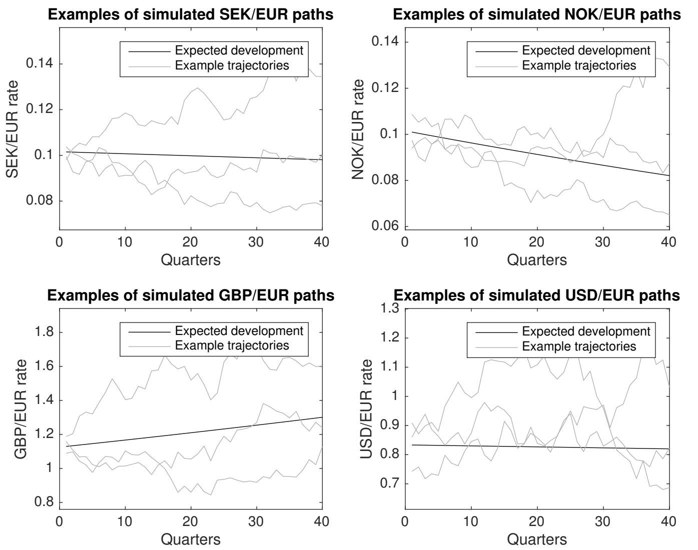


Figure B.12: Expected development and example paths of the FX rates under the Student's t copula simulation method

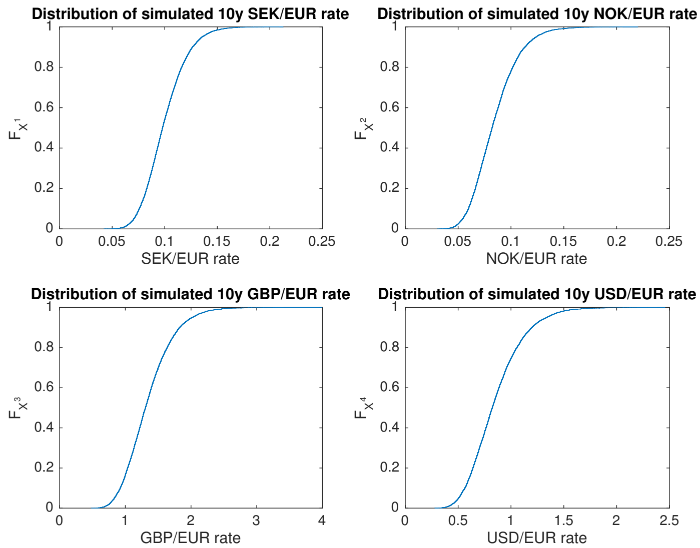


Figure B.13: Empirical distribution of $n = 10,000$ simulated FX rates in 10 years under the Student's t copula simulation method

B.8 Random Walk Simulation Method with Stochastic Short Rates

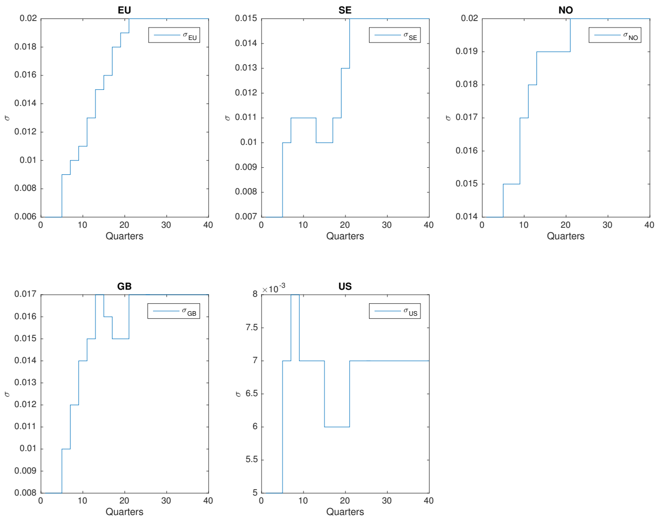


Figure B.14: Fitted piecewise constant σ for the Hull-White model

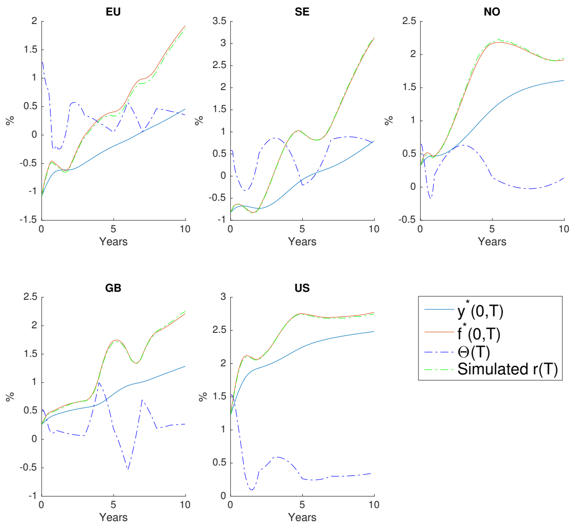


Figure B.15: Observed and fitted Hull-White curves for all markets

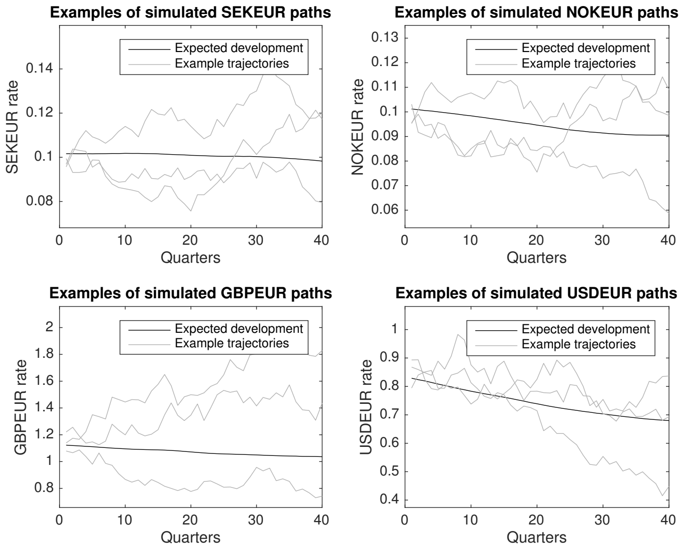


Figure B.16: Expected development and example paths of the FX rates under the Random Walk simulation method with stochastic short rates

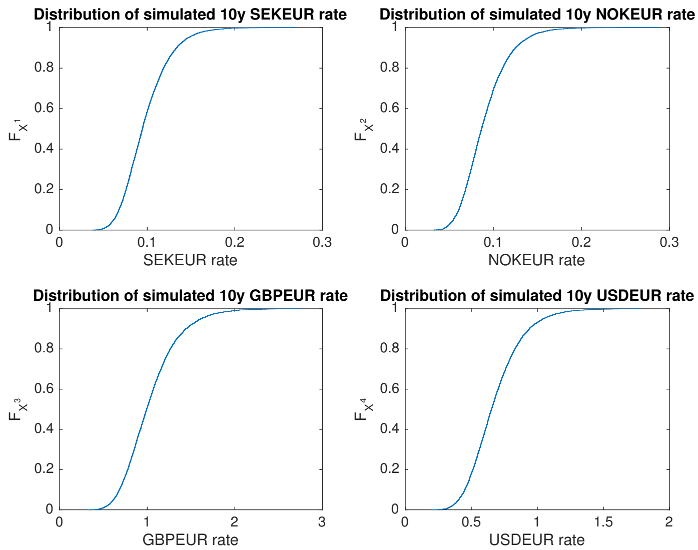


Figure B.17: Empirical distribution of $n = 10,000$ simulated FX rates in 10 years under the Random Walk simulation method with stochastic short rates

B.9 Random Walk Simulation Method with US as Domestic Market

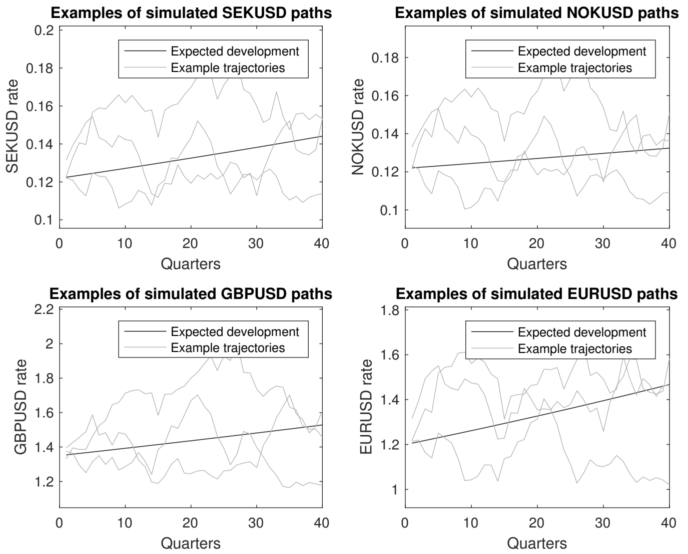


Figure B.18: Expected development and example paths of the FX rates under the random walk simulation method with US as domestic market

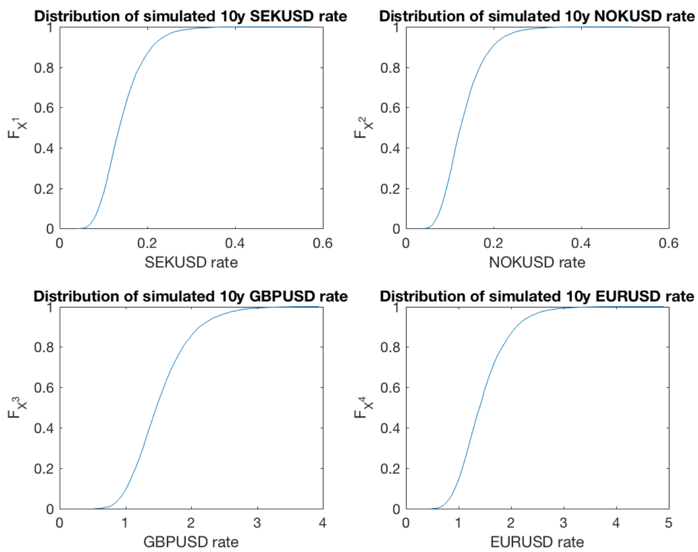


Figure B.19: Empirical distribution of $n = 10,000$ simulated FX rates in 10 years under the random walk simulation method with US as domestic market

C Performance of the Hedging Strategies - Base Model

C.1 Random Walk Simulation of FX Rates, Equity Funds

Fund	**Unh.**	**For.**	**Rol. for.**	**Call**$_{K_1^C}$	**Call**$_{K_2^C}$	**Call**$_{K_3^C}$	**Put**$_{K_1^P}$	**Put**$_{K_2^P}$	**Put**$_{K_3^P}$	**Str.**$_{K_1^S}$	**Str.**$_{K_2^S}$	**Str.**$_{K_3^S}$
GE A	19.43%	19.45%	19.45%	18.70%	18.90%	19.17%	18.59%	18.87%	19.24%	17.93%	18.38%	18.99%
GE B	19.69%	19.72%	19.71%	18.87%	19.10%	19.41%	18.94%	19.21%	19.54%	18.20%	18.66%	19.27%
GE C	18.99%	19.05%	19.05%	18.39%	18.56%	18.79%	18.09%	18.38%	18.78%	17.56%	17.99%	18.59%
LE A	19.46%	19.47%	19.49%	18.70%	18.91%	19.19%	18.59%	18.88%	19.26%	17.92%	18.38%	19.01%
LE B	19.70%	19.71%	19.72%	18.87%	19.10%	19.41%	18.94%	19.21%	19.55%	18.19%	18.66%	19.27%
LE C	19.30%	19.33%	19.34%	18.69%	18.87%	19.10%	18.35%	18.65%	19.07%	17.82%	18.26%	18.88%

Table C.1: IRR for the Equity Funds, Random Walk Simulation

Fund	**Unh.**	**For.**	**Rol. for.**	**Call**$_{K_1^C}$	**Call**$_{K_2^C}$	**Call**$_{K_3^C}$	**Put**$_{K_1^P}$	**Put**$_{K_2^P}$	**Put**$_{K_3^P}$	**Str.**$_{K_1^S}$	**Str.**$_{K_2^S}$	**Str.**$_{K_3^S}$
GE A	17.37%	18.08%	18.21%	16.36%	16.63%	17.02%	16.91%	17.12%	17.35%	16.00%	16.42%	16.99%
GE B	17.02%	18.35%	18.47%	15.92%	16.23%	16.65%	17.16%	17.28%	17.28%	16.10%	16.50%	16.91%
GE C	16.44%	17.70%	17.86%	15.74%	15.94%	16.21%	16.46%	16.69%	16.89%	15.71%	16.15%	16.63%
LE A	17.42%	17.63%	18.06%	16.41%	16.69%	17.08%	16.76%	17.02%	17.31%	15.82%	16.31%	16.97%
LE B	17.08%	17.79%	18.04%	15.95%	16.29%	16.71%	16.83%	17.02%	17.17%	15.78%	16.23%	16.79%
LE C	16.92%	17.24%	17.54%	16.12%	16.36%	16.67%	16.15%	16.45%	16.86%	15.43%	15.91%	16.59%

Table C.2: VaR for the Equity Funds, Random Walk Simulation

Fund	**Unh.**	**For.**	**Rol. for.**	**Call**$_{K_1^C}$	**Call**$_{K_2^C}$	**Call**$_{K_3^C}$	**Put**$_{K_1^P}$	**Put**$_{K_2^P}$	**Put**$_{K_3^P}$	**Str.**$_{K_1^S}$	**Str.**$_{K_2^S}$	**Str.**$_{K_3^S}$
GE A	16.90%	17.72%	17.90%	15.86%	16.16%	16.55%	16.53%	16.72%	16.90%	15.58%	16.00%	16.54%
GE B	16.45%	17.95%	18.16%	15.34%	15.67%	16.09%	16.76%	16.88%	16.87%	15.67%	16.08%	16.49%
GE C	15.90%	17.33%	17.58%	15.19%	15.41%	15.69%	16.07%	16.29%	16.48%	15.30%	15.72%	16.21%
LE A	16.92%	17.08%	17.67%	15.88%	16.17%	16.57%	16.31%	16.56%	16.86%	15.36%	15.84%	16.49%
LE B	16.48%	17.28%	17.72%	15.33%	15.67%	16.11%	16.39%	16.56%	16.68%	15.31%	15.76%	16.30%
LE C	16.36%	16.75%	17.20%	15.53%	15.78%	16.09%	15.69%	15.98%	16.37%	14.94%	15.42%	16.09%

Table C.3: ES for the Equity Funds, Random Walk Simulation

Fund	**Unh.**	**For.**	**Rol. for.**	**Call**$_{K_1^C}$	**Call**$_{K_2^C}$	**Call**$_{K_3^C}$	**Put**$_{K_1^P}$	**Put**$_{K_2^P}$	**Put**$_{K_3^P}$	**Str.**$_{K_1^S}$	**Str.**$_{K_2^S}$	**Str.**$_{K_3^S}$
GE A	14.97	22.86	25.76	12.26	12.83	13.78	16.58	16.15	15.51	13.18	13.44	14.06
GE B	11.39	22.65	25.93	9.06	9.56	10.40	14.61	13.73	12.36	10.97	11.03	11.12
GE C	11.51	22.54	26.65	9.84	10.25	10.87	15.62	15.04	13.68	12.47	12.60	12.56
LE A	15.58	17.44	22.97	12.91	13.49	14.45	15.54	15.61	15.74	12.83	13.28	14.35
LE B	12.24	16.46	18.64	9.70	10.25	11.16	13.91	13.61	12.91	10.96	11.20	11.66
LE C	13.17	15.13	17.23	11.24	11.71	12.41	12.73	13.07	13.54	11.16	11.63	12.62

Table C.4: SR for the Equity Funds, Random Walk Simulation

C.2 Random Walk Simulation of FX Rates, Infrastructure Funds

Fund	**Unh.**	**For.**	**Rol. for.**	**Call**$_{K_1^C}$	**Call**$_{K_2^C}$	**Call**$_{K_3^C}$	**Put**$_{K_1^P}$	**Put**$_{K_2^P}$	**Put**$_{K_3^P}$	**Str.**$_{K_1^S}$	**Str.**$_{K_2^S}$	**Str.**$_{K_3^S}$
GI A	11.41%	11.42%	11.42%	11.01%	11.12%	11.26%	11.16%	11.26%	11.38%	10.79%	10.99%	11.24%
GI B	11.65%	11.67%	11.67%	11.20%	11.32%	11.49%	11.42%	11.52%	11.63%	11.00%	11.20%	11.47%
GI C	11.04%	11.09%	11.09%	10.71%	10.80%	10.92%	10.83%	10.92%	11.03%	10.52%	10.70%	10.92%
LI A	11.43%	11.45%	11.46%	11.03%	11.14%	11.29%	11.18%	11.29%	11.40%	10.81%	11.00%	11.26%
LI B	11.65%	11.66%	11.67%	11.19%	11.32%	11.48%	11.41%	11.51%	11.62%	10.98%	11.19%	11.46%
LI C	11.20%	11.24%	11.25%	10.87%	10.97%	11.09%	10.97%	11.07%	11.19%	10.67%	10.85%	11.08%

Table C.5: IRR for the Infrastructure Funds, Random Walk Simulation

Fund	**Unh.**	**For.**	**Rol. for.**	**Call**$_{K_1^C}$	**Call**$_{K_2^C}$	**Call**$_{K_3^C}$	**Put**$_{K_1^P}$	**Put**$_{K_2^P}$	**Put**$_{K_3^P}$	**Str.**$_{K_1^S}$	**Str.**$_{K_2^S}$	**Str.**$_{K_3^S}$
GI A	9.76%	10.55%	10.72%	9.10%	9.28%	9.53%	9.95%	9.96%	9.90%	9.32%	9.50%	9.66%
GI B	9.41%	10.77%	10.95%	8.69%	8.90%	9.18%	10.09%	10.04%	9.81%	9.36%	9.51%	9.57%
GI C	8.92%	10.20%	10.41%	8.48%	8.61%	8.79%	9.60%	9.63%	9.55%	9.09%	9.26%	9.37%
LI A	9.81%	10.19%	10.62%	9.13%	9.32%	9.58%	9.81%	9.85%	9.87%	9.17%	9.37%	9.63%
LI B	9.54%	10.48%	10.80%	8.79%	9.03%	9.31%	9.95%	9.96%	9.82%	9.23%	9.42%	9.57%
LI C	9.31%	9.97%	10.29%	8.82%	8.97%	9.15%	9.48%	9.57%	9.62%	8.99%	9.19%	9.43%

Table C.6: VaR for the Infrastructure Funds, Random Walk Simulation

Fund	**Unh.**	**For.**	**Rol. for.**	**Call**$_{K_1^C}$	**Call**$_{K_2^C}$	**Call**$_{K_3^C}$	**Put**$_{K_1^P}$	**Put**$_{K_2^P}$	**Put**$_{K_3^P}$	**Str.**$_{K_1^S}$	**Str.**$_{K_2^S}$	**Str.**$_{K_3^S}$
GI A	9.40%	10.30%	10.54%	8.71%	8.91%	9.17%	9.69%	9.69%	9.60%	9.03%	9.20%	9.36%
GI B	8.92%	10.49%	10.77%	8.20%	8.41%	8.69%	9.82%	9.76%	9.52%	9.07%	9.22%	9.26%
GI C	8.48%	9.93%	10.24%	8.05%	8.19%	8.36%	9.33%	9.36%	9.27%	8.81%	8.97%	9.08%
LI A	9.41%	9.80%	10.38%	8.72%	8.92%	9.18%	9.50%	9.53%	9.53%	8.85%	9.05%	9.29%
LI B	9.05%	10.10%	10.61%	8.31%	8.53%	8.81%	9.65%	9.63%	9.47%	8.91%	9.09%	9.22%
LI C	8.87%	9.60%	10.07%	8.35%	8.51%	8.70%	9.16%	9.23%	9.25%	8.65%	8.85%	9.06%

Table C.7: ES for the Infrastructure Funds, Random Walk Simulation

Fund	**Unh.**	**For.**	**Rol. for.**	**Call**$_{K_1^C}$	**Call**$_{K_2^C}$	**Call**$_{K_3^C}$	**Put**$_{K_1^P}$	**Put**$_{K_2^P}$	**Put**$_{K_3^P}$	**Str.**$_{K_1^S}$	**Str.**$_{K_2^S}$	**Str.**$_{K_3^S}$
GI A	11.02	21.03	26.60	8.62	9.10	9.92	12.87	12.34	11.61	9.60	9.75	10.23
GI B	7.97	20.37	26.66	6.14	6.52	7.17	10.97	10.10	8.82	7.76	7.76	7.78
GI C	8.00	19.98	27.05	6.66	6.98	7.47	11.56	11.04	9.85	8.81	8.89	8.86
LI A	11.23	14.84	22.73	8.85	9.33	10.16	11.89	11.74	11.54	9.20	9.47	10.21
LI B	8.73	15.96	21.72	6.60	7.03	7.78	11.22	10.61	9.55	8.02	8.12	8.34
LI C	9.50	14.50	19.05	7.72	8.13	8.77	10.48	10.60	10.58	8.54	8.85	9.51

Table C.8: SR for the Infrastructure Funds, Random Walk Simulation

C.3 Random Walk Simulation of FX Rates with Zero Drift, Equity Funds

Fund	**Unh.**	**For.**	**Rol. for.**	**Call**$_{K_1^C}$	**Call**$_{K_2^C}$	**Call**$_{K_3^C}$	**Put**$_{K_1^P}$	**Put**$_{K_2^P}$	**Put**$_{K_3^P}$	**Str.**$_{K_1^S}$	**Str.**$_{K_2^S}$	**Str.**$_{K_3^S}$
GE A	20.10%	20.11%	20.11%	18.99%	19.27%	19.67%	19.43%	19.67%	19.97%	18.41%	18.89%	19.54%
GE B	20.08%	20.10%	20.11%	19.02%	19.30%	19.69%	19.44%	19.68%	19.97%	18.47%	18.95%	19.58%
GE C	20.06%	20.11%	20.10%	18.92%	19.20%	19.60%	19.40%	19.64%	19.93%	18.34%	18.82%	19.48%
LE A	20.09%	20.10%	20.12%	18.96%	19.25%	19.65%	19.40%	19.66%	19.96%	18.37%	18.86%	19.53%
LE B	20.10%	20.11%	20.13%	19.04%	19.32%	19.71%	19.46%	19.70%	19.99%	18.48%	18.96%	19.60%
LE C	20.10%	20.12%	20.13%	18.95%	19.24%	19.64%	19.42%	19.67%	19.97%	18.36%	18.84%	19.51%

Table C.9: IRR for the Equity Funds, Random Walk Simulation, Zero Drift

Fund	**Unh.**	**For.**	**Rol. for.**	**Call**$_{K_1^C}$	**Call**$_{K_2^C}$	**Call**$_{K_3^C}$	**Put**$_{K_1^P}$	**Put**$_{K_2^P}$	**Put**$_{K_3^P}$	**Str.**$_{K_1^S}$	**Str.**$_{K_2^S}$	**Str.**$_{K_3^S}$
GE A	18.02%	18.74%	18.87%	16.56%	16.92%	17.46%	17.74%	17.91%	18.03%	16.37%	16.82%	17.47%
GE B	17.42%	18.72%	18.87%	16.05%	16.44%	16.94%	17.64%	17.76%	17.70%	16.32%	16.76%	17.21%
GE C	17.42%	18.70%	18.85%	16.01%	16.38%	16.88%	17.55%	17.71%	17.73%	16.14%	16.61%	17.15%
LE A	18.02%	18.22%	18.68%	16.54%	16.92%	17.45%	17.54%	17.77%	18.01%	16.16%	16.68%	17.42%
LE B	17.48%	18.21%	18.46%	16.08%	16.46%	16.97%	17.41%	17.57%	17.64%	16.08%	16.56%	17.13%
LE C	17.61%	18.05%	18.45%	16.15%	16.51%	17.03%	17.29%	17.49%	17.69%	15.86%	16.38%	17.09%

Table C.10: VaR for the Equity Funds, Random Walk Simulation, Zero Drift

Fund	**Unh.**	**For.**	**Rol. for.**	**Call**$_{K_1^C}$	**Call**$_{K_2^C}$	**Call**$_{K_3^C}$	**Put**$_{K_1^P}$	**Put**$_{K_2^P}$	**Put**$_{K_3^P}$	**Str.**$_{K_1^S}$	**Str.**$_{K_2^S}$	**Str.**$_{K_3^S}$
GE A	17.52%	18.35%	18.55%	16.03%	16.41%	16.94%	17.35%	17.51%	17.61%	15.94%	16.41%	17.03%
GE B	16.78%	18.30%	18.56%	15.41%	15.80%	16.31%	17.27%	17.37%	17.27%	15.90%	16.34%	16.76%
GE C	16.90%	18.29%	18.54%	15.47%	15.84%	16.36%	17.16%	17.29%	17.30%	15.69%	16.16%	16.71%
LE A	17.53%	17.63%	18.27%	16.01%	16.39%	16.93%	17.09%	17.31%	17.53%	15.67%	16.20%	16.93%
LE B	16.83%	17.71%	18.14%	15.41%	15.80%	16.33%	16.96%	17.10%	17.11%	15.59%	16.06%	16.58%
LE C	17.09%	17.48%	18.07%	15.57%	15.96%	16.50%	16.80%	17.00%	17.18%	15.34%	15.86%	16.56%

Table C.11: ES for the Equity Funds, Random Walk Simulation, Zero Drift

Fund	**Unh.**	**For.**	**Rol. for.**	**Call**$_{K_1^C}$	**Call**$_{K_2^C}$	**Call**$_{K_3^C}$	**Put**$_{K_1^P}$	**Put**$_{K_2^P}$	**Put**$_{K_3^P}$	**Str.**$_{K_1^S}$	**Str.**$_{K_2^S}$	**Str.**$_{K_3^S}$
GE A	15.27	23.52	26.60	12.04	12.60	13.63	16.97	16.57	15.88	12.92	13.22	13.94
GE B	11.64	22.93	26.48	9.06	9.56	10.45	14.77	13.91	12.57	10.83	10.92	11.11
GE C	11.49	22.33	26.19	8.98	9.41	10.18	14.05	13.45	12.42	10.37	10.51	10.79
LE A	15.77	17.83	23.73	12.45	13.03	14.08	15.80	15.92	15.98	12.44	12.92	14.04
LE B	12.53	17.07	19.31	9.69	10.24	11.20	14.37	14.04	13.26	10.95	11.21	11.71
LE C	13.61	16.05	18.92	10.47	11.00	11.96	14.18	14.23	14.10	11.00	11.38	12.23

Table C.12: SR for the Equity Funds, Random Walk Simulation, Zero Drift

C.4 Random Walk Simulation with Zero Drift, Infrastructure Funds

Fund	**Unh.**	**For.**	**Rol. for.**	**Call**$_{K_1^C}$	**Call**$_{K_2^C}$	**Call**$_{K_3^C}$	**Put**$_{K_1^P}$	**Put**$_{K_2^P}$	**Put**$_{K_3^P}$	**Str.**$_{K_1^S}$	**Str.**$_{K_2^S}$	**Str.**$_{K_3^S}$
GI A	12.02%	12.03%	12.04%	11.41%	11.56%	11.78%	11.79%	11.89%	11.99%	11.21%	11.44%	11.75%
GI B	12.01%	12.03%	12.03%	11.42%	11.57%	11.78%	11.79%	11.88%	11.98%	11.24%	11.46%	11.76%
GI C	12.00%	12.03%	12.03%	11.35%	11.51%	11.73%	11.78%	11.87%	11.97%	11.17%	11.40%	11.71%
LI A	12.02%	12.03%	12.04%	11.40%	11.55%	11.77%	11.79%	11.89%	11.99%	11.20%	11.44%	11.75%
LI B	12.02%	12.04%	12.05%	11.43%	11.59%	11.80%	11.80%	11.90%	12.00%	11.25%	11.48%	11.78%
LI C	12.02%	12.04%	12.05%	11.38%	11.53%	11.76%	11.80%	11.89%	11.99%	11.19%	11.42%	11.73%

Table C.13: IRR for the Infrastructure Funds, Random Walk Simulation, Zero Drift

Fund	**Unh.**	**For.**	**Rol. for.**	**Call**$_{K_1^C}$	**Call**$_{K_2^C}$	**Call**$_{K_3^C}$	**Put**$_{K_1^P}$	**Put**$_{K_2^P}$	**Put**$_{K_3^P}$	**Str.**$_{K_1^S}$	**Str.**$_{K_2^S}$	**Str.**$_{K_3^S}$
GI A	10.33%	11.14%	11.32%	9.38%	9.62%	9.96%	10.52%	10.53%	10.47%	9.59%	9.81%	10.09%
GI B	9.74%	11.11%	11.32%	8.87%	9.12%	9.45%	10.42%	10.37%	10.12%	9.51%	9.70%	9.79%
GI C	9.80%	11.10%	11.31%	8.88%	9.12%	9.45%	10.38%	10.37%	10.21%	9.41%	9.63%	9.83%
LI A	10.36%	10.74%	11.19%	9.38%	9.63%	9.97%	10.38%	10.43%	10.45%	9.45%	9.70%	10.05%
LI B	9.92%	10.86%	11.18%	9.01%	9.26%	9.60%	10.36%	10.34%	10.18%	9.45%	9.66%	9.85%
LI C	10.05%	10.75%	11.17%	9.09%	9.33%	9.68%	10.29%	10.33%	10.26%	9.32%	9.57%	9.87%

Table C.14: VaR for the Infrastructure Funds, Random Walk Simulation, Zero Drift

Fund	**Unh.**	**For.**	**Rol. for.**	**Call**$_{K_1^C}$	**Call**$_{K_2^C}$	**Call**$_{K_3^C}$	**Put**$_{K_1^P}$	**Put**$_{K_2^P}$	**Put**$_{K_3^P}$	**Str.**$_{K_1^S}$	**Str.**$_{K_2^S}$	**Str.**$_{K_3^S}$
GI A	9.95%	10.87%	11.14%	8.97%	9.22%	9.57%	10.27%	10.26%	10.16%	9.30%	9.52%	9.77%
GI B	9.24%	10.80%	11.14%	8.35%	8.60%	8.93%	10.16%	10.11%	9.83%	9.23%	9.41%	9.49%
GI C	9.38%	10.79%	11.12%	8.47%	8.71%	9.04%	10.11%	10.08%	9.89%	9.11%	9.32%	9.49%
LI A	9.95%	10.33%	10.94%	8.95%	9.20%	9.56%	10.06%	10.10%	10.10%	9.09%	9.35%	9.70%
LI B	9.38%	10.49%	10.98%	8.48%	8.74%	9.07%	10.05%	10.02%	9.81%	9.11%	9.32%	9.46%
LI C	9.59%	10.35%	10.94%	8.62%	8.87%	9.22%	9.96%	9.97%	9.89%	8.95%	9.19%	9.48%

Table C.15: ES for the Infrastructure Funds, Random Walk Simulation, Zero Drift

Fund	**Unh.**	**For.**	**Rol. for.**	**Call**$_{K_1^C}$	**Call**$_{K_2^C}$	**Call**$_{K_3^C}$	**Put**$_{K_1^P}$	**Put**$_{K_2^P}$	**Put**$_{K_3^P}$	**Str.**$_{K_1^S}$	**Str.**$_{K_2^S}$	**Str.**$_{K_3^S}$
GI A	11.35	21.72	27.55	8.52	8.98	9.84	13.12	12.67	11.95	9.35	9.55	10.13
GI B	8.22	20.63	27.43	6.18	6.56	7.24	11.06	10.24	9.02	7.64	7.67	7.79
GI C	8.20	20.28	27.17	6.18	6.51	7.11	10.53	9.96	9.02	7.34	7.41	7.62
LI A	11.47	15.32	23.80	8.61	9.06	9.93	12.09	12.01	11.81	8.89	9.20	10.01
LI B	9.00	16.61	22.43	6.63	7.05	7.84	11.46	10.86	9.80	7.93	8.06	8.35
LI C	9.75	15.53	21.84	7.08	7.50	8.28	11.29	11.06	10.50	7.93	8.14	8.68

Table C.16: SR for the Infrastructure Funds, Random Walk Simulation, Zero Drift

C.5 Historical Simulation of FX Rates, Equity Funds

Fund	Unh.	For.	Rol. for.	Call$_{K_1^C}$	Call$_{K_2^C}$	Call$_{K_3^C}$	Put$_{K_1^P}$	Put$_{K_2^P}$	Put$_{K_3^P}$	Str.$_{K_1^S}$	Str.$_{K_2^S}$	Str.$_{K_3^S}$
GE A	19.62%	19.50%	19.45%	18.89%	19.10%	19.38%	18.59%	18.89%	19.33%	17.95%	18.42%	19.09%
GE B	19.47%	19.68%	19.69%	18.54%	18.80%	19.15%	18.63%	18.89%	19.24%	17.78%	18.25%	18.92%
GE C	20.49%	19.35%	19.17%	20.17%	20.28%	20.42%	18.80%	19.25%	19.95%	18.56%	19.09%	19.89%
LE A	19.85%	19.74%	19.59%	19.08%	19.30%	19.59%	18.79%	19.10%	19.54%	18.11%	18.60%	19.29%
LE B	19.56%	19.78%	19.74%	18.60%	18.87%	19.23%	18.72%	18.97%	19.33%	17.85%	18.32%	19.00%
LE C	20.16%	18.92%	19.14%	19.88%	19.98%	20.10%	18.44%	18.90%	19.61%	18.23%	18.76%	19.56%

Table C.17: IRR for the Equity Funds, Historical Simulation

Fund	Unh.	For.	Rol. for.	Call$_{K_1^C}$	Call$_{K_2^C}$	Call$_{K_3^C}$	Put$_{K_1^P}$	Put$_{K_2^P}$	Put$_{K_3^P}$	Str.$_{K_1^S}$	Str.$_{K_2^S}$	Str.$_{K_3^S}$
GE A	17.13%	18.14%	18.24%	16.12%	16.41%	16.78%	16.98%	17.18%	17.27%	16.05%	16.48%	16.94%
GE B	17.06%	18.39%	18.47%	15.93%	16.26%	16.70%	17.11%	17.25%	17.25%	16.06%	16.48%	16.87%
GE C	17.27%	17.76%	17.94%	16.39%	16.65%	16.98%	16.64%	16.88%	17.15%	15.87%	16.32%	16.88%
LE A	17.38%	17.97%	18.24%	16.33%	16.63%	17.03%	16.87%	17.13%	17.39%	15.96%	16.42%	17.02%
LE B	17.01%	17.95%	18.07%	15.82%	16.17%	16.62%	16.77%	16.96%	17.02%	15.73%	16.16%	16.65%
LE C	17.28%	16.75%	17.37%	16.40%	16.65%	16.99%	16.22%	16.56%	17.01%	15.52%	16.03%	16.74%

Table C.18: VaR for the Equity Funds, Historical Simulation

Fund	Unh.	For.	Rol. for.	Call$_{K_1^C}$	Call$_{K_2^C}$	Call$_{K_3^C}$	Put$_{K_1^P}$	Put$_{K_2^P}$	Put$_{K_3^P}$	Str.$_{K_1^S}$	Str.$_{K_2^S}$	Str.$_{K_3^S}$
GE A	16.60%	17.74%	17.94%	15.53%	15.84%	16.24%	16.59%	16.77%	16.87%	15.65%	16.06%	16.52%
GE B	16.48%	18.05%	18.16%	15.32%	15.66%	16.10%	16.76%	16.88%	16.81%	15.69%	16.09%	16.43%
GE C	16.56%	17.22%	17.64%	15.69%	15.95%	16.28%	16.22%	16.47%	16.70%	15.46%	15.90%	16.43%
LE A	16.78%	17.39%	17.86%	15.67%	15.99%	16.41%	16.40%	16.65%	16.88%	15.45%	15.93%	16.52%
LE B	16.43%	17.55%	17.76%	15.23%	15.58%	16.04%	16.36%	16.51%	16.55%	15.29%	15.72%	16.17%
LE C	16.57%	16.14%	17.01%	15.68%	15.95%	16.28%	15.77%	16.08%	16.47%	15.04%	15.52%	16.19%

Table C.19: ES for the Equity Funds, Historical Simulation

Fund	Unh.	For.	Rol. for.	Call$_{K_1^C}$	Call$_{K_2^C}$	Call$_{K_3^C}$	Put$_{K_1^P}$	Put$_{K_2^P}$	Put$_{K_3^P}$	Str.$_{K_1^S}$	Str.$_{K_2^S}$	Str.$_{K_3^S}$
GE A	12.80	22.97	26.57	10.45	10.95	11.74	17.15	16.16	14.30	13.45	13.35	12.95
GE B	12.86	24.52	26.60	10.71	11.34	12.19	18.12	16.63	14.20	14.34	14.18	13.37
GE C	9.74	18.69	25.63	7.64	7.94	8.52	11.52	11.01	10.25	8.64	8.68	8.86
LE A	12.90	18.21	24.09	10.53	11.01	11.79	15.14	14.87	13.97	12.40	12.56	12.66
LE B	12.72	17.25	18.72	10.70	11.28	12.07	15.19	14.77	13.59	12.85	13.01	12.86
LE C	11.28	14.46	17.17	8.60	8.98	9.71	12.75	12.44	11.81	9.58	9.70	10.07

Table C.20: SR for the Equity Funds, Historical Simulation

C.6 Historical Simulation of FX Rates, Infrastructure Funds

Fund	Unh.	For.	Rol. for.	Call$_{K_1^C}$	Call$_{K_2^C}$	Call$_{K_3^C}$	Put$_{K_1^P}$	Put$_{K_2^P}$	Put$_{K_3^P}$	Str.$_{K_1^S}$	Str.$_{K_2^S}$	Str.$_{K_3^S}$
GI A	11.58%	11.46%	11.43%	11.18%	11.30%	11.45%	11.16%	11.28%	11.45%	10.80%	11.02%	11.32%
GI B	11.46%	11.66%	11.66%	10.89%	11.04%	11.25%	11.14%	11.22%	11.35%	10.61%	10.83%	11.15%
GI C	12.35%	11.25%	11.14%	12.30%	12.34%	12.37%	11.40%	11.64%	12.04%	11.38%	11.64%	12.05%
LI A	11.72%	11.62%	11.52%	11.31%	11.43%	11.58%	11.29%	11.42%	11.59%	10.92%	11.14%	11.46%
LI B	11.51%	11.72%	11.69%	10.93%	11.09%	11.30%	11.19%	11.28%	11.40%	10.65%	10.87%	11.20%
LI C	12.17%	11.01%	11.12%	12.14%	12.17%	12.19%	11.20%	11.45%	11.85%	11.20%	11.47%	11.88%

Table C.21: IRR for the Infrastructure Funds, Historical Simulation

Fund	Unh.	For.	Rol. for.	Call$_{K_1^C}$	Call$_{K_2^C}$	Call$_{K_3^C}$	Put$_{K_1^P}$	Put$_{K_2^P}$	Put$_{K_3^P}$	Str.$_{K_1^S}$	Str.$_{K_2^S}$	Str.$_{K_3^S}$
GI A	9.52%	10.55%	10.74%	8.84%	9.03%	9.29%	9.98%	9.99%	9.83%	9.35%	9.52%	9.59%
GI B	9.45%	10.82%	10.96%	8.69%	8.92%	9.21%	10.06%	10.02%	9.75%	9.35%	9.51%	9.51%
GI C	9.59%	10.16%	10.45%	9.01%	9.18%	9.40%	9.73%	9.76%	9.72%	9.19%	9.38%	9.54%
LI A	9.67%	10.37%	10.72%	8.95%	9.16%	9.43%	9.91%	9.95%	9.89%	9.27%	9.47%	9.65%
LI B	9.48%	10.63%	10.83%	8.70%	8.93%	9.22%	9.93%	9.92%	9.70%	9.22%	9.39%	9.45%
LI C	9.66%	9.63%	10.15%	9.08%	9.26%	9.48%	9.58%	9.68%	9.71%	9.08%	9.29%	9.53%

Table C.22: VaR for the Infrastructure Funds, Historical Simulation

Fund	Unh.	For.	Rol. for.	Call$_{K_1^C}$	Call$_{K_2^C}$	Call$_{K_3^C}$	Put$_{K_1^P}$	Put$_{K_2^P}$	Put$_{K_3^P}$	Str.$_{K_1^S}$	Str.$_{K_2^S}$	Str.$_{K_3^S}$
GI A	9.04%	10.28%	10.56%	8.32%	8.53%	8.80%	9.72%	9.72%	9.54%	9.08%	9.24%	9.30%
GI B	8.96%	10.59%	10.77%	8.18%	8.41%	8.71%	9.83%	9.77%	9.46%	9.09%	9.24%	9.21%
GI C	8.98%	9.76%	10.27%	8.40%	8.58%	8.80%	9.44%	9.49%	9.40%	8.91%	9.10%	9.21%
LI A	9.16%	9.93%	10.51%	8.43%	8.65%	8.92%	9.55%	9.60%	9.52%	8.90%	9.11%	9.28%
LI B	9.02%	10.35%	10.64%	8.23%	8.47%	8.77%	9.66%	9.62%	9.38%	8.92%	9.09%	9.13%
LI C	9.08%	9.15%	9.94%	8.49%	8.67%	8.89%	9.25%	9.33%	9.33%	8.74%	8.94%	9.14%

Table C.23: ES for the Infrastructure Funds, Historical Simulation

Fund	Unh.	For.	Rol. for.	Call$_{K_1^C}$	Call$_{K_2^C}$	Call$_{K_3^C}$	Put$_{K_1^P}$	Put$_{K_2^P}$	Put$_{K_3^P}$	Str.$_{K_1^S}$	Str.$_{K_2^S}$	Str.$_{K_3^S}$
GI A	8.94	20.35	27.23	7.08	7.46	8.07	13.37	12.23	10.33	9.78	9.60	9.14
GI B	9.03	22.37	27.13	7.30	7.80	8.48	14.37	12.68	10.24	10.59	10.35	9.53
GI C	6.86	16.16	26.55	5.25	5.47	5.90	8.53	8.02	7.31	6.07	6.07	6.18
LI A	8.98	15.05	24.10	7.12	7.49	8.09	11.79	11.27	10.09	9.09	9.09	8.95
LI B	9.35	17.44	21.94	7.53	8.04	8.75	13.52	12.46	10.48	10.36	10.29	9.73
LI C	7.67	13.37	18.80	5.68	5.94	6.46	9.56	9.03	8.22	6.60	6.62	6.79

Table C.24: SR for the Infrastructure Funds, Historical Simulation

C.7 Gaussian Copula Simulation of FX Rates, Equity Funds

Fund	**Unh.**	**For.**	**Rol. for.**	**Call**$_{K_1^C}$	**Call**$_{K_2^C}$	**Call**$_{K_3^C}$	**Put**$_{K_1^P}$	**Put**$_{K_2^P}$	**Put**$_{K_3^P}$	**Str.**$_{K_1^S}$	**Str.**$_{K_2^S}$	**Str.**$_{K_3^S}$
GE A	20.14%	19.54%	19.48%	19.52%	19.70%	19.93%	18.82%	19.19%	19.75%	18.29%	18.80%	19.55%
GE B	20.05%	19.77%	19.73%	19.23%	19.45%	19.76%	18.92%	19.26%	19.74%	18.18%	18.70%	19.46%
GE C	20.20%	19.26%	19.14%	19.81%	19.93%	20.08%	18.58%	19.01%	19.69%	18.27%	18.79%	19.58%
LE A	20.19%	19.63%	19.55%	19.54%	19.72%	19.97%	18.85%	19.24%	19.80%	18.29%	18.81%	19.59%
LE B	20.02%	19.75%	19.73%	19.18%	19.40%	19.72%	18.89%	19.22%	19.71%	18.13%	18.65%	19.42%
LE C	20.11%	19.13%	19.24%	19.74%	19.86%	20.01%	18.46%	18.90%	19.59%	18.17%	18.69%	19.49%

Table C.25: IRR for the Equity Funds, Gaussian Copula Simulation

Fund	**Unh.**	**For.**	**Rol. for.**	**Call**$_{K_1^C}$	**Call**$_{K_2^C}$	**Call**$_{K_3^C}$	**Put**$_{K_1^P}$	**Put**$_{K_2^P}$	**Put**$_{K_3^P}$	**Str.**$_{K_1^S}$	**Str.**$_{K_2^S}$	**Str.**$_{K_3^S}$
GE A	18.21%	18.12%	18.26%	17.26%	17.53%	17.87%	17.09%	17.42%	17.87%	16.24%	16.75%	17.54%
GE B	17.88%	18.39%	18.50%	16.82%	17.13%	17.53%	17.25%	17.47%	17.72%	16.23%	16.73%	17.37%
GE C	17.48%	17.74%	17.94%	16.76%	16.97%	17.24%	16.61%	16.95%	17.35%	15.93%	16.41%	17.09%
LE A	18.15%	17.82%	18.17%	17.18%	17.44%	17.82%	16.86%	17.23%	17.78%	16.01%	16.55%	17.43%
LE B	17.72%	17.90%	18.07%	16.64%	16.96%	17.36%	16.88%	17.14%	17.49%	15.87%	16.40%	17.14%
LE C	17.73%	16.98%	17.47%	16.94%	17.16%	17.47%	16.29%	16.70%	17.28%	15.65%	16.18%	17.02%

Table C.26: VaR for the Equity Funds, Gaussian Copula Simulation

Fund	**Unh.**	**For.**	**Rol. for.**	**Call**$_{K_1^C}$	**Call**$_{K_2^C}$	**Call**$_{K_3^C}$	**Put**$_{K_1^P}$	**Put**$_{K_2^P}$	**Put**$_{K_3^P}$	**Str.**$_{K_1^S}$	**Str.**$_{K_2^S}$	**Str.**$_{K_3^S}$
GE A	17.76%	17.74%	17.95%	16.77%	17.05%	17.41%	16.70%	17.01%	17.45%	15.80%	16.33%	17.10%
GE B	17.35%	18.02%	18.19%	16.30%	16.61%	17.02%	16.87%	17.10%	17.30%	15.83%	16.34%	16.95%
GE C	16.93%	17.30%	17.64%	16.19%	16.41%	16.69%	16.20%	16.53%	16.94%	15.48%	15.98%	16.67%
LE A	17.67%	17.28%	17.79%	16.65%	16.93%	17.31%	16.40%	16.74%	17.29%	15.48%	16.04%	16.93%
LE B	17.21%	17.47%	17.74%	16.12%	16.44%	16.86%	16.47%	16.72%	17.03%	15.43%	15.95%	16.66%
LE C	17.11%	16.37%	17.09%	16.32%	16.55%	16.85%	15.86%	16.23%	16.78%	15.16%	15.70%	16.52%

Table C.27: ES for the Equity Funds, Gaussian Copula Simulation

Fund	**Unh.**	**For.**	**Rol. for.**	**Call**$_{K_1^C}$	**Call**$_{K_2^C}$	**Call**$_{K_3^C}$	**Put**$_{K_1^P}$	**Put**$_{K_2^P}$	**Put**$_{K_3^P}$	**Str.**$_{K_1^S}$	**Str.**$_{K_2^S}$	**Str.**$_{K_3^S}$
GE A	16.75	22.75	26.19	13.48	14.06	15.10	17.00	16.86	16.73	13.40	13.84	14.92
GE B	14.37	23.45	26.26	11.68	12.40	13.50	16.84	16.01	14.82	13.08	13.37	13.81
GE C	11.26	20.14	26.17	8.96	9.31	9.99	12.88	12.46	11.83	9.79	9.94	10.34
LE A	15.86	17.59	23.63	12.94	13.42	14.32	14.59	14.85	15.39	11.97	12.50	13.79
LE B	14.41	17.25	18.82	11.86	12.54	13.56	15.07	14.92	14.54	12.34	12.79	13.60
LE C	13.61	14.64	17.18	10.63	11.10	11.99	13.53	13.64	13.72	10.75	11.11	12.01

Table C.28: SR for the Equity Funds, Gaussian Copula Simulation

C.8 Gaussian Copula Simulation of FX Rates, Infrastructure Funds

Fund	Unh.	For.	Rol. for.	Call$_{K_1^C}$	Call$_{K_2^C}$	Call$_{K_3^C}$	Put$_{K_1^P}$	Put$_{K_2^P}$	Put$_{K_3^P}$	Str.$_{K_1^S}$	Str.$_{K_2^S}$	Str.$_{K_3^S}$
GI A	12.06%	11.48%	11.44%	11.77%	11.85%	11.96%	11.36%	11.55%	11.84%	11.11%	11.37%	11.75%
GI B	11.97%	11.71%	11.68%	11.52%	11.63%	11.80%	11.38%	11.54%	11.80%	10.96%	11.22%	11.63%
GI C	12.11%	11.20%	11.13%	11.99%	12.03%	12.08%	11.22%	11.45%	11.81%	11.13%	11.39%	11.79%
LI A	12.09%	11.54%	11.49%	11.79%	11.87%	11.99%	11.39%	11.58%	11.88%	11.12%	11.38%	11.77%
LI B	11.96%	11.70%	11.68%	11.49%	11.61%	11.78%	11.36%	11.53%	11.78%	10.93%	11.20%	11.61%
LI C	12.06%	11.13%	11.18%	11.95%	11.99%	12.04%	11.16%	11.39%	11.76%	11.08%	11.34%	11.74%

Table C.29: IRR for the Infrastructure Funds, Gaussian Copula Simulation

Fund	Unh.	For.	Rol. for.	Call$_{K_1^C}$	Call$_{K_2^C}$	Call$_{K_3^C}$	Put$_{K_1^P}$	Put$_{K_2^P}$	Put$_{K_3^P}$	Str.$_{K_1^S}$	Str.$_{K_2^S}$	Str.$_{K_3^S}$
GI A	10.53%	10.57%	10.74%	9.91%	10.08%	10.31%	10.08%	10.19%	10.37%	9.50%	9.75%	10.14%
GI B	10.19%	10.83%	10.97%	9.51%	9.71%	9.97%	10.17%	10.21%	10.19%	9.48%	9.71%	9.96%
GI C	9.81%	10.19%	10.45%	9.36%	9.50%	9.66%	9.74%	9.86%	9.96%	9.26%	9.50%	9.78%
LI A	10.49%	10.29%	10.67%	9.85%	10.02%	10.26%	9.93%	10.07%	10.30%	9.34%	9.61%	10.06%
LI B	10.23%	10.61%	10.83%	9.54%	9.74%	10.01%	10.05%	10.11%	10.17%	9.35%	9.61%	9.93%
LI C	10.12%	9.77%	10.22%	9.61%	9.76%	9.94%	9.63%	9.80%	10.03%	9.17%	9.44%	9.85%

Table C.30: VaR for the Infrastructure Funds, Gaussian Copula Simulation

Fund	Unh.	For.	Rol. for.	Call$_{K_1^C}$	Call$_{K_2^C}$	Call$_{K_3^C}$	Put$_{K_1^P}$	Put$_{K_2^P}$	Put$_{K_3^P}$	Str.$_{K_1^S}$	Str.$_{K_2^S}$	Str.$_{K_3^S}$
GI A	10.18%	10.30%	10.57%	9.52%	9.71%	9.95%	9.83%	9.93%	10.07%	9.21%	9.46%	9.84%
GI B	9.77%	10.57%	10.79%	9.10%	9.31%	9.56%	9.92%	9.96%	9.89%	9.20%	9.44%	9.66%
GI C	9.38%	9.86%	10.28%	8.93%	9.06%	9.23%	9.45%	9.56%	9.65%	8.95%	9.19%	9.47%
LI A	10.13%	9.90%	10.46%	9.46%	9.64%	9.90%	9.61%	9.74%	9.96%	8.98%	9.26%	9.72%
LI B	9.82%	10.30%	10.63%	9.12%	9.33%	9.60%	9.76%	9.81%	9.83%	9.04%	9.29%	9.58%
LI C	9.63%	9.32%	9.99%	9.13%	9.27%	9.46%	9.33%	9.47%	9.64%	8.84%	9.10%	9.46%

Table C.31: ES for the Infrastructure Funds, Gaussian Copula Simulation

Fund	Unh.	For.	Rol. for.	Call$_{K_1^C}$	Call$_{K_2^C}$	Call$_{K_3^C}$	Put$_{K_1^P}$	Put$_{K_2^P}$	Put$_{K_3^P}$	Str.$_{K_1^S}$	Str.$_{K_2^S}$	Str.$_{K_3^S}$
GI A	12.53	20.63	27.00	9.58	10.05	10.95	13.15	12.88	12.60	9.69	9.98	10.84
GI B	10.44	21.55	26.96	8.16	8.74	9.67	13.16	12.18	10.91	9.51	9.69	9.99
GI C	8.02	17.65	26.89	6.18	6.44	6.97	9.60	9.16	8.54	6.91	6.99	7.27
LI A	11.78	14.75	23.46	9.15	9.56	10.35	11.26	11.31	11.52	8.71	9.05	10.01
LI B	11.15	17.45	21.97	8.61	9.25	10.27	13.02	12.43	11.50	9.57	9.88	10.47
LI C	9.61	13.81	18.79	7.10	7.45	8.16	10.66	10.44	10.05	7.62	7.81	8.37

Table C.32: SR for the Infrastructure Funds, Gaussian Copula Simulation

C.9 Student's t Copula Simulation of FX rates, Equity Funds

Fund	**Unh.**	**For.**	**Rol. for.**	**Call**$_{K_1^C}$	**Call**$_{K_2^C}$	**Call**$_{K_3^C}$	**Put**$_{K_1^P}$	**Put**$_{K_2^P}$	**Put**$_{K_3^P}$	**Str.**$_{K_1^S}$	**Str.**$_{K_2^S}$	**Str.**$_{K_3^S}$
GE A	20.16%	19.53%	19.48%	19.55%	19.73%	19.96%	18.84%	19.22%	19.78%	18.31%	18.83%	19.58%
GE B	20.07%	19.77%	19.73%	19.25%	19.48%	19.78%	18.94%	19.28%	19.77%	18.20%	18.72%	19.48%
GE C	20.21%	19.26%	19.14%	19.83%	19.95%	20.10%	18.60%	19.04%	19.71%	18.30%	18.82%	19.60%
LE A	20.21%	19.63%	19.55%	19.57%	19.75%	19.99%	18.88%	19.26%	19.82%	18.32%	18.85%	19.62%
LE B	20.03%	19.74%	19.73%	19.20%	19.42%	19.74%	18.90%	19.24%	19.73%	18.15%	18.67%	19.44%
LE C	20.12%	19.12%	19.24%	19.76%	19.88%	20.02%	18.48%	18.93%	19.61%	18.19%	18.72%	19.51%

Table C.33: IRR for the Equity Funds, Student's t Copula Simulation

Fund	**Unh.**	**For.**	**Rol. for.**	**Call**$_{K_1^C}$	**Call**$_{K_2^C}$	**Call**$_{K_3^C}$	**Put**$_{K_1^P}$	**Put**$_{K_2^P}$	**Put**$_{K_3^P}$	**Str.**$_{K_1^S}$	**Str.**$_{K_2^S}$	**Str.**$_{K_3^S}$
GE A	18.21%	18.11%	18.25%	17.26%	17.53%	17.89%	17.10%	17.42%	17.89%	16.24%	16.76%	17.54%
GE B	17.89%	18.40%	18.50%	16.86%	17.18%	17.56%	17.28%	17.52%	17.77%	16.27%	16.76%	17.41%
GE C	17.39%	17.72%	17.92%	16.67%	16.89%	17.15%	16.62%	16.94%	17.36%	15.92%	16.42%	17.11%
LE A	18.11%	17.76%	18.13%	17.15%	17.42%	17.77%	16.87%	17.24%	17.75%	16.00%	16.56%	17.42%
LE B	17.74%	17.92%	18.07%	16.64%	16.96%	17.38%	16.90%	17.17%	17.52%	15.90%	16.44%	17.16%
LE C	17.65%	16.97%	17.46%	16.87%	17.08%	17.38%	16.28%	16.67%	17.26%	15.63%	16.19%	17.01%

Table C.34: VaR for the Equity Funds, Student's t Copula Simulation

Fund	**Unh.**	**For.**	**Rol. for.**	**Call**$_{K_1^C}$	**Call**$_{K_2^C}$	**Call**$_{K_3^C}$	**Put**$_{K_1^P}$	**Put**$_{K_2^P}$	**Put**$_{K_3^P}$	**Str.**$_{K_1^S}$	**Str.**$_{K_2^S}$	**Str.**$_{K_3^S}$
GE A	17.72%	17.71%	17.94%	16.72%	17.00%	17.37%	16.70%	17.00%	17.43%	15.80%	16.32%	17.09%
GE B	17.36%	18.01%	18.18%	16.31%	16.63%	17.03%	16.89%	17.12%	17.33%	15.84%	16.36%	16.97%
GE C	16.82%	17.26%	17.63%	16.10%	16.31%	16.58%	16.18%	16.50%	16.90%	15.46%	15.96%	16.63%
LE A	17.58%	17.17%	17.75%	16.56%	16.84%	17.22%	16.35%	16.72%	17.24%	15.45%	16.02%	16.88%
LE B	17.18%	17.45%	17.74%	16.09%	16.41%	16.83%	16.47%	16.74%	17.05%	15.43%	15.96%	16.68%
LE C	17.05%	16.38%	17.11%	16.26%	16.49%	16.78%	15.81%	16.19%	16.75%	15.12%	15.66%	16.48%

Table C.35: ES for the Equity Funds, Student's t Copula Simulation

Fund	**Unh.**	**For.**	**Rol. for.**	**Call**$_{K_1^C}$	**Call**$_{K_2^C}$	**Call**$_{K_3^C}$	**Put**$_{K_1^P}$	**Put**$_{K_2^P}$	**Put**$_{K_3^P}$	**Str.**$_{K_1^S}$	**Str.**$_{K_2^S}$	**Str.**$_{K_3^S}$
GE A	16.75	22.69	26.23	13.46	14.02	15.05	16.97	16.85	16.76	13.34	13.78	14.88
GE B	14.54	23.60	26.32	11.83	12.56	13.67	17.07	16.25	15.05	13.24	13.56	14.03
GE C	10.98	19.69	25.96	8.75	9.09	9.74	12.58	12.19	11.56	9.58	9.72	10.10
LE A	15.70	17.37	23.42	12.83	13.30	14.16	14.44	14.72	15.28	11.87	12.39	13.67
LE B	14.43	17.23	18.80	11.92	12.61	13.62	15.18	15.03	14.64	12.45	12.92	13.73
LE C	13.35	14.57	17.24	10.44	10.88	11.74	13.35	13.47	13.56	10.59	10.94	11.81

Table C.36: SR for the Equity Funds, Student's t Copula Simulation

C.10 Student's t Copula Simulation of FX rates, Infrastructure Funds

Fund	Unh.	For.	Rol. for.	Call$_{K_1^C}$	Call$_{K_2^C}$	Call$_{K_3^C}$	Put$_{K_1^P}$	Put$_{K_2^P}$	Put$_{K_3^P}$	Str.$_{K_1^S}$	Str.$_{K_2^S}$	Str.$_{K_3^S}$
GI A	12.07%	11.48%	11.44%	11.80%	11.88%	11.98%	11.38%	11.57%	11.86%	11.14%	11.39%	11.78%
GI B	11.99%	11.70%	11.68%	11.54%	11.66%	11.82%	11.39%	11.56%	11.82%	10.98%	11.24%	11.65%
GI C	12.12%	11.20%	11.13%	12.00%	12.05%	12.09%	11.23%	11.47%	11.83%	11.15%	11.41%	11.81%
LI A	12.11%	11.54%	11.49%	11.82%	11.90%	12.01%	11.42%	11.61%	11.90%	11.16%	11.42%	11.80%
LI B	11.97%	11.70%	11.68%	11.51%	11.63%	11.80%	11.38%	11.55%	11.80%	10.95%	11.22%	11.63%
LI C	12.07%	11.13%	11.18%	11.97%	12.01%	12.05%	11.18%	11.41%	11.78%	11.10%	11.36%	11.77%

Table C.37: IRR for the Infrastructure Funds, Student's t Copula Simulation

Fund	Unh.	For.	Rol. for.	Call$_{K_1^C}$	Call$_{K_2^C}$	Call$_{K_3^C}$	Put$_{K_1^P}$	Put$_{K_2^P}$	Put$_{K_3^P}$	Str.$_{K_1^S}$	Str.$_{K_2^S}$	Str.$_{K_3^S}$
GI A	10.54%	10.56%	10.74%	9.90%	10.07%	10.31%	10.08%	10.21%	10.39%	9.50%	9.75%	10.16%
GI B	10.22%	10.82%	10.97%	9.57%	9.77%	10.02%	10.18%	10.23%	10.23%	9.50%	9.73%	10.01%
GI C	9.77%	10.19%	10.44%	9.34%	9.46%	9.62%	9.72%	9.85%	9.94%	9.25%	9.49%	9.77%
LI A	10.45%	10.24%	10.65%	9.83%	10.00%	10.24%	9.93%	10.08%	10.30%	9.34%	9.62%	10.07%
LI B	10.24%	10.60%	10.82%	9.55%	9.75%	10.01%	10.05%	10.12%	10.19%	9.37%	9.62%	9.96%
LI C	10.05%	9.81%	10.22%	9.56%	9.69%	9.87%	9.62%	9.77%	9.99%	9.15%	9.41%	9.82%

Table C.38: VaR for the Infrastructure Funds, Student's t Copula Simulation

Fund	Unh.	For.	Rol. for.	Call$_{K_1^C}$	Call$_{K_2^C}$	Call$_{K_3^C}$	Put$_{K_1^P}$	Put$_{K_2^P}$	Put$_{K_3^P}$	Str.$_{K_1^S}$	Str.$_{K_2^S}$	Str.$_{K_3^S}$
GI A	10.15%	10.29%	10.57%	9.50%	9.68%	9.92%	9.83%	9.93%	10.06%	9.21%	9.46%	9.83%
GI B	9.78%	10.56%	10.79%	9.12%	9.32%	9.58%	9.93%	9.98%	9.92%	9.20%	9.45%	9.68%
GI C	9.29%	9.82%	10.27%	8.85%	8.98%	9.15%	9.44%	9.55%	9.62%	8.94%	9.17%	9.44%
LI A	10.05%	9.84%	10.43%	9.38%	9.57%	9.82%	9.57%	9.71%	9.92%	8.95%	9.23%	9.68%
LI B	9.79%	10.29%	10.63%	9.10%	9.31%	9.57%	9.76%	9.82%	9.84%	9.04%	9.29%	9.60%
LI C	9.55%	9.34%	10.00%	9.06%	9.20%	9.38%	9.27%	9.42%	9.60%	8.79%	9.05%	9.42%

Table C.39: ES for the Infrastructure Funds, Student's t Copula Simulation

Fund	Unh.	For.	Rol. for.	Call$_{K_1^C}$	Call$_{K_2^C}$	Call$_{K_3^C}$	Put$_{K_1^P}$	Put$_{K_2^P}$	Put$_{K_3^P}$	Str.$_{K_1^S}$	Str.$_{K_2^S}$	Str.$_{K_3^S}$
GI A	12.54	20.56	27.02	9.57	10.02	10.91	13.09	12.85	12.62	9.62	9.92	10.80
GI B	10.57	21.72	27.00	8.27	8.87	9.81	13.31	12.36	11.09	9.61	9.82	10.16
GI C	7.83	17.27	26.77	6.05	6.30	6.81	9.39	8.98	8.36	6.78	6.85	7.12
LI A	11.71	14.65	23.26	9.10	9.49	10.26	11.16	11.23	11.47	8.63	8.97	9.93
LI B	11.24	17.46	21.91	8.71	9.35	10.38	13.16	12.59	11.67	9.68	10.02	10.64
LI C	9.37	13.74	18.89	6.95	7.29	7.96	10.40	10.21	9.85	7.47	7.65	8.19

Table C.40: SR for the Infrastructure Funds, Student's t Copula Simulation

D Performance of the Hedging Strategies - Extended Model I

D.1 Random Walk Simulation of FX Rates with Stochastic Short Rates, Equity Funds

Fund	Unh.	For.	Rol. for.	Call$_{K_1^C}$	Call$_{K_2^C}$	Call$_{K_3^C}$	Put$_{K_1^P}$	Put$_{K_2^P}$	Put$_{K_3^P}$	Str.$_{K_1^S}$	Str.$_{K_2^S}$	Str.$_{K_3^S}$
GE A	19.36%	19.42%	19.39%	18.80%	18.99%	19.22%	18.57%	18.86%	19.22%	18.07%	18.52%	19.09%
GE B	19.69%	19.72%	19.73%	18.89%	19.13%	19.44%	18.97%	19.24%	19.56%	18.26%	18.72%	19.32%
GE C	18.88%	19.03%	18.93%	18.58%	18.72%	18.88%	18.10%	18.40%	18.77%	17.84%	18.26%	18.77%
LE A	19.39%	19.41%	19.42%	18.79%	19.00%	19.24%	18.59%	18.88%	19.25%	18.05%	18.52%	19.11%
LE B	19.68%	19.73%	19.72%	18.88%	19.12%	19.43%	18.97%	19.24%	19.56%	18.25%	18.72%	19.32%
LE C	19.21%	19.38%	19.24%	18.93%	19.07%	19.22%	18.38%	18.70%	19.09%	18.14%	18.57%	19.09%

Table D.1: IRR for the Equity Funds, Random Walk Simulation with Stochastic Short Rates

Fund	Unh.	For.	Rol. for.	Call$_{K_1^C}$	Call$_{K_2^C}$	Call$_{K_3^C}$	Put$_{K_1^P}$	Put$_{K_2^P}$	Put$_{K_3^P}$	Str.$_{K_1^S}$	Str.$_{K_2^S}$	Str.$_{K_3^S}$
GE A	17.26%	18.05%	18.14%	16.45%	16.72%	17.06%	16.90%	17.12%	17.31%	16.16%	16.58%	17.09%
GE B	16.97%	18.35%	18.49%	15.89%	16.22%	16.64%	17.19%	17.33%	17.31%	16.18%	16.60%	16.97%
GE C	16.47%	17.73%	17.75%	16.06%	16.24%	16.43%	16.48%	16.72%	16.92%	16.04%	16.45%	16.85%
LE A	17.35%	17.47%	18.00%	16.49%	16.78%	17.10%	16.74%	17.00%	17.30%	15.98%	16.45%	17.07%
LE B	17.04%	17.78%	18.04%	15.89%	16.25%	16.69%	16.86%	17.06%	17.18%	15.84%	16.30%	16.83%
LE C	16.83%	17.36%	17.44%	16.39%	16.59%	16.80%	16.18%	16.49%	16.87%	15.79%	16.23%	16.81%

Table D.2: VaR for the Equity Funds, Random Walk Simulation with Stochastic Short Rates

Fund	Unh.	For.	Rol. for.	Call$_{K_1^C}$	Call$_{K_2^C}$	Call$_{K_3^C}$	Put$_{K_1^P}$	Put$_{K_2^P}$	Put$_{K_3^P}$	Str.$_{K_1^S}$	Str.$_{K_2^S}$	Str.$_{K_3^S}$
GE A	16.78%	17.66%	17.83%	15.92%	16.20%	16.55%	16.53%	16.73%	16.91%	15.76%	16.19%	16.68%
GE B	16.40%	17.93%	18.18%	15.31%	15.64%	16.07%	16.81%	16.94%	16.91%	15.75%	16.17%	16.56%
GE C	15.89%	17.35%	17.45%	15.50%	15.69%	15.88%	16.10%	16.34%	16.53%	15.64%	16.06%	16.46%
LE A	16.84%	16.95%	17.60%	15.96%	16.24%	16.59%	16.28%	16.55%	16.85%	15.50%	15.99%	16.61%
LE B	16.40%	17.27%	17.72%	15.26%	15.61%	16.05%	16.42%	16.60%	16.67%	15.37%	15.82%	16.31%
LE C	16.26%	16.84%	17.10%	15.79%	16.00%	16.22%	15.75%	16.04%	16.38%	15.32%	15.78%	16.33%

Table D.3: ES for the Equity Funds, Random Walk Simulation with Stochastic Short Rates

Fund	Unh.	For.	Rol. for.	Call$_{K_1^C}$	Call$_{K_2^C}$	Call$_{K_3^C}$	Put$_{K_1^P}$	Put$_{K_2^P}$	Put$_{K_3^P}$	Str.$_{K_1^S}$	Str.$_{K_2^S}$	Str.$_{K_3^S}$
GE A	14.58	22.61	25.46	12.05	12.59	13.48	16.50	16.02	15.22	13.25	13.45	13.87
GE B	11.24	22.53	25.91	8.89	9.38	10.19	14.52	13.61	12.19	10.83	10.87	10.90
GE C	11.68	22.63	26.38	10.22	10.62	11.19	15.91	15.35	13.95	13.04	13.17	13.01
LE A	15.31	17.16	22.58	12.74	13.31	14.22	15.60	15.64	15.59	12.97	13.38	14.26
LE B	12.02	16.37	18.62	9.48	10.01	10.90	13.83	13.49	12.71	10.82	11.03	11.41
LE C	13.16	15.44	17.10	11.53	11.98	12.62	12.82	13.13	13.56	11.49	11.95	12.85

Table D.4: SR for the Equity Funds, Random Walk Simulation with Stochastic Short Rates

D.2 Random Walk Simulation of FX Rates with Stochastic Short Rates, Infrastructure Funds

Fund	Unh.	For.	Rol. for.	Call$_{K_1^C}$	Call$_{K_2^C}$	Call$_{K_3^C}$	Put$_{K_1^P}$	Put$_{K_2^P}$	Put$_{K_3^P}$	Str.$_{K_1^S}$	Str.$_{K_2^S}$	Str.$_{K_3^S}$
GI A	11.34%	11.40%	11.36%	11.05%	11.16%	11.28%	11.16%	11.26%	11.36%	10.89%	11.08%	11.29%
GI B	11.65%	11.68%	11.69%	11.21%	11.34%	11.51%	11.44%	11.54%	11.64%	11.04%	11.25%	11.50%
GI C	10.94%	11.08%	10.98%	10.81%	10.88%	10.96%	10.83%	10.93%	11.01%	10.72%	10.88%	11.03%
LI A	11.37%	11.40%	11.40%	11.07%	11.18%	11.30%	11.18%	11.29%	11.39%	10.90%	11.10%	11.32%
LI B	11.64%	11.69%	11.67%	11.20%	11.33%	11.50%	11.43%	11.53%	11.63%	11.03%	11.24%	11.50%
LI C	11.12%	11.29%	11.15%	11.00%	11.07%	11.14%	10.99%	11.10%	11.18%	10.88%	11.05%	11.21%

Table D.5: IRR for the Infrastructure Funds, Random Walk Simulation with Stochastic Short Rates

Fund	Unh.	For.	Rol. for.	Call$_{K_1^C}$	Call$_{K_2^C}$	Call$_{K_3^C}$	Put$_{K_1^P}$	Put$_{K_2^P}$	Put$_{K_3^P}$	Str.$_{K_1^S}$	Str.$_{K_2^S}$	Str.$_{K_3^S}$
GI A	9.66%	10.52%	10.66%	9.11%	9.29%	9.51%	9.93%	9.96%	9.89%	9.43%	9.60%	9.73%
GI B	9.37%	10.77%	10.97%	8.65%	8.87%	9.16%	10.10%	10.07%	9.82%	9.40%	9.56%	9.59%
GI C	8.91%	10.22%	10.30%	8.70%	8.82%	8.92%	9.63%	9.66%	9.57%	9.32%	9.47%	9.52%
LI A	9.74%	10.10%	10.55%	9.18%	9.36%	9.58%	9.82%	9.88%	9.87%	9.30%	9.50%	9.71%
LI B	9.47%	10.47%	10.81%	8.74%	8.97%	9.24%	9.97%	9.98%	9.82%	9.27%	9.47%	9.59%
LI C	9.27%	10.04%	10.19%	8.99%	9.12%	9.26%	9.52%	9.60%	9.62%	9.23%	9.43%	9.58%

Table D.6: VaR for the Infrastructure Funds, Random Walk Simulation with Stochastic Short Rates

Fund	Unh.	For.	Rol. for.	Call$_{K_1^C}$	Call$_{K_2^C}$	Call$_{K_3^C}$	Put$_{K_1^P}$	Put$_{K_2^P}$	Put$_{K_3^P}$	Str.$_{K_1^S}$	Str.$_{K_2^S}$	Str.$_{K_3^S}$
GI A	9.28%	10.26%	10.47%	8.72%	8.90%	9.13%	9.69%	9.70%	9.59%	9.16%	9.33%	9.43%
GI B	8.87%	10.48%	10.79%	8.15%	8.38%	8.65%	9.84%	9.80%	9.54%	9.11%	9.28%	9.31%
GI C	8.48%	9.95%	10.13%	8.26%	8.38%	8.49%	9.35%	9.40%	9.29%	9.04%	9.21%	9.25%
LI A	9.33%	9.73%	10.32%	8.75%	8.94%	9.17%	9.49%	9.55%	9.53%	8.95%	9.16%	9.37%
LI B	8.98%	10.10%	10.61%	8.24%	8.47%	8.76%	9.67%	9.66%	9.46%	8.95%	9.14%	9.22%
LI C	8.80%	9.68%	9.97%	8.52%	8.65%	8.79%	9.22%	9.30%	9.28%	8.93%	9.11%	9.24%

Table D.7: ES for the Infrastructure Funds, Random Walk Simulation with Stochastic Short Rates

Fund	Unh.	For.	Rol. for.	Call$_{K_1^C}$	Call$_{K_2^C}$	Call$_{K_3^C}$	Put$_{K_1^P}$	Put$_{K_2^P}$	Put$_{K_3^P}$	Str.$_{K_1^S}$	Str.$_{K_2^S}$	Str.$_{K_3^S}$
GI A	10.66	20.73	26.18	8.44	8.89	9.65	12.80	12.22	11.34	9.65	9.74	10.04
GI B	7.82	20.16	26.64	6.00	6.36	6.99	10.84	9.95	8.66	7.61	7.60	7.58
GI C	8.12	19.96	26.56	6.94	7.25	7.71	11.78	11.29	10.06	9.25	9.33	9.21
LI A	10.98	14.63	22.27	8.69	9.15	9.93	11.95	11.75	11.39	9.29	9.51	10.09
LI B	8.53	15.91	21.72	6.40	6.81	7.54	11.08	10.43	9.32	7.85	7.92	8.08
LI C	9.54	14.87	18.85	7.97	8.37	8.97	10.56	10.67	10.63	8.84	9.14	9.73

Table D.8: SR for the Infrastructure Funds, Random Walk Simulation with Stochastic Short Rates

E Performance of the Hedging Strategies - Extended Model II

E.1 Random Walk Simulation of FX Rates, USD Denominated Equity Funds

Fund	**Unh.**	**For.**	**Rol. for.**	**Call**$_{K_1^C}$	**Call**$_{K_2^C}$	**Call**$_{K_3^C}$	**Put**$_{K_1^P}$	**Put**$_{K_2^P}$	**Put**$_{K_3^P}$	**Str.**$_{K_1^S}$	**Str.**$_{K_2^S}$	**Str.**$_{K_3^S}$
GE A	21.02%	21.12%	21.10%	19.16%	19.50%	20.05%	20.64%	20.79%	20.96%	18.88%	19.33%	20.00%
GE B	21.08%	21.19%	21.17%	19.12%	19.46%	20.04%	20.71%	20.85%	21.02%	18.85%	19.30%	19.99%
GE C	21.23%	21.34%	21.32%	19.21%	19.57%	20.17%	20.89%	21.03%	21.18%	18.97%	19.42%	20.12%
LE A	20.84%	20.90%	20.91%	18.94%	19.28%	19.85%	20.44%	20.60%	20.77%	18.65%	19.10%	19.79%
LE B	20.89%	20.98%	20.97%	18.92%	19.27%	19.84%	20.52%	20.66%	20.83%	18.66%	19.10%	19.79%
LE C	20.98%	21.07%	21.06%	18.95%	19.31%	19.91%	20.64%	20.78%	20.93%	18.71%	19.17%	19.87%

Table E.1: IRR for the USD Denominated Equity Funds, Random Walk Simulation

Fund	**Unh.**	**For.**	**Rol. for.**	**Call**$_{K_1^C}$	**Call**$_{K_2^C}$	**Call**$_{K_3^C}$	**Put**$_{K_1^P}$	**Put**$_{K_2^P}$	**Put**$_{K_3^P}$	**Str.**$_{K_1^S}$	**Str.**$_{K_2^S}$	**Str.**$_{K_3^S}$
GE A	16.86%	19.53%	19.77%	14.28%	14.78%	15.58%	18.08%	17.97%	17.57%	15.58%	15.92%	16.28%
GE B	16.76%	19.59%	19.84%	14.07%	14.57%	15.39%	18.04%	17.92%	17.51%	15.42%	15.76%	16.10%
GE C	16.97%	19.73%	19.96%	14.18%	14.72%	15.57%	18.19%	18.05%	17.65%	15.47%	15.82%	16.24%
LE A	17.28%	18.80%	19.51%	14.61%	15.12%	15.94%	18.04%	18.04%	17.83%	15.52%	15.92%	16.46%
LE B	16.84%	18.91%	19.39%	14.05%	14.58%	15.43%	17.75%	17.70%	17.41%	15.08%	15.47%	15.98%
LE C	17.03%	19.04%	19.49%	14.13%	14.68%	15.57%	17.88%	17.78%	17.52%	15.10%	15.49%	16.05%

Table E.2: VaR for the USD Denominated Equity Funds, Random Walk Simulation

Fund	**Unh.**	**For.**	**Rol. for.**	**Call**$_{K_1^C}$	**Call**$_{K_2^C}$	**Call**$_{K_3^C}$	**Put**$_{K_1^P}$	**Put**$_{K_2^P}$	**Put**$_{K_3^P}$	**Str.**$_{K_1^S}$	**Str.**$_{K_2^S}$	**Str.**$_{K_3^S}$
GE A	15.97%	19.05%	19.44%	13.37%	13.88%	14.68%	17.64%	17.53%	17.09%	15.10%	15.44%	15.79%
GE B	15.85%	19.08%	19.51%	13.13%	13.65%	14.47%	17.59%	17.47%	17.02%	14.92%	15.26%	15.62%
GE C	16.12%	19.24%	19.64%	13.31%	13.85%	14.71%	17.74%	17.61%	17.14%	14.96%	15.32%	15.71%
LE A	16.52%	17.94%	19.11%	13.78%	14.32%	15.15%	17.53%	17.54%	17.31%	14.94%	15.37%	15.93%
LE B	15.92%	18.23%	19.04%	13.09%	13.63%	14.48%	17.22%	17.15%	16.82%	14.53%	14.91%	15.37%
LE C	16.12%	18.40%	19.14%	13.19%	13.76%	14.65%	17.35%	17.25%	16.90%	14.56%	14.94%	15.43%

Table E.3: ES for the USD Denominated Equity Funds, Random Walk Simulation

Fund	**Unh.**	**For.**	**Rol. for.**	**Call**$_{K_1^C}$	**Call**$_{K_2^C}$	**Call**$_{K_3^C}$	**Put**$_{K_1^P}$	**Put**$_{K_2^P}$	**Put**$_{K_3^P}$	**Str.**$_{K_1^S}$	**Str.**$_{K_2^S}$	**Str.**$_{K_3^S}$
GE A	7.71	20.64	25.44	5.56	5.82	6.33	9.97	9.36	8.46	6.75	6.73	6.80
GE B	7.44	20.43	25.55	5.33	5.57	6.05	9.55	8.99	8.14	6.42	6.41	6.49
GE C	7.71	20.76	25.41	5.48	5.73	6.25	9.70	9.14	8.33	6.49	6.49	6.63
LE A	9.20	16.87	24.99	6.24	6.56	7.22	11.92	11.27	10.19	7.59	7.61	7.79
LE B	8.25	16.69	21.01	5.68	5.97	6.55	10.48	9.93	9.04	6.83	6.86	7.02
LE C	8.47	17.07	21.26	5.79	6.09	6.69	10.55	10.00	9.16	6.84	6.89	7.11

Table E.4: SR for the USD Denominated Equity Funds, Random Walk Simulation

E.2 Random Walk Simulation of FX Rates, USD Denominated Infrastructure Funds

Fund	Unh.	For.	Rol. for.	Call$_{K_1^C}$	Call$_{K_2^C}$	Call$_{K_3^C}$	Put$_{K_1^P}$	Put$_{K_2^P}$	Put$_{K_3^P}$	Str.$_{K_1^S}$	Str.$_{K_2^S}$	Str.$_{K_3^S}$
GI A	12.85%	12.93%	12.92%	11.79%	11.98%	12.29%	12.78%	12.82%	12.86%	11.77%	11.98%	12.31%
GI B	12.91%	13.01%	12.99%	11.78%	11.98%	12.30%	12.84%	12.89%	12.92%	11.77%	11.99%	12.32%
GI C	13.04%	13.14%	13.12%	11.88%	12.09%	12.42%	12.97%	13.02%	13.05%	11.86%	12.09%	12.44%
LI A	12.74%	12.80%	12.80%	11.67%	11.86%	12.18%	12.66%	12.71%	12.75%	11.65%	11.86%	12.19%
LI B	12.82%	12.91%	12.90%	11.69%	11.88%	12.21%	12.76%	12.80%	12.83%	11.68%	11.89%	12.23%
LI C	12.92%	13.01%	13.00%	11.75%	11.96%	12.30%	12.85%	12.89%	12.93%	11.74%	11.96%	12.31%

Table E.5: IRR for the USD Denominated Infrastructure Funds, Random Walk Simulation

Fund	Unh.	For.	Rol. for.	Call$_{K_1^C}$	Call$_{K_2^C}$	Call$_{K_3^C}$	Put$_{K_1^P}$	Put$_{K_2^P}$	Put$_{K_3^P}$	Str.$_{K_1^S}$	Str.$_{K_2^S}$	Str.$_{K_3^S}$
GI A	9.18%	11.82%	12.13%	7.43%	7.77%	8.31%	10.70%	10.49%	9.97%	9.00%	9.10%	9.11%
GI B	9.09%	11.86%	12.20%	7.27%	7.63%	8.18%	10.64%	10.45%	9.90%	8.86%	8.96%	8.97%
GI C	9.31%	12.00%	12.32%	7.40%	7.77%	8.37%	10.76%	10.54%	10.01%	8.89%	9.01%	9.06%
LI A	9.51%	11.23%	11.97%	7.72%	8.06%	8.61%	10.68%	10.53%	10.14%	8.95%	9.12%	9.25%
LI B	9.19%	11.50%	12.04%	7.29%	7.65%	8.23%	10.54%	10.37%	9.92%	8.73%	8.85%	8.95%
LI C	9.37%	11.65%	12.14%	7.43%	7.82%	8.40%	10.65%	10.46%	9.99%	8.75%	8.89%	9.02%

Table E.6: VaR for the USD Denominated Infrastructure Funds, Random Walk Simulation

Fund	Unh.	For.	Rol. for.	Call$_{K_1^C}$	Call$_{K_2^C}$	Call$_{K_3^C}$	Put$_{K_1^P}$	Put$_{K_2^P}$	Put$_{K_3^P}$	Str.$_{K_1^S}$	Str.$_{K_2^S}$	Str.$_{K_3^S}$
GI A	8.43%	11.40%	11.93%	6.67%	7.02%	7.57%	10.38%	10.18%	9.62%	8.66%	8.77%	8.75%
GI B	8.32%	11.42%	12.00%	6.47%	6.83%	7.39%	10.33%	10.12%	9.55%	8.51%	8.62%	8.61%
GI C	8.56%	11.58%	12.12%	6.65%	7.02%	7.61%	10.44%	10.22%	9.65%	8.55%	8.67%	8.69%
LI A	8.78%	10.56%	11.73%	6.98%	7.34%	7.89%	10.31%	10.18%	9.76%	8.56%	8.73%	8.86%
LI B	8.38%	10.94%	11.83%	6.48%	6.85%	7.42%	10.16%	9.98%	9.48%	8.33%	8.46%	8.52%
LI C	8.59%	11.11%	11.94%	6.63%	7.01%	7.61%	10.26%	10.06%	9.56%	8.36%	8.50%	8.57%

Table E.7: ES for the USD Denominated Infrastructure Funds, Random Walk Simulation

Fund	Unh.	For.	Rol. for.	Call$_{K_1^C}$	Call$_{K_2^C}$	Call$_{K_3^C}$	Put$_{K_1^P}$	Put$_{K_2^P}$	Put$_{K_3^P}$	Str.$_{K_1^S}$	Str.$_{K_2^S}$	Str.$_{K_3^S}$
GI A	5.36	17.72	26.41	3.78	3.96	4.33	7.20	6.69	5.95	4.70	4.66	4.69
GI B	5.15	17.35	26.49	3.61	3.78	4.13	6.85	6.38	5.70	4.45	4.42	4.46
GI C	5.37	17.86	26.39	3.73	3.91	4.28	6.97	6.51	5.85	4.51	4.49	4.57
LI A	6.02	13.60	25.56	4.07	4.28	4.71	8.13	7.58	6.73	5.06	5.04	5.11
LI B	5.45	14.81	24.34	3.73	3.92	4.30	7.26	6.78	6.05	4.60	4.59	4.65
LI C	5.64	15.29	24.72	3.83	4.02	4.42	7.34	6.86	6.17	4.63	4.62	4.73

Table E.8: SR for the USD Denominated Infrastructure Funds, Random Walk Simulation

F Stochastic Calculus Results

In this section some important stochastic calculus results are presented. The theory is covered in Øksendal (1998) [25] and inspiration is to a large extent gathered from Björk (2009) [2].

F.1 Stochastic Processes

Let us begin with stating some important definitions required to develop the theory of asset pricing in continuous time. The theory used in the thesis builds on the concepts diffusion processes and SDEs.

Diffusion processes are stochastic processes, i.e. collections of random variables. They can be thought of as a series of values where each value depends on two separate terms, a locally deterministic velocity and a random disturbance term.

Definition F.1 ***Diffusion process***
X is said to be a diffusion processes if its local dynamics is on the form

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)Z_t, \tag{F.1}$$
$$X_0 = x, \tag{F.2}$$

where Z_t is a Gaussian disturbance term independent of events that have occurred before time t. Furthermore, μ, the drift of the process, and σ, the diffusion of the process, are given deterministic functions and x is a given constant.

The disturbance term of a diffusion process can be nicely modelled using Wiener processes. A Wiener process can be thought of as a random movement of a point in time, not depending on its previous position.

Definition F.2 ***Wiener process***
A stochastic process W_t is called a Wiener process (or Brownian motion) if

(i) $W_0 = 0$,

(ii) *W_t has independent increments, i.e. if $r < s \leq u < v$ then $W_v - W_u$ and $W_s - W_r$ are independent stochastic variables,*

(iii) *$W_s - W_r \sim N(0, s - r)$ if $r < s$,*

(iv) *W_t has continuous trajectories.*

Using a Wiener process the diffusion process defined in Definition F.1 can be stated in its commonly used form

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \tag{F.3}$$
$$X_0 = x. \tag{F.4}$$

Moreover, let us introduce the concept of correlated Wiener processes. Correlated Wiener processes are used to model the common movements of several dependent processes.

Definition F.3 ***Correlated Wiener processes***
Given a vector of d independent standard Wiener processes, $\tilde{W}_1, ..., \tilde{W}_d$, a vector of n correlated Wiener processes, $W_1, ..., W_n$ is defined by the matrix multiplication

$$W = \delta \tilde{W}, \tag{F.5}$$

where

$$\delta = \begin{bmatrix} \delta_{11} & \delta_{12} & \cdots & \delta_{1d} \\ \delta_{21} & \delta_{22} & \cdots & \delta_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{n1} & \delta_{n2} & \cdots & \delta_{nd} \end{bmatrix} \tag{F.6}$$

is a matrix of constant and deterministic elements. Furthermore, the correlation matrix of W is given by

$$\rho = \delta\delta^T. \tag{F.7}$$

F.2 Stochastic Integrals

The aim of this section is to define integrals of the form $\int_a^b g(s)dW_s$. The concept of information will play an important role in the theory to be developed. We denote by $\mathcal{F}_t^X$ the information generated by X on the interval $[0, t]$, i.e. $\mathcal{F}_t^X$ can be thought of as an object remembering all the values that X has taken over the interval $[0, t]$. To develop this further the two concepts of measurability and adaptedness will be required. Observe that a formal treatment of the concepts is out of the scope of this thesis, instead we give the heuristic definitions.

Definition F.4 ***Measurability***
A stochastic variable Z is said to be $\mathcal{F}_t^X$ measurable, $Z \in \mathcal{F}_t^X$, if the value of Z can be completely determined given observations of the trajectory X_s for $s \in [0, t]$.

Definition F.5 ***Adaptedness***
A stochastic process Y_t is said to be adapted to the filtration $\underline{\mathcal{F}} = \left\{\mathcal{F}_t^X\right\}_{t\geq 0}$ if $Y_t \in \mathcal{F}_t^X$ for all $t \geq 0$.

Next the construction and some important properties of stochastic integrals is considered. For the sake of completeness the following definition is important.

Definition F.6
The process g is said to belong to the class $\mathcal{L}^2[a, b]$ if

(i) $\int_a^b E[g^2(s)]ds < \infty$,

(ii) *The process g is adapted to the $\mathcal{F}_t^W$-filtration.*

Furthermore, the process g is said to belong to the class $\mathcal{L}^2$ if $g \in \mathcal{L}^2[0, t]$ for all $t > 0$.

Given a Wiener process W and a stochastic process $g \in \mathcal{L}^2[a, b]$ a stochastic integral can schematically be defined in the following steps. First, it is assumed that $g \in$

$\mathcal{L}^2[a,b]$ is simple, i.e. that there exists deterministic points $a = t_0 < t_1 < \cdots < t_n = b$ such that $g(s) = g(t_k)$ for $s \in [t_k, t_{k+1})$. Then the stochastic integral is defined as

$$\int_a^b g(s)dW_s = \sum_{k=0}^{n-1} g(t_k)[W(t_{k+1}) - W(t_k)]. \tag{F.8}$$

However, the process g needs not be simple. In the general case g is approximated with a sequence of simple processes g_n such that $\int_a^b E\left[\{g_n(s) - g(s)\}^2\right]ds \to 0$ in $\mathcal{L}^2$ sense as $n \to \infty$. Finally, the stochastic integral in the general case can be defined as follows.

Definition F.7 ***Stochastic integral***
The stochastic integral is defined as

$$\int_a^b g(s)dW_s = \lim_{n\to\infty} \int_a^b g_n(s)dW_s, \tag{F.9}$$

where $g \in \mathcal{L}^2[a,b]$ and $g_n \in \mathcal{L}^2[a,b]$ and are simple processes.

We finish up by stating some important results for stochastic integrals that will be useful later on.

Proposition F.1 ***Properties of stochastic integrals***
For the stochastic integral defined in Definition F.7 the following relations hold

$$E\left[\int_a^b g(s)dW_s\right] = 0, \tag{F.10}$$

$$E\left[\left(\int_a^b g(s)dW_s\right)^2\right] = \int_a^b E[g^2(s)]ds, \tag{F.11}$$

$$\int_a^b g(s)dW_s \text{ is } \mathcal{F}_b^W\text{- measurable.} \tag{F.12}$$

F.3 Martingales

The modern theory of financial derivatives is to a large extent based on martingale theory. Since a formal treatment of martingale theory requires abstract measure theory we limit ourselves to an informal treatment. A martingale is a stochastic process, X, for which the expectation of the future value of X, given today's information, is equal to the observed value of X today.

Definition F.8 ***Martingale***
A stochastic process X is called an $\mathcal{F}_t$-martingale if the following hold.

(i) *X is $\mathcal{F}_t$-measurable, i.e. X is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$,*

(ii) *X is integrable, i.e. $E[|X_t|] < \infty$ for all t,*

(iii) *X satisfies the martingale condition, i.e. $E[X_t|\mathcal{F}_s] = X_s$ for all s and t such that $s \leq t$.*

Returning to the discussion about stochastic integrals the following result can now be established.

Proposition F.2
For any process $g \in \mathcal{L}^2$ the process X defined by $X_t = \int_0^t g(s)dW_s$ is an $\mathcal{F}_t^W$-martingale.

Put in other words, given that the integrability condition is fulfilled, every stochastic integral is a martingale. The above relation can be expressed on differential form as $dX_t = g(t)dW_t$. It can thus be deduced that a stochastic process X is a martingale if its differential has no drift term.

F.4 Itô's Formula

In this section one of the most important formulas for the thesis, Itô's formula that originates from Itô (1944) [18], is presented. Itô's formula fills the role of the classical chain rule of differentiation in stochastic calculus.

An Itô process is a stochastic process that can be represented as a sum of a deterministic integral and a stochastic integral. That is, X_t on the following form is an Itô process

$$X_t = x + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW_s. \tag{F.13}$$

The stochastic differential and initial condition of X_t are thus given by

$$dX_t = \mu(t)dt + \sigma(t)dW(t), \tag{F.14}$$
$$X_0 = x, \tag{F.15}$$

where μ and σ are adapted processes and x is a constant.

Before we present Itô's formula some auxiliary properties are investigated. Note that the following reasoning leading up to Itô's formula is purely heuristic and not a formal proof in any sense. Setting $s < t$ and defining

$$\Delta t = t - s,$$

and

$$\Delta W_t = W_t - W_s,$$

the following relations follow from the fact that $\Delta W_t \sim N(0, t-s)$.

$$E[\Delta W_t] = 0,$$

$$Var(\Delta W_t) = \Delta t,$$
$$E\big[(\Delta W_t)^2\big] = \Delta t,$$
$$Var\big((\Delta W_t)^2\big) = 2(\Delta t)^2.$$

An important observation is that the variance of $(\Delta W_t)^2$ is negligible in comparison to its expected value. This means that as Δt tends to zero $(\Delta W_t)^2$ will tend to zero, however, the variance of $(\Delta W_t)^2$ will tend to zero much faster than the expected value of $(\Delta W_t)^2$. Thus, in the limit, as t tends to zero $(dW_t)^2 = dt$ or, using bracket notation, $d\langle W\rangle_t = dt$.

Theorem F.1 ***Itô's formula***
Define the process Z by $Z_t = f(t, X_t)$ where f is a $C^{1,2}$-function and X_t is the Itô process given in Eqs. F.14 and F.15. Then, Itô's formula states that Z has a stochastic differential given by

$$df(t, X_t) = \left\{\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}\right\}dt + \sigma\frac{\partial f}{\partial x}dW_t. \tag{F.16}$$

Using the auxiliary relations defined above Itô's formula can be expressed as

$$df(t, X_t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}d\langle X\rangle_t, \tag{F.17}$$

where $(dt)^2 = 0$, $dt \cdot dW = 0$ and $(dW)^2 = dt$.

F.5 The Girsanov Theorem

The Girsanov theorem describes how the dynamics of a stochastic processes change when the original probability measure P is changed to an equivalent probability measure Q. Most important, the process will be a martingale under the equivalent martingale measure. Girsanov's theorem allows us to convert the physical measure to a risk neutral measure which will be used for pricing derivatives on underlying instruments. For the reader wishing the full formal treatment, see Girsanov (1960) [13].

Theorem F.2 ***Girsanov's theorem***
Let W^P denote a standard P-Wiener process on $(\Omega, \mathcal{F}, P, \underline{\mathcal{F}})$ and let φ be an arbitrary adapted process. Assume that the Girsanov kernel, φ, fulfils the Novikov condition

$$E^P\left[\exp\left(\frac{1}{2}\int_0^T ||\varphi_t||^2 dt\right)\right] < \infty. \tag{F.18}$$

Moreover, define the likelihood process by

$$dL_t = \varphi_t L_t dW_t^P \tag{F.19}$$

$$L_0 = 1, \tag{F.20}$$

i.e.

$$L_t = \exp\left(\int_0^t \varphi_s dW_s^P - \frac{1}{2}\int_0^t ||\varphi_s||^2 ds\right). \tag{F.21}$$

Define the new probability measure Q as

$$dQ = L_T dP, \quad \text{on } \mathcal{F}_T. \tag{F.22}$$

Then,

$$dW_t^Q = dW_t^P - \varphi_t dt, \tag{F.23}$$

where W_t^Q is a Q-Wiener process.

F.6 Stochastic Differential Equations and Geometric Brownian Motions

We proceed our exploration of stochastic calculus by recalling the concept of SDE. The question which is set out to be answered is whether there exists a stochastic process X satisfying the stochastic differential equation (SDE)

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \tag{F.24}$$
$$X_0 = x_0. \tag{F.25}$$

The general SDE defined in Eqs. F.24-F.25 is complicated to solve in an explicit manner. Fortunately, there are exceptions, the most important one being the Geometric Brownian motion.

Definition F.9 ***Geometric Brownian motion***
The Geometric Brownian motion is defined as the SDE

$$dX_t = \alpha X_t dt + \sigma X_t dW_t, \tag{F.26}$$
$$X_0 = x_0. \tag{F.27}$$

Proposition F.3 ***Solution to the Geometric Brownian motion***
The solution to the Geometric Brownian motion defined in Eqs. F.26-F.27 is given by

$$X_t = x_0 \exp\left(\left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right). \tag{F.28}$$

Proof. In order to establish Proposition F.3 the Itô formula Eq. F.17 with $f(t,x) = \ln x$ will be used. The partial derivatives of $f(t,x)$ are given by

$$\frac{\partial f}{\partial t}(t,x) = 0, \quad \frac{\partial f}{\partial x}(t,x) = \frac{1}{x}, \quad \frac{\partial^2 f}{\partial x^2}(t,x) = -\frac{1}{x^2}.$$

Furthermore, applying Itô's formula to $f(t,x)$ gives

$$\begin{aligned} df(t,X_t) &= \frac{\partial f}{\partial t}(t,X_t)dt + \frac{\partial f}{\partial x}(t,X_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t)d\langle X\rangle_t \\ &= 0 \cdot dt + \frac{1}{X_t}(\alpha X_t dt + \sigma X_t dW_t) - \frac{1}{2}\frac{1}{X_t^2}\sigma^2 X_t^2 dt \\ &= (\alpha - \frac{1}{2}\sigma^2)dt + \sigma dW_t. \end{aligned}$$

Writing the expression in its integral form yields

$$\begin{aligned}\ln X_t &= \ln x_0 + \int_0^t \big(\alpha - \frac{1}{2}\sigma^2\big)ds + \int_0^t \sigma dW_s \\ &= \ln x_0 + \big(\alpha - \frac{1}{2}\sigma^2\big)t + \sigma W_t.\end{aligned}$$

And finally, by taking the exponent the desired result is obtained

$$X_t = x_0 \exp\Big(\big(\alpha - \frac{1}{2}\sigma^2\big)t + \sigma W_t\Big).$$

□

F.7 Partial Differential Equations and Feynman-Kač

In this section we present the general stochastic partial differential equation that returns frequently in the study of the Black-Scholes model. Furthermore, the Feynman-Kač formula for solving these types of problems is derived.

The general boundary value problem that will be considered is given by

$$\frac{\partial F}{\partial t}(t,x) + \mu(t,x)\frac{\partial F}{\partial x}(t,x) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x) - rF(t,x) = 0, \tag{F.29}$$

$$F(T,x) = \Phi(x), \tag{F.30}$$

where μ, σ and Φ are known scalar functions and r is a known constant. In order to solve the above boundary value problem it will be assumed that there exists a function F satisfying Eqs. F.29-F.30 on $[0,T] \times R$. We will then seek a stochastic representation formula which gives the solution to Eqs. F.29-F.30 in terms of the solution of an SDE associated to the problem at hand.

Suppose that X_s has the dynamics

$$\begin{aligned}dX_s &= \mu(s,X_s)ds + \sigma(s,X_s)dW_s, \\ X_t &= x.\end{aligned}$$

Using the technique of integrating factor and applying Itô's formula to $Z_s = e^{-\int_0^s rdu}F(s,X_s)$ we get

$$\begin{aligned}dZ_s = e^{-\int_0^s rdu}\Big(&\frac{\partial F}{\partial t}(s,X_s) + \mu(s,X_s)\frac{\partial F}{\partial x}(s,X_s) + \frac{1}{2}\sigma^2(s,X_s)\frac{\partial^2 F}{\partial x^2}(s,X_s) - rF(s,X_s)\Big)ds \\ &+ \sigma(s,X_s)e^{-\int_0^s rdu}\frac{\partial F}{\partial x}(s,X_s)dW_s.\end{aligned}$$

Using Eq. F.29 the above expression reduces to

$$dZ_s = \sigma(s,X_s)e^{-\int_0^s rdu}\frac{\partial F}{\partial x}(s,X_s)dW_s.$$

Writing the expression in its integral form yields

$$Z_T = Z_t + \int_t^T \sigma(s, X_s) e^{-\int_0^s r du} \frac{\partial F}{\partial x}(s, X_s) dW_s,$$

or more precisely

$$e^{-\int_0^T r du} F(T, X_T) = e^{-\int_0^t r du} F(t, X_t) + \int_t^T \sigma(s, X_s) e^{-\int_0^s r du} \frac{\partial F}{\partial x}(s, X_s) dW_s.$$

The initial condition, Eq. F.30, and the fact that $X_t = x$ allow us to rewrite the expression as

$$e^{-\int_0^T r du} \Phi(X_T) = e^{-\int_0^t r du} F(t, x) + \int_t^T \sigma(s, X_s) e^{-\int_0^s r du} \frac{\partial F}{\partial x}(s, X_s) dW_s.$$

Taking expectation conditional on $X_t = x$ yields

$$E_{t,x}\left[e^{-\int_0^T r du} \Phi(X_T)\right] = e^{-\int_0^t r du} F(t, x) + E_{t,x}\left[\int_t^T \sigma(s, X_s) e^{-\int_0^s r du} \frac{\partial F}{\partial x}(s, X_s) dW_s\right].$$

Finally, assuming that $e^{-\int_0^s r du} \sigma(s, X_s) \frac{\partial F}{\partial x}(s, X_s)$ fulfils the Novikov condition, Eq. F.18, and using property 1 of Proposition F.1 we get the following after rearranging

$$\begin{aligned} F(t, x) &= E_{t,x}\left[e^{-\int_t^T r du} \Phi(X_T)\right] \\ &= e^{-r(T-t)} E_{t,x}\left[\Phi(X_T)\right]. \end{aligned}$$

Formulating our findings in a proposition we arrive at the famous Feynman-Kač stochastic representation formula which concludes this section.

Proposition F.4 ***Feynman-Kač***
Assume that F is a solution to the boundary value problem

$$\frac{\partial F}{\partial t}(t, x) + \mu(t, x) \frac{\partial F}{\partial x}(t, x) + \frac{1}{2}\sigma^2(t, x) \frac{\partial^2 F}{\partial x^2}(t, x) - rF(t, x) = 0, \tag{F.31}$$

$$F(T, x) = \Phi(x). \tag{F.32}$$

Furthermore, assume that $e^{-rs} \sigma(s, X_s) \frac{\partial F}{\partial x}(s, X_s)$ fulfils the Novikov condition, Eq. F.18. Then F has the stochastic representation

$$F(t, x) = e^{-r(T-t)} E_{t,x}\left[\Phi(X_T)\right], \tag{F.33}$$

where the dynamics of X are given by

$$dX_s = \mu(s, X_s) ds + \sigma(s, X_s) dW_s, \tag{F.34}$$

$$X_t = x. \tag{F.35}$$

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