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Consolidating Multi-Factor Models of Systematic Risk with Regulatory Capital

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Abstract

To maintain solvency in times of severe economic downturns banks and financial institutions keep capital cushions that reflect the risks in the balance sheet. Broadly, how much capital that is being held is a combination of external requirements from regulators and internal assessments of credit risk. We discuss alternatives to the Basel Pillar II capital add-on based on multi-factor models for held capital and how these can be applied so that only concentration (or sector) risk affects the outcome, even in a portfolio with prominent idiosyncratic risk. Further, the stability and reliability of these models are evaluated. We found that this idiosyncratic risk can efficiently be removed both on a sector and a portfolio level and that the multi-factor models tested converge.

We introduce two new indices based on Risk Weighted Assets (RI) and Economic Capital (EI). Both show the desired effect of an intuitive dependence on the PD and LGD. Moreover, EI shows a dependence on the inter-sector correlation. In the sample portfolio, we show that the high concentration in one sector could be (better) justified by these methods when the low average LGD and PD of this sector were taken into consideration.

Keywords: *Economic Capital, Regulatory Capital, Basel Pillar II, Systematic Risk*

Konsolidering av flerfaktormodeller för systematisk risk med reglerande kapital

Sammanfattning

För att behålla solvens i tider av svår lågkonjunktur håller banker och finansiella institutioner buffertar med kapital som reflekterar risken i balansräkningen. I stora drag så är mängden kapital som hålls beroende av en kombination av externa krav från regulatorer och interna uppskattningar av kredit risken. Den här avhandlingen diskuterar alternativ till Basel pelare II kapital påslaget som är baserade på multifaktor modeller för kapital och hur dessa kan appliceras så att endast koncentration (eller sektor) risk påverkar resultat, även i en portfölj med tydlig idiosynkratisk risk. Utöver detta behandlas stabilitet och reliabilitet hos dessa modeller. Genom detta hittas att den idiosynkratisk risk kan effektivt tas bort på både portfölj- och sektornivå och att de multifaktor modeller som testas konvergerar.

Den här avhandlingen introducerar två nya index, baserat på Risk Weighted Assets (RI) och Economic Capital (EI). Båda visar på den önskade effekten av ett intuitivt beroende av PD och LGD. Dessutom visar EI ett beroende av inter-sektor korrelation. Med stickprovportföljen som används var det tydligt att hög koncentration i en sektor kunde (bättre) rättfärdigas av båda dessa metoder då LGD och PD för sektorn i fråga har beaktats.

Nyckelord: *Ekonomisk kapital, Regulatoriskt kapital, Basel pelare II, Systematisk risk*

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Notation and Abbreviations

PD - Probability of Default

EAD - Exposure at Default

LGD - Loss Given Default

D - Default variable

X - Systemic risk variable

A - Asset value

L - The Loss variable

ASRF - Asymptotic Single Risk Factor

IRB - Internal Ratings-Based

EL - Expected Loss

VaR - Value at Risk

ES - Expected Shortfall

UL - Unexposed Loss

EC - Economic Capital

RWA - Risk Weighted Asset

β - Average cross-sector correlation

CDI - Sector concentration

HHI - Herfindahl-Hirschman Index

CI - Concentration Index

EI - Economic capital Index

RI - Risk weighted asset Index

S² - PD dependent index

1 Introduction

Financial institutions hold capital cushions that reflect the risks in their balance sheets. The size of these is determined by a combination of external requirements from regulators and internal assessments of credit risk. However, in a competitive market holding unnecessary large capital cushions naturally reflects on the profitability by tying up capital that could otherwise be used to profit from (e.g. lending it). Consequently, an overestimation reduces profitability while an underestimation puts the bank at risk. Thus, any improvement in the accuracy of a credit risk models either reduce the risk or improves profitability.

Credit risk is the risk that an obligor (borrower) does not repay the loan in full and the lender may then lose the principal of the loan or the interest associated with it. A lender can in practice never be certain that a borrower will hold up its end of the contract and thus all lending is exposed to credit risk. Having said that, the estimated credit risk is reflected in the interest rate the lender is willing to give the borrower. A borrower with a low probability of not repaying the loan in full is most likely able to borrow to a lower interest rate than a borrower with higher probability. There are many different approaches to quantifying this estimate of the probability of not repaying; or as it is commonly referred to, the probability of default. A common way is using a logistic regression with both obligor specific and macroeconomic information or to infer the risk by the price of a Credit Default Swap or CDS [1]. A third frequently used approach is applying the credit rates provided by Moody's or Standard and Poors which are based on historical data. Whatever the approach, the move to analytical models from the historical "gut feeling approach" greatly improved (i.e. lowered) the defaults rates of financial institutions [2].

Despite the improvement, during the 2008 financial crisis, weaknesses of the then used analytically approach was uncovered. Firstly, the failure to differentiate the default behavior of obligors during a stressful time and during "normal" conditions [3] [4]. Secondly, the underestimation of the liquidity risk taken by financial institutions at the time [5]. To explore these mistakes one must first understand the importance of diversification and why liquidity reserves are needed.

A well-diversified a portfolio of loans refer to a portfolio of loans that are not highly concentrated to any industry or geographical location (referred to as sectors). Given the assumption that these loans are random and independent of each other a large enough portfolio would be sufficient to ensure that defaults follow that aggregated estimate. However, it is widely known that loans within a sector are correlated*. The rationale is that, entire sectors are able to experience stressful scenarios and that the entirety of loans within that sector are then exposed to a higher risk. Assuming that one cannot know beforehand which sectors that might experience this scenario it would make sense to not rely too heavily on any one of them, i.e. diversify. A natural way to quantify this sector risk is then by quantifying the correlations within each sector. Consequently, the historical correlation might seem as a sufficient indicator of the risk; however, as was uncovered during the financial crisis this approach greatly underestimated the actual correlation in a stressful scenario. This

*There is correlation between sectors as well although this is not as prominent as the intra-correlation

is because correlations are greater in such a scenario and since they are rare the data used will tend to estimate the correlations closer to those in a normal scenario [6]. This emphasizes the importance of differentiating between these normal and stressful scenarios and preparing for the correlation structure to change during the later one.

During a scenario with great stress to one or more sectors, liquidity could become an issue to financial institutions. The Liquidity of an asset refers to how quickly it can be sold without affecting the price of the asset. For the purpose of this thesis, liquidity risk occurs when an institution (or other entity) cannot repay its short-term debt. The institution cannot convert its assets to cash since there are no or too few buyers of the asset they want to sell. In order to combat this risk, there need to be a reserve of highly liquid assets that can be used in such a scenario. Naturally, there is a trade-off for any institution that trades in assets that are not highly liquid (illiquid). Simplified, to minimize risk only highly liquid assets should be held and to maximize profit (long term) the institution wants to invest in illiquid assets. As such, institutions seek the optimal reserve and this is of course, in part, dependent on the ability of their obligors to repay their loans. The risk of obligors defaulting (credit risk) is not alone responsible for the entirety of this reserve, other areas that contribute include market risk and operational risk [7]. This reserve is referred to differently depending on if it is calculated by the institution themselves or if it is a requirement. Economic capital refers to the reserve that the institution would themselves set if there were no regulations. In contrast, the Capital requirement is the reserve currently used since there are indeed regulations on how much should be held [8]. This is explored further in the next section.

1.1 The Basel Accords

The Basel Committee on Banking Supervision (BCBS) is a committee of banking supervisory authorities that was established in 1974 by the central bank governors from ten different countries [9]. Since then 45 institutions from 28 jurisdictions have joined the committee. The goal of the committee is to provide a supervisory standard for its members. This standard is not a regulatory demand since the committee does not enforce its standards. The local authorities of each member enforce the decisions made by the committee and as such there are some irregularities (see Section 1.2). Nevertheless, the total effect of the committee is a more uniform banking sector throughout its member states.

In terms of credit risk, the Basel accord of 1988 introduced the first capital requirements under what is referred to as Pillar I. Banks are required to keep capital for credit risk on an exposure by exposure basis*. The basic formula for computing the required capital is called the internal ratings-based model or IRB and this model is explained in more (mathematical) detail in Section 3.1. The IRB model is based on two simplifying assumptions:

Assumption 1.1 *The portfolio is infinitely fine-grained and thus all idiosyncratic risk is diversified away on a portfolio level.*

*Note that it is possible for an obligor to have more than one exposure. However, this thesis assumes that obligor only have one unique exposure to simplify calculations and notation

Assumption 1.2 *All default correlations are represented by one risk factor representing the state of the economy.*

Violation of the first assumption, i.e. there are large single exposures in the portfolio, gives rise to so-called name-concentration risk, whereas violation of the second assumption gives rise to so-called sector concentration. In the words of the Basel committee:

Sector concentration arises from the violation of the single systematic risk factor assumption which represents an elementary departure from the IRB model framework. It arises because business conditions and hence default risk may not be fully synchronized across all business sectors or geographical regions within a large economy. A bank's portfolio may be more or less concentrated on some of these risk factors leading to a discrepancy between the measured risk from a single-factor model and a model that allows for a richer factor structure. Given the calibration of the ASRF model for the IRB formulae, this discrepancy can be positive as well as negative. [10]

To remedy these shortcomings of the IRB model capital requirements for, respectively, name and sector risks are covered under Pillar II of the Basel accord which is the focus of this thesis. Furthermore, only models pertaining to the sector risk is explored in this thesis. The main reasoning behind this is that advanced models measuring name-concentration, like the granularity adjustment [11], is, in general, more accepted and in use while more simplistic models are used to measure the sector risk. Thus, the applicability of a more advanced (and hopefully more precise) model of sector risk is vital for banks using the IRB framework.

1.2 Sector Risk Today

Continuing on the point in the last section, there is much research on the subject of sector risk and much debate on how to accurately model it. This section aims to explain the current situation in this field of study.

To start with the IRB model is portfolio invariant and this is a very important property if a model is to be realistically used. A portfolio with this property is able to introduce or exclude any exposure to the model framework without recalculating for the entirety of the portfolio; in other words, it only needs to add or subtract the calculated change from the current portfolio. In theory, it would be desirable to have a model where an introduction of an exposure in a sector would affect the risk of other exposures within that sector, or even exposures that are not in the same sector to better reflect the actual change in risk for the portfolio. In practice, this would be far to calculation heavy to be realistically used since portfolios change on a day to day basis, if not on a minute to minute basis.

Sector risk is for regulatory purposes divided into two components: industry risk and geographical risk. Although, in the literature, sector risk will often only refer to the industry risk. Presumably because it is conceptually easier to quantify one specific exposure or obligor to a sector than to a country (or region). As such, there are advanced model estimations for the industry risk while there is not much discussion

on the estimation of geographical risk. Despite the fact that these models (presented in Section 4) are usable for the entirety of sector risk, authors neglect to estimate the effect of geographical risk. One is then left to assume that the uncertainties of calculating the correlation to a geographical area for each obligor mean that a more simplistic model is preferred in this case.

As mentioned in Section 1.1 each member state of the Basel accords are able to apply their own regulations [12]. Consequently, sector risk is treated differently within each member state. In Sweden, which is the focus area of this thesis, the Swedish Financial Supervisory Authority (Finansinspektionen) provide the models for calculating the sector risk. Both components have a unique model, albeit analogous. The models make use of the Herfindahl-Hirschman Index (HHI) which is defined as follows:

$$\text{HHI} = \sum_{k=1}^K s_k^2. \quad (1.1)$$

In the context of this thesis s_k is the proportion of each industry or geographical region (henceforth region) in the portfolio. This index punishes a high concentration in any sector and notably punishes this equally not taking into account the difference in the riskiness of each sector. Using the calculated HHI for industry and geographical region respectively an add-on for the capital requirement of pillar I is calculated to be:

$$\begin{aligned} \text{For Industry:} \quad \rho_{BK} &= 8 \cdot \left(1 - e^{-5 \cdot \text{HHI}^{1.5}}\right), \\ \text{For Region:} \quad \rho_{GK} &= 8 \cdot \left(1 - e^{-2 \cdot \text{HHI}^{1.7}}\right). \end{aligned} \quad (1.2)$$

Where ρ_{GK} and ρ_{BK} are the percentage units of the total capital requirement for credit risk in pillar I. The Swedish FSA use 12 industries and 15 regions which can be found on page 23 and 24 respectively in [13].

This "mapping" of HHI to a capital requirement takes a maximum value when HHI is equal to 1 and using 12 industries of SFSA (which is close to the 13 used in this thesis) yields a minimum value when HHI is $\sum_{k=1}^{12} \left(\frac{1}{12}\right)^2 \approx 0.0833$. From Fig. (1.1) it is evident that Banks that have a HHI between 0.10 and 0.20 are significantly more affected by a change than a bank with higher HHI.

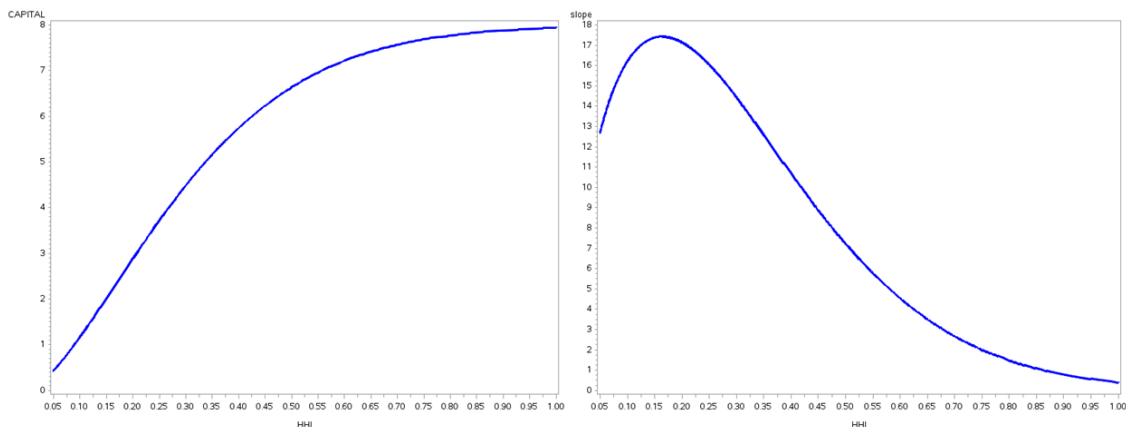


Figure 1.1: Left: Industry mapping. Right: First order derivative

1.3 Purpose

The purpose of this thesis is to evaluate different proposed multi-factor credit risk models and use these to find a tractable way to connect the multi-factor framework to a pillar II capital add-on. The aim is to bridge the gap between the academic models and those currently in use so that a variation of the models presented in this thesis could realistically be used in practice.

1.4 Limitations

The methods discussed in this thesis are applicable to any financial institution with implemented methods for PD, LGD, and EAD. However, the results are compared to the standard methodology currently in use in Sweden when the capital add-on is discussed and the Basel methodology when multi-factor models are discussed.

While concentration or sector risk includes both industry and geographical risk this thesis is limited to the industry component. As mentioned, this is a difference in the data used and not in the models. In order to keep consistency throughout the thesis "sector" is used exclusively while the geographical component is not included.

1.5 Outline

The remainder of this thesis is structured as follows: Section 2 covers the general framework in credit risk modeling and introduces the Vasicek formula, which is the basis of both single-factor and multi-factor models introduced in later sections. In Section 3 the currently used IRB model is explained as well as its connection to the capital requirements. Section 4 introduces the different multi-factor models developed to improve on the IRB model. Section 5 covers different proposals on connecting the economic capital to a capital requirement. Section 6 explains the choice of portfolio used in testing and some of the difficulties in applying the multi-factor framework. In Section 7 the results of this thesis are shown and finally in Section 8 a conclusion is presented.

2 Credit Risk Modelling

To model the credit risk of a portfolio a range of definitions and notations are introduced. Although definitions are consistent the notation tends to vary in the literature; this thesis only uses the notation presented in this section.

2.1 General Framework

While the theory behind this framework is extensive this thesis will not go further than to define the notation and some properties. Some explanations of how PD, EAD and LGD are calculated can be found in the Basel framework [14]. However, this is not a comprehensive list since banks and financial institutions are able to apply internal approaches to these calculations (see Section 1.2).

Probability of Default is the probability that an obligor default over some specified time horizon is denoted PD_i . There are two commonly used variations of PD; point-in-time (PIT) and through-the-cycle (TTC). TTC is the average probability of default over a business cycle which is in contrast to PIT where the state of the economy at that time is used. In other words, PIT uses all the available information at that time and TTC does not. It may then seem natural to only use PIT but since neither of them is known the reliability of each method needs to be taken into account.

The default variable often denoted D_i , is a random variable specified over some time horizon with a Bernoulli distribution and it takes the value 1 when obligor i has defaulted on 0 if it has not.

$$D_i = \begin{cases} 1, & \text{with probability } PD_i \\ 0, & \text{with probability } 1 - PD_i. \end{cases} \quad (2.1)$$

Note that by the properties of the Bernoulli distribution the expected value of D_i is the same as the PD_i .

Exposure at Default is the total value that the bank is exposed to at the time of the default. It is a random variable but for the purpose of this thesis, the expected value of EAD is easily found. Similarly to D_i the \widetilde{EAD}_i refers to the obligor specific exposure. To separate the random variable EAD with the expected value of it the following notation is introduced

$$\mathbf{E}[\widetilde{EAD}_i] = EAD_i. \quad (2.2)$$

When the nominal value of the exposure is not of interest the ratio of the exposure to the total portfolio of exposure is used as a replacement:

$$w_i = \frac{EAD_i}{\sum_{j=1}^n EAD_j}. \quad (2.3)$$

Both these terms are used interchangeably in this thesis to better align with the literature.

Loss given Default or LGD_i is the fraction of EAD_i that cannot be recovered in the event of a default. This random variable depends on many factors, one intuitive factor is if the bank owns collateral. This collateral could then be sold off after the default and thus some value is retained. In some literature the Recovery rate RR_i is used instead of LGD_i . The Recovery rate is defined as $1 - \text{LGD}_i$. Again, to not confuse the random variable with the expected value similar notation (to EAD) is introduced

$$\mathbf{E}[\widetilde{\text{LGD}}_i] = \text{LGD}_i. \quad (2.4)$$

Systemic Risk Variable At the heart of both the Asymptotic Single Risk Factor Model (ASRF) introduced in 2.3 and the multi-factor model introduced in 4 is the assumption that assets are correlated with some systemic risk factor denoted X in the single factor model or X_1, X_2, \dots, X_S where S is the number of sectors in the multi-factor model. X and X_s are assumed to be standard normal distributed $N(0, 1)$. The rationale in the model is that the default variable D_i is dependent on the state of the economy. In the single factor case X represents the state of the economy and in the multi-factor case, X_s represents the state of each sector.

The Loss variable The loss variable L_i is defined, over a specific time horizon, to be the outstanding amount that is not recovered due to a default within the time period. The loss of a credit portfolio with n obligors is defined as $L = \sum_{i=1}^n L_i$. Using the notation already introduced as well it can be expressed as

$$L = \sum_{i=1}^n \widetilde{\text{EAD}}_i \cdot \widetilde{\text{LGD}}_i \cdot D_i. \quad (2.5)$$

With some assumptions, the loss variables dependence on the systemic risk factor(s) can be deduced. To relax notation \mathbf{X} is used to represent both the single-factor and multi-factor model (interpreted as a vector in this case).

Assumption 2.1 *Both EAD_i and LGD_i are mutually independent i.e. $\text{LGD}_1, \dots, \text{LGD}_n$ and $\text{EAD}_1, \dots, \text{EAD}_n$ are independent of each other. Furthermore, these variables are independent of both D_1, \dots, D_n and \mathbf{X} .*

By this assumption it is trivial that

$$\mathbf{E}[L|\mathbf{X}] = \mathbf{E} \left[\sum_{i=1}^n \widetilde{\text{EAD}}_i \cdot \widetilde{\text{LGD}}_i \cdot D_i \middle| \mathbf{X} \right] = \sum_{i=1}^n \text{EAD}_i \cdot \text{LGD}_i \cdot \mathbf{E} \left[D_i \middle| \mathbf{X} \right]. \quad (2.6)$$

Where \mathbf{X} can be either the sector-specific systemic risk factor or the state of the economy as a whole.

2.2 Risk Measures

All information about the risk of a portfolio (at some future point in time) is contained in the Loss variable L defined in the previous section. A risk measure quantifies the risk associated with L so that one can, for example, calculate the required capital given a risk appetite. It is important to note that a risk measure, as the name suggests, only measures the risk and not the possible gain of a portfolio. To those familiar with financial mathematics, it might seem natural to try to estimate the expected utility. However, in the case of risk measures (and specifically their usage within the banking sector) the return is maximized given a target level of risk.

A more precise definition is that a risk measure maps a set of random variables to real numbers. Consider a random outcome Z viewed as an element of a linear space \mathcal{L} of measurable functions, defined on an appropriate probability space. A functional $\rho : \mathcal{L} \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be risk measure for \mathcal{L} if it satisfies the following three properties [15]

1. *Normalized* $\rho(0) = 0$. The risk of having an empty portfolio should be 0.
2. *Translative* If $\alpha \in \mathbb{R}$ and $Z \in \mathcal{L}$ then $\rho(Z + \alpha) = \rho(Z) - \alpha$. Since α here is a real number the interpretation of this property is that it is a deterministic addition to the portfolio. Thus, if a known amount is added to the portfolio naturally the risk should be lowered by the same amount.
3. *Monotone* If $Z_1, Z_2 \in \mathcal{L}$ and $Z_1 \leq Z_2$ then $\rho(Z_2) \leq \rho(Z_1)$. If it is known that Z_2 has better values than Z_1 (i.e. worth more but otherwise equal) then the risk measure should always have a lower risk for Z_2 .

Expected Loss, denoted EL, is a risk measure that shows the expected value of the sum of the total loss. it is defined as

$$\text{EL} = \mathbf{E}[L] = \sum_{i=1}^n \text{EAD}_i \cdot \text{LGD}_i \cdot \mathbf{E}[D_i], \quad (2.7)$$

this is again making use of Assumption (2.1). While EL by itself is of interest the variance of the loss distribution is not considered and as such a bank holding only the value of EL in capital risk defaulting if the losses during the time horizon are any greater than expected.

Value at Risk, to capture the behavior of the variance Value at Risk or VaR is introduced. VaR requires a specified probability (i.e. risk) threshold z at which a default is acceptable. Naturally one would like this risk to be 0% but as mention in the introduction, there is, of course, a trade-off in risk versus profit. As such, commonly used z are 95%, 99% and in the case of the IRB model 99.9% [6].

For a portfolio, X with a threshold of a default at x VaR at level z is defined as

$$\text{VaR}(X)_z = \inf\{x \in \mathbb{R} : 1 - F_X(-x) \geq z\}. \quad (2.8)$$

For F_X that is continuous and strictly increasing the VaR at level z is the quantile of the function F_X i.e. $\text{VaR}(X)_z = F_X^{-1}(z)$. In the case of IRB and this thesis, the portfolio X that is of interest is the portfolio of losses i.e. L .

Unexpected Loss is the risk measure that is used in the IRB framework [7]. It is defined as the difference between the Value at Risk for X and the Expected value of X : $\text{UL}(X)_z = \text{VaR}(X)_z - E[X]$. With the more relevant notation it is written:

$$\text{UL}_z = \text{VaR}(L)_z - \text{EL}. \quad (2.9)$$

The relationship between the three measures is shown in Fig. (2.1).

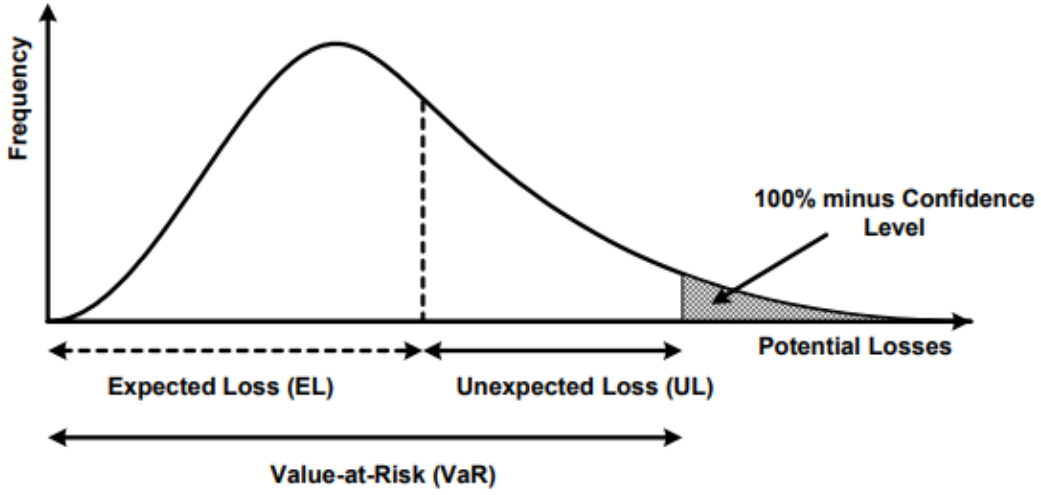


Figure 2.1: Depicting the relationship between EL, UL and VaR (Source: Basel [7])

Expected Shortfall A major critique of the VaR is that three properties presented earlier are not sufficient and that the risk measure used should be a *coherent risk measure*. This means that two new properties are added to list, namely:

4. *Positive homogeneity* If $\alpha \geq 0$ and $Z \in \mathcal{L}$, then $\alpha\rho(Z) = \rho(\alpha Z)$
5. *Sub-additivity* If $Z_1, Z_2 \in \mathcal{L}$ then $\rho(Z_1 + Z_2) \leq \rho(Z_1) + \rho(Z_2)$

Note that VaR fulfills the Positive homogeneity but that it does not fulfill the property of Sub-additivity [15]. If a risk measure does not follow the sub-additive property it may discourage diversification. Consequently, Some argue that instead of VaR a coherent risk measure should be used: Expected Shortfall or ES. This measure has an intuitive interpretation from Fig. (2.1) since it is the expected value of the area in grey. It can be defined both as the expected value conditional on the VaR at z :

$$\text{ES}(X)_z = E[X|X \geq \text{VaR}(X)_z], \quad (2.10)$$

or as the integral:

$$\text{ES}(X)_z = \frac{1}{1-z} \int_1^z \text{VaR}(X)_u du. \quad (2.11)$$

It should be noted that VaR does not necessarily lack sub-additivity but rather there are settings at which it does. The argument for using VaR often points to the difficulty of modeling the tail behavior and thus, ES introduces uncertainty by depending on it. Since both models are based on historical data a lack of data from extreme scenarios may inhibit the ability to estimate the behavior at the tail.

2.3 The Vasicek model

Merton introduced his model of pricing corporate debt and estimating the probability of default in 1974 [16]. Vasicek realized that with slight modifications it could instead be used to model the dependence of default events [17]. Since the single-factor model is a special case of the multi-factor model, only the multi-factor model is explained in this section. In the Vasicek model, the asset value A of obligor i at time t is modeled as multivariate geometric Brownian motion

$$dA_{i,t} = \mu_i A_{i,t} dt + A_{i,t} \sum_{k=1}^m \sigma_{i,k} dW_{k,t} + \eta_i A_{i,t} dB_{i,t}, \quad (2.12)$$

here μ_i , η_i and $\sigma_{i,1}, \dots, \sigma_{i,n}$ are constants which need to be estimated. $W_{1,t}, \dots, W_{m,t}$ and $B_{i,t}$ are mutually independent Wiener processes. $W_{1,t}, \dots, W_{m,t}$ represents the systemic risk factors introduced in Section 2.1, as such $W_{1,t}, \dots, W_{m,t}$ are not unique to each obligor. $B_{i,t}$ represent the idiosyncratic risk that each obligor faces. Björk explains the perhaps counter intuitive assumption that $W_{1,t}, \dots, W_{m,t}$ are mutually independent. After all, it seems natural that there should be some correlation between sectors, whether they represent geographical locations or industries one expects that there should be some dependence. However, if $W_{1,t}, \dots, W_{m,t}$ were correlated it would be possible to rewrite Eq. (2.12) so that they are independent by changing $\sigma_{i,1}, \dots, \sigma_{i,n}$ [18].

Solving the differential equation with $T = 1$ (i.e one year) Eq. (2.12) can be rewritten as

$$A_{i,1} = A_{i,0} \exp \left(\mu_i + \sum_{k=1}^m \left(\sigma_{i,k} X_k - \frac{1}{2} \sigma_{i,k}^2 \right) + \eta_i \epsilon_i - \frac{1}{2} \eta_i^2 \right), \quad (2.13)$$

where X_k and ϵ_i are i.i.d standard normal variables $N(0,1)$. If $A_{i,T}$ is below the value of its liabilities of obligor i then it is assumed to be in default, this threshold is denoted λ_i . Consequently, the probability of default of obligor i :

$$\begin{aligned}
PD_i &= \mathbb{P} \left(A_{i,0} \exp \left(\mu_i + \sum_{k=1}^m \left(\sigma_{i,k} X_k - \frac{1}{2} \sigma_{i,k}^2 \right) + \eta_i \epsilon_i - \frac{1}{2} \eta_i^2 \right) < \lambda_i \right) \\
&= \mathbb{P} \left(\sum_{k=1}^m \sigma_{i,k} X_k + \eta_i \epsilon_i \leq \ln \left(\frac{\lambda_i}{A_{i,0}} \right) + \frac{1}{2} \sum_{k=1}^m \sigma_{i,k}^2 + \frac{1}{2} \eta_i^2 - \mu_i \right) \\
&= \mathbb{P} \left(\frac{\sum_{k=1}^m \sigma_{i,k} X_k + \eta_i \epsilon_i}{\sqrt{\sum_{k=1}^m \sigma_{i,k}^2 + \eta_i^2}} \leq \frac{\ln \left(\frac{\lambda_i}{A_{i,0}} \right) + \frac{1}{2} \sum_{k=1}^m \sigma_{i,k}^2 + \frac{1}{2} \eta_i^2 - \mu_i}{\sqrt{\sum_{k=1}^m \sigma_{i,k}^2 + \eta_i^2}} \right).
\end{aligned} \tag{2.14}$$

Note that since X_k and ϵ_i are i.i.d the dividing factor introduced in the last step means that the probability is standard normal and that the condition for default is on the right-hand side of the last step in Eq. (2.14). Thus it has a clear relation to the PD_i

$$\Phi^{-1}(PD) = \frac{\ln \left(\frac{\lambda_i}{A_{i,0}} \right) + \frac{1}{2} \sum_{k=1}^m \sigma_{i,k}^2 + \frac{1}{2} \eta_i^2 - \mu_i}{\sqrt{\sum_{k=1}^m \sigma_{i,k}^2 + \eta_i^2}}. \tag{2.15}$$

To continue the left-hand side of Eq. (2.14) is rewritten so that it can be interpreted in terms of the correlation to the systemic risk factors. To achieve this the following notation is used

$$\rho_i = \frac{\sum_{k=1}^m \sigma_{i,k}^2}{\sum_{k=1}^m \sigma_{i,k}^2 + \eta_i^2} \quad \text{and} \quad \alpha_{i,k} = \frac{\sigma_{i,k}}{\sqrt{\sum_{k=1}^m \sigma_{i,k}^2}}. \tag{2.16}$$

The choice of ρ is deliberate since it will be shown that this in fact used (in combination with $\alpha_{i,k}$) as the correlation of obligor i to the systemic risk factors. With some clever rewriting it can be shown that

$$\begin{aligned}
\frac{\sum_{k=1}^m \sigma_{i,k} X_k + \eta_i \epsilon_i}{\sqrt{\sum_{k=1}^m \sigma_{i,k}^2 + \eta_i^2}} &= \frac{\sum_{k=1}^m \sigma_{i,k} X_k}{\sqrt{\sum_{k=1}^m \sigma_{i,k}^2 + \eta_i^2}} + \frac{\eta_i \epsilon_i}{\sqrt{\sum_{k=1}^m \sigma_{i,k}^2 + \eta_i^2}} \\
&= \sqrt{\rho_i} \cdot \left(\sqrt{\frac{\sum_{k=1}^m \sigma_{i,k}^2 + \eta_i^2}{\sum_{k=1}^m \sigma_{i,k}^2}} \cdot \frac{\sum_{k=1}^m \sigma_{i,k} X_k}{\sqrt{\sum_{k=1}^m \sigma_{i,k}^2 + \eta_i^2}} \right) + \sqrt{1 - \rho_i} \cdot \epsilon_i \\
&= \sqrt{\rho_i} \cdot \left(\sum_{k=1}^m \alpha_{i,k} X_k \right) + \sqrt{1 - \rho_i} \cdot \epsilon_i \\
&= \sqrt{\rho_i} \cdot \boldsymbol{\alpha}_i^\top \mathbf{X} + \sqrt{1 - \rho_i} \cdot \epsilon_i.
\end{aligned} \tag{2.17}$$

Where the last step introduces the vector notations of X_k and $\alpha_{i,k}$. Using the result from Eq. (2.15) with the notation from Eq. (2.17) it is apparent that obligor i defaults if

$$D_i = 1 \quad \text{if} \quad \sqrt{\rho_i} \cdot \boldsymbol{\alpha}_i^\top \mathbf{X} + \sqrt{1 - \rho_i} \cdot \epsilon_i \leq \Phi^{-1}(PD). \quad (2.18)$$

Some notable properties are

$$\rho_i \in [0, 1], \quad \sum_{k=1}^m \alpha_{i,k}^2 = 1, \quad \sqrt{\rho_i} \cdot \boldsymbol{\alpha}_i^\top \mathbf{X} + \sqrt{1 - \rho_i} \cdot \epsilon_i \sim N(0,1). \quad (2.19)$$

Finally, this condense to a model of the dependent probability of default

$$\begin{aligned} P(D_i = 1 | \mathbf{X} = \mathbf{x}) &= P\left(\sqrt{\rho_i} \cdot \boldsymbol{\alpha}_i^\top \mathbf{x} + \sqrt{1 - \rho_i} \cdot \epsilon_i \leq \Phi^{-1}(PD)\right) \\ &= P\left(\epsilon_i \leq \frac{\Phi^{-1}(PD) - \sqrt{\rho_i} \cdot \boldsymbol{\alpha}_i^\top \mathbf{x}}{\sqrt{1 - \rho_i}}\right) \\ &= \Phi\left(\frac{\Phi^{-1}(PD) - \sqrt{\rho_i} \cdot \boldsymbol{\alpha}_i^\top \mathbf{x}}{\sqrt{1 - \rho_i}}\right). \end{aligned} \quad (2.20)$$

This is known as the Vasicek formula and the single-factor version is used in the current framework. By Eq. (2.19), the single-factor case α_i is equal to 1 and it can then be written as

$$P(D_i = 1 | X = x) = \Phi\left(\frac{\Phi^{-1}(PD) - \sqrt{\rho_i} \cdot x}{\sqrt{1 - \rho_i}}\right). \quad (2.21)$$

The sign within this formula varies from paper to paper depending on if losses are defined as positive values and negative losses (i.e. profits) have a negative value or vice versa. The most common approach, which is used in the IRB model, is to view losses as positive values. For this reason the sign in Eq. (2.20) and (2.21) are flipped to:

$$P(D_i = 1 | \mathbf{X} = -\mathbf{x}) = \Phi\left(\frac{\Phi^{-1}(PD) + \sqrt{\rho_i} \cdot \boldsymbol{\alpha}_i^\top \mathbf{x}}{\sqrt{1 - \rho_i}}\right). \quad (2.22)$$

The factor loadings $\sqrt{\rho_i}$ (seen in Eq. 2.21) represent borrower's i sensitivity to systematic risk X . In the multi-factor case (Eq. 2.22) these show the sensitivity to the sector risk factors.

3 The Internal Ratings-Based Approach

In the currently used IRB model all obligors depend on a single risk factor. Conceptually this is regarded as the "state of the economy" with the reasoning that if this state worsens or improves all obligors are affected to some degree. This is in contrast to the multi-factor models presented in Section 4 where each obligor is dependent on the state of the individual sector of the obligor as well as other sectors.

3.1 The Asymptotic Single Risk Factor Model

The assumptions of the ASRF model are explained in the introduction (Section 1.1). However, to use them in the context of the mathematical model the assumption of an infinitely fine-grained portfolio needs to be more rigorously defined.

Definition 3.1 *A Portfolio is considered infinitely fine-grained if the portfolio consists of close to an infinite number of obligors and satisfies following the conditions*

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \text{EAD}_i \rightarrow \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \sum_{j=1}^n \left(\frac{\text{EAD}_j}{\sum_{i=1}^n \text{EAD}_i} \right)^2 < \infty. \quad (3.1)$$

Given Assumption 1.1 that the portfolio is infinitely granular it can be shown that the loss (L) almost surely converges to the expected loss conditional realization of the systematic risk factor [19].

$$L(n) - E[L(n)|X = x] \rightarrow 0 \quad a.s. \quad (3.2)$$

Consequently, this means that

$$\lim_{n \rightarrow \infty} \text{VaR}_z(L(n)) - \text{VaR}_z(E[L(n)|X = x]) = 0, \quad (3.3)$$

and given Assumption (1.2) of a single risk factor (X) it is shown that [20].

$$\text{VaR}_z(E[L|X = x]) = E[L|X = \text{VaR}_{1-z}(X)]. \quad (3.4)$$

Using the established definition of the Loss variable the Value at Risk is then, in the context of ASRF, found to be:

$$\text{VaR}_z^{\text{IRB}}(L) = \sum_{i=1}^n \text{EAD}_i \cdot \text{LGD}_i \cdot E[D_i|X = \text{VaR}_{1-z}(X)]. \quad (3.5)$$

Since X is a standard normal variable the $\text{VaR}_{1-z}(X)$ is trivial to calculate. Note that $\Phi^{-1}(1-z) = -\Phi^{-1}(z)$ so, as stated in the previous section, the sign is flipped in Eq. (2.22). As a next step, this is translated first to economic capital and then to a capital requirement.

3.2 Economic Capital and Pillar I Capital Requirements

As mentioned in Section 1 the economic capital is the amount of capital a bank would, in theory, hold if there were no regulations. In the framework of the ASRF model, this is named unexpected loss

$$UL^{IRB} = EC^{IRB} = VaR_z^{IRB}(L) - EL. \quad (3.6)$$

Inserting the result from Eq. (3.5) and (2.7) into (3.6) yields

$$EC^{IRB} = \sum_{i=1}^n EAD_i \cdot LGD_i \cdot (E[D_i|X = VaR_{1-z}(X)] - PD_i). \quad (3.7)$$

The risk appetite z is set by Basel to be 0.999. Using the Vasicek formula from Eq. (2.22) (and the fact that the expected value of the indicator function is its probability) the final expression is found to be:

$$EC^{IRB} = \sum_{i=1}^n EAD_i \cdot LGD_i \cdot \left[\Phi \left(\frac{\Phi^{-1}(PD_i) + \sqrt{\rho_i} \cdot \Phi^{-1}(0.999)}{\sqrt{1 - \rho_i}} \right) - PD_i \right]. \quad (3.8)$$

Here the ρ_i is defined (for corporate exposures) to be:

$$\rho_i = 0.12 \cdot \frac{1 - e^{-50 \cdot PD_i}}{1 - e^{-50}} + 0.24 \cdot \left(1 - \frac{1 - e^{-50 \cdot PD_i}}{1 - e^{-50}} \right). \quad (3.9)$$

However, capital requirements are not limited to this, instead, the Risk-Weighted Asset (RWA) is calculated for each obligor. This is done differently deepening on the asset class, for the purpose of this thesis only risk-weighted assets for corporate, sovereign, and bank exposures are relevant*. To calculate the RWA one must first find the aptly named "capital requirement" K by

$$K_i = SF \cdot LGD_i \cdot \left[\Phi \left(\frac{\Phi^{-1}(PD_i) + \sqrt{\rho_i} \cdot \Phi^{-1}(0.999)}{\sqrt{1 - \rho_i}} \right) - PD_i \right] \cdot MA. \quad (3.10)$$

Where the scaling factor (SF) and the maturity adjustment (MA) are

$$SF = 1.06, \quad MA = \frac{1 + (M_i - 2.5)b_i}{(1 - 1.5b_i)}. \quad (3.11)$$

The scaling factor was introduced to maintain the aggregate level of regulatory capital when Basel II was implemented. In the maturity adjustment, M_i is the time to maturity for each asset and b is defined as follows

* These are all affected by the same model.

$$b_i = (0.11852 - 0.05478 \cdot \ln PD_i)^2. \quad (3.12)$$

Note that the effect of the maturity adjustment increases with an increased maturity and decreases with an increase in PD. The reasoning behind more capital needed to be held with longer maturity is intuitive: an asset with longer maturity has a longer time of exposure and thus a longer time frame to default, or more commonly be downgraded on the PD scale. Consequently, an asset with a high PD has less room (potential) to be downgraded and as such does not increase its risk as much by increasing the time to maturity. The constants used in b and the maturity adjustment are derived by the Basel committee using a mark-to-market (MtM) model and the same underlying data as the ASRF model is based on [7].

The capital requirement (K) as laid out in the Framework is expressed as a percentage of the exposure. In order to derive RWA, it must be multiplied by EAD and the reciprocal of the minimum capital ratio of 8%, i.e. by a factor of 12.5

$$RWA_i = 12.5 \cdot K_i \cdot EAD_i. \quad (3.13)$$

Note that RWA is the Unexpected loss for each asset scaled by the reciprocal of the minimum capital ratio, the scaling factor and by the maturity adjustment. The total RWA (the sum of RWA per obligor) is used to determine different capital ratios that are regulated. The Common Equity Tier 1 or CET1 ratio broadly speaking measures the ratio of equity held and the risk-weighted assets. Banks are required to have a CET1 ratio greater than 4.50% by 2019 [21],

$$\frac{\text{CET1}}{\sum_{i=1}^n RWA_i} \geq 4.5\%. \quad (3.14)$$

As such, the RWA of each addition to the credit portfolio is of importance to a bank and therefore the accuracy of the models calculating the RWA is of great importance.

4 Multi-Factor Models

The Vasicek model introduced in Section 2.3 is applied, not only in the IRB model but in multi-factor models as well. As stated, the IRB make use of the special case of a single factor in the Vasicek model. The greatest advantage with this approach is that there is a tractable derivation of the VaR namely, the inverse of the standard normal function. In addition to this, the Swedish FSA argues that there is a greater model risk due to the introduction of sector correlations that are to be estimated [13] (revisited in Section 5). Leaving the argument of model risk for now, there is no simple way of determining the loss distribution of a multi-factor model. The different models in this section explore different ways of circumventing this issue.

The multi-factor Vasicek model shown in Section 2.22 is used throughout the section. However, there are some properties and different ways of writing this formula that was not shown here and is commonly used; equations (4.1), (4.2) and (4.3) are not conflicting with the Vasicek model but rather a rewritten form of it. The concept of inter-correlation and intra-correlation is introduced as well. The intra-correlation is the correlation within the sector that the exposure is in and the inter-correlation is the correlation between sectors (which is not obligor specific). From Eq. (2.17) the Asset value A_i is found to be (in the single factor case):

$$A_i = \sqrt{\rho_{\text{intra},i}} \cdot X + \sqrt{1 - \rho_{\text{intra},i}} \cdot \eta_i, \quad (4.1)$$

and in the general multi-factor case:

$$A_i = \sqrt{\rho_{\text{intra},i}} \cdot X_s + \sqrt{1 - \rho_{\text{intra},i}} \cdot \eta_i \quad \text{where} \quad X_s = \sum_{k=1}^K \alpha_{i(s),k} Z_k. \quad (4.2)$$

These $\alpha_{i(s),k}$ are found from the Cholesky decomposition of the inter-correlation (as seen in Eq. 2.16). Some literature inserts an extra subscript for each sector component of the asset (i.e. $A_{i,s}$) but since every obligor can only be in one sector this is deemed unnecessary. Instead the subscript $i(s)$ denotes the specific sector of obligor i . However, to relax the notation further, outside of this introduction $\alpha_{i(s),k}$ is denoted $\alpha_{i,k}$, observe that the i does not indicate that it takes values from 1 to n but rather that its sector defined by obligor i .

$$\rho_{i,j}^{\text{inter}} = \sum_{k=1}^K \alpha_{i(s),k} \cdot \alpha_{j(s),k} \quad \text{where} \quad \sum_{k=1}^K \alpha_{i(s),k}^2 = 1. \quad (4.3)$$

Consequently, the correlation between two assets is found by:

$$\text{corr}(A_i, A_j) = \begin{cases} \sqrt{\rho_{\text{intra},i}} \sqrt{\rho_{\text{intra},j}} & \text{if } i = j \\ \sqrt{\rho_{\text{intra},i}} \sqrt{\rho_{\text{intra},j}} \sum_{k=1}^K \alpha_{i(s),k} \alpha_{j(s),k} & \text{if } i \neq j, \end{cases} \quad (4.4)$$

where if i and j belong to the same sector the inter-correlation is 1 by Eq. (4.3). Finally, note that the economic capital referred to in this section (and the rest of the

thesis) is found in Eq. (3.6) where the value at risk is calculated by some multi-factor model.

4.1 The Monte Carlo Approach

The most mathematically simple method of approaching the Value at Risk for the loss distribution is a Monte Carlo (MC) simulation. In a MC simulation N portfolios are simulated and sorted by the loss. To find the quantile z of the loss distribution the z :th element is the VaR, at the level of z , for that portfolio (e.g. if $N=100$ and $z = 99\%$ then $\text{VaR}_{1-z}(L)$ is the 2nd largest loss). In order to find the loss of one iteration the asset value A_i for each obligor, i is simulated by drawing Z_k and η_i from the standard normal distribution. Inserting this in Eq. (2.18) it is determined if the indicator variable D_i is 1 or 0. Summation of these losses gives the total loss of that iteration:

$$L_{iter} = \sum_{i=1}^n \text{EAD}_i \cdot \text{LGD}_i \cdot \hat{D}_i, \quad (4.5)$$

where \hat{D}_i is the estimated default variable for obligor i in that specific iteration.

The models presented in the rest of this section either seek to mitigate the computational burden of the MC approach or propose a different framework for measuring risk in the multi-factor case.

4.2 Pykhtin - Analytic Value at Risk

This section introduces a multi-factor derivation of the value at risk. To do this a range of new notations are needed. Assume that all variables sharing notation with what was previously defined is the same. Pykhtin [23] offers a closed form calculation of the VaR to reduce the computational burden of a MC simulation. The general idea is to map the multi-factor model onto a single factor model, namely, the IRB model. To achieve this the systemic risk factors X_s of the multi-factor model are mapped to a single factor \bar{X} . This \bar{X} differs from the X in the IRB model; it is determined by the correlation between the new single factor (\bar{X}) and the old factors (X_s).

$$\bar{X} = \sum_{k=1}^K b_k \cdot X_k \quad \text{and} \quad \bar{L} = \sum_{i=1}^n w_i \cdot \text{LGD}_i \cdot \text{PD}_i(\bar{X}). \quad (4.6)$$

Here $\text{PD}_i(\bar{x})$ make use of the Eq. (2.21) but with the factor loading's of \bar{x} which are explained later on. Additionally, w_i is used instead of EAD_i with no loss of generality. Intuitively this problem of finding an optimal mapping to a single factor can be formulated as a maximization problem since the best choice of the set of b_k is where the correlation between \bar{X} to X_s is maximized over the entire portfolio. In other words, what is achieved by the optimization is replacing the inter-correlation

matrix in the multi-factor model by an average single risk factor by looking at the effect of this change on the entropy of the portfolio. By using the fact that $corr(X_s, \bar{X}) = \sum_{k=1}^K \alpha_{s,k} b_k$ the optimization is set up as:

$$\begin{aligned} & \underset{b_k}{\text{maximize}} \quad \sum_{i=1}^N \theta_i \sum_{k=1}^K \alpha_{i,k} b_k \\ & \text{subject to:} \quad \sum_{k=1}^K b_k^2 = 1. \end{aligned} \quad (4.7)$$

The squares of b_k need to sum to 1 to keep the standard normal properties. Using the Lagrange multiplier λ the solution to this problem is found to be [24]:

$$b_k = \sum_{i=1}^N \alpha_{i,k} \frac{\theta_i}{\lambda} \quad \implies \quad \lambda = \sqrt{\sum_{k=1}^K \left(\sum_{i=1}^N \alpha_{i,k} \theta_i \right)^2}. \quad (4.8)$$

Note that, as explained in Section (4), the inter-correlation matrix is the same for all obligors i so the $\alpha_{i,k}$ denotes index for the sector of obligor i and sector k . Since there is no unique solution to (4.8) Pykhtin proposed (after experimental testing) to use

$$\theta_i = w_i \cdot \text{LGD}_i \cdot \Phi \left(\frac{\Phi^{-1}(\text{PD}_i) + \sqrt{\rho_{\text{intra},i}} \cdot \Phi^{-1}(z)}{\sqrt{1 - \rho_{\text{intra},i}}} \right). \quad (4.9)$$

Where z is the confidence level. Furthermore, Pykhtin introduced the notation p_i as the factor loadings in the multi-factor case when correlation is between $corr(X_i, \bar{X})$ (in contrast to $corr(X_i, X_j)$ which gives the factor loading $\sqrt{\rho_{\text{intra},i}}$). Inserting \bar{X} in Eq. (4.2) and calculating the new factor loading gives:

$$A_i = p_i \bar{X} + \sqrt{1 - p_i^2} \cdot \eta_i, \quad p_i = \sqrt{\rho_{\text{intra},i}} \sum_{k=1}^K \alpha_{i,k} b_k. \quad (4.10)$$

The next step is to use this newly defined A_i and find an algebraic expression for the value at risk for the Loss L. To do this the VaR is approximated by the second order Taylor series around the solution for the IRB model, which loss is denoted \bar{L} . Additionally, the permutation U is introduced and defined as the difference between the multi-factor loss and the IRB loss (i.e. $U = L - \bar{L}$). Finally, to relax notation q_z is set to be the quantile (VaR) at level z :

$$q_z(L) \approx q_z(\bar{L}) + \left. \frac{dq_z(\bar{L} + \epsilon \cdot \bar{U})}{d\epsilon} \right|_{\epsilon=0} + \frac{1}{2} \cdot \left. \frac{dq_z^2(\bar{L} + \epsilon \cdot \bar{U})}{d\epsilon^2} \right|_{\epsilon=0} + \mathcal{O}(\epsilon^3). \quad (4.11)$$

It can be shown that the first order derivative in Eq. (4.11) is equal to zero [26]. Hence, the so-called multi-factor adjustment is explained by the second order derivative. This is rewritten as:

$$q_z(\mathbf{L}) \approx q_z(\bar{\mathbf{L}}) + \Delta q_z, \quad (4.12)$$

where Δq_z is the difference of the second order derivative

$$\Delta q_z = q_z(\mathbf{L}) - q_z(\bar{\mathbf{L}}) \approx -\frac{1}{2l'(\bar{x})} \cdot \left[v'(\bar{x}) - v(\bar{x}) \cdot \left(\frac{l''(\bar{x})}{l'\bar{x}} + \bar{x} \right) \right] \Big|_{\bar{x}=\Phi^{-1}(1-z)}. \quad (4.13)$$

Recall that there is $\Phi^{-1}(1-z)$ is the value at risk at level z in the single factor model and that $\bar{\mathbf{L}}$ is a single factor model. l' and l'' is the first and second order derivative of $\bar{\mathbf{L}}$ and v is the conditional variance of \mathbf{L} (and v' its derivative). The derivatives of l' and l'' are trivially found when the derivative of PD is known*. This variance of \mathbf{L} can be divided into two parts representing the systematic risk adjustment and the granularity adjustment (name-concentration) by making use of the following property:

$$\underbrace{Var[\mathbf{L}|\bar{X} = x]}_{v(\bar{x})} = \underbrace{Var[E[\mathbf{L}|X_s]|\bar{X} = x]}_{\text{Systematic}(\sigma_\infty^2)} + \underbrace{E[Var[\mathbf{L}|X_s]|\bar{X} = x]}_{\text{Granularity Adjustment}(\sigma_{GA}^2)}. \quad (4.14)$$

By the linearity of Eq. (4.13) these two terms (and their derivatives) can be calculated separately and then used to finally find an algebraic expression for $q_z(\mathbf{L})$. After some tedious calculation and by introducing some new notations, they are found to be:

$$\sigma_\infty^2 = \sum_{i=1}^n \sum_{j=1}^n w_i w_j \text{LGD}_i \text{LGD}_j \cdot \left[\Phi_2 \left(\Phi^{-1}(\text{PD}_i(\bar{x})), \Phi^{-1}(\text{PD}_j(\bar{x})), \rho_{i,j}^{\bar{X}} \right) - \text{PD}_i(\bar{x}) \text{PD}_j(\bar{x}) \right], \quad (4.15)$$

$$\frac{d}{dx} \sigma_\infty^2 = \sum_{i=1}^n \sum_{j=1}^n 2w_i w_j \text{LGD}_i \text{LGD}_j \text{PD}'_i(\bar{x}) \cdot \left[\Phi \left(\frac{\Phi^{-1}(\text{PD}_j(\bar{x})) - \rho_{i,j}^{\bar{X}} \Phi^{-1}(\text{PD}_i(\bar{x}))}{\sqrt{1 - (\rho_{i,j}^{\bar{X}})^2}} \right) - \text{PD}_j(\bar{x}) \right], \quad (4.16)$$

$$\sigma_{GA}^2 = \sum_{i=1}^n w_i^2 \left[\text{LGD}_i^2 \cdot \left[\text{PD}_i(\bar{x}) - \Phi_2 \left(\Phi^{-1}(\text{PD}_i(\bar{x})), \Phi^{-1}(\text{PD}_i(\bar{x})), \rho_{i,i}^{\bar{X}} \right) \right] \right], \quad (4.17)$$

$$\frac{d}{dx} \sigma_{GA}^2 = \sum_{i=1}^n w_i^2 \cdot \text{LGD}_i^2 \cdot \text{PD}'_i(\bar{x}) \cdot \left[1 - 2\Phi \left(\frac{\Phi^{-1}(\text{PD}_i(\bar{x})) - \rho_{i,i}^{\bar{X}} \Phi^{-1}(\text{PD}_i(\bar{x}))}{\sqrt{1 - (\rho_{i,i}^{\bar{X}})^2}} \right) \right]. \quad (4.18)$$

* $l' = \sum_{i=1}^n w_i \cdot \text{LGD}_i \cdot \text{PD}(\bar{x})'_i$ and $l'' = \sum_{i=1}^n w_i \cdot \text{LGD}_i \cdot \text{PD}(\bar{x})''_i$.

Here $\Phi_2(\cdot)$ is the bivariate normal distribution. To find the derivative of probability of default component ($\text{PD}'_i(x)$) the derivative of the standard normal distribution is used (i.e. $\frac{d}{dx}\Phi\left(\frac{\mu-x}{\sigma}\right) = -\frac{1}{\sigma}\phi(x)$). For the conditional asset correlation, $\rho_{i,j}^{\bar{X}}$ recall the factor loadings shown in earlier (4.10), using these in Eq. (4.2) and it yields the following three expressions:

$$\text{PD}'_i(\bar{x}) = -\frac{p_i}{\sqrt{1-p_i^2}}\phi\left(\frac{\Phi^{-1}(\text{PD}_i) - p_i\bar{x}}{\sqrt{1-p_i^2}}\right), \quad (4.19)$$

$$\text{PD}''_i(\bar{x}) = -\frac{p_i^2}{1-p_i^2} \cdot \frac{\Phi^{-1}(\text{PD}_i) - p_i\bar{x}}{\sqrt{1-p_i^2}} \cdot \phi\left(\frac{\Phi^{-1}(\text{PD}_i) - p_i\bar{x}}{\sqrt{1-p_i^2}}\right), \quad (4.20)$$

$$\rho_{i,j}^{\bar{X}} = \frac{\sqrt{\rho_{\text{intra},i}}\sqrt{\rho_{\text{intra},j}}\sum_{k=1}^K\alpha_{i,k}\alpha_{j,k} - p_i p_j}{\sqrt{(1-p_i^2)(1-p_j^2)}}. \quad (4.21)$$

Since $\text{PD}'_i(x)$ and $\text{PD}''_i(x)$ are known inserting these results in (4.13) gives an expression for the value at risk at level z for the multi-factor model by $\Delta q_z = \Delta q_z^\infty + \Delta q_z^{\text{GA}}$. This is then inserted into (4.12) to finally arrive at:

$$q_z(\text{L}) = q_z(\bar{\text{L}}) + \Delta q_z^\infty + \Delta q_z^{\text{GA}}. \quad (4.22)$$

4.3 Cespedes - Diversification Factor

Cespedes et al. [25] introduced a framework dependent on a diversification factor which is the connection between the economic capital of the single factor model and the multi-factor model. This factor takes in the accumulated differences of a range of portfolios to find a suitable level not just for one portfolio, as is the case in previously presented models but for all of them. This does, however, make the model very computationally heavy although in principle it should not be calculated as often (or maybe even just once) as it factors in different scenarios and thus should not change when the portfolio does. The diversification factor for a given set of portfolios needs to be able to be explained by some parameters that reflect the riskiness of the allocation. When this relationship is modeled the aim is to use this estimated diversification factor for any new portfolio. This means that only the single factor model need to be calculated when the relation between the two parameters and DF is known.

The diversification factor for one portfolio is defined as:

$$\text{DF} = \frac{\text{EC}^{\text{mf}}}{\text{EC}^{\text{sf}}}, \quad \text{DF} \leq 1 \quad \text{where} \quad \text{EC}^{\text{sf}} = \sum_{k=1}^K \text{EC}_k. \quad (4.23)$$

where EC^{mf} and EC^{sf} is the multi-factor EC and single factor EC respectively. EC_k is the economic capital attributed to each sector k using the single factor formula.

For a given percentile level (eg. $\alpha = 0.01\%$ used in the IRB model) the EC^{mf} is found by

$$EC^{mf}(\alpha; \cdot) \approx DF(\alpha; \cdot) \cdot EC^{sf}(\alpha). \quad (4.24)$$

$DF(\alpha; \cdot)$ is a scalar function with a small number of parameters. With this setup, the diversified capital is the product of the “additive” bottoms-up capital from a one-factor model. The authors go on to show that this factor can be estimated using two parameters:

1. Average cross-sector correlation, β
2. Sector concentration, CDI

They argue that these two broadly capture the effect of diversification. From 1. the correlation between sectors affect the efficiency of the diversification, if two sectors were perfectly correlated then there would be no diversification effect by allocating capital to both. This sector correlation β is the correlation of the systemic risk factors A_i in Eq. (4.2). To find this average the authors suggest using the following*

$$\beta = \frac{\sum_{k=1}^K \sum_{i \neq j} \rho_{i,j}^{inter} \cdot EC_k \cdot EC_i}{\sum_{k=1}^K \sum_{i \neq j} EC_k \cdot EC_i}. \quad (4.25)$$

By 2. there should be a parameter that captures the effect of allocating much capital to one sector with lower risk should not be punished, even if it is in the "naive way" of diversifying. Furthermore, the size of each exposure is captured in the parameter. The capital diversification index, CDI, is defined as the sum of squares of the SF capital weights in each sector

$$CDI = \frac{\sum_{k=1}^K (EC_k)^2}{(EC^{sf})^2} = \sum_{k=1}^K w_k^2. \quad (4.26)$$

Conceptually, w_k is the contribution to the single factor capital of sector k . This is analogous to the HHI and exactly the same as the RI index presented further on. Applying these parameters in Eq. (4.24):

$$EC^{mf}(\alpha; CDI, \beta) \approx DF(\alpha; CDI, \beta) \cdot EC^{sf}(\alpha). \quad (4.27)$$

Using equations (4.23), (4.25) and (4.26) a value of DF , β and CDI respectively is obtained for each simulated portfolio. Viewing this as a 3D surface (depicted in Fig. 7.4) a parameterization of DF is obtained by fitting it as follows:

$$DF = a_0 + a_1 \cdot (1 - CDI) \cdot (1 - \beta) + a_2 \cdot (1 - CDI)^2 \cdot (1 - \beta) + a_3 \cdot (1 - CDI) \cdot (1 - \beta)^2. \quad (4.28)$$

*to clarify it is the single factor economic capital that is used calculate β and CDI

Where $DF = DF(\alpha; CDI, \beta)$. Using this to estimate a_0, a_1, a_2 and a_3 the relationship between CDI, β and DF is found (as suggested by [26] [25]). In the framework presented by Cespedes et al. a_0 is set to 1^* since the diversification factor is bound by the single factor model and that the diversification factor can only lower the economic capital for the multi-factor model. In other words, there should be no values for CDI and β where DF is greater than 1.

Gürtler et al. [26] proposed that there should not be an upper barrier on risk set by the single factor model. The economic capital of a multi-factor model should be increased if it has a relatively risky allocation and conversely it should decrease if the allocation is not as risky. By changing the framework so that this holds they too calculate a diversification factor. Since there are no differences in the mathematical derivation of this factor (aside from the constraints on a_0 in DF) the model in its entirety is not reiterated in this section. To clarify, there is still a theoretical bound on DF by the multi-factor model being bound by the single factor but it is no longer bound by the fitting in the DF framework. In the next section, a proposal for circumventing the theoretical bound on the multi-factor model is shown.

This method is computational heavy, Gürtler et al. propose using 25000 portfolios for calculation of the multi-factor and single factor economic capital and with 100000 iterations for each MC simulation. They estimate the computational time to 30 days. This can be lowered to under a day using the Pykhtin method for value at risk instead of MC simulations.

4.4 Gürtler - Implied Correlation

To remove this upper barrier Gürtler et al. propose using a "relatively well diversified" portfolio to determine whether the multi-factor model should push the economic capital up or down (compared to the single factor). The obvious problem with this approach is defining what these should be relative to. There is no clear intuition on what this benchmark should reasonably be. It should be noted that if the data used in the Basel calculations were publicly available such a benchmark portfolio could be constructed since they implicitly defined it with the capital requirement. Nevertheless, the authors propose a definition of "implicit intra-sector correlation" $\rho_{intra}^{implied}$ found by:

$$EC^{mf}(\rho_{inter}, \rho_{intra}^{implied}) = UL^{ASRF}(\rho_{Basel}). \quad (4.29)$$

Here ρ_{Basel} is the correlation used for corporate exposures under the basel framework (see Eq. (3.9)). Analogously with this equation the authors propose re-configuring the scalars in the equation by:

$$\rho_{intra}^{implied} = a_0 \cdot \frac{1 - e^{-50 \cdot PD_i}}{1 - e^{-50}} + a_1 \cdot \left(1 - \frac{1 - e^{-50 \cdot PD_i}}{1 - e^{-50}}\right). \quad (4.30)$$

* This is achieved by using the constraints $DF(\alpha; CDI, \beta = 1) = 1$ and $DF(\alpha; CDI = 1, \beta) = 1$.

They go on to propose that these a_o and a_1 could be found using a grid search procedure [26]. They find that $a_o = 0.185$ and $a_1 = 0.34$ are suitable replacements. As it is not the purpose of this thesis to find a generalized solution to this problem but rather to find a fit to the data available the parameters are recalculated (fitted) to the data used in this thesis by using said grid search algorithm.

4.5 IMF - A Hybrid Approach

The International Monetary Fund (IMF) [12] propose a partial portfolio approach (PPA) where the aim is to minimize the computational time while maintaining accuracy. The model is based on the intuition that for smaller exposures the impact of the change of framework from analytical multi-factor to crude MC multi-factor is negligible. This is because this "granular" sub-portfolio is assumed to be diversified on name and sector level. Thus the IRB model is used for these exposures. The other sub-portfolio called the "non-granular" consists of exposures larger than some value m (where $m \ll n$) and for these obligors, a MC-simulation is used to find their contribution to the economic capital. Since both models are already presented they are not reiterated in this section.

4.6 Method Comparison

The models presented in this section are divided in their aim. The obvious outlier is the diversification factor model which relies on a number of either MC or analytical calculations of the economic capital and is designed to use these to accurately find the economic capital of a generic portfolio. In other words, the aim of the model is to reduce the computational burden (long-term) given an established multi-factor model.

In this section three different models for calculation, the economic capital using a multi-factor model was presented. The main advantage of the Pykhtin and IMF models is that the computational burden is significantly reduced. Additionally, the Pykhtin model is (can be) consistent with the assumption of no name-concentration. Importantly, the MC simulation has the major advantage of converging to the correct answer while the other two are estimations. Although using the term correctly here could be an overstatement as it does include name-concentration and the currently used IRB model does not.

5 Capital Requirements - The Pillar II Add-on

Both the scaling factor and the maturity adjustment are fitted to the single-factor model and there are no multi-factor alternatives. While both realistically have to be recalculated if a new multi-factor model was to be used, by assuming they keep their values some insight into (roughly) the changes in the capital requirement are found. In an academic sense, this does, however, offer little more than comparing the economic capital since the difference is just scaled by the same factor(s). Because of this the economic capital is used from both methods when comparing them in Section 7. Having said that, this method does offer an estimate of the difference to capital held in real terms.

Since there is no clear-cut way of moving from EC to regulatory capital the Swedish FSA has developed an internal method for sector and name-concentration risk respectively (see Section 1.2). While the other methods presented in this section point to the disadvantages of the Herfindahl–Hirschman Index currently in use the Swedish FSA points to the model risk that comes with a more advanced model:

The alternative methods available for assessing such risks are significantly more complicated and require, among other things, assumptions of correlations between industries and geographical areas. Correlations are difficult to estimate and there is often a high variance in correlation estimates. It is also difficult to validate correlation assumptions, and as a rule, the outcome of the model is influenced to a high degree by the correlation assumptions made. - Swedish FSA (2014) [13] (page 16-17)

Under pillar II of the Basel accords the risk of name and sector concentration must be addressed. However, banks are able to propose an internal methodology to replace methodology used granted that it is approved by the local authorities. The next two sections discuss different methods of calculating the regulatory add-on under Pillar II.

The indices presented in this section addresses the weakness of HHI that all sectors are treated as equal, even if it is known that some are riskier than others. Using HHI the optimal strategy, in terms of reducing the capital requirement, is to diversify the portfolio so that it has an equal amount of exposure in each sector (which is trivially proven). This is of course not representative of any actual collection of sectors although it has the advantage of not (wrongly) assuming any sector to be lower risk and thus placing more of the portfolio in that particular one.

5.1 The Concentration Index

The Concentration Index (CI) is an alternative to the Herfindahl–Hirschman Index proposed by Cabedo et al. [27]. Assuming that one can identify the covariance (or correlation) structure between these sectors (i.e. their riskiness) a measurement that encaptures this is an improvement on HHI.

To see the direct similarities between the two indexes note that HHI can be written in matrix form as follows:

$$\text{HHI} = S^T \cdot I \cdot S. \quad (5.1)$$

here S is the vector with each component s_k and I is the identity matrix. CI is similarly defined as $\text{CI} = S^T \cdot \text{VCM} \cdot S$ where VCM is the following matrix:

$$\text{VCM} = \begin{bmatrix} \hat{\sigma}_1^2 & \hat{\sigma}_{21} & \hat{\sigma}_{31} & \dots & \hat{\sigma}_{k1} \\ \hat{\sigma}_{12} & \hat{\sigma}_2^2 & \hat{\sigma}_{32} & \dots & \hat{\sigma}_{k2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{\sigma}_{1k} & \hat{\sigma}_{2k} & \hat{\sigma}_{3k} & \dots & \hat{\sigma}_k^2 \end{bmatrix} \quad (5.2)$$

and where $\hat{\sigma}_i^2$ and $\hat{\sigma}_{ij}$ are defined as

$$\hat{\sigma}_i^2 = \frac{\sigma_i^2}{\max_{l=1}^k(\sigma_l^2)} \quad \text{and} \quad \hat{\sigma}_{ij} = \max \left[0; \frac{\sigma_{ij}}{\max_{l=1}^k(\sigma_l^2)} \right]. \quad (5.3)$$

Here σ_{ij} is the covariance between i and j , since k is the number of sectors the divider is the maximum variance of all sectors and it is the same for each sector i . By this definition, any negative covariance is excluded. The authors explain this property by arguing that negative correlations should be nullified because the aim is not a full quantification of portfolio risk, but rather, a quantification of risk-weighted factors. Furthermore, HHI is a special case of CI where there are no correlations between sectors and the variance is equal to one for all sectors.

There are some noteworthy properties of the CI index (that are all proven in the paper by Cabedo et al):

Property 5.1 *The maximum of CI is when the entire portfolio is allocated in the sector with the maximum variance*

Property 5.2 *When introducing exposures in a new sector into a portfolio, the index will decrease only when the risk (variance) of the new sector is lower than the portfolio's original variance.*

Property 5.3 *CI is downwards bounded. In a scenario where sectors are uncorrelated and every change in the fraction invested in a sector involves a decrease in the index. For investments in a large enough number of sectors, CI will tend to 0.*

The third property is shared with the HHI which also tends to 0 for a large number of sectors. Since a common structure makes use of around 10-20 sectors the third property is perhaps not as important as the other two. The second property ensures that the insertion of a relatively (to the portfolio held) risky sector in the portfolio cannot lower the risk. This is also not relevant to this thesis as the number of sectors is fixed. Although, in the case where negative covariance is allowed the insertion of a new exposure could then be seen as negating the risk of other exposures. The first property is similar to the property of HHI with the critical difference that in the case of HHI which sector is chosen does not matter when 100% of the portfolio is allocated there, in the CI case each sector gives a scale on which sectors are worse in terms of variance and covariance, i.e. risk.

In the context of this thesis, CI has one fatal flaw, it does not yield a clear-cut view on how each sector contributes to its value. To remedy this I propose that it is redefined. In the case of HHI, the S from Eq. 5.1 is (as was already explained) the concentration of exposures expressed as the percentage of the total exposure. To be able to compare these models on a sector level and not a portfolio level CI is changed. By taking $VCM \cdot S$ as the risk-weighted exposure per sector and normalizing it with the total sum of risk-weighted exposure. This gives an expression that is similar to S and further this yields a new expression for CI:

$$\hat{CI} = \hat{C}^T \cdot I \cdot \hat{C} \quad \text{where} \quad \hat{C} = \frac{VCM \cdot S}{\mathbf{1}^T \cdot VCM \cdot S}. \quad (5.4)$$

Here $\mathbf{1}^T$ is a vector of ones. This \hat{CI} is comparable in the context of exposure per sector as well as the exposure of the total portfolio. There is, of course, no need to normalize VCM when using this adjusted CI but it does not change the outcome either way.

5.2 PD dependent sector concentration

Lefcaditis and Tsamis[28] argue that the add-on from sector concentration should be dependent on the probability of default as a heavy concentration in a sector with a, on average, high PD should be considered worse than one with lower average PD. To capture this effect they introduce the S^2 measurement:

$$S^2 = \frac{\sum_{k=1}^K \hat{S}_i^2}{\left(\sum_{k=1}^K \hat{S}_i\right)^2} \quad \text{where} \quad \hat{S}_i = S_i \cdot \overline{PD}_i \cdot (1 - \overline{PD}_i), \quad (5.5)$$

where \overline{PD}_i is the weighted average PD of sector i . One might think that an add-on should include both the PD and the inter-sector correlation but as Gürtler et al. [26] points out under the pillar II framework these are assumed to be interdependent. Combining these two will, therefore, produce an inconsistent sector concentration measure.

Concerning the properties of this measurement, it is similar to HHI. Exposure in sectors with a high PD is proportionally scaled up making allocating exposure to these more expensive compared to using the HHI measurement. Notably $\overline{PD}_i \cdot (1 - \overline{PD}_i)$ is not strictly increasing and has a maximum at $\overline{PD}_i = 0.5$. Consequently, a sector with an average PD of a number close to 1 (e.g. $a = 0.95$) will (all else equal) be treated the same as a sector with an average PD of $1 - a$. This is neglected due to the fact that no sector would realistically have an average PD greater than 50%.

5.3 Capital Add-on Alternatives

Analogously with authors of CI and S² I argue that only the using the exposure weight does not encapture the underlying risk structure of the sectors. Ultimately one would like a measurement that took all risk factors into account. CI and S² account for sector correlations and PD respectively but neither account for LGD. To this point, the sum of Risk-Weighted Assets (RWA) for each sector could be used in replacement of the sum of EAD. For notation purposes this is called RI and defined as:

$$RI = \sum_{k=1}^K q_k^2 \quad \text{where} \quad q_k = \frac{\sum_{i \in k} RWA_i}{\sum_{j=1}^n RWA_j}. \quad (5.6)$$

This proposal introduces a tractable way of relating the economic capital found to the capital requirement while not straying from models already in use. It should be noted that the Swedish FSA considered using the single factor RWA instead of EAD. They concluded that the model risk was in this case higher. This claim is not, however, backed up by any (shown) data and the multi-factor case was not considered*.

One way of circumventing the inconsistency of an add-on sensitive to both PD, LGD and the correlation of sectors is then to use the multi-factor RWA instead of EAD in the HHI measurement. Although, as stated earlier in the section there is no clear way to relate the multi-factor EC to RWA. To address this I propose to use the Economic Capital of the multi-factor model instead.

$$EI = \sum_{k=1}^K q_k^2 \quad \text{where} \quad q_k = \frac{EC_k}{\sum_{i=1}^K EC_i}. \quad (5.7)$$

As mentioned, if the scaling in RWA is neglected the proportion of EC and RWA in each sector is exactly the same this would also represent the risk-weighted exposure in each sector. This then leaves the problem of estimating EC_k for each sector k . It could be calculated using the analytical derivation of the multi-factor model. When the multi-factor model is mapped to a single factor it is made dependent on the inter-sector correlations and as mentioned it is already dependent on PD and LGD. This single factor model can then be divided into each sector analogously with how it is done using the IRB single factor. The issue using this method is allocating quantiles of the sector and name-concentrations (Δq_z^∞ and Δq_z^{GA}). From the equations for the granularity adjustment Eq. (4.17) and (4.18) it is clear that this quantile could be divided into sectors by using the sector of each obligor i and grouping these. This can be done similarly for equations (4.15) and (4.16) where the sector of obligor i places the partial sum under the associated sector. Using this the EC_k can be rewritten as:

* It is not explicitly stated that a multi-factor RWA was not considered. This is concluded from the fact that they (SFSA) do explicitly state that they did not consider multi-factor models, in general, it is then assumed that when speaking about RWA they refer to the single factor case.

$$\text{EC}_k = q_{z,k}(\bar{L}) + \Delta q_{z,k}^\infty + \Delta q_{z,k}^{\text{GA}} - \text{EL}_k. \quad (5.8)$$

Although it should be noted that the granularity adjustment is assumed to be 0 in Section 7.

The MC approach has no clear interpretation as to what the VaR (and consequently the EC) is for each sector. It could be seen as the allocation of capital found by dividing the loss in the z :th scenario so that the loss from obligors that defaulted under this scenario are ascribed to their respective sectors. This, however, may not give an accurate result as the sector factors are generated for each iteration and a large loss may depend mainly on a combination of sectors (see Eq. 4.2) that under-perform in this scenario. This could change to another combination of sectors moving just a step up or down in the list of sorted losses. Without referring to the equations one can think of this as that there is no one scenario that is representative of the z :th worst case and the loss attributed to this scenario could be similar to the total loss of all sectors to another scenario while completely different in which sectors this loss is found in. What is wanted is then a sort of expected VaR for each sector in the z :th scenario. A natural way of estimating this expected scenario could then be to redo the simulation a sufficient number of times and find the average distribution of loss per sector across all simulations. This would, however, require much in terms of computing power. Recall that in these capital methods the concentration of exposure is used and the actual value of the exposure (or in this case loss) is not relevant. If the aim is to answer the question of how the loss is distributed in a worst case scenario then this could be done using a different approach. Similarly to expected shortfall, one could use the average of the distribution of loss in each sector in the scenarios that have a total loss greater than the z :th scenario.

$$\hat{\text{EC}}_k = \frac{1}{n_{\mathbf{z}}} \sum_{i>z}^M \text{VaR}_{i,k} - \text{EL}_k. \quad (5.9)$$

Here M denote the number of scenarios in the MC simulation. $n_{\mathbf{z}}$ is the number of scenarios that have a loss greater than or equal to z . $\text{VaR}_{i,k}$ is sum of all obligors contribution to the value at risk for sector k in scenario i . EL_k is the sum of expected loss for all obligors within sector k . By the definition presented in Section 2.2 this measurement is not the unexpected loss and thus not the economic capital (which is why it is denoted as $\hat{\text{EC}}_k$). In the single factor case Grtler et al. [26] show that expected shortfall can be scaled so that it is theoretically the same as the value at risk (although more stable) but there is not a clear way to achieve this in the multi-factor case*. Using formula (5.9) as is would yield a slightly greater dependence on the ES term since it is naturally always greater than the VaR and since the EL is not scaled up accordingly this would skew the results. As was already mentioned the actual values of "Economic capital" would not be used, only their division between sectors which could result in this effect being negligible. To determine if this is accurate it will be compared to the accurate but more computational heavy method of determining the average VaR at level z using a bootstrap method. If a bootstrap

* Within the Pykhtin framework it is possible to derive an expected shortfall but this defeats the purpose since (as is shown) so is the value at risk.

method was not used the computational time would be unfeasible long. In short, the bootstrap algorithm is done in the following way:

1. Generate N loss scenarios on sector level from the sample portfolio.
2. Randomly select M of these scenarios and sort them by the sum of the loss on a sector level
3. Select the z :th loss scenario as the VaR
4. Repeat from step 2. k times and use the average sector specific VaR

The specifics of this method is shown in Section 7.2.

To make clear which of these methods is referred to the use of MC simulations and the analytical approach to finding EI is denoted EI MC and EI PY respectively.

5.4 Method Comparison

To summarize this section, 6 different methods of calculating a capital add-on were presented. The HHI methodology is the one currently in use and will serve as a benchmark for the performance of the other methods. In table 5.1 the dependencies of input variables of each model is shown. In essence, all methods strive to punish a high allocation of exposure to one sector while every method, apart from HHI, includes some more complexity (i.e. risk) into the model.

Method	Exposure (EAD)	PD	Sector corr.	LGD
HHI	✓			
\mathbf{S}^2	✓	✓		
CI	✓		✓	
RI	✓	✓		✓
EI MC	✓	✓	✓	✓
EI PY	✓	✓	✓	✓

Table 5.1: Methods dependency on input variables

In the context of evaluating different capital add-ons some measurements could arguably give meaningful results using a single portfolio (i.e. not comparing between portfolios). If one thinks of the calculations done by the Swedish FSA as the true converter of exposure per sector to a capital requirement one could make the case that using the methods presented in Section 5.3 should fit in the same model (i.e. no new estimation of parameters is needed). These models strive to provide a more accurate view of the exposure by weighing in other factors and thus, in some sense of the word, represent the true exposure per sector (in theory that is). This risk-weighted exposure changes the model so that it is no longer optimal to allocate capital evenly between sectors, the optimal strategy is to allocate the risk-weighted capital evenly. As such, it is of interest how these measurements perform within the framework presented by the Swedish FSA. Nevertheless, the opposite argument could be made as well, that these models are in fact not directly comparable to HHI in this framework. It then follows that their performance must be evaluated on the

change in their values for a range of portfolios since they cannot be converted to a capital add-on. While this is not done in this thesis the models are evaluated with different input intra- and inter-correlations.

6 Data Selection and Model Application

This section covers the selection process of the sample portfolio used as well as the correlation matrix created to match the sectors of this portfolio. Additionally, ways to lower the computational burden when applying the models is presented. Lastly, how to nullify name-concentration is discussed.

6.1 Portfolio selection

Analogously with similar studies, the portfolio size of 10 000 exposure was chosen in this thesis. These were drawn independently from a larger set of exposures and with some restrictions on size and handling of obligors with multiple exposures in order to make the model fit the requirement of one exposure per obligor. To keep the data anonymous only the concentration of exposure per sector is shown but some broad characteristics of each sector is discussed when they are compared in Section 7.2.3. There are 13 sectors used in this thesis and their concentration is as follows:

#	Sector Name	Concentration
1	A	10.1%
2	B	5.1%
3	C	2.8%
4	D	0.4%
5	E	3.7%
6	F	9.4%
7	G	3.6%
8	H	4.8%
9	I	39.8%
10	J	12.6%
11	K	3.3%
12	L	3.5%
13	M	0.9%
		100%

Table 6.1: Sector concentration in terms of exposure

The argument could be made that the results should show the models applied to a multitude of portfolios. Given that the characteristic of the sample portfolio is already altered by different correlation matrices and intra-correlations this is set to be outside the scope and only one sample portfolio is used. To clarify, Section 7.1.3 which presents the result of the DF calculations stands out from the rest of the results as they do not stem from a model designed to have the input as a single portfolio but rather a multitude of them. To avoid confusion the reasoning for portfolio selection, in this case, is presented in that section.

6.2 Model Application

The estimation of the characteristic of the portfolios is convoluted, to say the least. In the context of this thesis the complexity of the calculations of PD, EAD and LGD are all neglected and these values are taken at face value. The intra-sector correlation is provided in the IRB framework* which leaves only the inter-sector correlation to be estimated.

To estimate the correlation of default rates between sectors a natural first thought might be to use historical data. Per usual when measuring defaults there is a lack of data. In the context of capital requirements, severe economic downturns are most interesting but these are few in numbers and provide a poor basis for statistical analysis. Consequently, these historical default rates are often not preferred compared to using equity correlations as a proxy for default rates. This is identical to the reasoning behind the Vasicek model where it is assumed that the chance of a default is directly dependent on the equity of a company. Assuming that this model holds these should then be interchangeable. This connection has been shown by Düllmann et al. [29].

It is outside the scope of this thesis to try to find an estimate of these correlations. Instead, the correlation matrix produced by Moody's and SFSA sector definitions found in [30] is used. Since these sectors are not exactly the same as the sectors used I attempt to match these as well as possible and use the matrix as a benchmark when filling out missing values. Lastly, since this is not a calculated correlation matrix it may not be positive-definite (which a true correlation matrix should be). To remedy this Higham's algorithm [31] is used to find the nearest true correlation matrix. The resulting inter-correlation matrix is presented in Section 7.

6.2.1 Analytic Calculations of Value at Risk

Although it is not explicitly stated, the calculations of Pykhtin requires a small set of credit scores (PD) to be applied to the portfolio that was used. This is due to the matrix dimension of the calculations. Assume that in a portfolio of size n that each obligor has a unique probability of default then equations (4.15) and (4.16) require each combination of these obligors evaluated. While there is some symmetry, namely $w_i \cdot w_j = w_j \cdot w_i$ this takes the calculations from $n \times n$ to n over 2. For a portfolio of 10 000 obligors (as is used in this thesis) this means 49 995 000 combinations. While this could be done, it is computationally much heavier than the MC simulation and in that it becomes virtually useless as the argument for the analytic calculation is that it sacrifices some accuracy for better computational time.

If the credit scores (PDs) are not unique then the combinations can be drastically reduced. Consider a case where there are S sectors $\{S_1, S_2, \dots, S_S\}$ and M credit scores $\{PD_1, PD_2, \dots, PD_M\}$ and there is at least one obligor with each credit score in every sector. It is then, under the framework of Pykhtin and under the assumption that the intra-correlation analogously with the Basel framework dependent only on PD, accurate to think of each group of sectors and credit scores (e.g. PD_i and S_j) as

* Except for in Gürtler's implied correlation model

one larger obligor. It is then a matter of calculating the loss given default in terms of size of the group by $\sum_{i \in S \times M} w_i \cdot \text{LGD}_i$ (used in equations 4.15 and 4.16) and their squares $\sum_{i \in S \times M} w_i^2 \cdot \text{LGD}_i^2$ (used in equations 4.17 and 4.18). This new portfolio is of the size $M \times S$ and thus the calculations are, if done naively, $(M \times S)^2 \ll n \times n$.

6.2.2 Idiosyncratic Risk

As stated in Section 1.1 this thesis does not go into models concerning idiosyncratic risk. Although, Eq. (4.17) and (4.18) are closely related to the granularity adjustment which is the primary method used in measuring name-concentration. It's also something that is addressed (internally) in many banks while the sector concentration is not. Further, if the multi-factor model is compared to the IRB calculations of a portfolio then there is a discrepancy if the multi-factor is affected by name-concentration since the IRB assumes that there is none.

This does present a problem in evaluating the effectiveness of methods regarding sector concentration if the name-concentration of the portfolio is not negligible. This is not a problem if the data is simulated so that the size of each exposure is set to be the same and thus virtually no name-concentration will exist. However, it is desirable to have a method that is directly applicable to real data and that does not require a predefined model for name-concentration. In this thesis, this is achieved in two different ways. Firstly, using the Pykhtin framework the name-concentration is adjusted for or it can easily be assumed to be 0 similarly to the IRB model so no additional adjustments to the methodology are needed in this case. The question is then how to compare the result of a MC model of a multi-factor framework to IRB. One approach would be to use a MC model for the single factor (IRB) model as well. This means that the same name-concentration will affect the result of both models and thus the discrepancy between them should only be due to the effect of sector concentration. Furthermore, Since name-concentration is assumed to be negligible in the IRB framework the difference between this single factor MC simulation and IRB should only be due to name-concentration.

While this can be extended to sector level (which is needed in the index calculations) initial testing showed that the method does not work as intended when extended like this. Instead, the more computational heavy route of splitting the portfolio into a more granular one is taken. This means that to remove the factor of idiosyncratic risk all obligors with and EAD larger than $X\$$ are split into n obligors retaining PD, LGD while EAD is replaced with EAD/n . This is discussed in more detail in Section 7.2.1.

7 Results and Discussion

In this section, the results of the models presented in Section 4 and 5 are shown along with a discussion of their effectiveness, robustness and usefulness when applied. This section is divided into two parts: Section 7.1 discuss the results of multi-factor models and diversification factor while Section 7.2 relates these results to a capital add-on.

To estimate the results of the multi-factor models a sector correlation matrix is needed. Following the methodology described in Section 6.2 the following correlation matrix is found:

Sector	A	B	C	D	E	F	G	H	I	J	K	L	M
A	1												
B	0.80	1											
C	0.84	0.90	1										
D	0.75	0.77	0.81	1									
E	0.77	0.82	0.86	0.81	1								
F	0.78	0.89	0.85	0.83	0.80	1							
G	0.83	0.85	0.88	0.70	0.87	0.83	1						
H	0.73	0.69	0.72	0.67	0.66	0.72	0.71	1					
I	0.69	0.73	0.80	0.89	0.62	0.75	0.59	0.57	1				
J	0.56	0.66	0.64	0.65	0.52	0.51	0.58	0.70	0.61	1			
K	0.68	0.68	0.73	0.81	0.86	0.68	0.61	0.72	0.57	0.64	1		
L	0.75	0.87	0.79	0.83	0.83	0.83	0.72	0.83	0.66	0.77	0.91	1	
M	0.77	0.80	0.85	0.90	0.83	0.81	0.62	0.63	0.80	0.56	0.90	0.87	1

Table 7.1: Inter-sector correlations

If nothing else is stated it is safe to assume that this is the correlation matrix used in calculations. In the calculations of the concentration index (CI) the covariance matrix is used, this is found by using the following randomly generated variance (drawn from $U(0.1,1)$).

Sector	A	B	C	D	E	F	G	H	I	J	K	L	M
Std.	85%	24%	56%	89%	73%	71%	96%	14%	95%	17%	69%	17%	86%

Table 7.2: Standard deviation per sector

In order to test the robustness of result, it is interesting to investigate the effect of changing this matrix. To do this, two altered correlation matrices are used with higher and lower correlations. These are constructed by increasing the decreasing and decreasing the current correlation between two sectors by 10 and 20 percentage points respectively. This is not done if the change means that the correlations is greater than 1 or smaller than 0.

Sector	A	B	C	D	E	F	G	H	I	J	K	L	M
A	1												
B	0.89	1											
C	0.93	0.94	1										
D	0.85	0.89	0.93	1									
E	0.88	0.91	0.93	0.89	1								
F	0.89	0.96	0.94	0.91	0.91	1							
G	0.91	0.92	0.91	0.78	0.91	0.91	1						
H	0.82	0.8	0.81	0.77	0.77	0.81	0.8	1					
I	0.78	0.82	0.88	0.93	0.73	0.84	0.69	0.66	1				
J	0.66	0.75	0.73	0.74	0.63	0.63	0.66	0.8	0.7	1			
K	0.78	0.78	0.83	0.88	0.91	0.78	0.73	0.81	0.68	0.72	1		
L	0.85	0.93	0.9	0.91	0.9	0.9	0.83	0.9	0.78	0.83	0.92	1	
M	0.85	0.88	0.93	0.97	0.91	0.9	0.75	0.74	0.88	0.67	0.9	0.91	1

Table 7.3: High correlations

Sector	A	B	C	D	E	F	G	H	I	J	K	L	M
A	1												
B	0.60	1											
C	0.63	0.71	1										
D	0.55	0.57	0.60	1									
E	0.57	0.62	0.66	0.61	1								
F	0.58	0.68	0.65	0.63	0.59	1							
G	0.64	0.65	0.70	0.51	0.69	0.65	1						
H	0.53	0.49	0.53	0.47	0.45	0.52	0.51	1					
I	0.49	0.53	0.60	0.69	0.44	0.56	0.37	0.38	1				
J	0.36	0.46	0.45	0.45	0.30	0.30	0.39	0.49	0.42	1			
K	0.48	0.46	0.53	0.61	0.68	0.48	0.41	0.53	0.35	0.45	1		
L	0.55	0.69	0.56	0.63	0.65	0.65	0.50	0.65	0.45	0.59	0.71	1	
M	0.58	0.61	0.69	0.71	0.61	0.61	0.40	0.41	0.61	0.34	0.74	0.70	1

Table 7.4: Low correlations

Note that the values shown in the three correlation matrices are approximations. The actual matrices used include more precise estimations due to the change made by Higham's algorithm.

7.1 Multi-factor Performance

This section starts with the results of Grtler's method as this implied probability is used in the other methods. When this is found the results for Pykhtin's and MC models are shown respectively with individual discussions and finally, these results are compared. Throughout this section the economic capital EC is shown as percent of total EAD.

7.1.1 Implied Correlation

As stated in Section 4.4 This thesis does not aim to define what a well-diversified portfolio would entail in terms of sector concentration. Instead, the model is used with the assumption that the IRB-model gives the true result of EC and that a multi-factor model that is fitted to achieve the same result for the same portfolio would then better capture the effect of each obligor's part in this economic capital. This is admittedly a naive strategy and it would not be suitable to implement this in practice. This is merely the interpretation of what it means to use the sample portfolio in the Gürtler et al. framework. This is done for two reasons:

1. First and foremost one could argue that the sample portfolio is "relatively well-diversified" (which is true for any portfolio) and consequently, it is interesting to compare this to Gürtler's findings. For this model to realistically be implemented there would have to be some regulatory definitions on what a well-diversified portfolio is (as is implicitly done with the intra-correlation from IRB).
2. As is explained later on in the section the result from these calculations can be used to deduce the differences (or similarities) between the MC and Pykhtin models.

To find the parameters of Eq. (4.29) applied to the sample portfolio the aforementioned grid search was used. Each pair of a_0 and a_1 with a step length of 0.01 was tested and the squared distance between the calculated EC and IRB EC (i.e. UL) was used as a measurement on how well the pair performed*. These results are shown in Fig. 7.1.

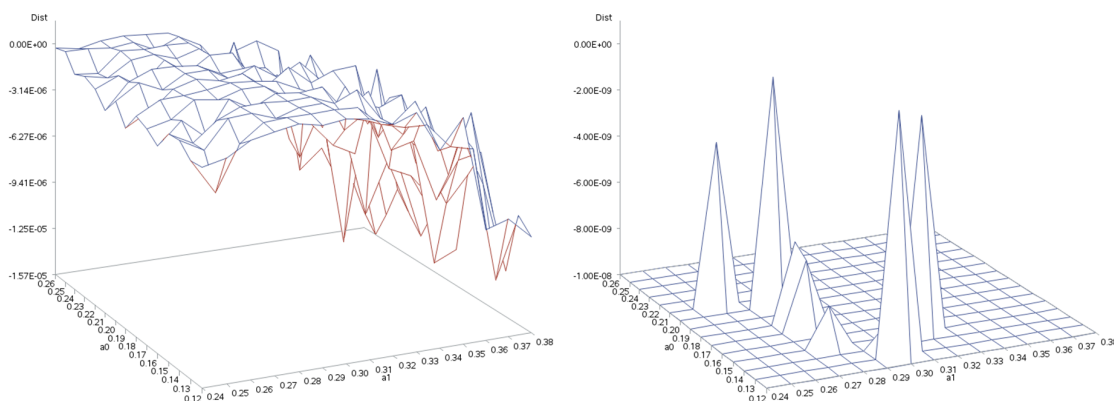


Figure 7.1: Left: Distance of every pair. Right: left graph zoomed in

As is evident from this test there is no clear "perfect pair". From the left graph, it is clear that increasing a_1 beyond 0.32 is too much while the same cannot be said for a_0 where there is no clear cut-off point. From the right graph, it is obvious that the pairs that have the shortest distance follow the pattern that if a_1 is large than a_0 is small and vice versa. This is, of course, to be expected while it also

* The following equation was used: $\text{Dist.} = - \left(\text{EC}^{\text{mf}} \left(\rho_{\text{inter}}, \rho_{\text{intra}}^{\text{implied}} \right) - \text{EC}^{\text{IRB}} \left(\rho_{\text{Basel}} \right) \right)^2$ and the maximum value of this is then (theoretically) the best fit.

demonstrates that this model would have difficulties in determining a "perfect pair" since from any such pair it could be possible to achieve the same result using a nudge in opposite direction for a_0 and a_1 . Having said that, except for one outlier of $a_0 = 0.14, a_1 = 0.31$ (which is the actual shortest distance) there is a concentration of well performing pairs around $a_0 = 0.20, a_1 = 0.30$. Consequently, the best of these (which is the second shortest distance) is chosen. This yield $a_0 = 0.21, a_1 = 0.29$ as the final result of the grid search. Note that even with 100 000 iterations per pair there is some randomness in the model and therefore the outlier is not chosen, if it was not there by a coincidence one would expect there to be other peaks (good pairs) close to the pair.

7.1.2 Multi-factor Models on the Sample Portfolio

When implementing Pykhtin's framework using the method described in Section 6.2.1 the computational time is significantly lowered to around 10 seconds for a portfolio that is not already conformed the specified sector/PD groups. The majority of this time is put in conforming the portfolio so that it is possible to insert it into the model. The actual calculations are done in less than a second. This is in a stark contrast to using the "naive" approach. While there is no time needed to conform or otherwise reshape the data the model takes up to 5 minutes to produce the result. It should be noted that the program used in this thesis, SAS System, specifically Interactive Matrix Language (IML) while efficient at matrix operations may not be best suited for the loops used to solve for the analytical value at risk. Loops had to be used since there was a problem with memory allocation for a portfolio with 10 000 exposures (i.e. the sample portfolio used). Most likely the computational time would have been significantly lowered if matrix operations were possible to use. This was not tested for two reasons: 1. this model must be possible to use of larger portfolios as well and if 10 000 exposures are computationally heavy than 1000 000 would not be feasible and 2. Conforming the portfolio is much more efficient in all plausible scenarios and when this is done the effect of loops versus matrix operations are negligible. Consequently, loops were used in all testing. To clarify, in a mathematical sense loops versus matrix operations are equivalent to calculating each of the partial sums of equations (4.15) - (4.18) versus rewriting the equations in matrix form.

In table 7.5 the results of the components when running the Pykhtin model on the sample portfolio is shown.

Variable	Impact
Δq_z^{GA}	0.39%
Δq_z^{∞}	0.01%
$q_z(\bar{L})$	1.32%
$q_z(L)$	1.89%
EL	0.09%
EC	1.80%

Table 7.5: Analytical calculation

It is clear that the name-concentration in the portfolio has a significant effect. By

spot-checking the model on a much larger portfolio (100 000 exposures) with similar characteristics it is found that this effect is indeed significantly reduced: testing revealed $\Delta q_z^{\text{GA}} = 0.02\%$ while the other variables remain relatively stable.

In contrast to name-concentration, the effect of sector concentration is relatively small. Recall that this sector concentration result stems from the discrepancy when the mapping from multi-factor to a single factor is done and is not the same as sector concentrations being negligible. To find the effect of sector concentrations different correlation matrices need to be used and the end result (EC) compared.

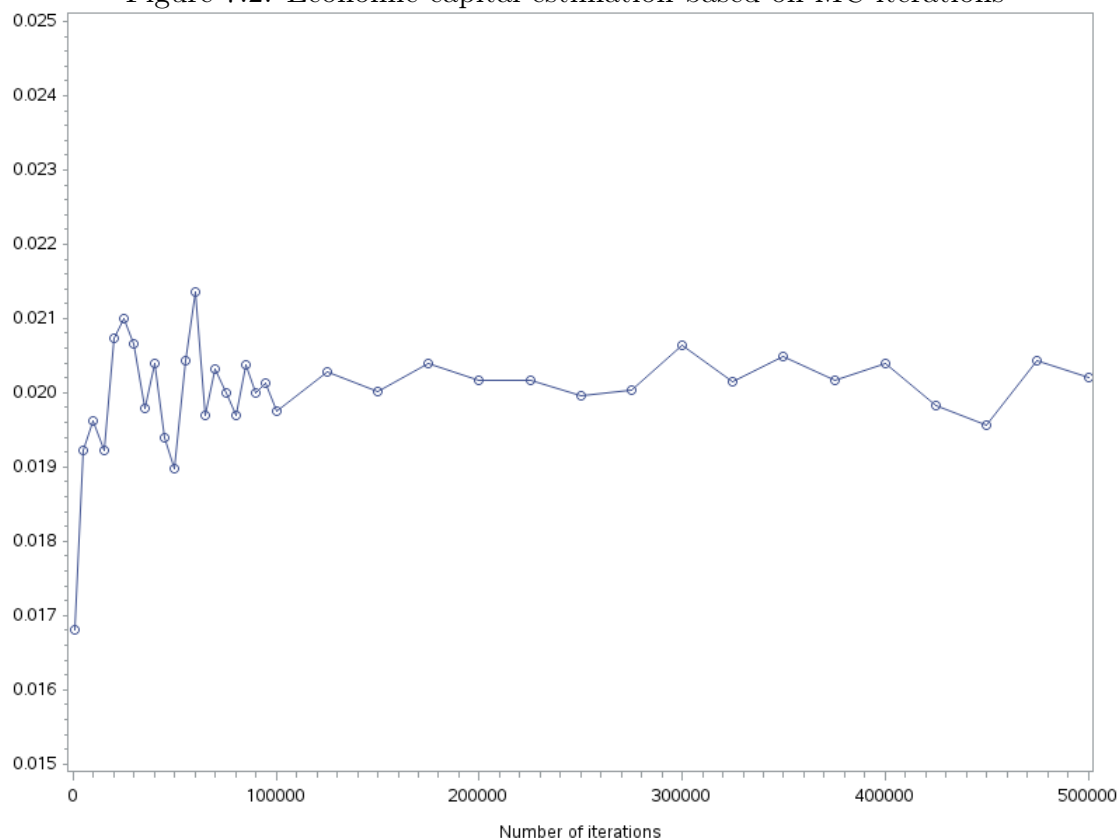
Variable	Low Corr	High Corr
Δq_z^{GA}	0.43%	0.37%
Δq_z^{∞}	0.02%	0.00%
$q_z(\bar{L})$	1.02%	1.46%
$q_z(L)$	1.66%	2.00%
EL	0.09%	0.09%
EC	1.34%	1.92%

Table 7.6: Analytical calculation - high and low correlations

The first apparent result is that the granularity adjustment is not stable. It seems that using the high concentration matrix the mapping into the single factor is more accurate which is why both the GA and infinity components are smaller. This could be due to the high correlation matrix being more homogeneous across sectors than the low correlation.

Before comparing the results from the Pykhtin model with the MC simulations it is important to note that as Pykhtin's model is analytical it will produce the same result when the underlying data has not changed. This is not true for the MC approach and as such the robustness of the model is of great importance. Most other academic reports on this subject use simulated data which could affect the convergence of the economic capital found when the number of iterations is increased. Nevertheless, when calculating the EC of a sample portfolio both IMF and Görtler et al. (see Section 7.1) use 500 000 iterations. Table 7.2 shows how the EC converges when iterations are sequentially increased to 500 000.

Figure 7.2: Economic capital estimation based on MC-iterations



As can be seen, there is still some variance at 500 000 iterations and it seems as the variance in the model is similar from 100 000 iterations and up. In terms of percent (OBS not percentage points) the difference in EC when comparing 475 000 iterations to 500 000 is around 1%. This is deemed sufficient for this thesis but in implementing this model this question may have to be revisited.

Finally, to evaluate the differences in the multi-factor models as well as their performance with a different intra-correlation table 7.7 shows a side by side comparison of the EC results.

EC model	EC as percent of total EAD		
	IRB intra-corr	Gürtler intra-corr	Estimated intra-corr
IRB	1.69%	1.69%	1.69%
Pykhtkin $-\Delta q_z^{\text{GA}}$	1.24%	1.90%	1.63%
MC single factor*	2.37%	2.37%	2.37%
MC multi-factor	1.96%	2.49%	2.36%
Pykhtkin	1.80%	2.37%	2.13%
IMF methodology**	1.84%	2.43%	2.17%

Table 7.7: Economic capital of single portfolio models

There are four key takeaways from these results:

* To clarify, the different intra-correlation is not inserted into the single factor models

** Here the 30 largest exposures is used as the cut-off point

1. When excluding the granularity adjustment (GA) from the Pykhtin model it seems as it yields similar results to the MC estimate. This is shown by the fact that economic capital of "Pykhtkin $-\Delta q_z^{GA}$ " is close to that of "IRB" when using the "Estimated intra-corr". Recall that in the grid search it was the MC method that was used to determine these and as can be seen the single factor and multi-factor MC are close as well. The conclusion is then that the granularity adjustment slightly underestimates the effect of name-concentration on this sample portfolio but that the analytical calculations of value at risk is otherwise similar to the MC simulations.
2. As explained in Section 6.2.2 with this table a tractable measurement of the effect of name-concentration is implicitly achieved. Since "IRB" assumes that the portfolio is perfectly diversified while the single factor MC does not and they are otherwise equal, the difference between the two is equal to the effect of name-concentration on the portfolio. This further demonstrates the point that the granularity adjustment underestimates the effect of name-concentration since it is 0.56% while it is found to be 0.69% with this method. While it would not be a perfect fit, replacing the GA variable with this value yields much closer results for the analytical calculations. Note also that the difference in MC multi-factor and Pykhtin with no granularity adjustment is 0.72%.
3. Using Gürtler's methodology for intra-correlation all models agree that the economic capital should be increased. This means that by the definition made by Gürtler et al. this sample portfolio is not well diversified on a sector level.
4. The IMF methodology yields similar results to the Pykhtin model. Based solely on these results it does not seem to improve the accuracy of Pykhtin.

Expanding on point 2. it can be shown that this measurement of name-concentration is accurate by splitting larger obligors so that no obligor made up more than 1/10000 part of the portfolio (as is suggested in Section 6.2.2). Using this granular sample portfolio, the following table is obtained:

EC model	EC as percent of total EAD		
	IRB intra-corr	Gürtler intra-corr	Estimated intra-corr
IRB	1.69%	1.69%	1.69%
MC single factor	1.69%	1.69%	1.69%
MC multi-factor	1.25%	1.96%	1.67%
Pykhtkin	1.24%	1.93%	1.66%
IMF methodology	1.24%	1.94%	1.66%

Table 7.8: Economic capital of single portfolio models

Since the economic capital of the models that should theoretically converge do so it is concluded that the name-concentration make up 0.69% of the economic capital when the portfolio is not "granularized". Admittedly there is some variation in the multi-factor models but in the context of this thesis, it is deemed negligible. Further, the IMF result is somewhat insipid since there are no larger exposures.

7.1.3 Diversification Factor Performance

The results in this section are based on 10 000 portfolios consisting of 100 obligors randomly drawn from a larger portfolio. The portfolio size is deliberately kept low to vary the characteristics of the portfolios and thus increasing the range (in terms of CDI and β) of which the fitted model is accurate. Furthermore, feeding the model data that is similar defeats the purpose of the model.

To calculate the multi-factor economic capital the Pykhtin model was used. This is firstly due to the computational time being unreasonable long using the crude MC model. Secondly, the possibility to assume that there is no name-concentration is especially useful in portfolios with 100 obligors.

In Fig. (7.3) the portfolio data is visualized. On the left plot, the diversification factor is shown on the z-axis, CDI on the y-axis and β on the x-axis. The right plot shows the concentration of CDI and β . They (roughly) span from $0.55 < \beta < 0.9$ and $0.1 < \text{CDI} < 1$. There is a lack of data in the top and bottom right corners which makes predictions made there uncertain.

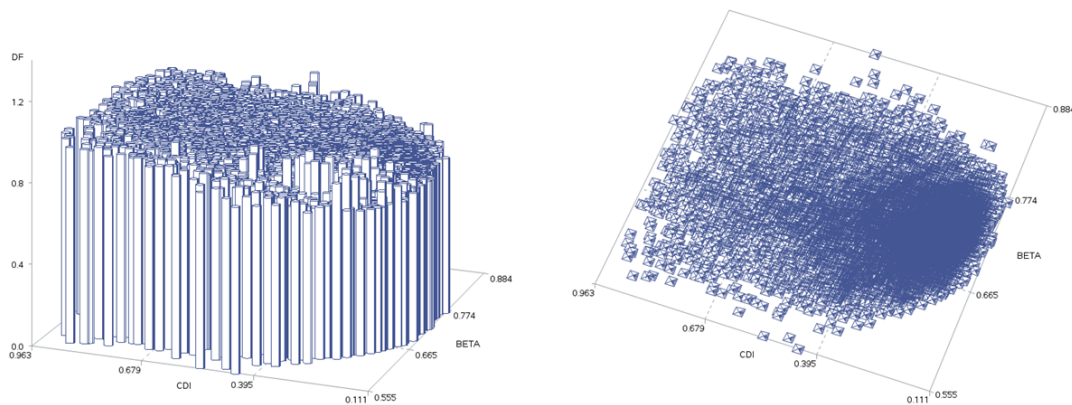


Figure 7.3: Scatter plot viewed in 3d and 2d of sampled data

Using the parameterization shown in Eq. (4.28) the following values are found:

Parameter	Estimate	Approx Std Error	95% Conf. Lower	95% Conf. Upper
a_0	1.00	0.00	1.00	1.00
a_1	-1.03	0.02	-1.06	-0.99
a_2	-0.26	0.01	-0.28	-0.24
a_3	0.73	0.02	0.68	0.78

Table 7.9: Estimation of parameters

In line with similar studies the $R^2 = 0.94$ value shows this parameterization is efficient. Finding $a_0 = 1$ is also in line with what was found by Cespedes et al. and this was notably done without any restrictions on it which were used in their study.

The shape of the parameterized surface kept the shape found in previous studies. In Fig. (7.4) this surface is shown along with the (linearly) extrapolated surface.

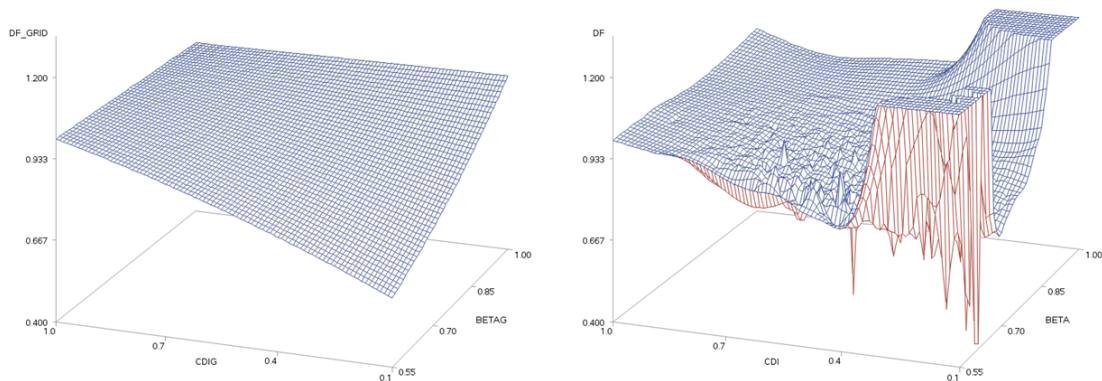


Figure 7.4: Grid found by using parameterization (left) and extrapolated grid from sample (right)

It is evident that the two are similar in areas where there is underlying data, although the extrapolated surface is not as smooth. As mentioned, the top and bottom right corners lack data and this explains the drastically different results there between the two graphs. The parameterized surface provides a shape that is more intuitively in this case. Finally, the fitted Eq. (4.28) is used to find the EC of the sample portfolio.

CDI	β	DF	EC
0.144	0.720	0.813	1.37%

Table 7.10: Using DF on sample portfolio

Comparing this with the value in table 7.7 there is notable difference, it seems this method overestimated the sample portfolio. Based on the R^2 this should have more to do with the sample portfolio used being an outlier than with an ineffective model.

7.2 Relating Economic Capital to Regulatory Capital

The aim of this section is to compare the different indexes that were presented in Section 5 while accounting for the underlying factors that can affect the outcome. There are two such factors brought up in the Section 5 and 6.

1. The effect of name-concentration may offset the result and need to be removed in order to ensure that only sector concentration affects the results (Section 7.2.1).
2. Is the expected shortfall estimator, shown in Section 5.3, is accurate and robust? Further, since the bootstrap model is assumed to yield the "true" answer the robustness of this answer gives insight into how accurate it is (Section 7.2.2).

When these factors are accounted for the result of each index are compared and the robustness of each individual factor is discussed (Section 7.2.3).

7.2.1 Idiosyncratic Risk Removal

The result from table 7.7 is not enough to determine how much of the economic capital is due to name-concentration. These results need to be divided into sector components in order to be useful in the context of EI MC index. Note that the name-concentration of the Pykhtin model can (again) be assumed to be 0 so no new calculations are needed to remove the idiosyncratic risk in this case.

As discussed earlier in this thesis the results for value at risk on a sector are not stable and analogously with what is done in the next section, the bootstrap method is used to determine the value at risk for the single factor MC simulation. By dividing the EC (or UL) of the IRB model into sectors and subtracting it from EC found by a single factor MC simulation the resulting effect of name-concentration on each sector is found. Since name-concentration is theoretically the same in the single and multi-factor models the resulting name-concentration is used to adjust the EC found by the multi-factor model. The results from this method are shown in table 7.11.

Sector	MC single factor	IRB	Initial diff.	Positive diff.	Major diff.
A	1.01E-03	1.28E-03	-2.05E-04		
B	1.81E-03	1.47E-03	4.25E-04	4.25E-04	
C	3.82E-04	4.59E-04	-6.02E-05		
D	1.01E-04	1.41E-04	-3.20E-05		
E	1.06E-03	1.05E-03	6.14E-05	6.14E-05	
F	6.74E-03	2.66E-03	4.23E-03	4.23E-03	4.23E-03
G	3.64E-03	8.89E-04	2.79E-03	2.79E-03	2.79E-03
H	7.92E-04	1.20E-03	-3.46E-04		
I	5.55E-03	4.93E-03	8.38E-04	8.38E-04	
J	4.41E-04	6.59E-04	-1.89E-04		
K	6.84E-04	1.02E-03	-2.79E-04		
L	8.38E-04	8.22E-04	5.22E-05	5.22E-05	
M	1.82E-04	2.71E-04	-7.44E-05		
	2.32%	1.68%	0.72%	0.84%	0.70%

Table 7.11: Economic capital of the single factor model on sector level

Recall the values presented in table 7.7, it is evident that the calculations when aggregated yield the same result (i.e. IRB 1.68% and MC sf 2.37%, even if MC sf deviate some due to the randomness of MC simulation). However, on a sector level, the contribution to EC is larger in some sectors using the IRB model. This should be due to fluctuations in the calculations since the limit if there is no name-concentration should be exactly IRB. Consequently, these values are ignored and treated as zero since there is no prominent name-concentration. Following this reasoning, sectors with smaller positive values are probably the result of the same fluctuations and assumed to have no name-concentration as well. This leaves two sectors that show prominent name-concentration. As mentioned, the difference between the two models could be used to adjust the multi-factor model. However, this

test reveals a weakness in this model since it gives a negative EC for the two sectors in question (compare sector-level results from table 7.11 and 7.12). The main advantage of this model is that it is much faster computationally compared to "splitting" the portfolio to a granular one. From this, I conclude that by applying this method, on the sample portfolio used, it is not possible to measure name-concentration on a sector level. This is in contrast to the fact that it is possible to measure the name-concentration accurately on a portfolio level using this method.

Using the aforementioned method of "granularizing" the portfolio the following results are obtained.

Sector	MC multi-factor	MC gran. Port.	Initial diff.	Positive diff.	Major diff.
A	4.19E-04	9.77E-04	-5.58E-04		
B	1.32E-03	1.26E-03	6.29E-05	6.29E-05	
C	1.31E-03	3.97E-04	9.14E-04	9.14E-04	
D	6.31E-05	1.16E-04	-5.31E-05		
E	1.12E-03	7.38E-04	3.84E-04	3.84E-04	
F	4.64E-03	2.14E-03	2.50E-03	2.50E-03	2.50E-03
G	2.17E-03	5.72E-04	1.60E-03	1.60E-03	1.60E-03
H	3.74E-04	7.31E-04	-3.57E-04		
I	7.39E-03	3.87E-03	3.52E-03	3.52E-03	3.52E-03
J	1.10E-05	3.40E-04	-3.29E-04		
K	2.83E-04	6.23E-04	-3.40E-04		
L	5.96E-04	6.42E-04	-4.63E-05		
M	1.18E-04	2.14E-04	-9.60E-05		
	1.96%	1.25%	0.72%	0.90%	0.76%

Table 7.12: Economic capital of multi-factor model on sector level

Although the total amount of name-concentration does deviate somewhat from the 0.69% found earlier this method offer answers at the correct magnitude. Of course, there is no point in adjusting the results when a portfolio with no name-concentration is already obtained. These results serve merely as a measurement on name-concentration on a sector level while the "granularized" portfolio is used for further calculations.

The key take away from these test is that the idiosyncratic risk of a sample portfolio (containing much of it) is efficiently estimated on a portfolio level using an MC simulation for a single factor model and comparing it to the analytical model. In contrast, this does not hold when it is divided into a sector level where it was overestimated and more importantly misleading in which sectors contain it.

7.2.2 Expected Shortfall Estimator

Following the terminology presented in Section 5.3 the number of loss scenarios, N is chosen to be 1 000 000 while the sample size in the bootstrap method M is set to 100 000 which is aligned with the size used in the previous section. Finally, this number of iterations k is chosen to be 5000 and this is done 2 times in order to

observe any fluctuations in the results. To achieve the estimate of economic capital the EL per sector is subtracted from the VaR. These 2 results are then normalized (so that they represent the percentage of EC in each sector) and compared to the normalized expected shortfall (which has the EL subtracted as well).

Sector	EC - Bootstrap 1	EC - Bootstrap 2	EC - Expected Shortfall
A	7.9%	7.7%	7.3%
B	10.0%	10.0%	9.6%
C	3.1%	3.1%	3.3%
D	0.9%	0.9%	0.9%
E	5.9%	5.9%	5.9%
F	17.2%	16.9%	17.7%
G	4.6%	4.5%	4.8%
H	5.8%	5.8%	5.6%
I	30.3%	30.7%	31.0%
J	2.6%	2.7%	2.4%
K	4.9%	4.9%	4.7%
L	5.1%	5.1%	5.1%
M	1.7%	1.7%	1.7%

Table 7.13: Economic capital per sector (name-concentration excluded)

The two trials with the bootstrap method yield similar results and on sector level the maximum difference is 0.4 ppt (percentage points) and the mean difference is 0.08 ppt. This is deemed sufficiently stable to draw conclusions from and with the reservation that if this was to be used in practice it may have to be more stable than this. Arbitrarily choosing Bootstrap 2 to compare with the expected shortfall (it does not make much difference which is chosen) the maximum difference is 0.80 ppt and the mean difference is 0.24 ppt. This is deemed too inaccurate to be used further in this section, it is, however, interesting that it does not deviate more from the bootstrap method. To clarify, henceforth EI MC denotes using the bootstrap method. Using percentage points and not the percentile difference may be misleading but the argument for using this is that in the sector with a larger part of the exposure contribute exponentially more (by definition) than the smaller ones. Percentage points capture this discrepancy in impact better than the percentile difference.

7.2.3 Index Performance

Having determined how to calculate EI through MC simulations the different Indices discussed in Section 5 can be compared side by side. This is leaning on the argument made that any index that measures the division of exposure and shares the basic properties of the Herfindahl-Hirschman Index (i.e. a measurement of the concentration of exposure) could be used in the SFSA's mapping. Note that the term exposure is used somewhat liberally here since the economic capital or risk-weighted assets are not the same as the exposure. Additionally, no further information is obtained on the performance of different indices by using SFSA's mapping (see Section 1.2) therefore only the HHI or equivalent is shown.

Method	Index value
HHI	0.204
CI	0.412
$\hat{C}I$	0.099
S^2	0.183
RI	0.144
EI MC	0.155
EI PY	0.158

Table 7.14: Regulatory add-on

Singling out the non-normalized concentration index all risk-weighted indices are lower than the HHI indicating that the risk is more diversified according to these indices. The normalized CI is significantly lower while S^2 is the largest one except for HHI, this shows that the portfolio is more well-diversified in terms of sector correlations than in terms of sector PD. As is to be expected EI MC and EI PY are close in their values while RI has the lowest value of this trio (henceforth, capital indices). Since these three are less abstract than CI or S^2 I argue that they are better suited for a HHI replacement. While S^2 produce a result closer to HHI which does mean the change would be smaller if this was used instead the PD scaling used in the measurement is somewhat arbitrary while the risk-weighted exposure found by EC or RWA are already established and in the case of RWA already in use in the IRB model.

To further compare the effect of the model table 7.15 show the exposure is divided between the sectors using the different indices.

Sector	HHI	$\hat{C}I$	S^2	RI	EI MC	EI PY
A	10.1%	10.6%	10.8%	7.6%	7.9%	7.4%
B	5.1%	3.1%	8.3%	8.7%	10.0%	9.7%
C	2.8%	7.5%	1.6%	2.7%	3.1%	3.1%
D	0.4%	12.4%	0.7%	0.8%	0.9%	1.0%
E	3.7%	8.6%	5.2%	6.2%	5.9%	5.9%
F	9.4%	9.3%	15.5%	15.8%	17.2%	17.9%
G	3.6%	11.0%	2.4%	5.3%	4.6%	4.7%
H	4.8%	1.5%	6.0%	7.1%	5.8%	5.6%
I	39.8%	13.5%	35.7%	29.3%	30.3%	30.9%
J	12.6%	1.7%	2.4%	3.9%	2.6%	2.4%
K	3.3%	7.4%	5.3%	6.0%	4.9%	4.8%
L	3.5%	2.1%	4.2%	4.9%	5.1%	5.0%
M	1.0%	11.2%	1.8%	1.6%	1.7%	1.7%

Table 7.15: Exposure (risk-weighted) per sector in add-on models

It is evident that while the correlations to other sectors are not greater in sector "F" the PD is and this effect is not mitigated by the effect of LGD since all models effected by PD show an increase of exposure here. In sector "I" the PD are slightly

below average as is evident by \mathbf{S}^2 but the effect of LGD lowers the exposure in the sector for the capital indices. The effect of sector correlations is apparent in sector "J" since it is known from table 7.1 that this sector has lower correlations than other sectors and RI is larger the both EI indices for this sector, in addition, the normalized CI is significantly lower in this sector. This effect is observed in "H" as well.

Applying the indices to the high and low correlation matrices points to their dependence on the correlation matrix (only showing indices affected).

Corr. Matrix	$\hat{C}I$	EI MC	EI PY
Low	0.102	0.192	0.194
High	0.096	0.149	0.150

Table 7.16: Changing inter-correlation for HHI replacement indices

This test shows a clear dependence on the inter sector correlation for the EI method. While this is not unexpected it differentiates from the CI method which does not fluctuate as much. Recall that it is the proportion of EC that is relevant in this method and as such a high correlation does not equate to a greater EI (while it does equate to a greater EC). When the intra sector correlation is changed instead the following results are found:

Intra-corr	RI	EI MC	EI PY
Estimated	0.144	0.158	0.158
Gürtler	0.145	0.159	0.160

Table 7.17: Changing intra-correlation for HHI replacement indices

Evidently, this change does not affect the methods as much. This change should theoretically be better distributed among all sectors given that their PDs are similar which would indicate that in this context the PDs of the sectors are relatively close.

8 Conclusion

The purpose of this thesis was to evaluate the multi-factor models and to find a suitable candidate for a pillar II capital add-on; this two-part question was evaluated in Section 7.1 and 7.2 respectively.

On the sample portfolio used in this thesis Pykhtin's analytic calculations accurately estimated the economic capital when there was no name-concentration present. The IMF methodology did not deviate much from the Pykhtin results and did similarly underperform when name-concentration was present. Multi-factor MC simulations did not clearly converge on the tested range while it did show relatively stable results.

Using the adjustment to the intra-correlation proposed by Gürtler et al. the sample portfolio should have its EC increased. Assuming instead that the portfolio was defined as "well-diversified" the resulting change in intra-correlation was not stable and no clear replacement for the calculations was found. The diversification factor model proved to be robust and predicted the EC of the sample portfolio relatively well. Consequently, by using a replacement intra-correlation it could be adjusted so that the DF is not bounded.

To isolate the effect of sector risk the name-concentration was removed from the calculations in two ways. For EC calculations on a portfolio level, it was sufficient to compare the single factor MC to the multi-factor MC model to accurately measure (and adjust for) the effect of name-concentration. On a sector level, this was not found to be accurate and the portfolio was instead "granularized".

This thesis introduced two new indices, RI and EI as alternatives to the Herfindahl-Hirschman index. Both show the desired effect of an intuitive dependence on the PD and LGD. Moreover, EI shows a dependence on the inter-sector correlation. In the sample portfolio, it was clear that the high concentration in one sector could be (better) justified by these methods when the low average LGD and PD of this sector was taken into consideration.

While both RI and EI remain relatively stable to changes in intra-sector correlation EI fluctuates when the sector-correlations change. This is not necessarily an argument against it but it does further cement the fact that this input needs to be accurate and as discussed in this thesis, this is challenging.

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