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Risk Modelling in Payment Guarantees

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Abstract

The Swedish Export Credit Agency (EKN) issues payment guarantees to Swedish companies who face the risk of non-payments in export transactions. Commitments are typically correlated, as defaults of companies are driven by other factors than factors specific to that company, such as the economic cycle or the regional conditions. In deciding upon how much capital to be reserved to remain solvent even in an unlikely scenario, this has to be accounted for in order to not underestimate financial risks. By studying models for credit risk and the research available in the area, the popular CreditRisk⁺ has been chosen as a suitable model for EKN to use in risk assessments. The model together with a few customizations are described in detail and tested on data from EKN.

Riskmodellering i Betalningsgarantier

Sammanfattning

Exportkreditnämnden (EKN) utfärdar betalningsgarantier till svenska exportörer som riskerar inställda betalningar. Fallissemang hos olika motparter är typiskt korrelerade. Vid bedömning av risken i portföljen av garantier måste detta tas i beaktning, för att inte underskatta risken väsentligt. Genom att studera befintliga kreditriskmodeller och tillgänglig forskning inom området har en modell föreslagits som kan användas i EKN:s riskbedömningar. Modellen beskrivs i detalj och testas på data från EKN.

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1 Introduction

The Swedish Export Credit Agency (Swedish: Exportkreditnämnden, EKN), henceforth EKN, is a Swedish government agency that issues payment guarantees to Swedish companies that face the risk of non-payment in export transactions. The transactions are typically large, often making a credit time over some years inevitable. During this time, the counter party may default on its payments, in which case EKN pays the remaining amount while the guarantee holder pays an insurance premium regardless of the outcome. Default events of different companies are typically correlated, something that has to be accounted for in financial decisions.

Consider a portfolio consisting of m obligors and denote by I_i the indicator function

$$I_i = \begin{cases} 1, \text{ if obligor i defaults} \\ 0, \text{ otherwise} \end{cases}$$

during the considered time period. Each obligor has a probability of default

$$p_i := \mathbb{P}(I_i = 1),$$

and default of obligor *i* results in a loss of $L_i = \text{LGD}_i * \text{EAD}_i$, where $\text{LGD}_i \in [0, 1]$ stands for Loss Given Default and EAD_i stands for Exposure at Default for obligor $i \in \{1, ..., m\}$. The total loss in the portfolio is then

$$X = \sum_{i=1}^{m} I_i L_i.$$

We wish to calculate various risk measures on this portfolio that are linked to capital requirement regulations and the vital task is to take dependency between commitments into account.

This thesis aims at developing a credit risk model for EKN, that can be used in risk assessments for their portfolio. The model has been chosen in consultation with EKN by studying the main industry models for credit risk. The proposed model is based on CreditRisk⁺, a model frequently used by practitioners in the financial industry, which has then been customized to become more efficient and adapted to EKN's needs and wishes.

The model is described in Section 3 after having introduced some necessary mathematical concepts in Section 2. In Section 4, the model is tested and analyzed on data provided by EKN followed by a summary and conclusions in Section 5.

2 Mathematical Background

As in the introduction, let

$$X = \sum_{i=1}^{m} I_i L_i$$

denote the total loss of a portfolio.

Probability transforms

A main ingredient in deriving the loss distribution in CreditRisk⁺ is the use of probability generating functions, defined as follows.

Definition 2.1. The Probability generating function (pgf) of a discrete random variable X taking values in \mathbb{N} , exists at least for |z| < 1 and is defined as

$$G_X(z) = \mathbb{E}[z^X].$$

The following proposition regarding probability generating functions will be used.

Proposition 2.1. The following properties hold for a random variable X with probability generating function $G_X(z)$

- (i) G_{kX}(z) = G_X(z^k), for k ∈ N
 and if X₁ and X₂ are independent random variables with probability generating functions G_{X1}
 and G_{X2},
- (*ii*) $G_{X_1+X_2}(z) = G_{X_1}(z)G_{X_2}(z).$

We shall take a slightly different approach and make use of characteristic functions, contrary to the original version of CreditRisk⁺, where probability generating functions are used.

Definition 2.2. The Characteristic function of a real-valued random variable X is defined as

$$\Phi_X(z) = \mathbb{E}[e^{izX}],$$

where *i* denotes the imaginary unit. The characteristic function exists for any real-valued r.v X and all $z \in \mathbb{R}$.

More generally, we say that $f : \mathbb{R} \to \mathbb{R}$ is \mathcal{L}^1 if

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty,$$

and for such functions we define its Fourier transform as follows.

Definition 2.3. If f is \mathcal{L}^1 , the Fourier transform of f exists and is defined as

$$\mathcal{F}f(z) = \int_{-\infty}^{\infty} e^{izx} f(x) dx.$$

In the special case when f is the probability density function of a random variable X, the Fourier transform of f can equivalently be stated as the characteristic function of X, i.e. $\mathcal{F}f(z) = \Phi_X(z)$.

We shall see that a closed-form expression of the characteristic function of the loss is available in CreditRisk⁺. Although the loss is not a continuous random variable and does not have a density function, the following theorem called the Fourier Inversion Theorem will be useful. It shows how to obtain the probability density function of a r.v. X, provided that both the density and the Fourier transform of X are \mathcal{L}^1 [1].

Theorem 2.2. Assume that both f and $\mathcal{F}f(z)$ are \mathcal{L}^1 . Then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-izx} (\mathcal{F}f)(z) dz.$$

We will also use the following result, relating the probability generating function and the characteristic function of a random variable X.

Proposition 2.3. If X has a characteristic function $\Phi_X(z)$ and a probability generating function $G_X(z)$, then

$$\Phi_X(z) = G_X(e^{iz})$$

for all z where $G_X(z)$ is defined.

Risk measures

Once we have knowledge about how different outcomes are distributed in the proposed model, we want to measure the risk in the portfolio. Particular focus will be devoted to the popular risk measures Value-at-Risk, Expected Shortfall and Unexpected Loss.

Definition 2.4.

$$\operatorname{VaR}_p(X) = \min\{l : \mathbb{P}(X \le l) \ge p\}$$

will be referred to as the Value-at-Risk at level p (or simply the p-quantile) for the random variable X [2].

Definition 2.5.

$$\mathrm{ES}_p(X) = \frac{1}{1-p} \int_p^1 \mathrm{VaR}_p(X) dp$$

will be referred to as the Expected shortfall for the random variable X at level p [2].

Definition 2.6.

$$\mathrm{EL} = \mathbb{E}[X]$$

will be referred to as the Expected Loss the portfolio.

Definition 2.7.

$$UL_p = VaR_p(X) - EL$$

will be referred to as the Unexpected Loss of the portfolio at level p [3].

Relevant Distributions

We end this section by stating some basic properties of the probability distributions that will be used most frequently.

Bernoulli distribution

A discrete random variable I is Bernoulli-distributed with parameter $p \in (0, 1)$ if it has the probability mass function

$$\mathbb{P}(I=k) = \begin{cases} p & \text{if } k = 1\\ 1-p & \text{if } k = 0\\ 0 & \text{otherwise} \end{cases}$$

and we write $I \sim Be(p)$. The mean and variance of a Be(p)-distributed r.v. is p and p(1-p) respectively, and its probability generating function is given by

$$G_I(z) = (1-p) + pz$$

Poisson distribution

A discrete random variable N is Poisson-distributed with intensity λ if it has the probability mass function

$$\mathbb{P}(N=n) = \frac{e^{-\lambda}\lambda^n}{n!}$$

and we write $X \sim \operatorname{Poi}(\lambda)$.

If $N_1, ..., N_n$ are independent Poisson-distributed random variables with intensities $\lambda_1, ..., \lambda_n$, then their sum is Poisson-distributed with intensity $\sum_{i=1}^n \lambda_i$, i.e.,

$$\sum_{i=1}^{n} N_i \sim \operatorname{Poi}\left(\sum_{i=1}^{n} \lambda_i\right). \tag{1}$$

The mean and variance of a $\text{Poi}(\lambda)$ -distributed random variable are both λ and its probability generating function is given by

$$G_N(z) = \exp\{\lambda(z-1)\}.$$

Gamma distribution

A random variable Γ is Gamma-distributed with parameters (α, β) , in which case we write $\Gamma \sim \text{Gamma}(\alpha, \beta)$, if it has the probability density function

$$f_{\Gamma}(\gamma) = \frac{\gamma^{\alpha - 1}}{\beta^{\alpha} \Gamma_0(\alpha)} e^{-\gamma/\beta},\tag{2}$$

where $\Gamma_0(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$ is the Gamma-function. A Gamma-distribution with shape α and scale β has mean $\alpha\beta$ and variance $\alpha\beta^2$.

3 Model

Throughout this thesis we will consider a portfolio of m obligors and it will be assumed that the following quantities are known or have been estimated beforehand:

- p_i Probability of default of obligor i during the considered time period
- EAD_i The amount of money possible to lose in case of default during the considered time period.
- $LGD_i \in [0, 1]$ Loss Given Default. The fraction of the exposure that is expected to be lost in case of default.

Apart from reporting risk figures regularly, a common question that arises at EKN is how the risk of the whole portfolio is affected by including a new commitment, possibly a large one, or how the risk figures are affected by a reinsurance contract that transfers some risk of a large commitment to a reinsurance firm. In the model used at EKN today, these questions can be answered easily, a convenient feature that has been a desire to keep.

A particular set of parameters used in the current model is left to expert judgment, which has been a desire to improve.

Well-documented and frequently used models for credit risk are **Portfolio Manager** by KMV, **CreditMetrics** by RiskMetrics Group, **Credit Portfolio View** by McKinsey & Company and **CreditRisk**⁺ by Credit Suisse Financial Products [4]. In both CreditMetrics and the KMV-model, default of a firm is decided by whether or not its asset value falls below some critical threshold before the end of the time period. Such models can be an attractive choice when the number of obligors in the portfolio is not too large, and typically publicly traded companies, where the asset value and volatility can be inferred from the stock market. EKN has about 2000 outstanding risks that in many cases are non-traded firms or sovereigns, making such a model hard to monitor. Technical details for these models can be found in [5] and [6].

In Credit Portfolio View, risk drivers are assumed to be macroeconomic factors which requires a quite complicated econometric framework and the benefits from implementing this model have not been considered to outweigh the extra implementational issues it entails in this case.

CreditRisk⁺ is best described as a Poisson-Mixture model, incorporating the Gamma-distribution to introduce variability and covariation in default intensities. This is the model that we have chosen to proceed with, mainly because of its transparent and straight-forward implementation. Credit Suisse expresses in the documenation of CreditRisk⁺ that, "The CreditRisk⁺ Model is highly scalable and hence is capable of handling portfolios containing large numbers of exposures. The low data requirements and minimum of assumptions make the CreditRisk⁺ Model easy to implement for a wide range of credit risk portfolios, regardless of the specific nature of the obligors" [7]. This is an accurate description, and a very appealing one from EKN's point of view since the number of commitments is large and have very different characteristics. The scalability and transparency of CreditRisk⁺ are the main reasons that it is so commonly used for insurance matters in credit risk [8].

Another appealing attribute of CreditRisk⁺ contrary to its competitors is that it provides an analytic form of the loss distribution. This makes it easy to calculate various functionals on the loss and Monte Carlo simulations are avoided.

CreditRisk⁺ is a default-only model, ignoring the fact that the credit quality of an obligor may have changed during the considered time period, even if it did not default. With this, the possibility to value the portfolio "marked-to-market" is lost. EKN does not consider this as a particularly big drawback since a separate procedure for this purpose is already in place, and the benefit of such an approach does not outweigh the extra complexity and uncertainty it implies insisting on such a model.

To summarize, CreditRisk⁺ fits the purpose well, while other popular models such as KMV, Credit-Metrics and Credit Portfolio View requires input that are difficult to monitor considering the size of EKN's portfolio. Through the rich literature devoted to CreditRisk⁺, guidance on parameter estimates are available and the loss distribution can be calculated efficiently, avoiding Monte Carlo simulations, and the marginal risk contribution of a single obligor can be easily determined.

CreditRisk⁺ will be described in detail below, after taking a detour in a general description of the family which CreditRisk⁺ belongs to.

Bernoulli Mixture Models

CreditRisk⁺ belongs to the family of mixture models, which is a convenient way to implicitly introduce correlation between defaults through a mixing distribution. Instead of being fixed numbers, default probabilities are considered random variables and the mixing distribution specifies the dynamics of the default probabilities.

Consider a portfolio of m obligors and define as in the introduction I_i for each $i \in \{1, ..., m\}$, the default indicator

$$I_i = \begin{cases} 1, \text{ if obligor i defaults} \\ 0, \text{ o.w.} \end{cases}$$

and let $I := \sum_{i=1}^{m} I_i$. We assume given a default probability $p_i = \mathbb{E}[I_i = 1]$ for each obligor. In the simplest case when defaults are independent from each other, the number of defaults would have expectation

$$\mathbb{E}[I] = \sum_{i=1}^{m} p_i$$

and variance

$$\operatorname{Var}[I] = \sum_{i=1}^{m} p_i (1 - p_i),$$

by basic properties of the Bernoulli distribution. The independence assumption is highly unrealistic and the modelling of a dependence structure is in fact a central challenge in credit risk modelling. One approach to introduce correlation between the default indicators is to introduce stochasticity in the default probabilities through a *mixing distribution*.

Assume now that there is a distribution F with support in $[0, 1]^m$ so that the now random default probabilities $(P_1, ..., P_m) \sim F$. We assume additionally that given an outcome of $p = (p_1, ..., p_m)$, the default indicators $I_1, ..., I_m$ are mutually independent, i.e.,

$$(I_i|P_i = p_i) \sim \text{Be}(p_i),$$

 $(I_i|P = p)_{i=1,\dots,m} \text{ independent }$

The full information of the default probabilities is then obtained by integrating over P,

$$\mathbb{P}(I_1 = i_1, ..., I_m = i_m) = \int_{[0,1]^m} \prod_{j=1}^m p_i^{i_j} (1 - p_i)^{1 - i_j} dF(p_1, ..., p_m).$$

In this setting, we have

 $\mathbb{E}[I_i] = \mathbb{E}[P_i].$

A well known result in statistics to be found in e.g. ([9], p. 37) states that

$$\operatorname{Cov}[X,Y] = \mathbb{E}[\operatorname{Cov}[X,Y|Z]] + \operatorname{Cov}[\mathbb{E}[X|Z],\mathbb{E}[Y|Z]].$$
(3)

From this the variance can be found to be

$$\operatorname{Var}[I_i] = \mathbb{E}[P_i](1 - \mathbb{E}[P_i]).$$

The covariance between the default indicators is

$$\operatorname{Cov}[I_i, I_j] = \mathbb{E}[I_i I_j] - \mathbb{E}[I_i] \mathbb{E}[I_j] = \operatorname{Cov}[P_i, P_j]$$
(4)

and hence the default correlation in this type of model is

$$\operatorname{Corr}[I_i, I_j] = \frac{\operatorname{Cov}[P_i, P_j]}{\sqrt{\mathbb{E}[P_i](1 - \mathbb{E}[P_i])}\sqrt{\mathbb{E}[P_j](1 - \mathbb{E}[P_j])}}.$$
(5)

We see that the dependence between the default indicators depends on how we specify F.

One particular choice of a mixing distribution is the Beta-distribution. We describe this in some detail since it serves as a pedagogical illustration of what effect default correlation can have on the distribution of defaults [2].

We assume that $I_1, ..., I_m$ are all Bernoulli-distributed with common default probability P. Let P

have a univariate Beta(a, b)-distribution which implies that it has the density function

$$f(p) = \frac{1}{\beta(a,b)} p^{a-1} (1-p)^{b-1}, \ a, b > 0, \ p \in (0,1),$$

where $\beta(a, b)$ can be given in terms of the Gamma function as

$$\beta(a,b) = \int_{0}^{1} p^{a-1} (1-p)^{b-1} dp = \frac{\Gamma_0(a)\Gamma_0(b)}{\Gamma_0(a+b)}.$$

Using that $\Gamma_0(p+1) = p\Gamma_0(p)$ we find the first and second order moments of P to be

$$\mathbb{E}[P] = \frac{1}{\beta(a,b)} \int_{0}^{1} p^{a} (1-p)^{b-1} dp = \frac{\beta(a+1,b)}{\beta(a,b)} = \frac{a}{a+b}$$

and

$$\mathbb{E}[P^2] = \frac{\beta(a+2,b)}{\beta(a,b)} = \frac{a(a+1)}{(a+b)(a+b+1)}$$

Following the outline of a Bernoulli Mixture Model, conditional on P, each obligor follows a Bernoullidistribution and the total number of defaults therefore follows a Binomial distribution with parameters m and P. The distribution of I, the total number of defaults, is therefore given by

$$\mathbb{P}(I=k) = \binom{m}{k} \int_0^1 p^k (1-p)^{m-k} f(p) dp$$
$$= \binom{m}{k} \frac{1}{\beta(a,b)} \int_0^1 p^{a+k-1} (1-p)^{m-k+b-1} dp$$
$$= \binom{m}{k} \frac{\beta(a+k,b+n-k)}{\beta(a,b)},$$

which we recognize as the Beta-Binomial distribution. The expected number of defaults in this model is

$$\mathbb{E}[I] = \mathbb{E}[\mathbb{E}[I|P]] = \mathbb{E}[mP] = m\frac{a}{a+b}.$$

Equation (4) can now be used to find that $\operatorname{Cov}[I_i, I_j] = \operatorname{Cov}[P, P] = \mathbb{E}[P^2] - \mathbb{E}[P]^2$. The correlation in this model is therefore by (5),

$$\frac{\mathbb{E}[P^2] - \mathbb{E}[P]}{\mathbb{E}[P] - \mathbb{E}[P]^2} = \frac{1}{a+b+1}.$$



Figure 1: Distribution functions and quantile functions in the Beta-Binomial model for c = 0.1%, 1%, and 5% as well as the case of independent obligors.

If the parameters (a, b) are set to

$$(a,b) = \frac{1-c}{c}(p,1-p), \ c \in (0,1),$$

we are ensured that the expected value of the random default probability P will be p on average and the correlation between any two obligors is c. We can therefore fix a value of c and see what effect this has on the distribution of defaults. We construct an example with $m = 10\ 000$ obligors, all with a mean default probability of 5% and let the correlation vary between 0 and 10%. The distribution functions and quantile functions are illustrated in Figure 1. As can be seen, mean default rates gives little information about the distribution of the number of defaults.

Poisson Mixture Models

Another form of mixture model is Poisson Mixture Models where default of obligor i is assumed to be Poisson-distributed with some parameter λ_i , i.e.,

$$N_i \sim \operatorname{Poi}(\lambda_i)$$

This builds on the observation that for small default probabilities, a Bernoulli distribution is well approximated by a Poisson-distribution. If we set $\lambda_i = p_i$,

$$\mathbb{P}(N_i \ge 1) = 1 - e^{-p_i} \approx p_i.$$

The Poisson-approximation introduces the possibility that an obligor defaults more than once. The likelihood of this is

$$\mathbb{P}(N_i \ge 2) = 1 - e^{-\lambda_i} (1 + \lambda_i),$$

which is a small number, at least for small default probabilities. There are at least three ways to calibrate λ_i so as to match the default probabilities in the case when defaults are Bernoulli-distributed [10]. If we choose $\lambda_i = -\log(1 - p_i)$, the probability of observing at least one default is the same as observing a default in the Bernoulli-case since $\mathbb{P}(N_i \geq 1) = 1 - e^{-\lambda_i}$ for a $\operatorname{Poi}(\lambda_i)$ -distributed random variable. If we simply choose $\lambda_i = p_i$, the expectations of I_i and N_i will match. As a last alternative, one can consider setting $\lambda_i = p_i(1 - p_i)$ in which the case the variances of I_i and N_i match. The choice depends on the situation but one can note that the three choices are ordered so that $-\log(1 - p_i) \geq p_i \geq p_i(1 - p_i)$. We shall introduce dependence between the default indicators $(N_1, ..., N_m)$ in the same way as for the Bernoulli Mixture Model and see what the correlation in this model looks like.

We proceed similarly as before, by replacing the default intensities $(\lambda_1, ..., \lambda_m)$ with the random variables $\Lambda = (\Lambda_1, ..., \Lambda_m)$ so that the outcome of each obligor is determined by a Poi (Λ_i) -distribution. Λ is distributed according to some distribution F with support in $[0, \infty)^m$ and as before, we make the assumption that N_i , i = 1, ..., m are conditionally independent given Λ , i.e.

$$(N_i | \Lambda_i = \lambda_i) \sim \text{Poi}(\lambda_i)$$

 $(N_i | \Lambda = \lambda)_{i=1,...,m}$ independent.

The full distribution of defaults is given by integrating over Λ ,

$$\mathbb{P}(N_1 = n_1, ..., N_m = n_m) = \int_{[0,\infty)^m} e^{-(\lambda_1 + ... + \lambda_m)} \prod_{i=1}^m \frac{\lambda_i^{n_i}}{n_i!} dF(\lambda_1, ..., \lambda_m).$$

The first and second moments are in this case given by

$$\mathbb{E}[N_i] = \mathbb{E}[\Lambda_i]$$

and

$$\operatorname{Var}[N_i] = \operatorname{Var}[\mathbb{E}[N_i|\Lambda]] + \mathbb{E}[\operatorname{Var}[N_i|\Lambda]] = \operatorname{Var}[\Lambda_i] + \mathbb{E}[\Lambda_i]$$

We also have that $\operatorname{Cov}[N_i, N_j] = \operatorname{Cov}[\Lambda_i, \Lambda_j]$ from which we can deduce that the correlation between

 N_i and N_j is given by

$$\operatorname{Corr}[N_i, N_j] = \frac{\operatorname{Cor}[\Lambda_i, \Lambda_j]}{\sqrt{\operatorname{Var}[\Lambda_i] + \mathbb{E}[\Lambda_i]}\sqrt{\operatorname{Var}[\Lambda_j] + \mathbb{E}[\Lambda_j]}}.$$
(6)

This can be compared to (5) where we found that

$$\operatorname{Corr}[I_i, I_j] = \frac{\operatorname{Cov}[P_i, P_j]}{\sqrt{\mathbb{E}[P_i](1 - \mathbb{E}[P_i])}\sqrt{\mathbb{E}[P_j](1 - \mathbb{E}[P_j])}}$$

The denominator in this expression is, when i = j,

$$\operatorname{Var}[P_i] + \mathbb{E}[P_i(1 - P_i)] = \operatorname{Var}[P_i] + \mathbb{E}[P_i] - \mathbb{E}[P_i^2],$$

while the denominator in (6) is

$$\operatorname{Var}[\Lambda_i] + \mathbb{E}[\Lambda_i].$$

Assuming that P_i and Λ_i have the same mean and variance, we will find a higher variance in the Bernoulli Mixture Model, which will result in fatter tails in the loss distribution. This illustrates that even if the Poisson-distribution might be accurate when approximating default events, these models are not exchangeable when it comes to the shape of the distribution.

One particular choice of the mixing distribution is the Gamma-distribution, which is the distribution that CreditRisk⁺ incorporates. This choice proves fruitful, in that we can obtain a closed-form expression of the probability generating function of the loss distribution, from which the loss distribution can be obtained.

3.1 CreditRisk⁺

In the CreditRisk⁺ model, it is assumed that the default probability of each obligor is a random variable p_i^{Γ} with

$$p_i = \mathbb{P}(I_i = 1) = \mathbb{E}[p_i^{\Gamma}],$$

where the expectation is taken over a Gamma-distributed vector

$$\Gamma = (\gamma_1, ..., \gamma_k)^T,$$

where the components of Γ are mutually independent and each having expected value 1. The default probability of obligor *i* is then set to

$$p_{i}^{\Gamma} = p_{i}(w_{0i} + \sum_{r=1}^{k} w_{ri}\gamma_{r}),$$

$$w_{0i} + \sum_{r=1}^{k} w_{ri} = 1,$$
(7)

where p_i is the mean default rate for obligor *i* and has been estimated beforehand. p_i^{Γ} is the probability of default for obligor *i*, given an outcome of Γ . The numbers $w_{0i}, w_{1i}, ..., w_{ki}, i = 1, ..., m$ are called sector weights or simply weights. The fact that the weights sum to 1 and that $\mathbb{E}[\gamma_r] = 1, r = 1, ..., k$, ensures that the default probability will be p_i on average. It is easy to see that obligors *i* and *j* will show a correlation if and only if both $w_{ri} > 0$ and $w_{rj} > 0$ for some $r \in \{1, ..., k\}$. w_{0i} is called the specific or idiosyncratic weight. It is further assumed that conditional on Γ , the default of any obligor is independent from the others.

The mixing distribution Γ can be interpreted as general economic conditions in different sectors or regions, or any other set of abstract latent variables, and the weights of each obligor describe to what extent that obligor is dependent on that variable. The idiosyncratic effect determines the part of the risk that can be mitigated through diversification effects.

Imagine that all necessary parameters are given. We would then have enough information to cast this model into a Monte Carlo-framework. We would simply generate N samples $\Gamma^{(n)} = (\gamma_1^{(n)}, ..., \gamma_k^{(n)}), n = 1, ..., N$ and for each sample determine if obligor *i* defaults by drawing a sample from

$$I_i^{(n)} \sim \operatorname{Be}(p_i^{\Gamma})$$
$$p_i^{\Gamma} = p_i \left(w_{0i} + \sum_{r=1}^k w_{ri} \gamma_r^{(n)} \right)$$

We then set the loss in sample n as

$$X_n = \sum_{i=1}^m I_i^{(n)} L_i,$$

and with the samples $(X_1, ..., X_N)$ we can draw inferences about the loss distribution. This would be an example of a Bernoulli mixture model. The probability that a given obligor defaults in each round is partly determined by its beforehand estimated mean default rate p_i , but also on the outcome of the sample $\Gamma^{(n)}$, and since obligors have part of their default probability allocated to the components of Γ , the default probabilities tend to move up and down in a positively correlated manner.

We shall go one step further towards the loss distribution via the use of probability generating functions.

Loss Distribution in CreditRisk⁺

The probability generating function of a $Be(p_i)$ -distributed random variable is

$$G_I(z) = (1-p) + pz.$$

The loss of a single obligor is given by $I_i L_i$ and conditional on Γ , the probability of a loss is p_i^{Γ} . An application of (i), Proposition 2.1 gives that the conditional probability generating function for a single obligor is, for $L_i \in \mathbb{N}$,

$$G_i(z|\Gamma) = (1 - p_i^{\Gamma}) + p_i^{\Gamma} z^{L_i}.$$
(8)

Conditional on Γ , the default events are assumed independent, so by (*ii*) of Proposition 2.1 we get the conditional probability generating function of the total loss as

$$G_X(z|\Gamma) = \prod_i G_i(z|\Gamma).$$

The unconditional probability generating function of the loss is then obtained by integrating over Γ ,

$$G_X(z) = \int G_X(z|\Gamma = \gamma) dF_{\Gamma}(\gamma).$$

There is no necessity to work with a Gamma-distribution, but as will be shown, this choice together with an approximation of the default indicators I_i by a Poisson-distribution, allows us to obtain a closed form expression for $G_X(z)$.

For small default probabilities, we can make the approximation

$$\log(1 + p_i^{\Gamma}(z^{L_i} - 1)) \approx p_i^{\Gamma}(z^{L_i} - 1).$$

Rewriting Equation (8) as

$$G_i(z|\Gamma) = \exp\{\log(1 + p_i^{\Gamma}(z^{L_i} - 1))\},\$$

together with this approximation we see that

$$G_i(z|\Gamma) \approx \exp\{p_i^{\Gamma}(z^{L_i} - 1)\}\tag{9}$$

is a good approximation of the conditional pgf of an obligor, assuming the default probabilities are small enough. We recognize (9) as the pgf of a random variable $L_i N_i$ where N_i is Poisson-distributed with parameter p_i^{Γ} . We therefore continue by replacing the default indicators $I_i \sim \text{Be}(p_i)$ with $N_i \sim \text{Poi}(p_i)$ and move from a Bernoulli Mixture Model to a Poisson Mixture Model.

By conditional independence between obligors, we obtain the conditional pgf of the total loss X by taking the product of each of the conditional pgfs,

$$G_X(z|\Gamma) = \prod_i G_i(z|\Gamma) = \exp\left\{\sum_i p_i^{\Gamma}(z^{L_i} - 1)\right\},\tag{10}$$

which we can write as

$$G_X(z|\Gamma) = \exp\left\{\sum_i \sum_{r=0}^k p_i w_{ri} \gamma_r (z^{L_i} - 1)\right\}$$
(11)

by using (7) and letting $\gamma_0 \equiv 1$. We define

$$\mu_r := \sum_i w_{ri} p_i$$

and

$$\mathcal{P}_r(z) := \frac{1}{\mu_r} \sum_i w_{ri} p_i z^{L_i}$$

which enables us to write (11) as

$$G_X(z|\Gamma) = \exp\left\{\sum_{r=0}^k \gamma_r \left(\sum_i p_i w_{ri}(z^{L_i} - 1)\right)\right\}$$

= $\exp\left\{\sum_{r=0}^k \gamma_r \mu_r (\mathcal{P}_r(z) - 1)\right\}.$ (12)

What remains in order to obtain the pgf of the loss X is to integrate (12) w.r.t Γ , i.e., we want to calculate

$$G_X(z) = \int G_X(z|\Gamma = \gamma) f_{\Gamma}(\gamma) d\gamma, \qquad (13)$$

where $f_{\Gamma}(\gamma)$ is the density function of $\Gamma = (\gamma_1, ..., \gamma_k)^T$. Let (α_l, β_l) be the parameters of γ_l , to be specified later on. Let us first find out what this density function is. The components are mutually independent which allows us to factorize the density. A $\text{Gamma}(\alpha_l, \beta_l)$ -distributed random variable has expectation and variance

$$\mathbb{E}[\gamma_l] = \alpha_l \beta_l,$$

$$\operatorname{Var}[\gamma_l] = \alpha_l \beta_l^2.$$
(14)

We have required that $\mathbb{E}[\gamma_l] = 1$ so we set $\beta_l = \frac{1}{\alpha_l}$ and get by (2) that the density of Γ is

$$f_{\Gamma}(\gamma) = \prod_{l=1}^{k} \frac{\gamma_{l}^{\alpha_{l}-1} \alpha_{l}^{\alpha_{l}}}{\Gamma_{0}(\alpha_{l})} e^{-\alpha_{l} \gamma_{l}},$$

where Γ_0 is the Gamma-function. Carrying out the integration in (13) with this density and first taking out what is known we get

$$G_X(z) = \int \exp\{\sum_{r=0}^k \gamma_r \mu_r(\mathcal{P}_r(z) - 1)\} \prod_{l=1}^k \frac{\gamma_l^{\alpha_l - 1} \alpha_l^{\alpha_l}}{\Gamma_0(\alpha_l)} e^{-\alpha_l \gamma_l} d\gamma_l$$

= $\exp\{\mu_0(\mathcal{P}_0(z) - 1)\} \prod_{r=1}^k \frac{\alpha_r^{\alpha_r}}{\Gamma_0(\alpha_r)} \int_0^\infty \prod_{l=1}^k \exp\{\gamma_l(\mu_l(P_l(z) - 1) - \alpha_l)\} \gamma_l^{\alpha_l - 1} d\gamma_l.$

By factorizing the integral, we get

$$G_X(z) = \exp\{\mu_0(\mathcal{P}_0(z) - 1)\} \prod_{r=1}^k \frac{\alpha_r^{\alpha_r}}{\Gamma_0(\alpha_r)} \int_0^\infty \exp\{\gamma_r(\mu_r(\mathcal{P}_r(z) - 1) - \alpha_r)\} \gamma_r^{\alpha_r - 1} d\gamma_r.$$
(15)

We take a closer look at the integral in (15). Let $x = \alpha_r - \mu_r(\mathcal{P}_r(z) - 1)$ so that the integral can be written

$$\int_{0}^{\infty} e^{-\gamma_r x} \gamma_r^{\alpha_r - 1} d\gamma_r,$$

or equivalently

$$\frac{1}{x^{\alpha_r-1}}\int\limits_0^\infty e^{-\gamma_r x} (\gamma_r x)^{\alpha_r-1} d\gamma_r.$$

Recall that the Gamma function is defined as $\Gamma_0(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy$ and make the substitution

$$y = \gamma_r x,$$
$$dy = x d\gamma_r$$

We then find that

$$\frac{1}{x^{\alpha_r-1}} \int_0^\infty e^{-\gamma_r x} (\gamma_r x)^{\alpha_r-1} d\gamma_r$$
$$= \frac{1}{x^{\alpha_r-1}} \int_0^\infty e^{-y} y^{\alpha_r-1} \frac{dy}{x}$$
$$= \frac{1}{x^{\alpha_r}} \int_0^\infty e^{-y} y^{\alpha_r-1} dy = \frac{1}{x^{\alpha_r}} \Gamma_0(\alpha_r).$$

Equation (15) then reduces to

$$G_X(z) = \exp\{\mu_0(\mathcal{P}_0(z) - 1)\} \prod_{r=1}^k \left(\frac{\alpha_r}{\alpha_r - \mu_r(\mathcal{P}_r(z) - 1)}\right)^{\alpha_r} = \exp\{\mu_0(\mathcal{P}_0(z) - 1)\} \prod_{r=1}^k \left(\frac{1}{1 - \frac{\mu_r}{\alpha_r}(\mathcal{P}_r(z) - 1)}\right)^{\alpha_r}.$$
(16)

We now replace μ_r and $\mathcal{P}_r(z)$ with $\mu_r = \sum_i w_{ri} p_i$ and $\mathcal{P}_r(z) = \frac{1}{\mu_r} \sum_i w_{ri} p_i z^{L_i}$ as previously defined and put this into (16) to get

$$G_X(z) = \exp\{\sum_i p_i w_{0i}(z^{L_i} - 1)\} \prod_{r=1}^k \left(\frac{1}{1 - \frac{1}{\alpha_r}\sum_i p_i w_{ri}(z^{L_i} - 1)}\right)^{\alpha_r}$$

With parameters of Γ chosen as $\{(\alpha_r, 1/\alpha_r)\}_{r=1}^k$, we still have to specify α_r . Defining $\sigma_r^2 = \operatorname{Var}[\gamma_r]$ we must necessarily have that $\alpha_r = \sigma_r^{-2}$ by (14) and σ_r will be estimated from obligor data, which will be described when the model is implemented. As of now, we simply write $\alpha_r = \sigma_r^{-2}$ and the pgf of X can be written

$$G_X(z) = \exp\{\sum_i p_i w_{0i}(z^{L_i} - 1)\} \prod_{r=1}^k \left(\frac{1}{1 - \sigma_r^2 \sum_i p_i w_{ri}(z^{L_i} - 1)}\right)^{\frac{1}{\sigma_r^2}}.$$
 (17)

The probability generating function can then be used to derive the distribution of losses. We shall not go further here, since we will use the pgf only to find the characteristic function of the loss, and from there derive its distribution. An efficient method if one wants to go through the pgf can be found in e.g. [11].

Moments of the Loss Distribution and Default events

Closed-form expressions of various moments of the loss distribution and default events in CreditRisk⁺ can be derived, and we have the necessary knowledge about CreditRisk⁺ to do this now.

We shall use the already stated formula

$$\operatorname{Cov}[X,Y] = \mathbb{E}[\operatorname{Cov}[X,Y|Z]] + \operatorname{Cov}[\mathbb{E}[X|Z],\mathbb{E}[Y|Z]].$$
(18)

Conditional on Γ , defaults of each obligor is $\operatorname{Poi}(p_i(w_{0i} + \sum_{r=1}^k w_{ri}\gamma_r))$ -distributed from which we can deduce that

$$\operatorname{Var}[N_i|\Gamma] = \mathbb{E}[N_i|\Gamma] = p_i(w_{0i} + \sum_{r=1}^k w_{ri}\gamma_r).$$

Using this together with (18) we find that, when $i \neq j$,

$$Cov[N_i, N_j] = \mathbb{E}\left[Cov[N_i, N_j | \gamma]\right] + Cov\left[\mathbb{E}[N_i | \Gamma], \mathbb{E}[N_j | \Gamma]\right]$$
$$= 0 + Cov\left[p_i(w_{0i} + \sum_{r=1}^k w_{ri}\gamma_r), p_j(w_{0j} + \sum_{r=1}^k w_{rj}\gamma_r)\right]$$
$$= p_i p_j \sum_{r=1}^k \sum_{l=1}^k w_{ri}w_{rj}Cov[\gamma_k, \gamma_l]$$
$$= p_i p_j \sum_{r=1}^k w_{ri}w_{rj}\sigma_r^2,$$

where conditional independence between N_i , N_j , as well as independence between the components of Γ has been used. When i = j, we get the variances,

$$\operatorname{Var}[N_i] = \mathbb{E}\left[\operatorname{Var}[N_i|\Gamma]\right] + \operatorname{Var}\left[\mathbb{E}[N_i|\Gamma]\right]$$
$$= p_i\left(\sum_{r=0}^k w_{ri}\right) + p_i^2\left(\sum_{r=1}^k w_{ri}\sigma_r^2\right).$$

Denoting by Cov[N] the covariance matrix of $N = (N_1, ..., N_m)$ we can write the elements of Cov[N] as

$$\operatorname{Cov}[N]_{i,j} = p_i p_j \sum_{r=1}^k w_{ri} w_{rj} \sigma_r^2 + \delta_{i,j} p_i \sum_{r=0}^k w_{ri},$$

where $\delta_{i,j} = 1$ if i = j and 0 if $i \neq j$. This can be used to calculate the variance of the loss as

$$Var[X] = Var[\sum_{i=1}^{m} N_i L_i] = \sum_{i=1}^{m} Var[N_i L_i] + 2\sum_{i < j} Cov[N_i L_i, N_j L_j]$$
$$= \sum_{i=1}^{m} \sum_{j=1}^{m} L_i L_j Cov[N]_{i,j} = \sum_{i=1}^{m} \sum_{j=1}^{m} L_i L_j \Big[p_i p_j \sum_{r=1}^{k} w_{ri} w_{rj} \sigma_r^2 + \delta_{i,j} p_i \sum_{r=0}^{k} w_{ri} \Big].$$

3.2 Customizations

Three customizations of CreditRisk⁺ are proposed and described below.

In the model currently used at EKN, a set of parameters that serve to introduce correlation between commitments is needed as input. These are today set manually which has been desired to improve. In the following section, a maximum likelihood approach to set the sector weights will be described. This will at least provide a useful reference for reasonable values of weight parameters that reflect covariation pattern in historical data.

It has been shown in [1] that the probability mass function of the loss distribution is particularly well suited for a Fast Fourier Transform, as will be shown in Section 3.2.2. The main advantages of using a Fourier Inversion are that the calculations are fast and stable, and that obligors don't need to be grouped in similar exposure sizes, as is necessary in the original formulation of CreditRisk⁺. In the original formulation, it is assumed that each exposure is an integer multiple of a basic loss unit. The choice of this unit may be problematic if the exposures are of very different sizes, as is the case for EKN [1]. With no need of a grouping on size and the computational efficiency of Fourier Inversion, this step can be avoided which simplifies the implementation and allows one to treat each risk separately. This is convenient for EKN who wish to be able to determine single risk contributions easily.

Lastly, the model shall be extended to allow for multi-period calculations, enabling hold-to-maturity analyses as well as taking account for the fact that the exposure decreases at the same rate that payments are made, not necessarily on a yearly basis.

3.2.1 Weight Parametrization

There are some methods available in the literature to set the sector weights from historical data. Lesko et al. [12] proposed a parametrization based on Principal Component Analysis. However, the output of the proposed algorithm can not make sure that the weights are all non-negative and sum to 1 as is needed in CreditRisk⁺. Vandendorpe et al. [13] proposed an algorithm based on Symmetric non-negative matrix factorization. While guaranteeing the desired conditions on the weights, the algorithm requires a default correlation matrix, grouped on clusters of obligors with similar default probability, which needs a non-trivial estimating procedure itself. Inspired by the theory of Generalized Linear Mixed Models applied to credit risk modelling, as described in e.g. ([14], [15]), a more direct approach through maximum likelihood estimation is proposed below to find values of sector weights.

Suppose we have default statistics available for a number of years, divided into groups of obligors with similar default probability, e.g. by taking their credit rating as an indicator of default probability. Moody's Investors Service publishes such reports regularly, where one can find, for each rating class ranging from the highest grade Aaa to the lowest Ccc-C, how many issuers that were involved in the study for each year and how many of those that defaulted during the year. We introduce the following notation, and give an illustration in Table 1 to clarify. The data set used in the implementation are shown in Table 7 in the Appendix.

Notation

- $r \in \{1, ..., k\}$ rating class indicator in decreasing credit-worthiness.
- $N_{j,r}$ the r.v. describing number of defaults in rating class r and year j.
- $m_{j,r}$ number of obligors in rating class r at the beginning of year j.
- $M_{j,r}$ observed number of defaults in rating class r during year j.
- $M_j = (M_{j,1}, ..., M_{j,k})$ vector of observed number of defaults during year j.
- $M = (M_1, ..., M_n)$ vector of total yearly observed number of defaults.
- λ_r mean default intensity for obligors in rating class r.
- $w_r = (w_{1r}, ..., w_{kr})$ vector of weights for obligors in rating class r (excluding the specific weight

 w_{0r} .

Year	A		Baa		Ba		В		Caa-C	
1	$m_{1,1}$	$M_{1,1}$	$m_{1,2}$	$M_{1,2}$	$m_{1,3}$	$M_{1,3}$	$m_{1,4}$	$M_{1,4}$	$m_{1,5}$	$M_{1,5}$
2	$m_{2,1}$	$M_{2,1}$	$m_{2,2}$	$M_{2,2}$	$m_{2,3}$	$M_{2,3}$	$m_{2,4}$	$M_{2,4}$	$m_{1,5}$	$M_{2,5}$
:										
j	$m_{i,1}$	$M_{i,1}$	$m_{j,2}$	$M_{i,2}$	$m_{j,3}$	$M_{i,3}$	$m_{i,4}$	$M_{i,4}$	$m_{j,5}$	$M_{i,5}$
:	5,	57	57	57	57-	J)-	5,	57	57-	J / -
• n	m .	М.	ma	M_{-2}	m	M_{-2}	m .	м.	- m	М -
n	$m_{n,1}$	$M_{n,1}$	$m_{n,2}$	$M_{n,2}$	$m_{n,3}$	$M_{n,3}$	$m_{n,4}$	$M_{n,4}$	$m_{n,5}$	$M_{n,5}$

Table 1: Number of obligors m and number of defaults M by rating category and year.

The goal is now to fit a k-factor CreditRisk⁺ model based on observations of the kind illustrated in Table 1 through a maximum likelihood estimation. The output is a weight matrix

$$W = \begin{bmatrix} w_{01} & w_{11} & \dots & w_{k1} \\ w_{02} & w_{12} & \dots & w_{k2} \\ & \vdots & & \\ w_{0k} & w_{1k} & \dots & w_{kk} \end{bmatrix}$$

where each row sum to 1 and are the weights for an obligor in rating class r.

Recall from (1) that the sum of Poisson-distributed random variables $N_1, ..., N_n$, each with intensity λ , is Poisson-distributed with intensity $n\lambda$. If we assume that each obligor belonging to the same rating class has the same default intensity λ_r , the probability that we observed $M_{j,r}$ defaults in group r and year j, given Γ , is then

$$\mathbb{P}(N_{j,r} = M_{j,r}|\Gamma) = \exp\{-m_{j,r}\lambda_r(w_{0r} + w_r\Gamma)\}\frac{(m_{j,r}\lambda_r(w_{0r} + w_r\Gamma))^{M_{j,r}}}{M_{j,r}!}.$$
(19)

Given Γ , we assume (as in CreditRisk⁺) that all obligors are independent. The probability of observing M_j is therefore the product of the number of defaults in each rating class, i.e.

$$\mathbb{P}(N_j = M_j | \Gamma) = \prod_{r=1}^k \mathbb{P}(N_{j,r} = M_{j,r} | \Gamma).$$
(20)

The likelihood function $L(M_j|W)$ is then obtained by integration over Γ ,

$$L(M_j|W) = \int_{\mathbb{R}^k} \mathbb{P}(N_j = M_j|\Gamma) dF_{\Gamma}.$$
(21)

If we furthermore assume independence between the years (this assumption is discussed further in

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Section 4.3), we get the full likelihood function as

$$L(M|W) = \prod_{j=1}^{n} L(M_j|W).$$
 (22)

Looking at the likelihood function $L(M_j|W)$ in (21), we start by moving out everything that does not depend on Γ from the integral,

$$\begin{split} L(M_{j}|W) &= \int_{\mathbb{R}^{k}} \prod_{r=1}^{k} \exp\{-m_{j,r}\lambda_{r}(w_{0r}+w_{r}\Gamma)\} \frac{(m_{j,r}\lambda_{r}(w_{0r}+w_{r}\Gamma))^{M_{j,r}}}{M_{j,r}!} dF_{\Gamma} \\ &= \prod_{r=1}^{k} \exp\{-m_{j,r}\lambda_{r}w_{0r}\} \frac{(m_{j,r}\lambda_{r})^{M_{j,r}}}{M_{j,r}!} \int_{\mathbb{R}^{k}} \prod_{r=1}^{k} \exp\{-m_{j,r}\lambda_{r}w_{r}\Gamma\} (w_{0r}+w_{r}\Gamma)^{M_{j,r}} dF_{\Gamma}. \end{split}$$

To get the full likelihood function we take the product of $L(M_j|W)$ over j = 1, ..., n,

$$L(M|W) = \prod_{j=1}^{n} \prod_{r=1}^{k} \exp\{-m_{j,r}\lambda_{r}w_{0r}\} \frac{(m_{j,r}\lambda_{r})^{M_{j,r}}}{M_{j,r}!} \prod_{j=1}^{n} \int_{\mathbb{R}^{k}} \prod_{r=1}^{k} \exp\{-m_{j,r}\lambda_{r}w_{r}\Gamma\}(w_{0r}+w_{r}\Gamma)^{M_{j,r}}dF_{\Gamma}.$$

We then take the logarithm of L(M|W),

$$\begin{split} l(M|W) &:= \log \left(\prod_{j=1}^{n} \prod_{r=1}^{k} \exp\{-m_{j,r} \lambda_{r} w_{0r}\} \frac{(m_{j,r} \lambda_{r})^{M_{j,r}}}{M_{j,r}!} \prod_{j=1}^{n} \int_{\mathbb{R}^{k}} \prod_{r=1}^{k} \exp\{-m_{j,r} \lambda_{r} w_{r} \Gamma\} (w_{0r} + w_{r} \Gamma)^{M_{j,r}} dF_{\Gamma} \right) \\ &= \sum_{j=1}^{n} \sum_{r=1}^{k} \log \left(\exp\{-m_{j,r} \lambda_{r} w_{0r}\} \frac{(m_{j,r} \lambda_{r})^{M_{j,r}}}{M_{j,r}!} \right) + \sum_{j=1}^{n} \log I_{j}, \end{split}$$

where I_j is the integral

$$\int_{\mathbb{R}^k} \prod_{r=1}^k \exp\{-m_{j,r}\lambda_r w_r \Gamma\} (w_{0r} + w_r \Gamma)^{M_{j,r}} dF_{\Gamma}$$

and we simplify l(M|W) further to

$$l(M|W) = \sum_{j=1}^{n} \sum_{r=1}^{k} M_{j,r} \left[\log(m_{j,r}) + \log(\lambda_r) - \log(M_{j,r}!) \right] - m_{j,r} \lambda_r w_{0r} + \sum_{j=1}^{n} \log I_j.$$
(23)

Our final optimization problem is then

$$\begin{array}{ll} \underset{W}{\operatorname{maximize}} & l(M|W) \\ \text{subject to} & W\mathbf{1} = \mathbf{1} \\ \text{and} & W_{i,j} \geq 0, \ i, j = 0, 1, ..., k. \end{array}$$

$$(24)$$

where W is the weight matrix described above and $W\mathbf{1} = \mathbf{1}$ meaning that the rows of W all sum to 1.

With the weight matrix at hand, each obligor i is put in the group r that minimizes $|p_i - \lambda_r|$. Each obligor will keep its estimated default probability but will share the same weights as all other obligors in the same group.

Details on how to solve the optimization problem is given in Section 4.1.

3.2.2 Loss Calculations Through Fourier Inversion

It was stated in Equation (17) of Section 3.1 that the probability generating function of the loss distribution is given by

$$G_X(z) = \exp\{\sum_i p_i w_{0i}(z^{L_i} - 1)\} \prod_{r=1}^k \left(\frac{1}{1 - \sigma_r^2 \sum_i p_i w_{ri}(z^{L_i} - 1)}\right)^{\frac{1}{\sigma_r^2}}.$$
(25)

Using Proposition 2.3, we get the characteristic function (Fourier transform) of X by the relation $\Phi_X(z) = G_X(e^{iz})$. Using this on (25) we get

$$\Phi_X(z) = \exp\{\sum_i p_i w_{0i}(e^{izL_i} - 1)\} \prod_{r=1}^k \left(\frac{1}{1 - \sigma_r^2 \sum_i p_i w_{ri}(e^{izL_i} - 1)}\right)^{\frac{1}{\sigma_r^2}}.$$
(26)

It was also stated in Section 2 that any real-valued r.v. has a characteristic function, which uniquely determines its distribution through Theorem 2.2: provided integrability of the characteristic function of X and its density f,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-izx} \Phi_X(z) dz.$$

In the case of CreditRisk⁺, the loss distribution does not have a density since it is not a continuous

random variable, so the conditions of Theorem 2.2 are not satisfied. However, in the case of a large portfolio, we can safely assume that the loss can be approximated by a continuous random variable that has a density f. We shall also assume that this density is nonzero only on a compact interval [a, b], which clearly is the case in our setting since the smallest possible loss is 0 and the largest is the sum of all exposures.

Proceeding under these assumptions, the Fourier transform of X is in any case available through (26) which we can calculate on a discrete set of points. We denote these points by $\{z_k : k = 0, 1, ..., Z - 1\}$ and make the following definitions

$$\Delta x := \frac{b-a}{Z-1}$$

$$\Delta z := \frac{2\pi}{Z\Delta x}$$

$$x_k := a + k\Delta x$$

$$f_k := f(x_k)$$

$$z_k := \begin{cases} k\Delta z & \text{if } k < \frac{z}{2} \\ (k-z)\Delta z & \text{otherwise} \end{cases}$$

$$E_{jk} := \exp\{2\pi i \frac{jk}{Z}\}$$

We see that the known function $\Phi_X(z)$ can be approximated as (c.f. Definition 2.3)

$$\Phi_X(z) = \int_{-\infty}^{\infty} e^{izx} f(x) dx \approx \Delta x e^{iza} \sum_{k=0}^{Z-1} e^{izk\Delta x} f(a+k\Delta x).$$

With the established notation, we can express this as

$$\Phi_X(z_k) = \Delta x e^{iaz_k} \sum_{j=0}^{Z-1} E_{kj} f_j$$
(27)

which is a linear system of equations, with the only unknown f_j :

$$\begin{pmatrix} \Phi_X(z_0) \\ \Phi_X(z_1) \\ \vdots \\ \Phi_x(z_{2-1}) \end{pmatrix} = \Delta x \begin{pmatrix} e^{iaz_0} E_{0,0} & e^{iaz_0} E_{0,1} & \dots & e^{iaz_0} E_{0,Z-1} \\ e^{iaz_1} E_{1,0} & e^{iaz_1} E_{1,1} & \dots & e^{iaz_1} E_{1,Z-1} \\ \vdots \\ e^{iaz_{2-1}} E_{Z-1,0} & e^{iaz_{2-1}} E_{Z-1,1} & \dots & e^{iaz_{2-1}} E_{Z-1,Z-1} \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{Z-1} \end{pmatrix}$$

The inverse of E_{jk} is

$$E_{jk}^{-1} = \frac{1}{Z} \exp\{-2\pi i \frac{jk}{Z}\},\$$

as can be verified by the calculation

$$\sum_{k=0}^{Z-1} E_{jk} E_{kl}^{-1} = \frac{1}{Z} \sum_{k=0}^{Z-1} (e^{2\pi i \frac{j-l}{Z}})^k = \begin{cases} 1 & j=l \\ \frac{1}{Z} \frac{1-e^{2\pi i (j-l)}}{1-e^{2\pi i (j-l)/Z}} = 0, & j \neq l. \end{cases}$$

Hence, the formula for f_j is

$$f_j = \frac{1}{Z\Delta x} \sum_{k=0}^{Z-1} \exp\left\{-2\pi i \frac{jk}{Z}\right\} e^{-iaz_k} \Phi_X(z_k).$$
(28)

With (28) and the characteristic function given in (26), we have all the information needed to calculate the density we seek.

We shall however give some extra attention to $\Phi_X(z_k)$ to make it better suited for numerical computations. To make the presentation of (26) more compact we define

$$\xi_r(z) := \sum_i p_i w_{ri} (e^{izL_i} - 1)$$

and hence

$$\Phi_X(z) = \exp\{\xi_0(z)\} \prod_{r=1}^k \left(\frac{1}{1 - \sigma_r^2 \xi_r(z)}\right)^{\frac{1}{\sigma_r^2}}.$$

We see that

$$\left(\frac{1}{1-\sigma_r^2\xi_r(z)}\right)^{\frac{1}{\sigma_r^2}} = \exp\left\{\log\left(\left(\frac{1}{1-\sigma_r^2\xi_r(z)}\right)^{\frac{1}{\sigma_r^2}}\right)\right\}$$
$$= \exp\left\{\frac{1}{\sigma_r^2}\left(\log(1) - \log(1-\sigma_r^2\xi_r(z))\right)\right\}$$
$$= \exp\left\{-\frac{1}{\sigma_r^2}\log\left(1-\sigma_r^2\xi_r(z)\right)\right\}$$

and hence

$$\Phi_X(z) = \exp\left\{\xi_0(z) - \sum_{r=1}^k \frac{1}{\sigma_r^2} \log(1 - \sigma_r^2 \xi_r(z))\right\}$$
(29)

Remark. The argument of $\Phi_X(z)$ is a complex number, which requires that we specify for which arguments the logarithm of $\Phi_X(z)$ is taken. For our purposes, we just note that the definition of z_k above ensures that we are referring to the *principal branch* of the complex logarithm and refer to any textbook on complex analysis for details, e.g. ([16], p. 97).

Remark. The case of independent obligors. From (10), we see that the pgf of an obligor whose default intensity is independent of Γ is

$$G_i(z) = \exp\{p_i(z^{L_i} - 1)\}\$$

and by independence, the pgf of the loss distribution is the product of the pgf of each obligor,

$$G_X(z) = \prod_i G_i(z) = \exp \left\{ \sum_i p_i(z^{L_i} - 1) \right\}.$$

We can then use the relation $\Phi_X(z) = G_X(e^{iz})$ to conclude that

$$\Phi_X(z) = \exp\Big\{\sum_i p_i(e^{izL_i} - 1)\Big\}.$$
(30)

This case will be included as a reference in Section 4, to see what effect the dependence structure has on the loss distribution.

3.2.3 Extension to a Multi-Period Framework

Extending the model to a Multi-period framework is not difficult. The payment guarantees that EKN issues are typically of hold-to-maturity type. A question that arises is therefore if the organization would remain solvent within reasonable confident bounds, were it to fulfill all its current commitments

without issuing any new guarantees (a "run-off scenario"). The longest guarantees extend over 20 years. Furthermore, even in a one-year perspective, payments from the counter-party are typically made quarterly or every half year. With a multi-period framework, calculations on a 1-year time horizon can be divided into quarters, say, so that the exposures decrease by the amount paid every quarter.

Consider a number of periods t = 1, ..., T. We interpret them as years but they can be quarters, months or any other time unit. We shall define $p_{i,t}$ to be the marginal probability of default at time tsimply the probability that obligor i defaults (exactly) in period t. As previously, these probabilities are assumed given exogenously. We shall repeat the first steps towards the probability generating function of the loss that we took in Section 3.1. Under Bernoulli-distributed defaults, the conditional probability generating function for an obligor was said to be

$$G_i(z|\Gamma) = (1 - p_i^{\Gamma}) + p_i^{\Gamma} z^{L_i}.$$

Since the marginal default probabilities $p_{i,t}$ are necessarily mutually exclusive, we must have that the probability generating function in this case must be

$$G_i(z|\Gamma) = 1 + \sum_{t=1}^T p_{i,t}^{\Gamma}(z^{L_{i,t}} - 1),$$

where $L_{i,t}$ is the outstanding exposure for obligor *i* in period *t*. Making the approximation

$$\log\left(1 + \sum_{t=1}^{T} p_{i,t}^{\Gamma}(z^{L_{i,t}} - 1)\right) \approx \sum_{t=1}^{T} p_{i,t}^{\Gamma}(z^{L_{i,t}} - 1),$$

we find that

$$\exp\left\{\log(1+\sum_{t=1}^{T}p_{i,t}^{\Gamma}(z^{L_{i,t}}-1))\right\} \approx \exp\left\{\sum_{t=1}^{T}p_{i,t}^{\Gamma}(z^{L_{i,t}}-1)\right\}$$

is a good approximation, provided that the marginal default probabilities are small enough. Proceeding with this approximation, we find by conditional independence that

$$G_X(z|\Gamma) = \prod_{i=1}^m G_i(z|\Gamma) = \prod_{i=1}^m \exp\Big\{\sum_{t=1}^T p_{i,t}^{\Gamma}(z^{L_{i,t}}-1)\Big\},\$$

which can be written (setting $\gamma_0 \equiv 1$),

$$\exp\Big\{\sum_{i}\sum_{t=1}^{T}\sum_{r=0}^{k}p_{i,t}w_{ri}\gamma_{r}(z^{L_{i,t}}-1)\Big\}.$$

Defining

$$\mu_r := \sum_i \sum_{t=1}^T p_{i,t}$$

and

$$\mathcal{P}_r(z) := \frac{1}{\mu_r} \sum_i \sum_{t=1}^T w_{ri} p_{i,t} z^{L_{i,t}},$$

the conditional pgf can be written more compactly as

$$G_X(z|\Gamma) = \exp\Big\{\sum_{r=0}^k \gamma_r\Big(\sum_i \sum_{t=1}^T p_{i,t} w_{ri}(z^{L_{i,t}} - 1)\Big)\Big\}$$
$$= \exp\Big\{\sum_{r=0}^k \gamma_r \mu_r(\mathcal{P}_r(z) - 1)\Big\}.$$

We see now that this has exactly the same form as (12), only with different definitions of μ_r and $\mathcal{P}_r(z)$. In the 1-period case, we went on to integrate this over Γ . This case works no differently, so we realize that we arrive at (16), namely

$$G_X(z) = \exp\left\{\mu_0(\mathcal{P}_0(z) - 1)\right\} \prod_{r=1}^k \left(\frac{1}{1 - \frac{\mu_r}{\alpha_r}(\mathcal{P}_r(z) - 1)}\right)^{\alpha_r}$$

from which we can replace μ_r and $\mathcal{P}_r(z)$ with their actual expressions and α_r with σ_r^{-2} to get

$$G_X(z) = \left\{ \sum_{i} \sum_{t=1}^T w_{0i} p_{i,t} (z^{L_{i,t}} - 1) \right\} \prod_{r=1}^k \left(\frac{1}{1 - \sigma_r^2 \sum_i \sum_{t=1}^T p_{i,t} w_{ri} (z^{L_{i,t}} - 1)} \right)^{\frac{1}{\sigma_r^2}} dz^{T_{i,t}} dz$$

With the probability generating function in place, we can follow the exact same procedure as in Section 3.2.2 to arrive at its characteristic function

$$\Phi_X(z) = \exp\left\{\xi_0(z) - \sum_{r=1}^k \frac{1}{\sigma_r^2} \log(1 - \sigma_r^2 \xi_r(z))\right\},\tag{31}$$

with

$$\xi_r(z) = \sum_i \sum_{t=1}^T p_{i,t} w_{ri} (e^{izL_{i,t}} - 1).$$

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The conclusion from this is that the model extends to a multi-period framework and that the Fourier inversion technique described in 3.2.2 works in the same way, with a slightly different expression for the characteristic function.

4 Case Study

We shall now implement the proposed model and perform some tests on a portfolio that resembles that of EKN. We begin by providing some implementational details as well as numerical values on parameters.

We also put some attention on the assumptions that have been made to derive the model and parameters, that can deserve some special attention for the practitioner who seeks to use the model.

In the Analysis section we shall give examples of outputs from the model, as well as testing its sensitivity to input parameters.

4.1 Implementation

The method of calculating the loss density using the Fourier inversion technique described in 3.2.2 will be denoted CR^+ and results will be compared to the case where no dependence structure is included in the model. We will call this method CR^+_{indep} .

The results from the Fourier inversion is likely to exhibit a saw tooth pattern due to periodic perturbations [1]. A remedy to this problem is to set

$$\hat{f}_k = \frac{1}{2}(f_k + f_{k-1}), \ k = 1, ..., Z - 1.$$

This mid-point interpolation smoothens the density (and reduces the number of sample points by 1). It is also possible to find singularities at some points where the density is small, in which case the density will oscillate and perhaps become negative. For each such point where $f_k < 0$, the following procedure can be useful:

$$\xi := (f_{k-1})^{+} + (f_{k+1})^{+},$$

$$f_{k-1} \leftarrow f_{k-1} + \frac{(f_{k-1})^{+}}{\xi} f_{k},$$

$$f_{k+1} \leftarrow f_{k+1} + \frac{(f_{k+1})^{+}}{\xi} f_{k},$$

$$f_{k} \leftarrow 0,$$

where $y^+ = \max(y, 0)$.

We shall also compare the above methods with a Monte Carlo approach where defaults are determined by Bernoulli-indicators. $N = 600\ 000$ simulations were used for this purpose. This method is denoted CR^+_{MC} .

The output is N samples of the loss $(X_1, ..., X_N)$ which we can assume to be sorted so that $X_1 \leq X_2 \leq ... \leq X_N$. The empirical cumulative distribution function is then calculated as

$$\hat{F}(x) = \frac{1}{N} \sum_{k=1}^{N} \mathbf{1}\{X_k \le x\}.$$
(32)

The empirical Value-at-Risk and Expected shortfall can be esitmated using

$$\widehat{\operatorname{VaR}}_{p}(X) = X_{[Np]+1}$$

$$\widehat{\operatorname{ES}}_{p}(X) = \frac{1}{1-p} \Big(\sum_{k=[Np]}^{N} \frac{X_{k}}{N} + \Big(p - \frac{[Np]}{N}\Big) X_{[Np]+1} \Big).$$
(33)

In (33), [x] denotes the integer part [2].

The weight parameters were estimated by solving the minimization problem (24) with a gradientbased non-linear solver in the MATLAB toolbox, fmincon. Data was collected from Moody's 2017 Annual Default Study [17]. The data set contains default statistics between the years 2000 and 2016. Only credit ratings A, Baa, Ba, B, and Caa-C were included since no defaults for classes of higher quality were recorded. More about the definitions of each rating class can be found in [18]. The default intensities $\lambda_1, ..., \lambda_r$ and the parameters of Γ can be included in the optimization problem (24) as well. However, we are mainly interested in the weights and not the default intensities, since these have already been estimated more carefully for each obligor. We instead aim to estimate the weights and put each obligor in the "closest group". Furthermore, it is necessary that the parameters of Γ are chosen so that they all have expectation 1. This greatly complicates the formulation of the optimization problem. A pragmatic choice is to set the default intensities to the average default rates:

$$\lambda_r = \frac{1}{n} \sum_{j=1}^n \frac{M_{j,r}}{m_{j,r}}$$

and choose the parameters of γ_r to be $(1/\sigma_r^2, \sigma_r^2)$, where

$$\sigma_r^2 = \frac{1}{n-1} \sum_{j=1}^n (\frac{M_{j,r}}{m_{j,r}} - \lambda_r)^2.$$

The mean and variance of γ_r will then be 1 and σ_r^2 respectively. The estimates of σ_r are used in the actual implementation of the full model as well.

A 3-factor model using the three largest components of σ was chosen, although the data set enables us to use 5 factors.

The optimization requires an initial guess to get started. By experience, if the initial guess is too far away, the solver might arrive at a local maximum that is not the global maximum, but the algorithm arrives at the same result for points close to the initial guess.

To arrive at such a guess, we solve the very same problem, but treating all rating classes as independent from each other, and using k = 1 factor only.

After this procedure we are given $\{\hat{w}_{0i}, \hat{w}_{ri}\}_{r=1}^k$ and set

$$W^{(0)} = \begin{bmatrix} \hat{w}_{01} & \hat{w}_{11} & 0 & 0 & 0 & 0 \\ \hat{w}_{02} & 0 & \hat{w}_{22} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \hat{w}_{0k} & 0 & 0 & 0 & 0 & \hat{w}_{kk} \end{bmatrix}$$

as our initial guess for the full optimization problem. Intuitively, this procedure should at least calibrate the idiosyncratic weight w_0 , so that the default rate volatility for each rating class reflects the data, ignoring the covariation between rating classes. When the full optimization problem is carried out, the non-idiosyncratic weight is spread out so as to reflect the covariation in the data.

All calculations were carried out in MATLAB.

4.2 Parameters

Numerical values of mean default rates and default rate volatilities from Moody's were found to be

$$\lambda = \begin{bmatrix} 0.0011 & 0.0027 & 0.0068 & 0.0224 & 0.1182 \end{bmatrix}$$

and

$$\sigma = \begin{bmatrix} 0.0012 & 0.0037 & 0.0070 & 0.0265 & 0.0846 \end{bmatrix}.$$

The last three components of σ , which we call σ_1 , σ_2 , σ_3 from now on, were used to estimate the weight matrix according to the procedure described in Section 3.2.1, and the values were found to be

$$\begin{bmatrix} \mathbf{w_1} \\ \mathbf{w_2} \\ \mathbf{w_3} \\ \mathbf{w_4} \\ \mathbf{w_5} \end{bmatrix} = \begin{bmatrix} 0.8770 & 0.0489 & 0.0444 & 0.0297 \\ 0.8508 & 0.0343 & 0.0642 & 0.0507 \\ 0.7726 & 0.0586 & 0.1142 & 0.0546 \\ 0.6523 & 0.0517 & 0.1544 & 0.1416 \\ 0.6121 & 0.0178 & 0.1760 & 0.1941 \end{bmatrix}$$

Each row represents the weights for an obligor in the corresponding group, going from the highest quality to the lowest. The group of each obligor is determined by its probability of default compared to the mean probability of default for each rating class, i.e., obligor i is put in the group r that minimizes $|\lambda_r - p_i|$, r = 1, ..., 5, where λ_r is the mean default rate in group r. The left-most column are the idiosyncratic weights, w_0 , and the following columns are the factors that creates the correlation in the model.

The outcome of each obligor is then determined by $N_i \sim \text{Poi}(p_i^{\Gamma})$, where $p_i^{\Gamma} = p_i(w_{0i} + \sum_{r=1}^3 w_{ri}\gamma_r)$ and

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} \sim \begin{bmatrix} \text{Gamma}(1/\sigma_1^2, \sigma_1^2) \\ \text{Gamma}(1/\sigma_2^2, \sigma_2^2) \\ \text{Gamma}(1/\sigma_3^2, \sigma_3^2) \end{bmatrix}$$

The remaining parameters that make up the portfolio are given in the Analysis section.

4.3 Assumptions and Expert Judgments

The model relies on the assumptions that default probabilities are small enough to be approximated by a Poisson distribution. We shall explore this effect to some extent in the Analysis-section.

Historical default data from Moody's has been used to calculate default rate volatilities as well as estimating weight parameters. It was found sufficient to use 17 years although more data are available from the data source.

Only credit ratings A, Baa, Ba, B, and Caa-C have been included, as other authors referenced on the subject have done as well.

Using all 5 factors has not been judged to be justified considering the extra complexity it entails, since a three-factor model gives virtually the same results.

The weight parameters were derived under the assumptions that the observations of defaults between years are independent. This might not be a valid assumption, but taking this dependency into account would complicate the formulation to computationally prohibitive levels. In fact, interpreting the sectors from an economic point of view, the assumptions that the components of Γ are mutually independent is questionable as well. We take the view that the risk factors should not be taken as a precise econometric model for default rates, but merely statistical objects that serve to introduce a dependence structure in the model that reflects historical data.

Incorporating dependence between the risk factors have been studied e.g. in [19], [20], [21], and [22].

Loss Given Default for all obligors are assumed fixed and independent from each other. Modelling LGD is an area of research itself that we do not touch upon here. A paper by Bürgisser describes one way to incorporate non-deterministic LGD's into the CreditRisk⁺ framework [23].

4.4 Analysis

We shall now implement the proposed model and perform some tests on a portfolio that resembles that of EKN. We have m = 2100 obligors, all with a LGD of 1 for illustrative reasons. The default probabilities are set at random, in the intervals given in Table 2.

As a first example, the total exposure of the portfolio is set to 181 billion. 10 billion are distributed

PD-group	$p_i~(\%)$	% of obligors
1	0.01 - 0.05	5
2	0.05 - 0.10	5
3	0.1 - 0.5	10
4	0.5 - 1	10
5	1 - 1.5	20
6	1.5-2	20
7	2-4	10
8	4-6	10
9	6-8	5
10	8-10	5
	mean: 2.5%	sum: 2100

Table 2: Probability of default allocated into groups.

evenly among 10 different obligors, one in each group of Table 2. We also put 36 billion, about one fifth of the total exposure, on one single obligor in group 6. The remaining obligors get a random share of the remaining 135 billion.

We illustrate the calculated density function for this example in Figure 2.



Figure 2: Density function of the loss in the introductory example.

The density exhibits a bump which is due to the large concentration of the exposure on one obligor. Hereafter, we use the same portfolio as described in this example but exclude this obligor.

Excluding this large risk, we continue and illustrate the density function of the loss in Figure 3. Here we have also included the case where all obligors are independent from each other to illustrate the

effect the dependence structure of CreditRisk⁺ has on the shape of the distribution. In connection to Figure 3 we also present the Value-at-Risk, Expected Shortfall and Unexcpected Loss calculated in the two cases (in billions), as well as the fraction between the calculations with the two methods.



Figure 3: Density functions of the loss, with random default intensity (solid line) and with independent obligors (dashed line). The vertical lines mark the Expected Loss, and the 99.5% quantiles of the distributions.

	CR^+	CR^+_{indep}	Δ
EL	5.39	5.39	1
$\mathrm{VaR}_{99.5\%}$	9.00	8.34	1.08
$\mathrm{ES}_{99.5\%}$	9.59	8.76	1.10
$\mathrm{UL}_{99.5\%}$	3.61	2.95	1.23

Table 3: Selected risk measures calculated with CreditRisk⁺ with random default intensity and independent obligors, as well as the difference between the two methods expressed as a fraction.

Sensitivity Analysis

To test the sensitivity with respect to input parameters, we vary default volatilities and the idiosyncratic weight as in Table 4. In each column, the original parameters have been multiplied with the numbers indicated at the top. Note that decreasing the idiosyncratic weight increases the variance and covariance in the portfolio, so the two parameters move in the same direction from left to right in the table. Each entry is then divided by the same measures calculated with the original parameter setting, i.e., the value in the leftmost column.

We shall do a similar test as in Table 4 but where we change all default rates in the portfolio between

σ	1	1.05	1.1	1.15	1.2	1.25
w_0	1	0.95	0.9	0.85	0.8	0.75
$\mathrm{VaR}_{99.9\%}$	9.9658	1.0151	1.0355	1.0595	1.0870	1.1199
$\mathrm{VaR}_{99.5\%}$	9.0011	1.0079	1.0197	1.0344	1.0511	1.0728
$\mathrm{VaR}_{99\%}$	8.5674	1.0052	1.0124	1.0227	1.0351	1.0506
$\mathrm{ES}_{99.9\%}$	10.5279	1.0195	1.0449	1.0745	1.1081	1.1475
$\mathrm{ES}_{99.5\%}$	9.5861	1.0126	1.0299	1.0508	1.0744	1.1035
$\mathrm{ES}_{99\%}$	9.1677	1.0100	1.0229	1.0396	1.0592	1.0826
$\mathrm{UL}_{99.9\%}$	4.5771	1.0329	1.0773	1.1296	1.1895	1.2610
$\mathrm{UL}_{99.5\%}$	3.6124	1.0196	1.0490	1.0858	1.1274	1.1813
$\mathrm{UL}_{99\%}$	3.1787	1.0139	1.0334	1.0613	1.0947	1.1364

Table 4: Various risk measures compared with the original parameter setting (left-most column) as the parameters vary.

-15 and 15%. The middle column represents the risk figure in question with the original parameter setting, and the columns to the left and right are fractions compared to the middle column. Results for this test with CR⁺ are shown in Table 5.

The same test has been done for the Monte Carlo method. Apart from stress testing the model through perturbation of parameters, financial institutions may in some cases be required to quantify how sensitive their risk figures are to model assumptions. CreditRisk⁺ relies on the assumption that the default events can be approximated by a Poisson distribution, which gets less accurate as the default rates increase. While no analytic representation is available if we insist on Bernoullidistributed defaults, we can use the Monte Carlo approach with the "true" Bernoulli-indicators. The same test for CR_{MC}^+ can therefore be a useful reference and we show the results in Table 6. Note that the two models are not exchangeable, not only due to their different methodologies, but because of the systematic differences in default correlations as explained in the beginning of Section 3. Nevertheless, the results show that the methods show very similar behaviour.

\mathbf{C}	R.	ł
0		

p	0.85	0.90	0.95	1	1.05	1.1	1.15
VaR _{99.9%}	0.8730	0.9156	0.9583	9.9658	1.0417	1.0844	1.1261
$VaR_{99.5\%}$	0.8712	0.9145	0.9567	9.0011	1.0423	1.0855	1.1278
$VaR_{99\%}$	0.8698	0.9132	0.9566	8.5674	1.0434	1.0857	1.1291
$\mathrm{ES}_{99.9\%}$	0.8737	0.9160	0.9584	10.5279	1.0416	1.0840	1.1256
$\mathrm{ES}_{99.5\%}$	0.8723	0.9152	0.9571	9.5861	1.0420	1.0848	1.1268
$\mathrm{ES}_{99\%}$	0.8713	0.9142	0.9571	9.1677	1.0429	1.0849	1.1278
UL _{99.9%}	0.9001	0.9340	0.9680	4.5771	1.0320	1.0660	1.0980
$\mathrm{UL}_{99.5\%}$	0.9028	0.9360	0.9668	3.6124	1.0308	1.0640	1.0948
$\mathrm{UL}_{99\%}$	0.9035	0.9356	0.9678	3.1787	1.0322	1.0616	1.0938

Table 5: Various risk measures calculated with CR^+ in relation to the original parameter setting (middle column) as the default rates vary.

				CR^+_{MC}			
p	0.85	0.90	0.95	1	1.05	1.1	1.15
$VaR_{99.9\%}$	0.8704	0.9147	0.9597	10.1396	1.0426	1.0839	1.1281
$VaR_{99.5\%}$	0.8678	0.9125	0.9572	9.0853	1.0401	1.0839	1.1308
$\mathrm{VaR}_{99\%}$	0.8668	0.9112	0.9568	8.6081	1.0412	1.0863	1.1324
ES _{99.9%}	0.8736	0.9124	0.9566	10.7756	1.0425	1.0794	1.1247
$\mathrm{ES}_{99.5\%}$	0.8699	0.9128	0.9585	9.7379	1.0411	1.0829	1.1291
$\mathrm{ES}_{99\%}$	0.8686	0.9121	0.9572	9.2814	1.0408	1.0837	1.1300
UL _{99.9%}	0.8924	0.9305	0.9701	4.8665	1.0345	1.0665	1.1044
$\mathrm{UL}_{99.5\%}$	0.8923	0.9299	0.9672	3.8123	1.0265	1.0618	1.1043
$\mathrm{UL}_{99\%}$	0.8933	0.9290	0.9674	3.3351	1.0274	1.0646	1.1047

Table 6: Various risk measures calculated with CR_{MC}^+ in relation to the original parameter setting (middle oclumn) as the default rates vary.

Multi Period Calculations

We end this section by illustrating a calculation of the density in a multi-period setting. We assume that the marginal default probabilities are given exogenously and that each guarantee has a time to maturity between 1 and 20 years. The exposure for each obligor then evolves so that $L_{i,1} = L_i$ and decreases by equal amounts to become 0 at the maturity date for that obligor.

The density calculated in this setting is shown in Figure 4, with the independent case included as a reference.



Figure 4: Density functions calculated with CreditRisk⁺, in the 20-year framework.

It should be noted that compared to a Monte Carlo setting, large quantiles tend to be overestimated and small quantiles tend to be underestimated (it is more pessimistic). Here one has to take a closer look on the sizes of the default probabilities and their corresponding exposures to decide if the assumptions the model relies on can be justified (just as in the 1-period setting).

5 Conclusions

By studying the available industry models for credit risk, CreditRisk⁺ with some customizations has been proposed as a candidate for EKN to use in risk assessments on their portfolio. With adaptations from available research, parameter estimates can be obtained from historical data, that at least may provide useful references for plausible values of parameters. A Fourier inversion technique has been proposed, offering fast and stable calculations of the loss distribution, from which various functionals can be calculated.

With the Fourier inversion technique, grouping risks with similar exposure size is no longer necessary as is the case in the original CreditRisk⁺. This enables quick calculations of the risk contribution of single commitments on the whole portfolio, which is a convenient feature when considering new commitments or reinsurance contracts.

The model has proven to be fairly stable when input parameters are perturbed.

The model extends to a multi-period framework with little effort. This enables calculations when payments are made at a higher frequency than one year, as well as hold-to-maturity analyses. When calculating over long time horizons, a Monte Carlo setting can be considered a more reliable option, at the cost of increased computational effort.

6 Appendix

6.1 External Data

	A		Baa		Ba		В		Caa-C	
Year	$m_{j,1}$	$M_{j,1}$	$m_{j,2}$	$M_{j,2}$	$m_{j,3}$	$M_{j,3}$	$m_{j,4}$	$M_{j,4}$	$\mid m_{j,5}$	$M_{j,5}$
2000	1298	3	1157	3	622	8	1110	103	309	93
2001	1302	3	1249	13	601	9	893	42	359	97
2002	1290	0	1209	0	591	6	873	24	359	74
2003	1276	0	1269	0	569	3	966	8	314	38
2004	1326	0	1291	3	560	0	1050	9	346	26
2005	1385	0	1280	0	567	2	1117	12	375	23
2006	1379	0	1281	0	623	0	1145	0	423	21
2007	1301	6	1244	13	629	15	1020	41	655	71
2008	1291	4	1233	12	602	11	861	63	757	198
2009	1232	3	1354	2	586	0	834	4	723	62
2010	1222	0	1447	6	640	2	968	4	744	44
2011	1191	0	1480	2	721	2	1000	6	839	66
2012	1166	2	1663	3	708	5	1055	10	946	59
2013	1162	2	1725	2	739	2	1107	6	1140	53
2014	1237	0	1724	0	747	3	1101	25	1348	92
2015	1290	0	1713	0	763	2	1043	17	1414	128
2016	1260	0	1702	0	831	3	1009	4	1426	92
λ_r	0.	10	0.1	27	0.68		2.24		11.82	
σ_r	0.	12	0.3	37	0.	70	2.0	65	8.	46

The data used from Moody's [17] used in parameter estimates are given in Table 7.

Table 7: Number of issuers and number of defaults by credit rating and year (λ , σ in %).

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