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Reflected Stochastic Differential Equations on a Time-Dependent Non-Smooth Domain

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Abstract

In this thesis we prove existence and uniqueness for reflected stochastic differential equation on a specific non-smooth, time-dependent domain. The domain is the intersection of a finite number of smooth domains that are allowed to vary in time. The reflection is oblique to the domain and at the corners more than one direction of reflection is allowed.

The time restrictions on the domain is firstly the existence of a semi-concave family of sets that are $C^{1,+}$ in time. Secondly that the distance function to the domain is in $\mathcal{W}^{1,p}$.

The first part of the proof is to construct of three kinds of test functions with desired properties. Using these test functions, existence is proved to the Skorokhod problem. Finally uniqueness is proved for the reflected stochastic differential equation.

Keywords: Reflected stochastic differential equations, non-smooth, time-dependent, Skorokhod problem

Sammanfattning

I den här mastersuppsatsen så bevisar vi existens och entydighet för reflekterade stokastiska differentialekvation på ett icke slätt, tidsberoende område. Området är snittet mellan ett ändligt antal släta områden som tillåts variera i tiden. Reflektionen är ej nödvändigtvis vinkelrät till området och i hörnen finns det mer än en tillåten riktning.

Tidsrestriktionen på området är dels existensen av en familj av semi-konkava mängder som är $C^{1,+}$ i tiden. Dessutom att avståndet till området är $\mathcal{W}^{1,p}$ i tiden.

Första delen av beviset är att konstruera tre hjälp funktioner med eftersökta egenskaper. Med hjälp av de här funktionerna så bevisas sedan existens av lösningar till Skorokhod problemet. Slutligen så bevisas entydighet av den reflekterade stokastiska differentialekvationen.

Keywords: Reflekterande stokastiska differentialekvationer, icke-slät, tidsberoende, Skorokhod problemet

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Nomenclature

$B(a, b)$ Closed ball around a with radius b

$S(a, b)$ Sphere around a with radius b

$|\lambda|(t)$ The total variation of the process $\lambda(s)$ at the time t

$|\lambda(t)|$ The norm of $\lambda(s)$ at the time t

$\langle \cdot, \cdot \rangle$ The inner product

$a.e$ Almost everywhere

$a.s$ Almost surely

$C(A)$ Continuous functions on A

$C^p(A)$ Functions on A with p continuous derivatives

$C^{p,1}(A)$ Functions on A which p derivatives is Lipschitz

I_A Indicator function on the set A

Chapter 1

Introduction

In this master thesis we prove the existence and uniqueness of a strong solution to a stochastic differential equation (SDE) with an oblique reflection, on a non-smooth time dependent domain. Dupuis and Ishii[1] proved existence and uniqueness for SDE for two different kinds of non-smooth time independent domain with oblique reflection, denoted Case 1 and Case 2 in. Önskog and Lundström later generalized the results in Case 1 to a time dependent domain in [2]. This thesis will generalize the result for Case 2 to a time-dependent domain. The approach, which is similar to [2] and [1], rely on the Skorokhod problem (SP). This approach was first used by Tanaka in [3] to solve SDE with reflection in convex regions. The method was later used by Lions and Sznitman in [4] to generalize the proof to non-smooth domains. This proof will also make use of a series of functions adapted to the domain and reflection. A time-independent variant of these functions are used in [1] and their existence is verified in [5], where they are used to prove existence and uniqueness of a viscosity solution to certain partial differential equation on a non-smooth domain.

Chapter 2

Background

2.1 Mathematical theory

2.1.1 Stochastic calculus

We begin our treatise of stochastic calculus by defining Wiener process and Ito integral which are vital parts of the SDE theory. First we define the Wiener process, sometimes called Brownian motion, given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$.

Definition 2.1. *A Wiener process, denoted $W(s)$, is an \mathcal{F}_t -adapted stochastic process with the following properties*

$$W_0 = 0 \text{ a.s.},$$
$$W(t) - W(s) \sim N(0, t - s) \text{ and independent of } \mathcal{F}_s.$$

Now we move to the construction of the Ito integral. We will first define the integral for simple processes, defined below.

Definition 2.2. *A stochastic process $X^n(s)$ on $[0, T]$ is simple if there exist times $0 = t_0 < t_1 < t_2 < \dots < t_n = T$ and stochastic variables Y_1, Y_2, \dots, Y_n such that*

$$X^n(s) = \sum_{i=0}^{n-1} Y_i I_{[t_i, t_{i+1}]}(t).$$

Definition 2.3. *Let $X^n(s)$ be a simple process and $W(s)$ a Wiener process. Then we define the Ito integral of $X^n(s)$ with respect to $W(s)$ to be*

$$I(X^n(s))(t) = \int_0^t X^n(s) dW(s) = \sum_{i=0}^{n-1} Y_i (W(t_{i+1}) - W(t_i)).$$

Next we define the Ito integral for bounded processes by approximating the process with a simple one. The following theorem, Theorem 2.4 in [6], will therefore be important.

Theorem 2.4. *Let X_t be a bounded process. Then there exists a series of simple processes X_t^n such that*

$$\lim_{n \rightarrow \infty} \int_0^t E[|X(s) - X^n(s)|^2] ds = 0. \quad (2.1)$$

Definition 2.5. *Let X_t be a process and X_t^n be a series of simple processes satisfying (2.1). Then the Ito integral of X_t with respect to the Wiener process W_t is defined as follows*

$$I(X(t)) = \int_0^t X(s) dW(s) = \lim_{n \rightarrow \infty} \int_0^t X^n(s) dW(s).$$

Although the Ito integral is constructed in a similar manner as the Riemann integral, the Ito integral does not behave exactly the same. For example

$$\int_0^t W(s) dW(s) = \frac{W^2(t)}{2} - \frac{t}{2}.$$

2.1.2 Stochastic differential equation

Given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ and a Wiener process W , an equation on the form

$$X(t) = x + \int_0^t b(s, X(s)) ds + \int_0^t \sigma(s, X(s)) dW(s),$$

where $b(t, x)$ and $\sigma(t, x)$ are Borel measurable functions, is called a Stochastic differential equation (SDE).

Definition 2.6. *A strong solution to a SDE with coefficients $b(t, x)$ and $\sigma(t, x)$, driven by a Wiener process $W(t)$ is a (\mathcal{F}_t) adapted stochastic process $X(t)$ satisfying*

$$X(t) = x + \int_0^t b(s, X(s)) ds + \int_0^t \sigma(s, X(s)) dW(s), \quad (2.2)$$

a.s.

Usually a strong solution is hard to explicitly found. The following theorem gives existence and uniqueness for solutions to SDE

Theorem 2.7. *If the functions $b(t, x)$ and $\sigma(t, x)$ fulfills the Lipschitz condition*

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| < K|x - y|, \quad (2.3)$$

for some constant K and fulfills the linear growth condition

$$|b(t, x)|^2 + |\sigma(t, x)|^2 < C(1 + |x|^2), \quad (2.4)$$

for some constant C , then there exists a unique strong solution to (2.2).

A useful Lemma that can be used as a change of variable formula in stochastic calculus is Ito's Lemma

Lemma 2.8. *Let $f(t, x)$ be a twice differentiable function and $X(t)$ a stochastic process on the following form*

$$X(t) = x_0 + \int_0^t b(t, X(t))dt + \int_0^t \sigma(t, X(t))dW(s).$$

Then we have

$$f(t, X(t)) = f(0, x_0) + \int_0^t \left(\frac{\partial f}{\partial t} + b(s, X(s)) \frac{\partial f}{\partial x} + \frac{\sigma(s, X(s))^2}{2} \frac{\partial^2 f}{\partial x^2} \right) ds + \int_0^t \sigma(s, X(s)) \frac{\partial f}{\partial x} dW(s).$$

2.1.3 Reflected Stochastic differential equation

Now consider an SDE confined in a domain $\bar{\Omega}$ with a reflecting boundary. The solution should be reflected along a certain direction when the solution is on the boundary of the domain but should otherwise behave ordinary. To resolve this we add a reflection term to equation (2.2) which only increases when the solution is at the boundary.

Definition 2.9. *A strong solution to a reflected stochastic differential equation (RSDE) with coefficients $b(t, x)$ and $\sigma(t, x)$, on a domain $\bar{\Omega}$ driven by a Wiener process $W(t)$, reflected along γ is a pair of stochastic processes $(X(t), \Lambda(t))$ satisfying*

$$X(t) = x + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dW(s) + \Lambda(t), \quad (2.5)$$

where $X(t) \in \bar{\Omega}$ and

$$|\Lambda|(t) = \int_0^t I_{\{X(s) \in \partial\Omega\}} d|\Lambda|(s), \quad (2.6)$$

$$\Lambda(t) = \int_0^t \gamma(s) d|\Lambda|(s). \quad (2.7)$$

We will assume that the functions $b(t, x)$ and $\sigma(t, x)$ satisfy the Lipschitz continuity condition (2.3). The Λ function compensates so that X stays within the domain. Equation (2.6) means that the compensation function only increases when X is at the boundary and equation (2.7) means that Λ pushes X along the direction of the reflection γ . In this thesis Ω will time-dependent. This means that the boundary $\partial\Omega$ and γ will not be fixed in time. Closely related to RSDE is the Skorokhod problem.

Definition 2.10. Given a function ψ , a pair of functions (ϕ, λ) is solution to the Skorokhod problem (SP) on a domain Ω with directions of reflection $r(x) \subset S(0, 1)$, $\forall x \in \partial\Omega$, if

$$\phi(t) = \psi(t) + \lambda(t), \quad (2.8)$$

$$\phi(t) \in \bar{\Omega}, \quad (2.9)$$

$$|\lambda|(t) < \infty, \quad (2.10)$$

$$|\lambda|(t) = \int_0^t I_{\{\phi(s) \in \partial\bar{\Omega}\}} d|\lambda|(s), \quad (2.11)$$

$$\lambda(t) = \int_0^t \gamma(s) d|\lambda|(s), \quad (2.12)$$

where $\gamma(t) = \gamma(t, \phi(t)) \in r(x)$ a.s.

We see that a solution to an RSDE (X, Λ) should solve the SP with $\psi = x + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dW(s)$ on a path-wise basis almost surely. For more information on RSDE and the connection to SP see [6].

2.1.4 Axillary theorems

Here we define and state theorems and definitions that will be useful in the rest of this thesis. First we define Hilbert, L^p and Sobolev spaces

Definition 2.11 (Hilbert space). A Hilbert space is a complete real or complex vector space with an inner product, here denoted $\langle \cdot, \cdot \rangle$. The inner product induces a norm $\|x\| = \sqrt{\langle x, x \rangle}$ and a metric $d(x, y) = \sqrt{\langle x - y, x - y \rangle}$.

Definition 2.12 (L^p). An $L^p(\Omega)$ space is the vector space of functions on a domain Ω , with finite p -norm i.e.

$$\|f\|_p = \left(\int_{\Omega} |f|^p dx \right)^{\frac{1}{p}} < \infty.$$

Definition 2.13 ($\mathcal{W}^{k,p}$). A Sobolev space, denoted $\mathcal{W}^{k,p}$, is the vector space of functions, whose k :th order derivatives are in L^p . Function in $\mathcal{W}^{k,p}$ have the norm

$$\|f\| = \sum_{i=0}^k \|D^i f\|_p.$$

Definition 2.14. A sequence x_n in a Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ converges weakly to $x \in H$ if

$$\langle x_n, u \rangle \rightarrow \langle x, u \rangle \quad \forall u \in H. \quad (2.13)$$

Definition 2.15 (Precompact). *A subset $X \subset Y$ is precompact, or relative compact, if the closure \overline{X} is compact.*

Theorem 2.16. *If a subset $F \subset C(X)$ is equicontinuous i.e $\forall \epsilon > 0 \exists \delta > 0$ such that*

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon \quad \forall x, y \in X, \quad \forall f \in F,$$

and all $f \in F$ are bounded, then F is precompact.

Definition 2.17 (Martingale). *A stochastic process $X(s)$, on $[0, T]$, adapted to a filtration \mathcal{F}_t is called a martingale if*

$$\begin{aligned} E[|X(t)|] &< \infty, \\ E[X(t)|\mathcal{F}_s] &= X(s) \quad 0 \leq s \leq t \leq T. \end{aligned}$$

Lemma 2.18 (Doob's Martingale inequality). *Let $M(t)$ be a martingale satisfying $M(t) \geq 0$ a.s and $E[M(t)] < \infty$ then*

$$E \left[\left(\sup_{0 \leq s \leq t} M(s) \right)^p \right] \leq \left(\frac{p}{p-1} \right)^p E [M^p(s)]. \quad (2.14)$$

Theorem 2.19 (Gronwall's inequality). *Let α, β and u be real valued functions on the interval $[a, b]$. Let β be non-negative and u satisfy*

$$u(t) \leq \alpha(t) + \int_a^t \beta(s)u(s)ds \quad \forall t \in [a, b].$$

Then we have the following inequality

$$u(t) \leq \alpha(t) + \int_0^t \alpha(s)\beta(s)e^{\int_s^t \beta(r)dr} ds \quad \forall t \in [a, b].$$

2.1.5 Superdifferential

Since the domain we will define the RSDE on is non-smooth we will have to introduce superdifferentials for the test functions and for the definition of the domain.

Definition 2.20. *The superdifferentials of first order to a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ at the point x are*

$$D^+ f(x) = \{p \in \mathbb{R}^n : f(x+h) \leq f(x) + \langle p, h \rangle + o(|h|) \text{ as } h \rightarrow 0\}.$$

The superdifferentials of second order are

$$D^{2,+} f(x) = \left\{ (p, A) \in \mathbb{R}^n \times S^n : f(x+h) \leq f(x) + \langle p, h \rangle + \frac{1}{2} \langle Ah, h \rangle + o(|h|^2) \text{ as } h \rightarrow 0 \right\},$$

where S^n is the set of all symmetric $n \times n$ matrices.

We next define the sets of functions $C^{1,+}$ and $C^{2,+}$

Definition 2.21. A function $f : \Omega \rightarrow \mathbb{R}^n$ is in $C^{1,+}$ if $D^+f(x)$ is nonempty $\forall x \in \Omega$.

Definition 2.22. A function f is in $C^{2,+}$ if for each compact set K there exists a constant $C < \infty$ such that $(p, CI) \in D^{2,+}f(x) \forall x \in K$ for some p , depending on x , where I is the identity matrix.

We also define what it means for a family of sets to be in class $C^{2,+}$. This will be necessary for the definition of the domain.

Definition 2.23. A family of sets $\{B(x) : x \in U \subset \mathbb{R}^n\}$ is of class $C^{2,+}$ if for all $y \in \mathbb{R}^n$ the distance function to the set B , defined as $d(x, B) = \inf(|x - y| : y \in B)$ satisfy $d(y, B(x))^2 \in C^{2,+}(U \times \mathbb{R}^n)$.

2.2 Domain

We consider the domain in Case 2 in [1] but the domain is here allowed to vary in time. Let I be a finite index set, and for each $i \in I$, let G_i be an open bounded subset of \mathbb{R}^{n+1} and let $\Omega_i = G_i \cap [0, T] \times \mathbb{R}^n$. The domain Ω is then defined as

$$\Omega = \bigcap_{i \in I} \Omega_i.$$

We define Ω_t and $\Omega_{i,t}$ to be the time sections of the domain, e.g $\Omega_t = \{x \in \mathbb{R}^n : (t, x) \in \Omega\}$. We assume that for each time the boundary $\partial\Omega_{i,t}$ is of class C^1 . We define $I(t, x) = \{i \in I : x \notin \Omega_{i,t}\}$, so the set $I(t, x)$ corresponds to those $\partial\Omega_{i,t}$ on which x is located at time t . $I(t, x)$ will be assumed to be upper semi-continuous which means that $\forall x \in \partial\Omega_t$ and $\forall t \in [0, T]$ there exist a neighborhood V surrounding x and a interval W surrounding t , such that $I(s, y) \subset I(t, x) \forall y \in V$ and $\forall s \in W$. To define the directions of reflections $r(t, x)$ we assume that there exist vector fields $\gamma_i(t, x) \in C^{0,1}([0, T] \times \mathbb{R}^n, \mathbb{R}^n)$, where $C^{0,1}$ is the space of continuous Lipschitz functions. Let $\eta_i(t, x)$ denote the inward normal to $\partial\Omega_{i,t}$. Then we assume that

$$\langle \gamma_i(t, x), \eta_i(t, x) \rangle > 0 \quad \forall x \in \partial\Omega_{i,t}, \quad \forall t \in [0, T], \quad \forall i \in I(t, x),$$

implying that γ_i points inwards. We also assume that at the corners of the domain the convex hull of $\{\gamma_i(t, x) : i \in I(t, x)\}$ does not contain the origin. Now we are ready to define the directions of reflections as

$$r(t, x) = \left\{ \sum_{i \in I(t, x)} a_i \gamma_i(t, x), a_i \geq 0 : \sum_{i \in I(t, x)} a_i |\gamma_i(t, x)| = 1 \right\}. \quad (2.15)$$

We will also assume there exists a $\gamma \in r(t, x)$ such that γ points inward to Ω_t , that is

$$\langle \gamma(t, x), \eta_i(x, t) \rangle > 0 \quad \forall i \in I(t, x), \quad \forall x \in \partial\Omega_t, \quad \forall t \in [0, t]. \quad (2.16)$$

Our final assumption is that for each $t \in (0, T)$, $x \in \partial\Omega_t$, there exist a neighborhood V to x and a neighborhood W to t , and a family of compact convex subsets $\{B(s, y) : y \in \Omega_s\}$ containing the origin, such that for $\forall y \in V \cap \partial\Omega_s$, $s \in W$ and $p \in \partial B(s, y)$ the following holds

$$\langle \gamma(t, y), n \rangle \begin{cases} \geq 0 & \text{If } \langle p, \eta_i(s, y) \rangle \geq -1, \\ \leq 0 & \text{If } \langle p, \eta_i(s, y) \rangle \leq 1, \end{cases} \quad (2.17)$$

where n is the inward normal to $B(s, y)$ at p . The family of sets is jointly $C^{1,+}$ in time and space and $C^{2,+}$ for fixed a time.

Remark 2.24. Condition (2.17) and its consequences is discussed further in [5] and [7]. In [7] Lipschitz continuity is proved for the mapping $\phi = \Gamma(\psi)$ where $(\phi, \phi - \psi)$ solves the SP for ψ . The following sufficient condition for (2.17) is proved in [5]

Lemma 2.25. If there for each $x \in \partial\Omega_t$, $t \in [0, T]$, exist a set of positive numbers b_i , $i \in I(t, x)$, such that

$$b_j \langle \gamma_j(t, x), n_i(t, x) \rangle > \sum_{j \in I(t, x) \setminus \{i\}} b_j(t, x) |\langle \gamma_j(t, x), n_i(t, x) \rangle| \quad \forall i \in I(t, x),$$

then (2.17) holds.

For the time variation we require that there exists a p such that for all fixed $x \in \mathbb{R}^n$ the following holds

$$f(t) = d(x, \Omega_{i,t}) \in \mathcal{W}^{1,p}([0, T], [0, \infty]), \quad (2.18)$$

meaning that the distance function to each set $\Omega_{i,t}$, for fixed x , should have a weak first derivative in time that is in $L^p([0, T], [0, \infty])$. This also implies that the distance function to the domain Ω_t is in $\mathcal{W}^{1,p}([0, T], [0, \infty])$.

Remark 2.26. Property (2.18) implies that there exists a Hölder exponential α and a constant K such that

$$|d(x, \Omega_s) - d(x, \Omega_t)| \leq K |s - t|^\alpha. \quad (2.19)$$

Remark 2.27. The regularity of $\partial\Omega_t$ implies that there exists a $\delta > 0$ and a $\theta > 0$ such that for all $x \in \partial\Omega_t$, $y \in \bar{\Omega}_t$, $t \in [0, T]$, it holds that

$$|x - y| \leq \delta \Rightarrow \langle y - x, n_i \rangle \geq -\theta |x - y|.$$

See Remark 3.3 in [1].

Chapter 3

Outline

The Theorem we want to prove is

Theorem 3.1. *Assume that Ω and γ are domain and directions of reflections as described in Section 2.2. Then there exists a unique solution (ϕ, Λ) to the RSDE on the domain Ω with reflection γ .*

The proof will follow the following outline

- Prove existence of solutions to the SP for C^1 functions by a penalty method
- Prove that the solutions to the SP are of bounded variation and is a relatively compact set.
- Prove existence of solutions to the SP for C functions. This is done by approximating the continuous function by a series of C^1 functions and use the previous result
- Prove uniqueness for solutions to the RSDE.
- The solutions to the SP problem makes it possible to prove existence of a solution to the RSDE by a fix point iteration

The first three steps are carried out in Section 5 and the final two are done in Section 6. A key ingredient in these proofs will be a series of test functions, which will be stated and constructed in Section 4.

Chapter 4

Test functions

4.1 g and f functions

In this section we will state and construct the test functions g and f used in Lemma 5.3 and 6.1. The test functions are similar to those in [5] but with a time-dependence.

Theorem 4.1. *There exists a function $g(t, x, r) : [0, T] \times W \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, where W is a open set containing Ω_t for all $t \in [0, T]$, with the following properties*

$$g(t, x, r) \text{ is } C^{1,+}, \quad (4.1)$$

$$\text{For fixed time } t \text{ } g(t, \cdot, \cdot) \text{ is } C^{2,+}, \quad (4.2)$$

$$\text{For fixed time } t \text{ and } x \text{ } g(t, x, \cdot) \text{ is } C^1, \quad (4.3)$$

$$g(t, x, 0) = 0, \quad (4.4)$$

$$g(t, x, r) \geq |r|^2, \quad (4.5)$$

$$\langle D_r g(t, x, r), \gamma_i(t, x) \rangle \geq 0, \quad \text{if } \langle r, n_i(t, x) \rangle \geq -\theta|r|, \quad (4.6)$$

$$|u| \leq C|r|^2, \quad |p| \leq C|r|^2, \quad |q| \leq C|r|, \quad \text{for } (u, p, q) \in D^+g(t, x, r), \quad (4.7)$$

and for fixed time t there exists $(p, q) \in D^+g(t, x, r)$ such that

$$\left((p, q), C \begin{bmatrix} |r|^2 I & 0 \\ 0 & I \end{bmatrix} \right) \in D^{2,+}g(t, x, r). \quad (4.8)$$

To prove the rest we use the following lemmas. They are the $C^{1,+}$ variants of Lemma 4.2 and Lemma 4.3 in [5]

Lemma 4.2. *Let U be a subset of \mathbb{R}^n and V an open interval. $H \in C^{1,+}(U \times V)$ and $f : U \rightarrow V$ is $C^{0,1}$ so that $H(x, f(x)) = 0$. Assume also that for each compact set $K \subset U$ there exists a $\delta > 0$ such that if $x \in K$ and $(p, q) \in D^+H(x, f(x))$ then $q \leq -\delta$. Then $f(x) \in C^{1,+}$*

Proof. Let $(p, q) \in D^+H(x, f(x))$. Then we have as $h \rightarrow 0$

$$\begin{aligned} H(x+h, f(x+h)) - H(x, f(x)) &\leq \\ \langle p, h \rangle + q(f(x+h) - f(x)) + o(|f(x+h) - f(x)|). \end{aligned}$$

Using the fact that $H(x, f(x)) = 0$ and rearranging gives us

$$\begin{aligned} -q(f(x+h) - f(x)) &\leq \langle p, h \rangle + o(|f(x+h) - f(x)|) \Rightarrow \\ f(x+h) - f(x) &\leq \langle \delta^{-1}p, h \rangle + o(h), \end{aligned}$$

where we used that $q \leq -\delta$ and f is Lipschitz in the ordo term. This proves that $f \in C^{1,+}$. \square

Lemma 4.3. *Let $g(x) \in C^{1,+}$ and $f(x) \in C^{0,1}$. Then $g(f(x)) \in C^{1,+}$.*

Proof. Let $p \in D^+g(f(x))$ and let C be the Lipschitz constant to f . Then

$$\begin{aligned} g(f(x+h)) - g(f(x)) &\leq \langle p, f(x+h) - f(x) \rangle + o(|f(x+h) - f(x)|) \leq \\ &\langle Cp, h \rangle + o(|h|). \end{aligned}$$

This proves that $g(f(x)) \in C^{1,+}$. \square

Now we can prove Theorem 4.1

Proof of Lemma 4.1. The properties (4.2), (4.3), (4.5), (4.6) are (4.8) are time-independent so the proof of those statements follows Lemma 4.4 in [5]. Here we will prove (4.1), (4.4) and the time part of (4.7). First we will construct g satisfying (4.1) with the help from the sets defined in (2.17) on the punctured space. Then extending the functions to include the origin will let us prove (4.4). Let $\{B(t, x)\}$ be the family of sets defined in (2.17) and define the function $d(t, x, \xi) = (\text{dist}(\xi, B(s, x)))^2$ on $[0, T] \times \bar{\Omega} \times \mathbb{R}^n$. We know from the definition of the domain that the family of sets $\{B(t, x)\}$ is $C^{1,+}$ and for fixed time is $C^{2,+}$. This means that $d(t, x, \xi)$ is $C^{1,+}$ and for fixed time is $C^{2,+}$. We refer to [5] Lemma 3.2 for the proof that for fixed t and x the function $\xi \rightarrow d(t, x, \xi)^2$ is in $C^{1,1}$ with derivative

$$D_\xi d(t, x, \xi) = 2(\xi - P_{B(t,x)}(\xi)),$$

where $P_{B(t,x)}(\xi)$ is the closest point on $B(t, x)$ to ξ . We first define the functions on the punctured plane. Let $U = \mathbb{R}^n \setminus \{0\}, \delta > 0$ and define the function $g(t, x, \xi) = r$ where r is the scalar satisfying $d(t, x, r^{-1/2}\xi) = -\delta^2$. First lets prove that g is well defined, e.g. that r exists and is unique. Since $d(t, x, s\xi) \rightarrow \infty$ as $s \rightarrow \infty$ and $d(t, x, 0) = 0$, r exists. To check uniqueness we look at the derivative of $d(t, x, s\xi)$

$$\begin{aligned} D_s d(t, x, s\xi) &= 2\langle s\xi - P_{B(t,x)}(s\xi), \xi \rangle = \\ &\frac{2}{s}\langle s\xi - P_{B(t,x)}(s\xi), P_{B(t,x)}(s\xi) \rangle + \frac{2}{s}d(t, x, s\xi) \geq \frac{2}{s}d(t, x, s\xi) \geq 0, \end{aligned}$$

where we have used the fact that $B(t, x)$ is convex and contains the origin. This means that when $d(t, x, s\xi)$ is larger than zero it is a strictly increasing function in s . This implies that r is unique and therefore g is well defined. To prove (4.2) we define the function $H(t, x, \xi, r) = d(t, x, r^{-1/2}\xi) - \delta^2$. Since d is a $C^{1,+}$ function and $(t, x, \xi, r) \rightarrow (t, x, r^{-1/2}\xi)$ is smooth (for $r > 0$), H is a $C^{1,+}$ function by Lemma 4.3. To show that g is a $C^{1,+}$ function we use Lemma 4.2. So we need to show that g is continuous, Lipschitz continuous and that $g \leq -\delta$ where $(s, p, q) \in D^+(t, x, \xi, g(t, x, \xi))$. Since H is differentiable in r we calculate the derivative

$$\frac{\partial H}{\partial r} = -\frac{r^{-3/2}}{2} \langle \xi, D_\xi d(t, x, r^{-1/2}\xi) \rangle \leq -r^{-1} d(t, x, r^{-1/2}\xi) = -\frac{\delta^2}{r}.$$

Now since $r \rightarrow -r^{-1}d(t, x, r^{-1/2}\xi)$ is a increasing function we get for $0 < r \leq g(t, x, \xi)$

$$\frac{\partial H}{\partial r} \leq -\frac{\delta^2}{g(t, x, \xi)}.$$

Since in Lemma 4.2 the bound is necessary for all compact sets K , so if g is continuous and therefore bounded on each K the bound holds in Lemma 4.2. So the only thing left to prove is that g is continuous and Lipschitz continuous. Continuity follows from the uniqueness of r . To see this let $(t_n, x_n, \xi_n) \rightarrow (t, x, \xi)$ then

$$d(t_n, x_n, g(t_n, x_n, \xi_n)^{-1/2}\xi_n) \rightarrow d(t, x, g(t, x, \xi)^{-1/2}\xi),$$

since $d(t, x, g(t, x, \xi)^{-1/2}\xi)$ is constant. But since r is unique then $g(t_n, x_n, \xi_n) \rightarrow g(t, x, \xi)$. So g is continuous. To prove Lipschitz continuity we fix an $a \in (0, T) \times W \times U$ and an $\epsilon > 0$ such that

$$\frac{\delta^2}{g(x)} \geq \epsilon, \quad \forall x \in B(a, \epsilon).$$

Let L be the compact image by g of $B(a, \epsilon)$. Since H is $C^{1,+}$ it is Lipschitz continuous on the compact domain $B(a, \epsilon) \times L$. Let $x, y \in B(a, \epsilon)$ and without loss of generality assume that $g(x) \geq g(y)$

$$\begin{aligned} 0 &= H(x, g(x)) - H(y, g(y)) = \\ &H(x, g(x)) - H(y, g(x)) + H(y, g(x)) - H(y, g(y)), \end{aligned}$$

then by Lipschitz continuity of H and mean value theorem

$$\begin{aligned} H(x, g(x)) - H(y, g(x)) + H(y, g(x)) - H(y, g(y)) &\leq \\ M|x - y| + \frac{\partial H}{\partial r}|g(x) - g(y)| &\leq \\ M|x - y| - \frac{\delta^2}{g(x)}|g(x) - g(y)| &\leq \\ M|x - y| - \epsilon|g(x) - g(y)|. \end{aligned}$$

Rearranging gives us Lipschitz for g on $B(a, \epsilon)$. Since $B(a, \epsilon)$ creates an open cover for every compact set $K \subset (0, T) \times W \times U$ then there exists a finite subcover and therefore g is Lipschitz for every compact set K . Applying Lemma 4.2 we get that $g \in C^{1,+}$ and we have proved (4.1). Next we want to prove (4.4), we do this by extend g when $\xi = 0$. We observe that for $s > 0$

$$d(t, x, (s^2 g(t, x, \xi))^{-1/2} s \xi) = \delta^2 \Rightarrow s^2 g(t, x, \xi) = g(t, x, s \xi). \quad (4.9)$$

Then it is clear that $g(t, x, 0) = 0$ gives a continuous extension and (4.4) is proved. To prove the time part of (4.7), we first assume $\xi \neq 0$. From the definition of $C^{1,+}$ we have

$$g(t + j, x + h, \xi + |\xi|k) \leq g(t, x, \xi) + uj + \langle p, h \rangle + \langle q, |\xi|k \rangle + o(|j| + |h| + |k|).$$

Now multiplying the above with $|\xi|^{-2}$ and using (4.9) we get

$$\begin{aligned} g(t + j, x + h, |\xi|^{-1}\xi + k) &\leq \\ g(t, x, |\xi|^{-1}\xi) + |\xi|^{-2}uj + \langle |\xi|^{-2}p, h \rangle + \langle \xi^{-1}q, k \rangle + o(|j| + |h| + |k|), \end{aligned}$$

which proves that $(|\xi|^{-2}u, |\xi|^{-2}p, \xi^{-1}q) \in D^+g(t, x, |\xi|^{-1}\xi)$. This means that the above must hold for $j < 0, h = 0, k = 0$. Together with the Lipschitz continuity for g , we obtain

$$\begin{aligned} g(t + j, x, |\xi|^{-1}\xi) &\leq |\xi|^{-2}uj + o(|j|) \Rightarrow |\xi|^{-2}u + o(|j|) \leq \\ &\frac{1}{j}(g(t + j, x, |\xi|^{-1}\xi) - g(t, x, |\xi|^{-1}\xi)) \leq C. \end{aligned}$$

Multiplying by $|\xi|^2$ and sending $j \rightarrow 0$ proves (4.7). The rest of the proof is the same as the time independent part and we refer to [5]. \square

Next we define the family of functions $\{f_\epsilon(t, x, y) : \epsilon > 0\}$

Theorem 4.4. *There exists a family of functions $\{f_\epsilon(t, x, r) : \epsilon > 0\}$ and constants $C, \theta > 0$ with the following properties*

$$f(t, x, y) \text{ is } C^{1,+}, \quad (4.10)$$

$$\text{For fixed time } t \text{ } f(t, \cdot, \cdot) \text{ is } C^{2,+}, \quad (4.11)$$

$$f(t, x, y) \geq \frac{|x - y|^2}{\epsilon}, \quad (4.12)$$

$$f(t, x, y) \leq C \frac{|x - y|^2}{\epsilon}, \quad (4.13)$$

for $(u, p, q) \in D^{1,+}f_\epsilon(t, x, y)$

$$\langle p, \gamma_i(t, x) \rangle \geq 0 \quad \text{if } \langle y - x, n_i(t, x) \rangle \geq -\theta|x - y|, \quad (4.14)$$

$$\langle q, \gamma_i(t, x) \rangle \geq 0 \quad \text{if } \langle x - y, n_i(t, x) \rangle \geq -\theta|x - y|, \quad (4.15)$$

$$|u| \leq C \frac{|x - y|^2}{\epsilon}, \quad |p + q| \leq C \frac{|x - y|^2}{\epsilon}, \quad \max(|p|, |q|) \leq C \frac{|x - y|}{\epsilon}. \quad (4.16)$$

For fixed t , $\forall x, y \exists (p, q) \in D^{1,+} f_\epsilon(t, x, y)$

$$\left((p, q), \frac{C}{\epsilon} \begin{bmatrix} I & -I \\ -I & I \end{bmatrix} + \frac{C|x-y|^2}{\epsilon} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \right) \in D^{2,+} f_\epsilon(t, x, y). \quad (4.17)$$

The functions f_ϵ are constructed by setting $f_\epsilon(t, x, y) = \epsilon g(t, x, \frac{x-y}{\epsilon})$. So we omit the proof that is identical to the proof of Theorem 4.1 in [5].

4.2 h function

We also need a function $h(t, x)$ in the proof of Lemma 6.1. The function is constructed so that its derivative is aligned with the reflection.

Lemma 4.5. *There exists a function $h(t, x) \geq 0$ defined on $\bar{\Omega}$ which is C^1 in time and C^2 in space and fulfills*

$$\langle D_x h(t, x) \gamma(t, x) \rangle \geq 1.$$

The main part of the proof will follow the same idea as Lemma 3.2 in [5]. For the time extension we will prove that the functions in [5] are robust to small changes in time and this will able us to define the time-dependent functions. We will need the following Lemma, for proof see Lemma A.3 in [5]

Lemma 4.6. *Let $q = \sum_i \beta_i \gamma_i$ with $\beta_i \geq 0$. Then $\max_i \langle n_i, q \rangle > 0$.*

This Lemma follows from Lemma A.1 in [5] which we state here

Lemma 4.7. *Let $B(t, x)$ be a set from the family of sets defined in (2.17). Let $p \in B(t, x)$ satisfy $\langle p, n_i(t, x) \rangle < 1$. Then there exist a ϵ such that $p + \epsilon \gamma_i(t, x) \in B(t, x)$.*

This Lemma follows from the properties of the family of sets $\{B(t, x)\}$ defined in Section 2.2.

Proof of Lemma 4.5. Since $\bar{\Omega}_t$ and $[0, T]$ is compact it is enough to prove that for each $s \in [0, T]$ and $z \in \partial\Omega_s$ there exists a function u with the following properties

$$\langle D_x u(t, x), \gamma_i(t, x) \rangle > 0 \quad \text{for } |x - z| < \epsilon_1, |t - s| < \epsilon_2 \text{ and } i \in I(s, z). \quad (4.18)$$

$$\langle D_x u(t, x), \gamma_i(t, x) \rangle \geq 0 \quad \text{for } x \in \partial\Omega_t \text{ and } i \in I(t, x). \quad (4.19)$$

Define the following compact convex set

$$K = \left\{ - \sum_{i \in I(s, z)} \beta_i \gamma_i(s, z), \beta_i \geq 0 \sum_{i \in I(s, z)} \beta_i = 1 \right\}.$$

Using Lemma 4.6, we see that

$$\min_{i \in I(s,z)} \langle n_i(s,z), p \rangle < 0 \quad \forall p \in K.$$

Next we define the following sets from K

$$K_\epsilon = \{p \in \mathbb{R}^n : \text{dist}(p, K) \leq \epsilon\} \text{ and } L_\epsilon = \bigcup_{t \geq 0} tK_\epsilon.$$

From the above we get that there exists a $\delta > 0$ such that

$$\min_{i \in I(s,z)} \langle n_i(s,z), p \rangle < 0 \quad \forall p \in K_{2\delta}.$$

This means that $0 \notin K_{2\delta}$ and that $L_{2\delta}$ is a closed, convex cone with 0 as vertex. The above inequality implies that there exists a constant θ such that

$$\min_{i \in I(s,z)} \langle n_i(s,z), p \rangle \leq -\theta|p| \quad \forall p \in L_{2\delta}. \quad (4.20)$$

Now due to the upper semi-continuity of $I(t, x)$, the regularity of γ_i and $\partial\Omega_i$ and (2.18), there exists a open neighborhood V around z and open interval W around s with the following properties

$$I(t, x) \subset I(s, z), \quad \forall x \in V \cap \partial\Omega_t, \quad \forall t \in W, \quad (4.21)$$

$$-\gamma_i(t, x) \in K_\delta \quad \forall x \in V, \quad \forall t \in W, \quad (4.22)$$

$$\bar{\Omega}_s \cap \bar{V} \subset \left\{ z + p : p \in \mathbb{R}^n : \min_{i \in I(s,z)} \langle n_i(s,z), p \rangle \geq -\frac{\theta}{2}|p| \right\}. \quad (4.23)$$

Combining (4.20) and (4.23) we get

$$(z + L_{2\delta}) \cap \bar{\Omega}_s \cap \bar{V} = \{z\}.$$

This means that there exists an ϵ such that

$$\{x : \text{dist}(x, z + L_{2\delta}) \leq 3\epsilon\} \cap \bar{\Omega}_s \cap \partial V = \emptyset.$$

Define the set $M = z + q + L_{2\delta}$ for $q \in L_{2\delta} \cap \partial B(o, \epsilon)$ and $M_\eta = \{p : p \in \mathbb{R}^n, \text{dist}(x, M) \leq \eta\}$. From the above we notice that

$$M \cap \bar{\Omega}_s \cap \bar{V} = \emptyset,$$

$$M_{3\epsilon} \cap \bar{\Omega}_s \cap \partial V = \emptyset.$$

Define a function $h_1(x) = \text{dist}(x, M)$, on $\mathbb{R}^n \setminus M$ $h_1 \in C^{1,1}$ with derivative

$$Dh_1(x) = \frac{x - P_M(x)}{|x - P_M(x)|}.$$

Next we prove that

$$\langle DH_1(x), p \rangle < 0 \quad \forall x \in \mathbb{R}^n \setminus M, \quad \forall p \in K_\delta.$$

Notice that $\langle x - P_M(x), q - P_M(x) \rangle \leq 0$ for $q \in M$. Then from the construction of L_δ and M we see that $p + P_M(x) \in M$ for $p \in K_{2\delta}$, which yields $\langle x - P_M(x), p \rangle \leq 0$. Now choosing $p \in K_\delta$ we get that $\langle x - P_M(x), p \rangle < 0$, since $p + P_M(x)$ will be an internal point to M . This proves the statement above. Now take a function $\zeta \in C^1(\mathbb{R})$ with the following properties, $\zeta'(x) \geq 0$ $x \in \mathbb{R}^n$, $\zeta'(x) > 0$ for $x \leq \epsilon$ and $\zeta(x) = 0$ for $x \geq 2\epsilon$. Set $h_2(x) = \zeta(h_1(x))$. We conclude that for $p \in K_\delta$

$$\begin{aligned} \langle D_x h_2(z), p \rangle &< 0, \\ \langle D_x h_2(x), p \rangle &\leq 0 \quad \forall x \in \mathbb{R}^n \setminus M, \end{aligned}$$

And that $\text{supp}(h_2) \subset M_{2\epsilon}$. Set η such that $\eta < \text{dist}(\overline{\Omega_s} \cap \overline{V}, M)$, observe that $\eta < \epsilon$, and then an approximation argument yields that there exists a function $h_3 \in C^2(\mathbb{R}^n)$, $\text{supp } h_3 \subset M_{3\epsilon}$ which fulfills

$$\begin{aligned} \langle D_x h_3(z), p \rangle &< 0, \\ \langle D_x h_3(x), p \rangle &\leq 0 \quad \forall x \in V \setminus M_\eta. \end{aligned}$$

Combining the above with (4.22) yields

$$\begin{aligned} \langle D_x h_3(z), \gamma_i(t, x) \rangle &> 0 \quad \text{for } i \in I(s, x), \\ \langle D_x h_3(x), \gamma_i(t, x) \rangle &\geq 0 \quad \text{for } x \in V \setminus M_\eta, \quad i \in I(s, z). \end{aligned}$$

Define a function $\alpha \in C^2(\mathbb{R}^n)$ with the following properties

$$\begin{aligned} \text{supp } \alpha &\subset V, \\ \alpha &= 1 \text{ on } \overline{\Omega_s} \cap \overline{V} \cap \text{supp } h_3. \end{aligned}$$

Finally set $u(t, x) = h_3(x)\alpha(x)$. It is easy to check that $u(s, \cdot)$ fulfills (4.18) and (4.19) for the spatial variables with $\epsilon_1 = \eta$ because of the Lipschitz continuity on γ_i and $D_x u$. The idea for the time variable is that if $-\gamma_i(t, z) \in K_\delta$ for t in some small interval W around s the above proof works for all times in that interval. Since we have upper semi continuous for $I(t, x)$ we can assume that for $t \in W$, $I(t, z) \subset I(s, z)$. Then we want to prove that

$$\left\{ - \sum_{i \in I(t, z)} t_i \gamma_i(t, z), t_i \geq 0 \sum_{i \in I(t, z)} \beta_i = 1 \right\} \subset K_\delta.$$

This is the same as to prove that there exists a $p \in K$ such that

$$| - \sum_{i \in I(t, z)} \beta_i \gamma_i(t, z) - p | \leq \delta. \quad (4.24)$$

From the definition of K and the semi upper continuity of $I(t,x)$ we can represent p by $-\sum_{i \in I(t,z)} \beta_i \gamma_i(s, z)$. This and the Lipschitz continuity of γ_i yields

$$\begin{aligned} \left| -\sum_{i \in I(t,z)} \beta_i \gamma_i(t, z) + \sum_{i \in I(t,z)} \beta_i \gamma_i(s, z) \right| &\leq \sum_{i \in I(t,z)} \beta_i |\gamma_i(t, z) - \gamma_i(s, z)| \leq \\ &\sum_{i \in I(t,z)} C \beta_i |t - s| = C |I(t, z)| |t - s| \leq C |I(s, z)| |t - s|. \end{aligned}$$

Setting $\epsilon_s = \frac{\delta}{C |I(s, z)|}$ and then choosing the interval $t \in (s - \epsilon_s, s + \epsilon_s)$ we have that $-\gamma_i(t, z) \in K_\delta$. Now (4.18) and (4.19) holds. This means that we can find an open cover for $\bar{\Omega}$ with functions that fulfills (4.18) and (4.19). Since $\bar{\Omega}$ is compact, due to Tychonoff's theorem, there exists a finite open cover. This together with a partition of unity argument, adding and scaling with some constants yields the function h . \square

Chapter 5

Skorokhod Problem

In this sections we will prove the existence of a solution to the SP for continuous functions under our assumptions on the domain and direction of reflection.

5.1 SP for C^1 functions

The first part of this will be to prove the existence of a solution to the SP for C^1 functions. To do this we first need a Lemma.

Lemma 5.1. *Let $b > 0$ be chosen so that $d(x, \Omega_{i,t}) < b$, $i \in I(t, x)$, implies that $d(x, \Omega_{i,t}) \in C^1$. Then there exists a constant $c > 0$ and functions $a_i(t, x) \in C^i$ satisfying $a_i \geq 0$ and $a_i(t, x) = 0 \ \forall x \notin I(x)$. Such that*

$$\left\langle \sum_{i \in I} a_i(t, x) \gamma_i(t, x), D_x d(x, \Omega_{i,t}) \right\rangle < -\nu \quad \forall j \in I(t, x), \quad (5.1)$$

for all x satisfying $\sum_{i \in I} d(x, \Omega_{i,t}) < c$.

Lemma 5.1 follows from (2.15), (2.16) the compactness of Ω and a partition of unity argument, see [1]. Now we are ready to prove the following theorem.

Theorem 5.2. *Let $\psi \in C^1([0, T], \mathbb{R}^n)$ with $\psi(0) \in \Omega_0$. Then there exists a solution $(\phi, \lambda) \in \mathcal{W}^{1,p}([0, T], \mathbb{R}^n) \times \mathcal{W}^{1,p}([0, T], \mathbb{R}^n)$ to the Skorokhod problem for (Ω, r, ψ)*

The proof is based on a penalty method similar to the one used in [1], [2] and [4].

Proof. Set $\epsilon > 0$ and consider the ordinary differential equations

$$\dot{\phi}_\epsilon(t) = \frac{1}{\epsilon} \tilde{d}(t, \phi_\epsilon(t)) \left(\sum_{i \in I} a_i(t, \phi_\epsilon(t)) \gamma_i(t, \phi_\epsilon(t)) \right) + \dot{\psi}(t) \quad \phi_\epsilon(0) = \psi(0), \quad (5.2)$$

where $\tilde{d}(t, x) = \sum_{i \in I} d(x, \Omega_{i,t})$ and $a_i(t, x)$ are the functions in Lemma 5.1. The ordinary differential equation have a penalty term that pushes the solution ϕ_ϵ back into Ω in the direction of reflection. The idea is then to prove that $\phi_\epsilon \rightarrow \phi$ as $\epsilon \rightarrow 0$. We define a function $\zeta(t)$ with the following properties

$$\zeta \in C^\infty \quad \zeta(t) = \begin{cases} t & t < \frac{c}{3} \\ \frac{2c}{3} & t > c \end{cases} \quad 0 \leq \dot{\zeta}(t) \leq 1 \quad \forall t \in \mathbb{R}^+, \quad (5.3)$$

where c is the constant in Lemma 5.1. Then define $\tilde{v}(t, x) = \tilde{d}(t, x)^p$ and $V(t) = \zeta(\tilde{v}(t, \phi_\epsilon(t)))$. Note that if $\tilde{d}(t, \phi_\epsilon(t)) > c^{1/p}$ then $\zeta'(\tilde{v}(t, \phi_\epsilon(t))) = 0$. Next we look at the derivative to $V(t)$ and note that the weak derivative $D_t \tilde{d}(t, x)$ exists due to (2.18). We get

$$\dot{V}(t) = \dot{\zeta}(\tilde{v}(t, \phi_\epsilon(t))) \left(D_t \tilde{v}(t, \phi_\epsilon(t)) + \langle D_x \tilde{v}(t, \phi_\epsilon(t)), \dot{\phi}_\epsilon(t) \rangle \right).$$

We have that $D_x \tilde{v}(t, \phi_\epsilon(t)) = p \tilde{d}(t, \phi_\epsilon(t))^{p-1} D_x \tilde{d}(t, \phi_\epsilon(t))$. With this, (5.2), Lemma 5.1 and the Cauchy-Schwartz inequality we have the following

$$\begin{aligned} \dot{V}(t) &= \dot{\zeta}(\tilde{v}(t, \phi_\epsilon(t))) (D_t \tilde{v}(t, \phi_\epsilon(t))) \\ &+ \left\langle D_x \tilde{v}(t, \phi_\epsilon(t)), \frac{1}{\epsilon} \tilde{d}(t, \phi_\epsilon(t)) \left(\sum_{i \in I} a_i(t, \phi_\epsilon(t)) \gamma_i(t, \phi_\epsilon(t)) + \dot{\psi}(t) \right) \right\rangle \\ &\leq \dot{\zeta}(\tilde{v}(t, \phi_\epsilon(t))) (D_t \tilde{v}(t, \phi_\epsilon(t)) + |D_x \tilde{v}(t, \phi_\epsilon(t))| |\dot{\psi}(t)| - \frac{p\nu}{\epsilon} \tilde{v}(t, \phi_\epsilon(t))). \end{aligned}$$

Next, we rearrange and integrate

$$\begin{aligned} V(t) - V(0) + \frac{\nu}{\epsilon} \int_0^t \dot{\zeta}(\tilde{v}(s, \phi_\epsilon(s))) \tilde{v}(s, \phi_\epsilon(s)) ds &\leq \\ \underbrace{\int_0^t \dot{\zeta}(\tilde{v}(s, \phi_\epsilon(s))) D_t \tilde{v}(s, \phi_\epsilon(s)) ds}_{I_1} &+ \underbrace{\int_0^t \dot{\zeta}(\tilde{v}(s, \phi_\epsilon(s))) |D_x \tilde{v}(s, \phi_\epsilon(s))| |\dot{\psi}(s)| ds}_{I_2}. \end{aligned}$$

We first consider I_1 and notice that

$$\begin{aligned} |D_t \tilde{v}(s, \phi_\epsilon(s))| &= p \tilde{d}(s, \phi_\epsilon(s))^{p-1} |D_t \tilde{d}(s, \phi_\epsilon(s))| = \\ &= p \tilde{v}(s, \phi_\epsilon(s))^{\frac{p-1}{p}} |D_t \tilde{d}(s, \phi_\epsilon(s))|. \end{aligned}$$

Using this and Hölder's inequality we get

$$I_1 \leq p \left(\int_0^t \dot{\zeta}(\tilde{v}(s, \phi_\epsilon(s))) \tilde{v}(s, \phi_\epsilon(s)) ds \right)^{\frac{p-1}{p}} \left(\int_0^t \dot{\zeta}(\tilde{v}(s, \phi_\epsilon(s))) |\tilde{d}(s, \phi_\epsilon(s))|^p ds \right)^{\frac{1}{p}}.$$

Now since $\tilde{d}(t, x) \in \mathcal{W}^{1,p}$ and $\dot{\zeta}(t) \leq 1$ the second integral on the right hand side is bounded by a constant $C(T)$. For the I_2 integral we notice that the

derivative $|D_x d(t, x, G_i)| \leq 1$ a.e. This means that $|D_x \tilde{d}(t, x)| \leq k$, where k is the size of the finite index set I . Using this we get $|D_x \tilde{v}(t, x)| \leq pk \tilde{d}(t, x)^{p-1}$. Applying Hölder's inequality again

$$I_2 \leq pk \left(\int_0^t \dot{\zeta}(\tilde{v}(s, \phi_\epsilon(s))) \tilde{v}(s, \phi_\epsilon(s)) ds \right)^{\frac{p-1}{p}} \left(\int_0^t \dot{\zeta}(\tilde{v}(s, \phi_\epsilon(s))) |\dot{\psi}(t)|^p \right)^{\frac{1}{p}}.$$

Since $\psi \in C^1$, and $\dot{\zeta}(t) \leq 1$ so the second integral is bounded by a constant $C(T)$. Collecting the terms, we obtain

$$V(t) + \frac{\nu}{\epsilon} \int_0^t \dot{\zeta}(\tilde{v}(s, \phi_\epsilon(s))) \tilde{v}(s, \phi_\epsilon(s)) ds \leq C(t) \left(\int_0^t \dot{\zeta}(\tilde{v}(s, \phi_\epsilon(s))) \tilde{v}(s, \phi_\epsilon(s)) ds \right)^{\frac{p-1}{p}}. \quad (5.4)$$

Since both terms on the left hand side are positive, both are bounded by the constant and hence

$$\frac{\nu}{\epsilon} \left(\int_0^t \dot{\zeta}(\tilde{v}(s, \phi_\epsilon(s))) \tilde{v}(s, \phi_\epsilon(s)) ds \right)^{\frac{1}{p}} \leq C(T). \quad (5.5)$$

Substituting (5.5) into (5.4) we get

$$V(t) + \frac{\nu}{\epsilon} \int_0^t \dot{\zeta}(\tilde{v}(s, \phi_\epsilon(s))) \tilde{v}(s, \phi_\epsilon(s)) ds \leq C(T) \epsilon^{p-1}. \quad (5.6)$$

Choosing ϵ so small that $V(t) = \zeta(\tilde{v}(s, \phi_\epsilon(s))) = \tilde{v}(s, \phi_\epsilon(s))$ and $\dot{\zeta}(\tilde{v}(s, \phi_\epsilon(s))) = 1$, we obtain

$$\frac{1}{\epsilon^{p-1}} \tilde{d}(t, \phi_\epsilon(t))^p + \frac{\nu}{\epsilon^p} \int_0^t \tilde{d}(s, \phi_\epsilon(s))^p ds \leq C(T). \quad (5.7)$$

Now define the functions

$$l_\epsilon(t) = \frac{1}{\epsilon} \tilde{d}(t, \phi_\epsilon(t)), \quad (5.8)$$

$$\lambda_{i,\epsilon}(t) = \int_0^t l_\epsilon(s) a_{i,\epsilon}(t, \phi_\epsilon(t)) \gamma_{i,\epsilon}(s, \phi_\epsilon(t)) ds, \quad (5.9)$$

$$\lambda_\epsilon(t) = \sum_{i \in I} \lambda_{i,\epsilon}(t). \quad (5.10)$$

We can assume that the functions $a_{i,\epsilon}$ from Lemma 5.1 fulfills $|\sum_{i \in I} a_{i,\epsilon} \gamma_{i,\epsilon}| = 1$. Now by (5.7) l_ϵ is bounded in L^p and $\lambda_{i,\epsilon}, \lambda_\epsilon$ are bounded in $\mathcal{W}^{1,p}$. Therefore they converge weakly to l, λ_i, λ in L^p and $\mathcal{W}^{1,p}$, respectively, as $\epsilon \rightarrow 0$. We see from (5.2) that $\phi_\epsilon = \psi + \lambda_\epsilon$ which means that ϕ_ϵ converges weakly to $\phi = \psi + \lambda$. This proves (2.8). (2.9) follows from (5.7) since the first term

must be bounded as $\epsilon \rightarrow 0$. (5.9) implies that $\dot{\lambda}_{i,\epsilon} = l_\epsilon a_{i,\epsilon} \gamma_{i,\epsilon}$ and therefore $\dot{\lambda}_i = l a_i \gamma_i$ hence

$$|\lambda|(t) = \int_0^t \left| \sum_{i \in I} \dot{\lambda}_i(s) \right| ds = \int_0^t \left| \sum_{i \in I} l(s) a_i(\phi(s)) \gamma_i(\phi(s)) \right| ds = \int_0^t l ds < \infty, \quad (5.11)$$

which proves (2.10). To prove that (2.12) holds, we look at

$$\lambda(t) = \int_0^t \sum_{i \in I} l(s) a_i(\phi(s)) \gamma_i(\phi(s)) ds = \int_0^t \sum_{i \in I} a_i(\phi(s)) \gamma_i(\phi(s)) d|\lambda|(s). \quad (5.12)$$

From Lemma 5.1 we have that $a_i \geq 0$ and $a_i > 0$ implies that $i \in I(t, x)$. Since we also have that $|\sum_{i \in I} a_i \gamma_i| = 1$ we see that $\sum_{i \in I} a_i \gamma_i \in r(x)$ and therefore (2.12) holds. To prove (2.11) we look at (5.8). For $\phi(t) \in \cap G_i$ $\tilde{d}(t, \phi(t)) = 0$ so $l(t) = 0$. This means that

$$|\lambda|(t) = \int_0^t l(s) ds = \int_0^t I_{\{\phi(s) \in \partial \bar{\Omega}\}} l(s) ds = \int_0^t I_{\{\phi(s) \in \partial \bar{\Omega}\}} d|\lambda|(s). \quad (5.13)$$

This proves that (ϕ, λ) solves the Skorokhod problem. \square

5.2 Compactness of solutions for SP

To go from continuous differentiable functions to only continuous functions we prove that the solution set to the SP is compact and bounded. This will also come in handy in Section 6 when we prove tightness of the solutions to the RSDE. The following lemma is a generalization of Lemma 4.7 in [1] Case 2 and the time-dependent approach is similar in [2].

Lemma 5.3. *Let A be a compact subspace of $C([0, T], \mathbb{R}^n)$. Then the following holds for the set $\{(\phi, \lambda) : (\phi, \lambda) \text{ solves the SP for } (\Omega, r, \psi), \psi \in A\}$*

1. $|\lambda|(T) < L$ for a constant $L < \infty$
2. The set $\{\phi : (\phi, \lambda) \text{ solves the SP for } (\Omega, r, \psi), \psi \in A\}$ is relative compact

First we define the modulus of continuity of a function.

Definition 5.4. *The modulus of continuity of a function f is defined as*

$$\|f\|_{s,t} = \sup_{s \leq t_1 \leq t_2 \leq t} |f(t_1) - f(t_2)|. \quad (5.14)$$

Proof. There exists a vector field $v(t, x)$ such that for some constant $c > 0$ the following holds

$$\langle \gamma(s, y), v(t, x) \rangle > c \quad \forall y \in \times B(x, c) \cap \partial\Omega_s, \quad \forall s \in [t, t + c], \quad \forall \gamma \in r(s, y). \quad (5.15)$$

Next for a solution (ϕ, λ) to the SP for $\psi \in A$, we define T_1 as the smallest of c and $\inf\{t \in [0, T] : \phi(t) \notin B(\phi(0), c)\}$. In other words at the time T_1 the solution ϕ have moved a distance less than c from the initial value and $T_1 < c$. Define T_m analogously, as $T_m = \min(T, T_{m-1} + c, \inf\{t \in [T_{m-1}, T] : \phi(T_{m-1}) \notin B(\phi(T_{m-1}), c)\})$. Now we will show that $|\lambda|$ is bounded on $[T_{m-1}, T_m]$ by looking at the inner product with respect to the vector field v .

$$\begin{aligned} & \langle \phi(T_m) - \phi(T_{m-1}), v(T_{m-1}, \phi(T_{m-1})) \rangle \\ & \quad - \langle \psi(T_m) - \psi(T_{m-1}), v(T_{m-1}, \phi(T_{m-1})) \rangle = \\ & \quad \int_{T_{m-1}}^{T_m} \langle \gamma(s, \phi(s)), v(T_{m-1}, \phi(T_{m-1})) \rangle d|\lambda|(s) \geq c(|\lambda|(T_m) - |\lambda|(T_{m-1})). \end{aligned}$$

In the first equality we use (2.12) and the fact that $d|\lambda|$ is zero unless $\phi \in \partial\Omega$. The left hand side is bounded since A is bounded and $\phi \in \Omega$. Therefore we get that

$$|\lambda|(T_m) - |\lambda|(T_{m-1}) < M, \quad (5.16)$$

for some constant $M < \infty$. Now we will show a bound to the modulus of continuity for λ for all $\tau \in [T_{m-1}, T_m]$

$$\|\lambda\|_{T_{m-1}, \tau} \leq R(\|\psi\|_{T_{m-1}, \tau}^{\frac{1}{2}} + \|\psi\|_{T_{m-1}, \tau}^{\frac{3}{2}} + (\tau - T_{m-1})^{\frac{\alpha}{2}}), \quad (5.17)$$

where $\psi \in A$ and α is the Hölder coefficient in (2.19). Since we only look at one time-step $[T_{m-1}, T_m]$ we simplify the notation by setting $\phi(T_{m-1}) = z$ and $T_{m-1} = 0$. To prove the bound we define some help functions. We use the function g in Theorem 4.1 but use a sub-convolution to define functions g^β which have more regularity

$$\begin{aligned} g^\beta(t, x, r) &= \sup\{g(s, y, w) - \frac{1}{2\beta}(|t - s|^2 + |x - y|^2 + |r - w|^2) : \\ & \quad (s, y, w) \in [0, T] \times W \times B(0, R + 1)\}. \end{aligned} \quad (5.18)$$

We refer to [1] that the following properties holds for β sufficiently small

$$g^\beta \in C^{1,1}([0, T] \times \bar{\Omega} \times B(0, R)), \quad (5.19)$$

$$g^\beta \rightarrow g \text{ in } C, \quad (5.20)$$

$$(u, p, q) = Dg^\beta(t, x, r) \Rightarrow (u, p, q) \in D^+g(t + \beta u, x + \beta p, r + \beta q), \quad (5.21)$$

$$|u| \leq 4C|r|^2, \quad |p| \leq 4C|r|^2, \quad |q| \leq 2C|r|. \quad (5.22)$$

$C^{1,1}$ is the space of once differentiable functions with Lipschitz continuous derivatives. Now we define the functions $B_\epsilon^\beta(t)$ and $E(t)$ as follow

$$\begin{aligned} B_\epsilon^\beta(t) &= \epsilon g^\beta(t, z, -\lambda(t)/\epsilon), \\ E(t) &= e^{-C(2|\lambda(t)|+4t)}. \end{aligned}$$

Since $g^\beta(t, x, 0) = 0$, we get

$$\begin{aligned} B_\epsilon^\beta(\tau)E(\tau) &= B_\epsilon^\beta(0)E(0) + \int_0^\tau E(s)dB_\epsilon^\beta(s) + \int_0^\tau B_\epsilon^\beta(s)dE(s) \\ &= \int_0^\tau E(s)dB_\epsilon^\beta(s) - 2C \int_0^\tau B_\epsilon^\beta(s)E(s)d|\lambda|(s) - 4C \int_0^\tau B_\epsilon^\beta(s)E(s)ds. \end{aligned}$$

Now we look at the first integral and use (5.22) and (4.5)

$$\begin{aligned} \int_0^\tau E(s)dB_\epsilon^\beta(s) &= \int_0^\tau E(s)\epsilon D_t g_\epsilon^\beta(s, z, -\lambda(s)/\epsilon)ds \\ &\quad - \int_0^\tau E(s)\langle D_r g^\beta(s, z, -\lambda(s)/\epsilon), d\lambda(s) \rangle \leq 4C \int_0^\tau E(s)\epsilon|\lambda(s)/\epsilon|^2 ds \\ &\quad - \int_0^\tau E(s)\langle D_r g^\beta(s, z, -\lambda(s)/\epsilon), d\lambda(s) \rangle \\ &\leq 4C \int_0^\tau E(s)B_\epsilon^\beta(s)ds - \int_0^\tau E(s)\langle D_r g^\beta(s, z, -\lambda(s)/\epsilon), d\lambda(s) \rangle. \end{aligned}$$

For the second integral on the right hand side we must be careful because of the time-dependent domain. Since the domain changes over time it is not certain that $z \in \overline{\Omega_s} \forall s$. If $z \notin \overline{\Omega_s}$ we define y_s as the point in $\overline{\Omega_s}$ which fulfills $|y_s - z| = d(s, z)$. In the integrals below if $z \in \Omega_s$ then we can replace y_s with z . We decompose the integral in three parts

$$\begin{aligned} - \int_0^\tau E(s)\langle D_r g^\beta(s, z, -\lambda(s)/\epsilon), d\lambda(s) \rangle &= I_1 + I_2 + I_3, \\ I_1 &= - \int_0^\tau E(s)\langle D_r g^\beta(s, \phi(s), (y_s - \phi(s))/\epsilon), \gamma(s, \phi(s)) \rangle d|\lambda|(s), \\ I_2 &= \int_0^\tau E(s)\langle D_r g^\beta(s, \phi(s), -\lambda(s)/\epsilon) - D_r g^\beta(s, z, -\lambda(s)/\epsilon), d\lambda(s) \rangle, \\ I_3 &= \int_0^\tau E(s)\langle D_r g^\beta(s, \phi(s), (y_s - \phi(s))/\epsilon) - D_r g^\beta(s, \phi(s), -\lambda(s)/\epsilon), d\lambda(s) \rangle. \end{aligned}$$

We first consider I_1 , and from (5.21) we get

$$I_1 = - \int_0^\tau E(s)\langle D_r g(s + \beta u, \phi(s) + \beta p, (y_s - \phi(s))/\epsilon + \beta q), \gamma(s, \phi(s)) \rangle d|\lambda|(s),$$

Where

$$\begin{aligned} u &= D_t g^\beta(s, \phi(s), (y_s - \phi(s))/\epsilon), \\ p &= D_x g^\beta(s, \phi(s), (y_s - \phi(s))/\epsilon), \\ q &= D_r g^\beta(s, \phi(s), (y_s - \phi(s))/\epsilon). \end{aligned}$$

Now we want to use (4.6), and notice that

$$\int_0^\tau E(s) \langle D_r g(s + \beta u, \phi(s) + \beta p, (y_s - \phi(s))/\epsilon + \beta q), \gamma(s + \beta u, \phi(s) + \beta p) \rangle d|\lambda|(s) \geq 0.$$

Adding this term to I_1

$$\begin{aligned} I_1 \leq & - \int_0^\tau E(s) \langle D_r g(s + \beta u, \phi(s) + \beta p, (y_s - \phi(s))/\epsilon + \beta q), \\ & \gamma(s, \phi(s)) - \gamma(s + \beta u, \phi(s) + \beta p) \rangle d|\lambda|(s). \end{aligned}$$

But since γ is continuous $\gamma(s, \phi(s)) - \gamma(s + \beta u, \phi(s) + \beta p)$ tends to zero as $\beta \rightarrow 0$ and therefore $I_3 \leq 0$ as $\beta \rightarrow 0$. For I_2 and I_3 we will use the Lipschitz property of the derivatives and (5.22). For fixed r we can choose $2C|r|$ as a Lipschitz constant for $D_r g^\beta(t, x, r)$ and for fixed x we can choose $4C$ as a Lipschitz constant. Using this, the Cauchy-Schwartz inequality and the fact that $|\gamma| = 1$ when $\phi \in \partial\Omega_s$ and $d|\lambda| = 0$ when $\phi \notin \partial\Omega_s$, we obtain

$$\begin{aligned} I_2 \leq & \int_0^\tau E(s) |D_r g^\beta(s, \phi(s), -\frac{\lambda(s)}{\epsilon}) - D_r g^\beta(s, z, -\lambda(s)/\epsilon)| |\gamma(s, \phi(s))| d|\lambda|(s) \leq \\ & \int_0^\tau \frac{2C}{\epsilon} E(s) |\lambda(s)| |z - \phi(s)| d|\lambda|(s) \leq \\ & \int_0^\tau \frac{2C}{\epsilon} E(s) |\lambda(s)| (|z - \psi(s)| + |\lambda(s)|) d|\lambda|(s) \leq \\ & \int_0^\tau 2CE(s) \epsilon \left| \frac{\lambda(s)}{\epsilon} \right|^2 d|\lambda|(s) + \int_0^\tau \frac{2C}{\epsilon} E(s) |z - \psi(s)| |\lambda(s)| d|\lambda|(s) \leq \\ & \int_0^\tau 2CE(s) B_\epsilon^\beta(s) d|\lambda|(s) + \int_0^\tau \frac{2C}{\epsilon} E(s) |z - \psi(s)| |\lambda(s)| d|\lambda|(s) \leq \\ & \int_0^\tau 2CE(s) B_\epsilon^\beta(s) d|\lambda|(s) + \int_0^\tau \frac{2C}{\epsilon} E(s) |z - \psi(s)|^2 d|\lambda|(s). \end{aligned}$$

The same argument for I_3 yields

$$\begin{aligned}
I_3 &\leq \\
\int_0^\tau E(s) &|D_r g^\beta(s, \phi(s), \frac{(y_s - \phi(s))}{\epsilon}) - D_r g^\beta(s, \phi(s), -\frac{\lambda}{\epsilon})| |\gamma(s, \phi(s))| d|\lambda|(s) \leq \\
&\int_0^\tau \frac{4C}{\epsilon} E(s) |y_s - \phi(s) + \lambda(s)| d|\lambda|(s) = \\
&\int_0^\tau \frac{4C}{\epsilon} E(s) |y_s - \psi(s)| d|\lambda|(s) \leq \\
&\int_0^\tau \frac{4C}{\epsilon} E(s) (|z - \psi(s)| + |z - y_s|) d|\lambda|(s) = \\
&\int_0^\tau \frac{4C}{\epsilon} E(s) (|z - \psi(s)| + d(s, z)) d|\lambda|(s).
\end{aligned}$$

Collecting all the terms and letting $\beta \rightarrow 0$

$$B_\epsilon(\tau)E(\tau) \leq \frac{2C}{\epsilon} \int_0^\tau E(s) (|z - \psi(s)|^2 + 2|z - \psi(s)| + 2d(s, z)) d|\lambda|(s).$$

Now with the modulus of continuity and using (2.19) we get

$$B_\epsilon(\tau)E(\tau) \leq \frac{4C}{\epsilon} \int_0^\tau E(s) (\|\psi\|_{0,\tau}^2 + \|\psi\|_{0,\tau} + K\tau^\alpha) d|\lambda|(s).$$

Only $E(s)$ is dependent of s in the integral and it can be estimated easily

$$\int_0^\tau E(s) d|\lambda|(s) \leq \int_0^\tau e^{-2C|\lambda|(s)} d|\lambda|(s) \leq \frac{1}{2C}.$$

Now we get

$$B_\epsilon(\tau) \leq \frac{2}{\epsilon} (\|\psi\|_{0,\tau}^2 + \|\psi\|_{0,\tau} + K\tau^\alpha) e^{C(2|\lambda|(\tau)+4t)}.$$

A use of the inequality

$$x \leq \frac{1}{2}(\epsilon + \frac{1}{\epsilon}x^2),$$

and (4.5) gives us

$$|\lambda(\tau)| \leq \frac{1}{2}(\epsilon + \frac{1}{\epsilon}|\lambda(\tau)|^2) \leq \frac{1}{2}(\epsilon + B_\epsilon) \leq \frac{\epsilon}{2} + \frac{1}{\epsilon}(\|\psi\|_{0,\tau}^2 + \|\psi\|_{0,\tau} + K\tau^\alpha) e^{C(2M+4T)},$$

where in the exponential we have used (5.16) and $\tau \leq T$. Now set $\epsilon = \min(\|\psi\|_{0,\tau}^{1/2}, \tau^{\alpha/2})$, which implies that $\epsilon < \|\psi\|_{0,\tau}^{1/2} + \tau^{\alpha/2}$ and $\frac{1}{\epsilon} \leq \|\psi\|_{0,\tau}^{-1/2}$ and $\frac{1}{\epsilon} \leq \tau^{-\alpha/2}$. With this we get (5.17) since the above hold for all $\tau \in [T_{m-1}, T_m]$. Since A is compact we know that $\|\psi\|$ is bounded. This implies there exists a time $\hat{\tau} > 0$ such that

$$\max(\|\psi\|_{T_{m-1}, T_{m-1}+\hat{\tau}}, \|\lambda\|_{T_{m-1}, T_{m-1}+\hat{\tau}}) \leq \frac{c}{3}.$$

By the definition of SP this implies that $\|\phi\|_{T_{m-1}, T_{m-1}+\hat{\tau}} \leq \frac{2c}{3}$. From the definition of T_m we now have a lower bound on the time partition $T_m - T_{m-1} \geq \min(c, \hat{\tau})$. From (5.16) we have a bound of λ for each time step so with the bound on the time step we get that $|\lambda(T)| < M(\frac{T}{\min(c, \hat{\tau})} + 1)$ which proves 1 in Lemma 5.3. To prove 2 we use the Arzela-Ascoli theorem. Since $\phi(t)$ is bounded by Ω_t and (5.16) together with the lower bound of the time partition gives us that the set $\{\phi : (\phi, \lambda) \text{ is a solution to SP for } \psi \in A\}$ is equicontinuous. Since we have a bounded and equicontinuous set, the Arzela-Ascoli theorem implies that the set is relatively compact. \square

5.3 SP for continuous functions

Now we are ready to prove the main lemma of this section.

Lemma 5.5. *Let $\psi \in C[0, T]$ with $\psi(0) \in \Omega_0$. Then there exists a solution (ϕ, λ) to the SP for ψ .*

The proof is the same as for Theorem 4.8 in [1], and is similar to the proof of Theorem 3.1 in [8] and Theorem 5.1 in [9]. For completeness we include the proof

Proof. Let $\psi_n \in C^1([0, T])$ be a series of functions that converge uniformly to ψ . By Lemma 5.2 there exists a solution to the SP for ψ_n denoted (ϕ_n, λ_n) . From Lemma 5.3 we know that λ_n is bounded for all n and that λ_n is equicontinuous, therefore by Arzela-Ascoli theorem there exists a convergent subsequence which converges to λ . Set $\phi = \psi + \lambda$ and the pair (ϕ, λ) fulfills (2.8), (2.9) and (2.10) in the Skorokhod problem. To show (2.11) and (2.12) we define the measures μ_n on $\bar{\Omega} \times S(0, 1)$ as

$$\mu_n([0, t] \times A) = \int_{[0, t]} I_{(s, \phi_n(s), \gamma(s, \phi_n(s))) \in A} d|\lambda_n|(s),$$

for every Borel set $A \subset \bar{\Omega} \times S(0, 1)$. Define $\bar{\Omega}_{[0, t]} = \bar{\Omega} \cap ([0, t] \times \mathbb{R}^n)$ and $\Omega_{[0, t]} = \Omega \cap ([0, t] \times \mathbb{R}^n)$. This means that

$$|\lambda_n|(t) = \mu_n(\bar{\Omega}_{[0, t]} \times S(0, 1)),$$

which together with the definition of a solution to the SP gives

$$\lambda_n = \int_{\bar{\Omega}_{[0, t]} \times S(0, 1)} \gamma d\mu_n.$$

Since $|\lambda_n|(T) < \infty$, we have, by the Banach-Alaoglu theorem that μ_n converges to a measure μ in the weak*-topology. By the weak* convergence we have that

$$\lambda(t) = \int_{\bar{\Omega}_{[0, t]} \times S(0, 1)} \gamma d\mu. \quad (5.23)$$

Next we define the following sets

$$\begin{aligned}\Sigma_1 &= \Omega_{[0,T]} \times S(0,1), \\ \Sigma_2 &= \{(s,x,y) \mid s \in [0,T], x \in \partial\Omega_s, y \notin r(s,x)\}, \\ \Sigma_3^\delta &= \{(s,x) \mid s \in [0,T], |x - \phi(s)| > \delta\} \times S(0,1), \\ \Sigma_3 &= \{(s,x) \mid s \in [0,T], x \neq \phi(s)\} \times S(0,1).\end{aligned}$$

From the definition of μ_n and (2.11) we conclude that for large enough n

$$\mu_n(\Sigma_1) = \mu_n(\Sigma_2) = \mu_n(\Sigma_3^\delta) = 0.$$

From the weak* convergence we then have

$$\mu(\Sigma_1) = \mu(\Sigma_2) = \mu(\Sigma_3) = 0. \quad (5.24)$$

We define a new measure ν on $[0, T]$ by

$$\nu([0, t]) = \mu(\bar{\Omega}_{[0,t]} \times S(0,1)).$$

Combining (5.23) and (5.24) yields

$$\lambda(t) = \int_{(s,x,y) \mid \phi(s) \in \partial\Omega_s, y \in r(s,\phi(s))} \gamma d\mu = \int_0^t I_{\phi(s) \in \partial\Omega_s} \int_{r(s,\phi(s))} \gamma p(s, d\gamma) d\nu, \quad (5.25)$$

where $p(\cdot, A)$ is a non-negative ν -measurable function for all Borel sets A . From (5.23) we see that λ is of bounded variation and therefore

$$\lambda(t) = \int_0^t \tilde{\gamma}(s) d|\lambda|(s). \quad (5.26)$$

From (5.25) and (5.26) we conclude that $|\lambda|$ is absolutely continuous with respect to ν . This fact and (5.25) gives us (2.11). The absolute continuity of $|\lambda|$ w.r.t ν , (5.25) and the Radon-Nikodym theorem gives that the existence of a positive function $l(t)$ such that

$$l(s)\tilde{\gamma}(s) = \int_{r(s,\phi(s))} \gamma p(s, d\gamma). \quad (5.27)$$

Then using the fact that the set $\{\alpha\gamma, \alpha \geq 0, \gamma \in r(s,x)\}$ is convex for $x \in \partial\Omega_s$ allows us to conclude that $\tilde{\gamma}(s, \phi(s)) \in r(s, \phi(s))$. This fact and (5.26) finally gives us (2.12). \square

Chapter 6

Stochastic differential equation

6.1 Uniqueness of RSDE

Now we will prove the existence and uniqueness of solutions to RSDE by using the existence of solutions to the SP problem and a Picard iteration scheme, see [10]. The only thing left to prove is then uniqueness for the RSDE, Assume the following

$$Y(t) = x + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dW + \Lambda(t), \quad (6.1)$$

$$X(t) \in \bar{\Omega} \quad Y(t) \in \bar{\Omega},$$

$$|\Lambda|(t) = \int_{0,t} I_{Y(s) \in \partial\Omega} d|\Lambda|(s) < \infty, \quad \Lambda(t) = \int_{0,t} \gamma(s, Y(s))d|\Lambda|(s),$$

where b, σ is Lipschitz continuous. Now let Y' be defined as the above but with x replaced by x' , X replaced with X' and $\gamma' = \gamma(s, Y'(s))$, then we have the following lemma

Lemma 6.1. *Let $Y(s)$ and $Y'(s)$ be defined as (6.1). Then we have the following inequality*

$$E \left[\sup_{0 \leq s \leq t} |Y(s) - Y'(s)|^2 \right] \leq C \left(|x - x'|^2 + \int_0^t E \left[\sup_{0 \leq s \leq t} |X(s) - X'(s)|^2 ds \right] \right).$$

The main tool in the proof of Lemma 6.1 is a function v which will be a time extension of the function v in [1] with the correction [11]. Let $\phi(x, r) = (|x|^2 + A)|r|^2$ and choose B large enough so that

$$h(t, x, r) = g(t, x, r) - B\phi(x, r),$$

is convex in the spatial variables. Now to get required smoothness we sub-convolute h

$$h^\beta(t, x, r) = \sup_{(s, z, w) \in [0, T] \times W \times W} \left\{ h(s, y, w) - \frac{1}{2\beta} (|t - s|^2 + |x - y|^2 + |r - w|^2) \right\},$$

and then $g^\beta(t, x, r) = h^\beta(t, x, r) + B\phi(x, r)$. We set $f_\epsilon^\beta = \epsilon g(t, x, \frac{x-y}{\epsilon})$ and see that $f_\epsilon^\beta \rightarrow f_\epsilon$ in $C([0, T] \times \bar{W} \times \bar{W})$. As in the proof of Lemma 5.3 we have that for $(u, p, q) \in Df_\epsilon^\beta(t, x, y)$

$$f_\epsilon^\beta \in C^{1,1}([0, T] \times \bar{\Omega} \times \bar{\Omega}), \quad (6.2)$$

$$|u| \leq \frac{4C}{\epsilon} |x - y|^2, \quad |p + q| \leq C \frac{|x - y|^2}{\epsilon}, \quad \max(|p|, |q|) \leq C \frac{|x - y|}{\epsilon} \quad (6.3)$$

We also know since f_ϵ^β is Lipschitz continuous and \bar{W} is compact that (p, q) are bounded independently from x, y . In the proof we will use Ito's lemma and for that we require the functions to be twice differentiable in space. To achieve this we define the mollifier ρ_α

$$\rho_\alpha \geq 0 \quad \text{supp } \rho_\alpha = B(0, \alpha) \quad \int_{\mathbb{R}^n} \rho_\alpha = 1 \quad \rho_\alpha \in C^\infty,$$

and the convolution $f_\epsilon^{\beta, \alpha} = f_\epsilon^\beta * \rho_\alpha$. For fixed time t we refer to [11] for the following inequality

$$D^2 f_\epsilon^{\beta, \alpha}(t, x, y) \leq \frac{C}{\epsilon} \begin{bmatrix} I & -I \\ -I & I \end{bmatrix} + \frac{C|x - y|^2}{\epsilon} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \quad (6.4)$$

Now using the function h defined in Lemma 4.2 we define the function $u(t, x, y) = e^{-\lambda(h(t, x) + h(t, y))}$, $\lambda > 0$ constant, and then define

$$v_\epsilon^{\beta, \alpha}(t, x, y) = u(t, x, y) f_\epsilon^{\beta, \alpha}(t, x, y).$$

We will specify λ later, until then we will denote $C(\lambda)$ a constant that depends on λ . Before we prove Lemma 6.1 we state a lemma that will be helpful.

Lemma 6.2. *For fixed time t we have that the second derivative of the spatial variables satisfies*

$$D^2 v_\epsilon^{\beta, \alpha}(t, x, y) = C(\lambda) \left(\frac{1}{\epsilon} \begin{bmatrix} I & -I \\ -I & I \end{bmatrix} + \frac{|x - y|^2}{\epsilon} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \right).$$

The proof is identical to the proof of Lemma 5.7 in [1] with f_ϵ exchanged for $f_\epsilon^{\beta, \alpha}$. An easy extension of Lemma 6.2, using the Lipschitz continuity of σ is

$$\begin{bmatrix} \sigma(\nu) \\ \sigma(\xi) \end{bmatrix}^T D^2 v_\epsilon^{\beta, \alpha}(t, x, y) \begin{bmatrix} \sigma(\nu) \\ \sigma(\xi) \end{bmatrix} \leq \frac{C(\lambda)}{\epsilon} (|\nu - \xi|^2 + |x - y|^2) I. \quad (6.5)$$

Now we are ready to prove Lemma 6.1. We will denote $v_\epsilon^{\beta,\alpha}(s, Y(s), Y'(s)) = v_\epsilon^{\beta,\alpha}$ and the same with u and $f_\epsilon^{\beta,\alpha}$.

Proof of Lemma 6.1. We prove the Lemma for $t < \tau$ where τ is a stopping time defined as

$$\tau = \inf\{s \in [0, T], |Y(s) - Y'(s)| < \delta\},$$

and δ is the constant in Remark 2.27. This is without loss of generality since

$$E \left[\sup_{0 \leq s \leq t} |Y(s) - Y'(s)|^2 \right] \leq \left(\frac{B}{\delta} \right)^4 E \left[\sup_{0 \leq s \leq \tau} |Y(s) - Y'(s)|^2 \right],$$

where B is the diameter of the smallest ball containing Ω_t for all t . An application of Ito's lemma yields

$$\begin{aligned} v_\epsilon^{\beta,\alpha}(t, Y(t), Y'(t)) &= v_\epsilon^{\beta,\alpha}(0, x, x') + \int_0^t D_s v_\epsilon^{\beta,\alpha} ds \\ &\quad + \int_0^t \langle D_x v_\epsilon^{\beta,\alpha}, b(s, X(s)) \rangle ds + \int_0^t \langle D_y v_\epsilon^{\beta,\alpha}, b(s, X'(s)) \rangle ds \\ &\quad + \int_0^t \langle D_x v_\epsilon^{\beta,\alpha}, \sigma(s, X(s)) dW(s) \rangle + \int_0^t \langle D_y v_\epsilon^{\beta,\alpha}, \sigma(s, X'(s)) dW(s) \rangle \\ &\quad + \int_0^t \langle D_x v_\epsilon^{\beta,\alpha}, \gamma(s, Y(s)) d|\Lambda|(s) \rangle + \int_0^t \langle D_y v_\epsilon^{\beta,\alpha}, \gamma(s, Y'(s)) d|k|(s) \rangle \\ &\quad + \int_0^t \text{tr} \left(\begin{bmatrix} \sigma(X(s)) \\ \sigma(X'(s)) \end{bmatrix}^T D^2 f_\epsilon^{\beta,\alpha}(t, x, y) \begin{bmatrix} \sigma(X(s)) \\ \sigma(X'(s)) \end{bmatrix} \right) ds. \end{aligned}$$

Now using (6.5) to get rid of the second order derivative, we get

$$\begin{aligned} v_\epsilon^{\beta,\alpha}(t, Y(t), Y'(t)) &\leq v_\epsilon^{\beta,\alpha}(0, x, x') + \int_0^t D_s v_\epsilon^{\beta,\alpha} ds \\ &\quad + \int_0^t \langle D_x v_\epsilon^{\beta,\alpha}, b(s, X(s)) \rangle ds + \int_0^t \langle D_y v_\epsilon^{\beta,\alpha}, b(s, X'(s)) \rangle ds \\ &\quad + \int_0^t \langle D_x v_\epsilon^{\beta,\alpha}, \sigma(s, X(s)) dW(s) \rangle + \int_0^t \langle D_y v_\epsilon^{\beta,\alpha}, \sigma(s, X'(s)) dW(s) \rangle \\ &\quad + \int_0^t \langle D_x v_\epsilon^{\beta,\alpha}, \gamma(s, Y(s)) d|\Lambda|(s) \rangle + \int_0^t \langle D_y v_\epsilon^{\beta,\alpha}, \gamma(s, Y'(s)) d|k|(s) \rangle \\ &\quad + \int_0^t \frac{C(\lambda)}{\epsilon} (|X(s) - X'(s)|^2 + |Y(s) - Y'(s)|^2) ds. \end{aligned}$$

Now since $f_\epsilon^\beta \in C^1$, the function $v_\epsilon^{\beta,\alpha}$ and its first derivatives converge to v_ϵ^β and its first derivatives when we let α in the mollifier go to zero. Since we no longer have the second derivative on the right hand side we let $\alpha \rightarrow 0$.

So we drop the α from $v_\epsilon^{\beta,\alpha}$. Now we look at the integrals and bound them. First the time derivative part is evaluated

$$\int_0^t D_s v_\epsilon^\beta ds = \int_0^t f_\epsilon^\beta D_s u ds + \int_0^t u D_s f_\epsilon^\beta ds. \quad (6.6)$$

For first term in (6.6) we use the regularity of u to bound the derivative and use (4.13) for β small enough. For the second term in (6.6) we use (6.3) and the fact that $u \leq 0$, to conclude that

$$\int_0^t D_s v_\epsilon^\beta ds \leq C(\lambda) \frac{|Y(s) - Y'(s)|^2}{\epsilon}.$$

Next we look at the integrals involving $b(s, X(s))$, and get

$$\begin{aligned} & \int_0^t \langle D_x v_\epsilon^\beta, b(s, X(s)) \rangle ds + \int_0^t \langle D_y v_\epsilon^\beta, b(s, X'(s)) \rangle ds = \\ & \int_0^t \langle u D_x f_\epsilon^\beta + f_\epsilon^\beta D_x u, b(s, X(s)) \rangle ds + \int_0^t \langle u D_y f_\epsilon^\beta + f_\epsilon^\beta D_y u, b(s, X(s)) \rangle ds = \\ & \int_0^t \langle u (D_x f_\epsilon^\beta + D_y f_\epsilon^\beta), b(s, X'(s)) \rangle ds + \int_0^t \langle u D_y f_\epsilon^\beta, b(s, X'(s)) - b(s, X(s)) \rangle ds \\ & + \int_0^t f_\epsilon^\beta (\langle D_x u, b(s, X(s)) \rangle + \langle D_y u, b(s, X'(s)) \rangle) ds. \end{aligned} \quad (6.7)$$

Du and b are continuous on a compact domain and can therefore be bounded by a constant $C(\lambda)$. Then setting β small enough to use (4.13), using (6.3) and the Lipschitz continuity on b , we obtain

$$\begin{aligned} & \int_0^t \langle D_x v_\epsilon^\beta, b(s, X(s)) \rangle ds + \int_0^t \langle D_y v_\epsilon^\beta, b(s, X'(s)) \rangle ds \leq \\ & C(\lambda) \int_0^t \left(\frac{|Y(s) - Y'(s)|^2}{\epsilon} + \frac{|X(s) - X'(s)| |Y(s) - Y'(s)|}{\epsilon} \right) ds. \end{aligned}$$

Now for the integrals with respect to $d|\Lambda|$, we start by expanding Dv_ϵ^β as in the previous cases

$$\begin{aligned} & \int_0^t \langle D_x v_\epsilon^\beta, \gamma(s, Y(s)) \rangle d|\Lambda|(s) + \int_0^t \langle D_y v_\epsilon^\beta, \gamma(s, Y'(s)) \rangle d|\Lambda'(s)| = \\ & \int_0^t \langle u D_x f_\epsilon^\beta, \gamma(s, Y(s)) \rangle d|\Lambda|(s) + \int_0^t \langle f_\epsilon^\beta D_x u, \gamma(s, Y(s)) \rangle d|\Lambda|(s) \\ & + \int_0^t \langle u D_y f_\epsilon^\beta, \gamma(s, Y'(s)) \rangle d|\Lambda'(s)| + \int_0^t \langle f_\epsilon^\beta D_y u, \gamma(s, Y'(s)) \rangle d|\Lambda'(s)| = \\ & \int_0^t \langle u D_x f_\epsilon^\beta, \gamma(s, Y(s)) \rangle d|\Lambda|(s) + \int_0^t \langle u D_y f_\epsilon^\beta, \gamma(s, Y'(s)) \rangle d|\Lambda'(s)| \\ & - \lambda \int_0^t \langle f_\epsilon^\beta u D_x h, \gamma(s, Y(s)) \rangle d|\Lambda|(s) - \lambda \int_0^t \langle f_\epsilon^\beta u D_y h, \gamma(s, Y'(s)) \rangle d|\Lambda'(s)|. \end{aligned}$$

For sufficiently small β we can use (4.12), this and Lemma 4.2 yields

$$\begin{aligned} & -\lambda \int_0^t \langle f_\epsilon^\beta u D_x h, \gamma(s, Y(s)) \rangle d|\Lambda|(s) - \lambda \int_0^t \langle f_\epsilon^{\beta, \alpha} u D_y h, \gamma(s, Y'(s)) \rangle d|\Lambda'|(s) \leq \\ & -\lambda \int_0^t u \frac{|Y(s) - Y'(s)|^2}{\epsilon} d|\Lambda|(s) - \lambda \int_0^t u \frac{|Y(s) - Y'(s)|^2}{\epsilon} d|\Lambda'|(s). \end{aligned}$$

We now state a lemma that is Lemma 2 in [11]

Lemma 6.3. *There exists a $\tau > 0$ and a function $w_R(x)$, $w(0) = 0$, such that for all $R > 0$*

$$\langle D_r g^\beta(t, x, r), \gamma_i(t, x) \rangle > -w_R(\beta),$$

for $x \in \partial\Omega$, $i \in I(t, x)$, $|r| < R$, $\langle r, n_i \rangle > -\tau|r|$

Choosing $R = \frac{B}{\epsilon}$ and by Remark 2.27, the assumptions in Lemma 6.3 is fulfilled. Applying this to the term $\langle D_y f_\epsilon^\beta(s, Y(s), Y'(s)), \gamma(s, Y'(s)) \rangle$ yields

$$\begin{aligned} & \langle D_y f_\epsilon^\beta(s, Y(s), Y'(s)), \gamma(s, Y'(s)) \rangle \\ & = -\langle D_r g^\beta(s, Y(s), Y'(s)), \gamma(s, Y'(s)) \rangle \\ & = -\langle D_r g^\beta(s, Y(s), Y'(s)), \gamma(s, Y'(s)) - \gamma(s, Y(s)) \rangle \\ & \quad - \langle D_r g^\beta(s, Y(s), Y'(s)), \gamma(s, Y(s)) \rangle \\ & \leq C \frac{|Y(s) - Y'(s)|^2}{\epsilon} + C_2 w_R(\beta), \end{aligned}$$

where we also have used the Lipschitz condition of γ and that $|D_r g| \leq C|r|$. For the D_x term, we see that to use Lemma 6.3, $\langle r, n_i \rangle > -\tau|r|$ must hold. The assumptions in Lemma 6.3 is fulfilled with $r = \frac{Y'(s) - Y(s)}{\epsilon}$, since we have $r = \frac{Y(s) - Y'(s)}{\epsilon}$ we define $r_1 = -r$ and get

$$\begin{aligned} & \langle D_x f_\epsilon^\beta(s, Y(s), Y'(s)), \gamma(s, Y(s)) \rangle \\ & = \langle \epsilon D_x g^\beta(s, \frac{Y(s) - Y'(s)}{\epsilon}) \rangle + \langle D_r g^\beta(s, \frac{Y(s) - Y'(s)}{\epsilon}) \rangle \\ & \leq C \frac{|Y(s) - Y'(s)|^2}{\epsilon} - \langle D_{r_1} g^\beta(s, \frac{Y(s) - Y'(s)}{\epsilon}) \rangle \\ & \leq C \frac{|Y(s) - Y'(s)|^2}{\epsilon} + C_2 w_R(\beta). \end{aligned}$$

Collecting all the $d|\Lambda|$ terms, we obtain

$$\begin{aligned} (C - \lambda) \int_0^t \frac{|Y(s) - Y'(s)|^2}{\epsilon} d|\Lambda|(s) + (C - \lambda) \int_0^t \frac{|Y(s) - Y'(s)|^2}{\epsilon} d|\Lambda'|(s) \\ + C_2 w_R(\beta) |\Lambda|(t) + C_2 w_R(\beta) |\Lambda'|(t). \quad (6.8) \end{aligned}$$

Setting $\lambda = C$ and letting $\beta \rightarrow 0$, the terms in (6.8) disappears. The terms containing dW define a martingale which we denote by $N(t)$, that is

$$N(t) = \int_0^t \langle D_x v_\epsilon^\beta, \sigma(s, X(s)) dW(s) \rangle + \int_0^t \langle D_y v_\epsilon^\beta, \sigma(s, X'(s)) dW(s) \rangle.$$

Collecting all terms and using (4.12), we obtain

$$\begin{aligned} \frac{|Y(t) - Y'(t)|^2}{\epsilon} &\leq v_\epsilon(t, Y(t), Y'(t)) \leq \\ v(0, Y(0), Y'(0)) &+ N(t) + C \int_0^t \left(\frac{|Y(s) - Y'(s)|^2}{\epsilon} + \frac{|X(s) - X'(s)|^2}{\epsilon} \right) ds. \end{aligned}$$

Next we multiply by ϵ , square, applying Hölder's inequality, take supremum, use (4.12) and take expectation, which yields

$$\begin{aligned} E \left[\sup_{0 \leq s \leq t} (|Y(s) - Y'(s)|^4) \right] &\leq C \left(|x - x'|^4 + \epsilon^2 E \left[\sup_{0 \leq s \leq t} (N(t)^2) \right] \right. \\ &\quad \left. + \int_0^t (E [(|Y(s) - Y'(s)|^4) + (|X(s) - X'(s)|^4)]) ds \right). \end{aligned}$$

For the martingale term we use Doob's maximal inequality

$$E \left[\sup_{0 \leq s \leq t} (N(t)^2) \right] \leq 4E [(N(t)^2)].$$

Now using the same method as in (6.7) we get

$$\epsilon^2 E [(N(t)^2)] \leq C \int_0^t (E [(|Y(s) - Y'(s)|^4) + (|X(s) - X'(s)|^4)]) ds.$$

Finally we apply Gronwall's inequality and get the desired inequality

$$E \left[\sup_{0 \leq s \leq t} (|Y(s) - Y'(s)|^4) \right] \leq C \left(|x - x'|^4 + \int_0^t (E [(|X(s) - X'(s)|^4)]) ds \right).$$

□

6.2 Existence and Uniqueness of RSDE

Now we are ready to prove Theorem 3.1. The proof follows the outline of the proof of Corollary 5.2 in [1] which is based on the proof of Theorem 4.3 in [4].

Proof of Theorem 3.1. Assume that for a continuous \mathcal{F}_t adapted martingale X_t there are continuous \mathcal{F}_t adapted processes (Y_t, Λ) that fulfills (2.2). Then

a Picard iteration scheme, see [10], converges to a fix-point and with the uniqueness Lemma 6.1 we have existence and uniqueness for the RSDE. So the only thing to check is the existence of Y_t and Λ . Since strong uniqueness and weak existence implies strong existence [12] and since we have strong uniqueness for Y_t from Lemma 6.1, we only need to prove existence for Y_t in distribution. Set

$$\psi = x + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dW,$$

and then let ψ_n be a continuous, bounded variation, \mathcal{F}_t -adapted semimartingales converging uniformly to ψ . Let (Y_n, Λ_n) be the solution to the SP for ψ_n , which exists due to Lemma 5.2. Since ψ_n is of bounded variation, Lemma (5.3) gives us that the solution (Y_n, Λ_n) have bounded variation. Since bounded variation implies measurably we have that Y_n is \mathcal{F}_t -adapted. Finally from Lemma 5.3 we know that (Y_n) is relatively compact and by Prokhorov's theorem tight. This together with the fact that S_n converges to S gives us that $Y_n \rightarrow Y$ in distribution where Y fulfills the SP for S . But this is what we required and concludes the proof of Theorem 3.1. \square

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