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Hedging Error in CVA

Impact of inconsistency between simulation and pricing models

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Abstract

The aim of this thesis is to investigate the hedging error in Credit Value Adjustment (CVA) produced by using a model for the simulation of the risk factors different from the one used in the pricing of the derivative contract. The hypothesis is that this inconsistency between simulation and pricing models affects the CVA leading to an error in the hedging of credit counterparty risk. When computing the CVA, market factors are simulated forward in time and the portfolio is priced in each scenario to obtain the Expected Positive Exposure (EPE). To hedge the market risk of CVA we use a dynamic Delta-hedging strategy. We investigate the hedging error for a default free portfolio and for its CVA and how it is affected by the mismatch between the models.

Keywords: Hedging Error, Counterparty Credit Risk, Model Mismatch, Credit Value Adjustment, Expected Positive Exposure, Monte Carlo.

Hedging fel i CVA

Effekten av missanpassning mellan simulerings- och prissättningsmodeller

Sammanfattning

Denna studie ämnar att analysera hedgingfelet i kreditvärdesjustering (CVA) som uppstår när simuleringsmodellen för riskfaktorerna är annorlunda än den som används för derivatets prissättning. Hypotesen är att diskrepansen i modellerna påverkar CVA och leder till en hedgingportfölj med ett avvikande värde från det egentliga som krävs för att optimalt hedga motpartens kredit-risk. För att beräkna CVA simulerar vi marknadsfaktorerna framåt i tiden och portföljen är prissatt i varje scenario för att beräkna förväntade positiva exponeringen (EPE). För att hedga marknadsrisken av CVA använder vi en dynamisk Delta hedging strategi. Vi undersöker hedgingfelet för en portfölj utan och med kreditrisk och hur det påverkas av diskrepansen mellan modellerna.

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0.1 Notation

Symbol	Description	Value
r	Risk-free yearly return	0.05
μ	Mean of the log-return of the risky asset	0.06
σ	Volatility	—
σ_H	Volatility used in the pricing model	—
σ_S	Volatility used for the simulation	—
σ_P	Volatility used to simulate in the pricing model	—
T	Maturity of the financial contract (in years)	1
Z	Standard normally distributed random variable	—
W_t	Wiener process at time t	—
N_t	Poisson process at time t	—
S_t	Geometric Brownian Motion, Risky asset at time t	—
S_0	Initial value of the underlying	100
K	Strike price of the European Option	100
$D(t, T)$	Stochastic Discount Factor	—
$B(t, T)$	Expected value of the stochastic discount factor	—
$P(t, T)$	Default Probability	—
$\hat{P}(t, T)$	Survival Probability	—
π	Recovery Rate	0.4
λ, h	Intensity of the Poisson Process	0.03
τ	Default time	—
$Q_D(t)$	Cumulative distribution function of the default time τ	—
Δt	Time interval used in the time discretisation	—
\mathbb{P}	Real world probability	—
\mathbb{Q}	Risk neutral probability	—
$\mathbf{1}_{\{\cdot\}}$	Indicator function	—
$V(t)$	Portfolio value at time t	—
$V_{rf}(t)$	Risk free portfolio value at time t	—
N	Number of Monte Carlo simulations	—
N_{out}	Number of Monte Carlo simulations (outer loop)	—
n	Number of time intervals	—

0.2 Abbreviations

	Description
a.s.	almost surely
BM	Brownian Motion
BS	Black-Scholes
CCR	Credit Counterparty Risk
CDS	Credit Default Swap
CI	Confidence Interval
CVA	Credit Value Adjustment
DVA	Debit Value Adjustment
EPE	Expected Positive Exposure
GBM	Geometric Brownian Motion
i.e.	id est
iff	if and only if
LGD	Loss Given Default
MC	Monte Carlo
OTC	Over The Counter
P&L	Profit and Loss
SDE	Stochastic Differential Equation
s.t.	such that
STD	Standard Deviation
VaR	Value at Risk
w.r.t.	with respect to

Chapter 1

Introduction

There are different theories of the etymological origin of the word risk. Some declare that the word risk comes from the Greek, either from "tò rizikó", which means destiny, fate, or from "riza" which means cliff and is related to the fact that cliffs have always been considered a great danger for sailors. Another idea is that it comes from the Ancient Roman word "resecare" which meant to swim against the tide referring to the ones who dared to challenge the sea and embrace adventure. Or maybe from the Arab word "risq" which is something divine from which you can make a profit. Generally speaking, a risk can be seen at the same time as a danger and as an opportunity. Most human actions can be considered risky as they generate more or less unpredictable effects. That's the way it is in finance, a field where the word risk mainly refers to the possibility of loosing some or all of the original investment. Financial risk can be divided into many components and the more of them a financial institution manages to handle, the safer it will be. Completely avoiding taking risks is not considered a good strategy so financial institutions usually hedge and protect themselves against the most risky events they are likely to face, such as changes in the market factors or a default of a counterparty.

Before the year 2008 financial institutions mainly focused on Market Risk, which is the risk of portfolio losses resulting from movements in the market prices. The fact that a counterparty could default was considered almost impossible and financial institutions had little incentives to invest money and time into a risk that was considered small, hard to compute and expensive to manage. However the default cascade experienced in 2008 proved that anyone can potentially default: even if your counterparty is triple-A rated, you take counterparty and related risks. The bankruptcy of Lehman Brothers and of institutions that were considered too big to fail gave rise to the need of quantifying these risks and taking them into consideration when pricing OTC derivatives (for more see [2]).

This is where the Credit Value Adjustment (CVA) comes into play. The CVA is an adjustment to the fair value (or price) of derivative instruments to account for Counterparty Credit Risk (CCR): it is considered as the price of this risk. CCR is the risk that the counterparty (the entity with whom we have stipulated a financial contract) will not be able to fulfill its side of the contractual agreement because it defaults. It looks like an option on the residual value of the portfolio, with a random maturity given by the default time of the counterparty. If the counterparty defaults and the present value of the portfolio is positive to the surviving party, this one only gets a recovery fraction of the portfolio value from the defaulted entity. If instead the present value is negative to the surviving

party, the surviving party has to pay it in full to the liquidators of the defaulted entity. This generates an asymmetry that suggests that the value of the deal under counterparty risk is the value without counterparty risk minus this positive adjustment, called CVA.

But what happens from the point of view of the risky counterparty? Let us suppose that a default risk-free bank and a risky counterparty make a deal. The bank will compute the CVA to account for the possible default of the counterparty, subtracting it from the risk-free price while the counterparty itself will add it. This means that the CVA of the bank is the Debit Value Adjustment (DVA) for the risky counterparty. This is a positive quantity added to the default risk-free price in order to consider that an early default of the client itself would imply a "discount" on the client payment obligations. This is in a certain way a gain for the client, who marks a positive adjustment over the risk-free price by adding a positive amount called DVA ([15]). For the purpose of this thesis, we will focus only on the CVA.

After the year 2008 the industry began to realise the importance of treating the CVA seriously. As a matter of fact, two thirds of the losses that the banks suffered during the financial crisis did not come from counterparty defaults but from fair value adjustments on derivatives. Under the Basel II market risk framework (2004), firms were required to hold capital against the variability in the market value of their derivatives in the trading book, but there was no requirement to capitalise against variability in the CVA. In the year 2006 the value of OTC derivatives started to account for CCR and finally in 2011 Basel III introduced a CVA-VaR capital charge and increased the CCR charge. The accounting standards now require to report in the balance sheet both asset and liabilities sides of CVA which, as a consequence, can produce fluctuation in the balance sheet as long as the CVA changes.

To reduce the risk the CVA is carrying and to have a stable balance sheet, it is now common practice for each financial institution to have a so called CVA Desk, with the purpose to handle that risk. A financial institution usually tries to minimise the capital charge but must at the same time fulfill the rules settled by the Basel committee. This trade-off suggests that it is very important to be precise when computing the CVA, even if these calculations often are computationally heavy.

In this thesis we will act as a CVA Desk of a financial institution. First we will compute the CVA: this computation is itself challenging since we have to simulate the risk factors forward in time and evaluate the portfolio in each scenario, in each time instant, to obtain the Expected Positive Exposure (EPE). Then, since we also want to hedge the CVA, we need another Monte Carlo loop that generates different sample paths, and in each time instant of each path we will hedge both the market and the credit component of the CCR. For the market risk, we need to compute the CVA sensitivity to hedge out the described risk, so a new MC loop is started in each time instant, whereas for the credit part we can rebalance the portfolio so that we hold a suitable amount of a Credit Default Swap (CDS) contract each time (as explained in Chapter 3), however the main focus will be on the market risk carried by the CVA.

In practice, it can happen that a financial institution is pricing and simulating (in the inner MC loop) with a model that actually doesn't reflect the real movements of the underlying. This first type of hedging error could be for instance due to a calibration error. Secondly, it is interesting to analyse the error produced when banks consciously use one model (reflecting the real movements of market prices) to price and hedge the portfolio,

but a less advanced model for the simulation of the market factors. We will analyse these examples in Chapter 6.

For the purpose of this thesis we have not calibrated the parameters from the market. We have instead changed them within a range of reasonable values analysing each case. All the code was written in C++ and the plots were created with Gnuplot. The work was carried out at TriOptima.

After a mathematical background (Chapter 2), the definition of the CVA (Chapter 3) and a description of the methods used in this thesis (Chapter 4), we analyse the impacts of inconsistency between simulation and pricing models in hedging error for a chosen portfolio (Chapter 5) and present some comments and conclusions (Chapter 6).

Chapter 2

Mathematical Background

This chapter introduces the main mathematical tools for the development and understanding of this thesis. For more on stochastic calculus or probability theory see respectively [4], [6] and [12].

2.1 Stopping Times and the Poisson Process

In mathematical finance you must often deal with Counterparty Credit Risk (CCR). For this purpose it is useful to model a random point in time with a random variable τ representing the time of the default of a certain counterparty. In the range of possible values for τ we will include infinity in order to model events that may never occur. But this is not enough. It is also very important to link this random variable with the filtration of interest, as we can see in the following definition.

Definition 2.1.1. *Let $(\mathcal{F}_t)_{t \in T}$ be a filtration. A random variable $\tau : \Omega \rightarrow T \cup \{\infty\}$ is a **stopping time** if $\forall t \in T, \{\tau \leq t\} \in \mathcal{F}_t$.*

Intuitively, this means that it is not enough just to have the random variable τ , we must be sure that at time t it should be possible to say whether $\tau \leq t$ or not. For instance the first time on which a Wiener process ([4]) comes out of an open set is a stopping time. On the other hand the last time of visit of the Wiener process on an open set is not a stopping time since we would also need to know its future positions.

A very useful tool in credit modeling is the so called hazard rate, which intuitively can be seen as an *odds ratio*, i.e. the (expected) number of events divided by the (expected) number of non events. Mathematically:

Definition 2.1.2. *Let τ be a stopping time and $F(T) := \mathbb{P}[\tau \leq T]$ its distribution function. Assume that $F(T) < 1$ for all T , and that $F(T)$ has density $f(T)$. The **hazard rate function** h of τ is defined as:*

$$h(T) := \frac{f(T)}{1 - F(T)}. \quad (2.1)$$

At later points in time $t > 0$ with $\tau > t$, the **conditional hazard rate** is defined as:

$$h(t, T) := \frac{f(t, T)}{1 - F(t, T)}, \quad (2.2)$$

where $F(t, T) := \mathbb{P}[\tau \leq T | \mathcal{F}_t]$ is the conditional distribution of τ given the information at time t , and $f(t, T)$ is the corresponding density ([3]).

The conditional distribution function of τ , in function of the hazard rate $h(t)$ is

$$F(t, T) = 1 - e^{-\int_t^T h(t,s) ds}. \quad (2.3)$$

Thus the hazard rate helps us to define the default probability. In this thesis we will consider a constant hazard rate, thus the default probability between t and T simply becomes

$$P(t, T) = 1 - e^{-h(T-t)}.$$

It is interesting to see how the survival probability can be derived using the homogeneous Poisson process with parameter λ . We will see that this parameter takes the role of the hazard rate h just described.

The Poisson process is an important example of point process, which means that any sample path of the process consists of a set of separate points. Observe that the Poisson process is connected to the Poisson distribution in the same way as the Wiener process is connected to the normal distribution: namely as the distribution of independent increments ([12]). A process $N(t)$ representing the number of occurrences of some event in a certain period $(0, t]$ (i.e. it has non-negative integer values) is commonly referred to as **counter process**. Mathematically:

Definition 2.1.3. $N = \{N(t) | t \geq 0\}$ is a Poisson process with parameter $\lambda > 0$ if

- $N(0) = 0$.
- For all partitions $0 = t_0 \leq \dots \leq t_n = T$, the increments $N(t_k) - N(t_{k-1})$ are independent and Poisson distributed with parameter $\lambda(t_k - t_{k-1})$.

We now define the memoryless property that will help us in the computation of the default probabilities. Intuitively it usually refers to the cases when the distribution of a "waiting time" until a certain event, does not depend on how much time has elapsed already. Mathematically:

Definition 2.1.4. A random variable X with $\mathbb{P}(X > 0) = 1$ has the so called **Memoryless Property** if for every $x \geq 0$ and $t \geq 0$,

$$\mathbb{P}(X > t + x) = \mathbb{P}(X > x) \mathbb{P}(X > t).$$

Exponentially distributed random variables all satisfy the memoryless property and this fact can be used to derive the distribution for the first jump of a Poisson Process. This is of big interest for this thesis since it help us to model the default event of a counterparty. T_n , i.e. the sum of the jump times, was defined as a sum of independent exponentially distributed random variables (all with parameter λ), so if we consider Δt sufficiently small then the probability to have a jump in $(t, t + \Delta t)$ is $\lambda \Delta t$. Let us compute the survival probability $\hat{P}(t, T)$ between t and T for a generic intensity λ_t . Considering constant time intervals $\Delta t = (T - t)/n$, we get

$$\hat{P}(t, T) = \prod_{i=1}^n (1 - \lambda(t) \Delta t) = e^{\sum_{i=1}^n \ln(1 - \lambda(t_i) \Delta t)},$$

which for a very small time interval, considering the Taylor expansion, can be written as

$$\hat{P}(t, T) = e^{-\sum_{i=1}^n \lambda(t) \Delta t}.$$

Letting Δt go to zero we get the following default probability

$$P(t, T) = 1 - \hat{P}(t, T) = 1 - e^{-\int_t^T \lambda(s) ds}.$$

Getting back to our constant intensity, we have

$$P(t, T) = 1 - e^{-\lambda(T-t)}, \quad (2.4)$$

which corresponds to what we previously computed with the hazard rate.

2.2 The Geometric Brownian Motion (GBM)

Most models used in financial engineering can be described through a Stochastic Differential Equation (SDE). An example of a stochastic process satisfying an SDE is the so called geometric Brownian motion (GBM) (also known as exponential Brownian motion), i.e. a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion with drift. If it starts from a positive value we are sure this price process stays positive through time and this is one of the reasons why it is suitable to model prices. In the Black–Scholes it is used to model the price of the underlying asset as we will see in section 2.3. Mathematically:

Definition 2.2.1. *A stochastic process $S = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0:T]}, (S_t)_{t \in [0:T]}, P)$ is said to follow a **GBM** if it satisfies the following SDE*

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad (2.5)$$

where W_t is a Wiener process and μ and σ are constants.

It can easily be shown that a stochastic process satisfies a GBM iff

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right). \quad (2.6)$$

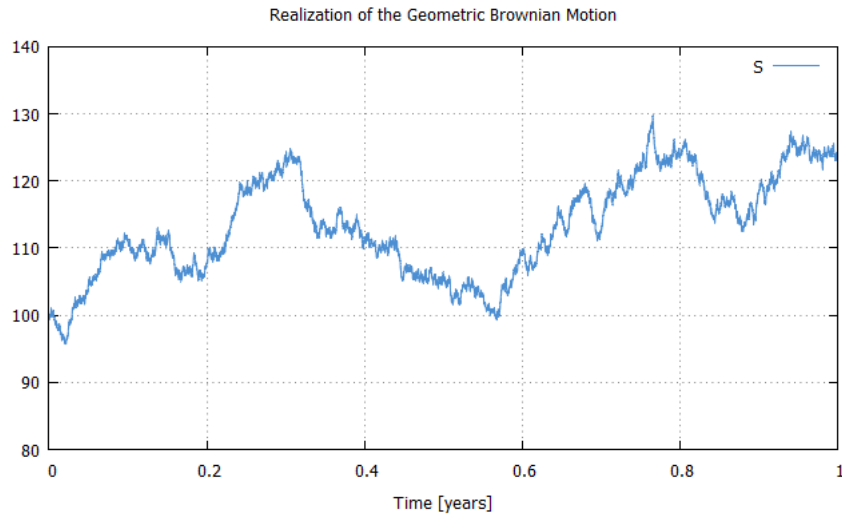


Figure 2.1: One simulated path of the Geometric Brownian Motion with $n = 10000$ intervals over a time horizon of one year.

Note that it is very rare to be able to solve an SDE in an explicit manner so the GBM represents a very nice exception within the family of stochastic differential equations.

2.3 The Black-Scholes Framework

Let us assume that the following conditions are satisfied:

- *Perfect liquidity*: it is possible to purchase or sell any amount of stock or options or their fractions at any given time (also short positions are allowed).
- *Infinite liquidity of cash*: we can borrow or lend any amount of cash whenever we want at the risk-free rate.
- *Frictionless market*: no commissions or transaction costs for buying or selling options and stocks.
- *Gaussian asset returns*: the underlying evolves according to a geometric Brownian motion.
- *Risk-free rate*: there are assets out there that are risk free, that is they will deliver a rate of return r (i.e. the risk-free rate) for sure, without uncertainty.
- *No arbitrage*: a portfolio of riskless assets always returns the risk-free rate.

Let $W = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, (W_t)_{t \in [0, T]}, \mathbb{P})$ be a Brownian Motion w.r.t. the natural filtration \mathcal{F}_t (i.e. intuitively the filtration that reflects all the available information up to time t). Let us consider a financial market consisting only in one risky asset with price process S_t following a GBM with constant parameters μ and σ as in definition (2.2.1) and a risk-free savings account with a continuously compounded interest rate r for riskless borrowing

and lending (i.e. the risk-free rate). Everything takes place in the interval $[0, T]$. Then

$$\begin{aligned} dB_t &= r_t B_t dt, & B(t_0) &= 1, \\ dS_t &= \mu S_t dt + \sigma S_t dW_t, & S(t_0) &= S. \end{aligned}$$

Note that a great difference between the risk free asset and the stock price is that the first one is locally deterministic in the sense that at time t we have complete knowledge of the return by simply observing the value of the short rate r while S_t has a stochastic rate of return.

In the given framework, i.e. the so called Black-Scholes (BS) setting, the price Π_t of a European Call Option with strike price K and time to maturity T is given by the formula $\Pi_t = C(t, S_t)$, where

$$C(t, s) = s\Phi[d_1(t, s)] - e^{-r(T-t)}K\Phi[d_2(t, s)] \quad (2.7)$$

$\Phi(\cdot)$ is the cumulative distribution function for the standard normal distribution $\mathcal{N}(0, 1)$ and

$$\begin{aligned} d_1(t, s) &= \frac{1}{\sigma\sqrt{T-t}} \left(\ln \frac{s}{K} + \left(r + \frac{1}{2}\sigma^2 \right) (T-t) \right), \\ d_2(t, s) &= d_1(t, s) - \sigma\sqrt{T-t}. \end{aligned}$$

The Put Option can be computed via the Put-Call parity

$$P(t, S_t) - C(t, S_t) = S_t - Ke^{-r(T-t)} \quad (2.8)$$

or by using the Black-Scholes Put formula

$$P(t, s) = e^{-r(T-t)}K\Phi[-d_2(t, s)] - s\Phi[-d_1(t, s)] \quad (2.9)$$

where $d_1(t, s)$ and $d_2(t, s)$ are the same as in (2.7).

Black-Scholes hedging strategy. The idea of the BS setting is that we can replicate any derivative contract by buying and selling the underlying assets and by borrowing money at the risk-free rate. In this way we can build a portfolio made by the derivative and its hedging positions that doesn't fluctuate in value regardless the fluctuations of the underlying. Furthermore the price of the derivative contract is equal to the cost of creating the hedging positions since, if not, it would be possible to create an arbitrage by selling that portfolio at the wrong price and hedging it, making a return higher than the risk-free rate without any risk. This is called *risk neutral valuation* framework.

Pricing under the risk-neutral measure \mathbb{Q} . Statistical properties of random objects such as future losses, mean and variance, depend on the probability measure we are using so it is important to point out the probability measure with respect to which the expectations are taken. \mathbb{P} usually refers to the historical or physical probability measure, also called real world probability measure under which we do historical estimation of financial variables. This measure reflects the true value of the financial quantities. When we simulate the financial variables up to the risk horizon we use this measure. But if we want to price a financial product, we want to use the so called risk neutral framework,

which means we take expectations of the discounted future cashflows under the risk neutral measure \mathbb{Q} . This is the measure associated with the locally risk-free bank account numeraire B_t , evolving according the risk-free rate r_t

$$dB_t = r_t B_t dt, \quad B_0 = 1 \quad (2.10)$$

Under this measure all tradable assets divided by B_t are martingales (see [6] for more details).

In practice the choice of measure influences how the underlying is simulated and, in the case of a geometric Brownian motion (i.e. in the Black-Scholes setting) it affects how the drift parameter μ should be chosen. In pricing under risk neutral measure basically we discount a payoff to be received at time t back to time 0 by dividing by $B(0, t) = e^{-rt}$ (we have considered a constant interest rate r in this thesis). In a risk neutral world, investors would not demand a higher rate of return for holding risky assets which implies that all assets would have the same average rate of return, so the drift parameter of the underlying is r , i.e.

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

The two measures are not in conflict, they just represent different things.

Now, since the CVA is a price, when we compute it we simulate the market factors under risk neutral measure \mathbb{Q} . But when we simulate the hedging strategy, we simulate the possible scenarios of our portfolio with the probability measure \mathbb{P} .

Chapter 3

Credit Value Adjustment

After a brief introduction to Credit Counterparty Risk (CCR) we present the mathematical concepts of CVA and the derivation of a formula for the hedging error due to a mismatch in the models for simulation and pricing.

As said in the introduction, the CVA is a positive quantity to be subtracted from the risk-free price in order to account for the counterparty default risk in the valuation. Its computation consists in pricing the risk that a counterparty defaults before the final maturity of the deal. Since we are dealing with pricing, when we compute the CVA we work under the risk neutral probability measure \mathbb{Q} . As already pointed out, we will focus on the unilateral counterparty risk case, so the investor can consider itself to be default free, and the defaultable counterparty agrees with that. For more see [15].

3.1 Counterparty Credit Risk

In the introduction we said that counterparty credit risk (or simply counterparty risk) is the risk that the counterparty will not be able to fulfill its side of the contractual agreement because it defaults. CCR is not a trivial area to deal with since it involves the most complex financial instruments, derivatives, and is driven by the intersection of some of the different financial risk components that we briefly review next.

- *Market risk* is the risk that comes from the (short-term) movement of market prices. It can arise from an exposure to the direction of movement of underlying variables such as stock prices, interest rates, foreign exchange rates, commodity prices or credit spreads. But it can also come from the exposure to market volatility. To hedge against market risk one can take opposite financial positions in regards to the original opening position. If these contracts are written with other counterparties this generates a counterparty credit risk [2].
- *Credit risk* is the risk that a counterparty may be unable or unwilling to make a payment or fulfill contractual obligations. This may be characterised in terms of an actual default which may result in an actual and immediate loss or, less severely, it may be characterised by deterioration in a counterparty's credit quality (i.e. credit migration) [2]. Even if credit and counterparty risk are highly related, they are not the same. The first difference is that while credit risk is one sided, counterparty risk is taken by both parties, i.e. it is bilateral (and can be positive or negative). The second is that while the credit risk is typically known at transaction, counterparty

credit risk is stochastic and can't be estimated directly since we do not know how much we could potentially lose during the lifetime of the contract.

- The concept of *Liquidity risk* includes the risk that a transaction cannot be executed at market prices, perhaps due to the size of the position and/or relative illiquidity of the underlying and for instance the ability of a financial institution to meet its operational and debt obligations (such as collateral obligations) without incurring severe losses or defaulting. Even if collateralization¹ is used to reduce CCR it may lead to liquidity risk if the collateral has to be sold at some point due to a credit event.
- *Operational risk* arises from people, systems, internal and external events. It includes human error, model risk such as inaccurate or badly calibrated models, fraud and legal risk. Some techniques to mitigate counterparty risk, such as collateralization give rise to operational risks (for more see [2]).

Thus counterparty risk represents mainly an interaction between credit risk, which represents the counterparty's credit quality, and market risk, which reflects the exposure (i.e. the potential value of the contract(s) with that counterparty at the point at which the credit quality deteriorates).

To model counterparty we will focus on three main building blocks:

- $B(t, t_i) = \mathbb{E}_t[D(t, t_i)]$, i.e. the expected value of the stochastic discount factor.
- The recovery value π : it models the recovery risk which represents the uncertainty of severity of the losses if a default occurs. It is the amount (usually expressed in percentage) we can get back from our counterparty even if it has defaulted. In this thesis we have considered a constant recovery $\pi = 40\%$.
- The survival probability $\hat{P}(t, T) = \mathbb{E}[\mathbb{1}_{\tau > T}]$: it represents the probability that the counterparty will survive between t and T . In the notation τ represents the default time of the counterparty. Thus the default probability will simply be $P(t, T) = 1 - \hat{P}(t, T)$, and is obviously related to the credit quality of a company over the entire lifetime of transactions with that counterparty. Future default probabilities will in general have a tendency to decrease due to the fact that the more time passes, the more likely it is that the default event already happened. Thus the fact that the default probability usually decreases with time doesn't necessary mean that the company is becoming more credit-worthy but simply that it is unlikely to survive until that period. Furthermore a counterparty with an expectation of deterioration of credit quality will have an increasing default probability over time even if the phenomenon just written might reverse it. Finally if on the other hand we expect our counterparty to become more credit-worthy its default probability will decrease both for this reason and the first phenomenon pointed out [3].

¹Collateral management is used to reduce CCR and begun in the 1980s with Bankers Trust and Salomon Brothers taking collateral against credit exposures. The fundamental idea is that cash or securities are passed from one counterparty to another as security for a credit exposure. For more see [2].

3.2 Mathematical Definition of CVA

Let $(\Omega, \mathcal{G}, \mathcal{G}_t, \mathbb{Q})$ be a filtered probability space. This space is endowed with a right-continuous and complete sub-filtration \mathcal{F}_t representing all the observable market quantities but the default events. Mathematically $\mathcal{F}_t \subseteq \mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$, where $\mathcal{H}_t = \sigma(\tau \leq u : u \leq t)$ is the right-continuous filtration generated by the default event. Intuitively \mathcal{G}_t is the filtration modelling the market information up to time t whereas \mathcal{F}_t contains the default-free market information. We set $\mathbb{E}_t := \mathbb{E}[\cdot | \mathcal{G}_t]$ ([15]).

Suppose that at time t we have a portfolio of derivative contract with a risky counterparty up to maturity T . Let τ be the stopping time modelling the default event of the counterparty. To derive the CVA we follow this structure:

- If $\tau > T$ there is no default by the counterparty during the life of the product and the counterparty will fulfill its obligations repaying the investors.
- If $\tau \leq T$ the counterparty cannot repay its investors and the Net Present Value (V) of the residual payoff until maturity is computed:
 - If this V is negative for the investor it is completely paid by the investor.
 - If it is positive for the investor, only a recovery fraction π of the V is received by the investor.

In an arbitrage-free complete financial market, if we follow a risk-neutral valuation approach and denoting a future claim of a derivative at time t_i in the portfolio by X_{t_i} , the value of the portfolio at time t is

$$V(t) = \sum_{t_i \in (t, T]} \mathbb{E}_t^{\mathbb{Q}}[D(t, t_i)X_{t_i}], \quad (3.1)$$

where $D(t, t_i)$ is the stochastic risk-free discount factor between t and t_i .

A useful quantity to define when dealing with the CVA is the **Credit Exposure** (or simply exposure), which is the loss in the event of a counterparty defaulting assuming zero recovery value. The most common measure to quantify credit exposure is the **Expected Positive Exposure (EPE)**, i.e. the average of the positive exposure at a particular time t_i (computed at t). This quantity is, as we will see, very related to the CVA and is computed mathematically as follows

$$\text{EPE}(t_i) = \mathbb{E}^{\mathbb{Q}}[V(t_i)^+ | \mathcal{F}_t]. \quad (3.2)$$

Let π the constant recovery rate. Then at time t the discounted loss is

$$L(t) = \mathbf{1}_{\{\tau \leq T\}}(1 - \pi)D(t, \tau)[V(\tau)]^+. \quad (3.3)$$

The CVA is then the risk-neutral expectation of the loss

$$\text{CVA}(t) = \mathbb{E}_t^{\mathbb{Q}}[\mathbf{1}_{\{\tau \leq T\}}(1 - \pi)D(t, \tau)(V(\tau))^+]. \quad (3.4)$$

As explained in the Introduction, counterparty risk thus adds an optionality level to the original payoff. This makes the counterparty risky payoff model dependent even when the

original payoff is model independent. This implies, for example, that while the valuation of swaps without counterparty risk is model independent, requiring no dynamical model for the term structure (no volatility and correlations in particular), the valuation of swaps under counterparty risk will require an interest rate model. This implies that quick fixes of existing pricing routines to include counterparty risk are difficult to obtain ([15]).

After the CVA computation we can write

$$V(t) = V_{rf}(t) - \text{CVA}(t), \quad (3.5)$$

where $V_{rf}(t)$ is the value of the derivative at time t , assuming a default-free framework, whereas $V(t)$ is the value of the contract taking into consideration counterparty credit risk. It is interesting to notice that the risk free price and the CVA can be seen as two distinct quantities that are computed separately. This means that they can also be hedged separately. As a matter of fact, as we describe in the next section, the CVA desk acts as a separated entity and deals with this challenge.

Let us now rewrite Equation (3.4) conditioning the expected value on the default time (using the so called "Tower property" of conditional expectation, see [12] for more), using the risk neutral cumulative distribution function for the default time τ denoted by $Q_D(t) = \mathbb{Q}(\tau \leq t)$ and assuming independence between the credit exposure and the default probability. We get

$$\begin{aligned} \text{CVA}(t) &= \mathbb{E}_t^{\mathbb{Q}}[\mathbf{1}_{\{\tau \leq T\}} \mathbb{E}_{\tau}^{\mathbb{Q}}[(1 - \pi)D(0, \tau)(V(\tau))^+]] \\ &= \int_t^T \mathbb{E}^{\mathbb{Q}}[(1 - \pi)D(t, s)(V(s))^+ | \tau = s] dQ_D(s) \end{aligned} \quad (3.6)$$

By assuming constant recovery rate π and constant short rate r one can simplify Equation (3.6) with

$$\begin{aligned} \text{CVA}(t) &= (1 - \pi) \int_t^T B(t, s) \mathbb{E}^{\mathbb{Q}}[(V(s))^+ | \mathcal{F}_t] dQ_D(s) \\ &= (1 - \pi) \int_t^T B(t, s) \text{EPE}(s) dQ_D(s), \end{aligned} \quad (3.7)$$

where $B(t, s) = \mathbb{E}_t[D(t, s)]$ is the expected value of the stochastic discount (i.e. the price of a zero coupon bond). When computing this value numerically it is necessary to discretise the time interval of interest and choose a suitable quadrature rule to solve the integral as shown in Section 4.1.

3.3 Hedging strategy

As described in the introduction financial institutions have started to take very seriously the problems related with CVA and work to hedge this quantity. Basically the CVA desk aims at having a zero $P\&L$ so it takes responsibility for the counterparty credit risk in the book of OTC derivatives in the organization and it hedges it out, so that at each time instant the losses from CVA corresponds to profits from the hedge and vice versa [16].

Hedging counterparty risk has two components:

- *Cash Hedging*, which is the hedging of actual default events. That is, we do something so when one of our counterparties defaults, our losses are limited or in the best case

we do not suffer any loss. This can be done using some so called credit derivatives (in this thesis we have used a basket of Credit Default Swaps (CDSs) with different maturities, i.e. a Contingent CDS). This type of financial contract is a derivative security that has a payoff which is conditioned on the occurrence of a credit event. If the credit event has occurred, the default payment has to be made by one of the counterparties. Besides the default payment a credit derivative can have further payoffs that are not default contingent [3].

- *Paper Hedging*: This type of hedging consists in hedging the market price of potential future default events. As a matter of fact, since CVA is a price to a risk, like any other price that is marked periodically it will fluctuate and give a *P&L*. To hedge this part one can follow a Delta hedging strategy rebalancing the portfolio in each time instant. This is the type of hedging we have put the biggest focus on in this thesis.

In this thesis we have hedged both the market and the credit risk. For the first hedging we have followed a dynamic delta hedging strategy while to hedge against credit risk we have used a basket of CDS.

3.4 Hedging the market risk

Let us consider a contingent claim $V(t, S_t)$ with maturity T . Assuming a Black-Scholes (BS) setting, the underlying S_t follows a Geometric Brownian Motion with constant drift and diffusion. We want to hedge our portfolio against the movements of the market factors (which in our case is just the underlying S_t). In a BS setting (as described in section 2.3), if the hedger knows the future volatility of the stock and hedges continuously, he can replicate the option payoff by rebalancing his portfolio with the underlying stock and the bank account. On one hand, it is impossible to hedge continuously, so the hedger is forced to rebalance the portfolio only in some discrete time instants, which implies that the final P&L can take values different from zero (here we will not focus on this discretisation error but we will analyse it a bit in the numerical results section). On the other hand, it is interesting to investigate what happens due to a mismatch between the real model used for the simulation of the underlying and the one used for hedging and pricing. Supposing we are short in the claim $V(t, S_t)$, we can write the final hedging error at maturity as

$$\text{Final P\&L} = \text{B-S hedge at } T - \text{payoff}$$

In section 3.4.1 we derive the hedging error generated by the mismatch in the models, supposing that we are able to hedge continuously.

3.4.1 Hedging error for a contingent claim $V(t, S_t)$

Theorem 3.4.1. *Consider a portfolio where we are short in one unit of a square integrable claim $V(t, S_t)$ with maturity $t = T$. Let r be the risk free interest rate and $\Gamma_t = \frac{\partial^2 V_t}{\partial x^2}(t, S_t)$ the second order Greek of our claim. Suppose that we sell the claim at the price computed considering a volatility σ_H and that we are able to hedge continuously with a delta hedging strategy using a model for the underlying with drift μ_H and diffusion σ_H . Suppose that the underlying S_t actually evolves lognormally but with drift $\mu_S(t)$ and diffusion $\sigma_S(t)$, i.e. it*

has a different dynamic from the one supposed when hedging. Then the hedging error due to the mismatch in the models is given by

$$Z_T - Z_0 e^{rT} = \int_0^T e^{r(T-t)} \frac{1}{2} S_t^2 \Gamma_t (\sigma_H^2 - \sigma_S(t)^2) dt.$$

Proof. Note that $V(t, S_t)$ is the process representing the considered claim and $V(t, x)$ is the corresponding function.

If we assume

$$dS_t = \mu_H S_t dt + \sigma_H S_t dW_t, \quad dB_t = r B_t dt$$

then we can show, using some basic stochastic calculus, that the price of the claim at time t satisfies the PDE

$$\frac{\partial V}{\partial t} + rx \frac{\partial V}{\partial x} + \frac{1}{2} \sigma_H^2 x^2 \frac{\partial^2 V}{\partial x^2} - rV = 0 \quad (3.8)$$

with boundary condition $V(T, x) = V_T$. Following a Black-Scholes Delta hedging strategy, i.e. choosing to rebalance the portfolio such that at every time t we hold $\frac{\partial V}{\partial S_t}$ units of the underlying, we can get a perfect hedge. Basically this means that if the underlying has the same parameters μ_H and σ_H we have used to hedge the position, and we follow a delta hedging strategy we can find a portfolio that follows perfectly the movements of our contingent claim.

Now, according to the hypothesis we suppose that the underlying follows a different price process with parameters $\mu_S(t)$ and $\sigma_S(t)$ (not necessarily constant in time) and that X_t is the self-financing hedging portfolio (i.e. there is no infusion or withdrawal of money, so the purchase of a new asset must be financed by the sale of an old one, or by the money put into the bank account).

The value of X_t at time t is given by $X_0 = V(0, S_0)$ and

$$dX_t = \frac{\partial V}{\partial x}(t, S_t) dS_t + \left(X_t - \frac{\partial V}{\partial x}(t, S_t) S_t \right) r dt. \quad (3.9)$$

Thus the hedging error $Z_t = X_t - V_t$ of our portfolio follows the equation

$$dZ_t = \frac{\partial V}{\partial x}(t, S_t) dS_t + \left(X_t - \frac{\partial V}{\partial x}(t, S_t) S_t \right) r dt - dV. \quad (3.10)$$

Using Itô's formula we get that the claim $V(t, S_t)$ satisfies

$$\begin{aligned} dV &= \left(\frac{\partial V}{\partial t}(t, S_t) + \mu_S(t) S_t \frac{\partial V}{\partial x}(t, S_t) + \frac{1}{2} \sigma_S(t)^2 S_t^2 \frac{\partial^2 V}{\partial x^2}(t, S_t) \right) dt + \sigma_S(t) S_t \frac{\partial V}{\partial x}(t, S_t) dW_t \\ &= \left(\frac{\partial V}{\partial t}(t, S_t) + \frac{1}{2} \sigma_S(t)^2 S_t^2 \frac{\partial^2 V}{\partial x^2}(t, S_t) \right) dt + \frac{\partial V}{\partial x}(t, S_t) dS_t. \end{aligned}$$

Thus hedging error Z_t satisfies

$$dZ_t = \left(X_t - \frac{\partial V}{\partial x}(t, S_t) S_t \right) r dt - \left(\frac{\partial V}{\partial t}(t, S_t) + \frac{1}{2} \sigma_S(t)^2 S_t^2 \frac{\partial^2 V}{\partial x^2}(t, S_t) \right) dt \quad (3.11)$$

Since we are hedging with a wrong model for the underlying, we are actually assuming

that the claim V satisfies the B-S equation (3.8). We can therefore use the equality $-\frac{\partial V}{\partial t}(t, S_t) - rS_t \frac{\partial V}{\partial x}(t, S_t) = \frac{1}{2}\sigma_H^2 S_t^2 \frac{\partial^2 V}{\partial x^2}(t, S_t) - rV(t, S_t)$, in Equation (3.11) getting that

$$\begin{aligned} dZ_t &= (X_t - Y_t)r dt + \frac{1}{2}S_t^2 \Gamma_t (\sigma_H^2 - \sigma_S(t)^2) dt \\ &= Z_t r dt + \frac{1}{2}S_t^2 \Gamma_t (\sigma_H^2 - \sigma_S(t)^2) dt, \end{aligned} \quad (3.12)$$

where $\Gamma_t = \frac{\partial^2 V}{\partial x^2}(t, S_t)$. To solve (3.12) we can use the general formula for linear SDEs or simply notice that

$$\begin{aligned} \frac{d}{dt}(e^{-rt} Z_t) &= -re^{-rt} Z_t + e^{-rt} dZ_t \\ &= \frac{1}{2}e^{-rt} S_t^2 \Gamma_t (\sigma_H^2 - \sigma_S(t)^2) \end{aligned}$$

Integrating between 0 and T we get

$$e^{-rT} Z_T - Z_0 = \int_0^T e^{-rt} \frac{1}{2} S_t^2 \Gamma_t (\sigma_H^2 - \sigma_S(t)^2) dt.$$

We obtain the following formula for the hedging error at maturity T

$$Z_T - Z_0 e^{rT} = \int_0^T e^{r(T-t)} \frac{1}{2} S_t^2 \Gamma_t (\sigma_H^2 - \sigma_S(t)^2) dt. \quad (3.13)$$

□

Remark 3.4.1. In our case we have constructed the hedging portfolio s.t. $Z_0 = 0$, so Equation (3.13) becomes

$$Z_T = \int_0^T e^{r(T-t)} \frac{1}{2} S_t^2 \Gamma_t (\sigma_H^2 - \sigma_S(t)^2) dt. \quad (3.14)$$

Remark 3.4.2. If we assume that also the volatility used in the true model σ_S is constant, (3.13) becomes

$$Z_T = (\sigma_H^2 - \sigma_S^2) \int_0^T e^{r(T-t)} \frac{1}{2} S_t^2 \Gamma_t dt. \quad (3.15)$$

Remark 3.4.3. If for example the claim $V(t, S_t)$ is a European Put Option, using that $\nu_t = \sigma_H(T-t)S_t^2 \Gamma_t$ (where $\nu_t = \frac{\partial V}{\partial \sigma_H}(t, S_t)$) (3.15) becomes

$$Z_T = \frac{\sigma_H^2 - \sigma_S^2}{\sigma_H} \int_0^T \frac{1}{2(T-t)} e^{r(T-t)} \nu_t dt. \quad (3.16)$$

In the put case, which implies $\Gamma_t > 0$, the hedger is lucky if $\sigma_H^2 > \sigma_S^2$ since it means he is making a profit with probability one even though the true price model is different from the assumed one. On the other hand if $\sigma_H^2 < \sigma_S^2$ the hedger will loose money for sure. This doesn't mean we are creating an arbitrage, since we have assumed that we have sold the claim at the "hedging price", not at the price computed using the dynamics of the underlying with parameters μ_S and σ_S .

This suggests that if there is the possibility to use the correct model in the hedging strategy, one should definitively do so but a hedging strategy is still possible if the difference in volatility or the convexity Γ are small.

Hedging Error in CVA. The formula derived in the previous section was for any derivative contract that is a square integrable contingent claim $V(t, S_t)$. A typical example of contingent claim is a European option but it can be generalised to many others. For instance, for the purpose of this thesis, it is interesting to apply this formula to the CVA. As a matter of fact, if we have a long position in a derivative contract, as previously discussed in the CVA section, this implies we are "short" in the CVA. We can thus directly apply the computations of section 3.4.1 to this case getting the formula

$$Z_T = \int_0^T e^{r(T-t)} \frac{1}{2} S_t^2 \Gamma_t^{CVA} (\sigma_H^2 - \sigma_S(t)^2) dt. \quad (3.17)$$

where $\Gamma_t^{CVA} = \frac{\partial^2 CVA}{\partial x^2}(t, S_t)$. This formula explains the error in the hedging of the market risk.

3.5 Hedging the credit risk

We now define the credit derivative we use to hedge the credit risk of our portfolio and explain a bit more of the chosen hedging strategy.

CDS. A Credit Default Swap (CDS) is one of the most used instruments to hedge against credit risk. As any credit derivative, it is a financial contract that is used to transfer the credit risk and works as an insurance against the fact that our counterparty could default. Basically a CDS allows you to get a cash flow in case of default of your counterparty, equal to the "Loss Given Default" (LGD), i.e. $(1 - \pi)$, where π is the recovery value as already pointed out. This cash flow is balanced by some periodic cash flows that the buyer of the CDS has to pay and which are calibrated s.t. the net present value of the contract at the initial time is zero. This type of contract is usually entered at par so there are no cash flows in t_0 . Actually there is also a CCR embedded in CDSs themselves, but for the purpose of this thesis we have neglected any extra counterparty risk arising from the usage of a CDS.

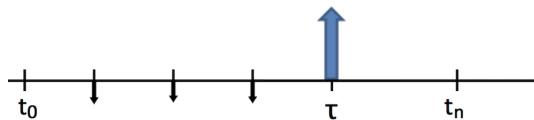


Figure 3.1: Cash flows for a CDS contract. The black arrows represent the fixed legs the buyer of the CDS has to pay in return of the Contingent leg (the Blue arrow), that is paid to the buyer of the protection in case of default of the counterparty.

Supposing that our default probability is given by Equation (2.4), let us derive the Jarrow Turnbull approximation for the spreads the holder of the CDS is going to pay,

considering a time grid $0 = t_0 < \dots < t_N = T$ with equispaced time intervals Δt . Since we are interested in our discrete time cashflows we consider that case directly but one could derive the approximation also for a continuous case just by considering the intergral instead of the summation. Since the contract is entered at par, the problem becomes to calibrate the CDS spreads \bar{s} (i.e. black arrows in Figure 3.1) s.t. the net present value of the contract is 0. For the conditional expectation we have used this notation $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot | \mathcal{F}_t]$. Let us consider equispaced time intervals $\Delta t = (T - t_0)/n$ and the following quantities:

- the survival probability $\hat{P}(t, T) = \mathbb{E}_t[\mathbf{1}_{\tau > T}]$,
- the expected value of the stochastic discount factor
 $B(t, T) = \mathbb{E}_t[D(t, T)] = \mathbb{E}_t[e^{-\int_t^T r_s ds}]$,
- the defaultable ZC bond with no recovery $\bar{B}(t, T) = \mathbb{E}_t[e^{-\int_t^T r_s ds} \mathbf{1}_{\tau > T}]$
- $e(t, T, T + \Delta T) = \mathbb{E}_t[e^{-\int_t^T r_s ds} (\mathbf{1}_{\tau > T} - \mathbf{1}_{\tau > T + \Delta T})]$.

Let us start computing the net present value in t of the fee part is

$$NPV_t^{fee} = \bar{s} \Delta t \sum_{i=1}^n \mathbb{E}_t[D(t, t_i) \mathbf{1}_{\tau > t_i}] = \bar{s} \Delta t \sum_{i=1}^n \bar{B}(t, t_i). \quad (3.18)$$

The net present value for the protection leg is

$$NPV_t^\pi = (1 - \pi) \sum_{i=1}^n \mathbb{E}_t[D(t, t_i) (\mathbf{1}_{\tau > t_{i-1}} - \mathbf{1}_{\tau > t_i})] = (1 - \pi) \sum_{i=1}^n e(t, t_{i-1}, t_i). \quad (3.19)$$

One of our key assumptions is the independency between rates and defaults, so

$$\bar{B}(t, T) = \mathbb{E}_t[e^{-\int_t^T r_s ds} \mathbf{1}_{\tau > T}] = \mathbb{E}_t[e^{-\int_t^T r_s ds}] \mathbb{E}_t[\mathbf{1}_{\tau > T}] = B(t, T) \hat{P}(t, T),$$

and

$$e(t, T, T + \Delta T) = \mathbb{E}_t[e^{-\int_t^T r_s ds}] \mathbb{E}_t[(\mathbf{1}_{\tau > T} - \mathbf{1}_{\tau > T + \Delta T})] = B(t, T) (\hat{P}(t, T) - \hat{P}(t, T + \Delta T)).$$

Imposing that $NPV_t^{fee} = NPV_t^\pi$, we get

$$\bar{s} \Delta t \sum_{i=1}^n B(t, t_i) \hat{P}(t, t_i) = (1 - \pi) \sum_{i=1}^n B(t, t_i) (\hat{P}(t, t_{i-1}) - \hat{P}(t, t_i)),$$

which, considering as survival probability $\hat{P}(t, T) = 1 - P(t, T)$, where $P(t, T)$ is given by Equation (2.4) leads to

$$\bar{s} \Delta t \sum_{i=1}^n B(t, t_i) e^{-\lambda(t_i - t)} = (1 - \pi) \sum_{i=1}^n B(t, t_i) e^{-\lambda(t_i - t)} (e^{\lambda \Delta t} - 1).$$

For small values of Δt we can use a Taylor expansion and get

$$\bar{s} \Delta t \sum_{i=1}^n B(t, t_i) e^{-\lambda(t_i - t)} \approx (1 - \pi) \sum_{i=1}^n B(t, t_i) e^{-\lambda(t_i - t)} \lambda \Delta t. \quad (3.20)$$

so we can conclude that an approximation for the CDS spread \bar{s} is

$$\bar{s} \approx (1 - \pi)\lambda, \quad (3.21)$$

which is one of the most famous thumb rules in finance.

Strategy. Let us consider again the discretised time grid $0 = t_0 < \dots < t_N = T$ from the initial time to maturity. When hedging the credit risk of a portfolio, the challenge is that in general, parties' credit exposure in derivative transactions varies on a day-to-day basis depending on the termination value of those transactions. This means the potential loss depends on time, so taking a certain position in t_0 will not ensure that we are protected by the potential default of our counterparty, since, to hedge perfectly, we would like to hold a CDS giving us back an amount that depends on which instant our counterparty defaults in.

A solution to this problem is to rebalance our portfolio s.t. at each time instant we pay an amount equal to the average between the present value of the portfolio and the EPE in the following time instant, i.e. $\bar{N}\bar{s}\Delta t$, where $\bar{N}(t_{i+1}) = \frac{V(t_i) + EPE(t_{i+1})}{2}$. Assuming that the hypothesis of perfect liquidity hold also for the CDSs, this strategy can be seen as "holding at each time instant a CDS with a notional that would give us perfect protection if the default occurred in the coming time instant and rebalance the portfolio at the following time if the default hasn't occurred". So it is a dynamic strategy in order to get time varying cashflows. Alternatively we can interpret it as buying in each time instant t_i a CDS with maturity equal to the following time instant t_{i+1} of the amount $\bar{N}(t_{i+1}) = \frac{V(t_i) + EPE(t_{i+1})}{2}$. In this way we hold exactly a CDS that expires the time instant after, so basically we are holding n different CDSs along the life of our derivative instrument.

Chapter 4

Methods

This chapter presents the methods used for the computation of the CVA and its hedging. It also includes the algorithms followed to replicate the portfolio in the hedging strategies.

4.1 CVA Computation

Scenario Generation. A *scenario* consists in the realization of the fundamental market factors at a predefined set of simulation dates. The risk factors are simulated forward in time and the portfolio is priced in each scenario to obtain the EPE which allows us to compute the CVA for each simulated path ([14]). A fundamental assumption is that we suppose independence between the exposure and the default probability. The risk factor S_t is easy to simulate for any increasing sequence of times $t_0 < \dots < t_N = T$. We have considered equispaced time intervals i.e. $\Delta t := t_{i+1} - t_i$ for $i = \{0, \dots, N - 1\}$ and simulated under the risk neutral measure \mathbb{Q} obtaining at a generic time instant t_{i+1}

$$S_{t_{i+1}} = S_{t_i} \exp\left((r - \sigma_P^2/2)\Delta t + \sigma_P\sqrt{\Delta t}X\right)$$

where X is a standard normal, r the short rate, σ_P the volatility we use in the model chosen for the simulation of the risk factor and $S_{t_0} = S_0$ given.

Trapezoidal Quadrature Rule. From Equation (3.7) we know that the CVA can be written as

$$\text{CVA}(t) = (1 - \pi) \int_t^T B(t, s) \text{EPE}(s) dQ_D(s)$$

where $\text{EPE}(t)$ is the expected positive exposure at time t and X_t is the future claim of the derivative contract, π is the recovery rate and $Q_D(t)$ is the cumulative distribution function for the default time τ as pointed out in Section (3.2).

The integral inside the expectation in the CVA formula is a standard Riemann-Stieltjes integral. Considering the same equispaced time intervals shown in the scenario generation (3.7) can be written as

$$(1 - \pi) \sum_{k=1}^n B(t, t_k) \text{EPE}(t_k) (Q_D(t_k) - Q_D(t_{k-1})). \quad (4.1)$$

Using the trapezoidal rule in our discrete time setting, (3.7) becomes

$$(1 - \pi) \sum_{k=0}^{n-1} \frac{1}{2} \left(B(t, t_k) \text{EPE}(t_k) + B(t, t_{k+1}) \text{EPE}(t_{k+1}) \right) (Q_D(t_{k+1}) - Q_D(t_k)). \quad (4.2)$$

Equation (4.2) can be rewritten as

$$(1 - \pi) \mathbb{E}_t^{\mathbb{Q}} \left[\sum_{k=0}^{n-1} \frac{1}{2} \left(B(t, t_k) (V(t_k))^+ + B(t, t_{k+1}) (V(t_{k+1}))^+ \right) (Q_D(t_{k+1}) - Q_D(t_k)) \right],$$

so to get an estimate of the CVA we compute it for each MC simulation and take the average using a MC method.

In our case we have considered constant short rate and as default probability the homogeneous Poisson process with parameter h as explained in Section 3.5. The last equation thus becomes

$$(1 - \pi) \mathbb{E}_t^{\mathbb{Q}} \left[\sum_{k=0}^{n-1} \frac{1}{2} \left(B(t, t_k) (V_{t_k})^+ + B(t, t_{k+1}) (V_{t_{k+1}})^+ \right) e^{-h(k\Delta t)} (1 - e^{-h\Delta t}) \right]. \quad (4.3)$$

4.2 CVA Hedging

Scenario Generation. When hedging, we need to generate different scenarios of the possible evolutions of the market factors forward in time, so this is done under the real world measure \mathbb{P} . We consider an equispaced time grid $0 = t_0 < \dots < t_N = T$ as described in Section 4.1 and simulate the underlying in the same way, with the difference that we consider μ as drift instead of r since we are simulating under the real world measure, obtaining at a generic time instant t_{i+1}

$$S_{t_{i+1}} = S_{t_i} \exp \left((\mu - \sigma_S^2/2) \Delta t + \sigma_S \sqrt{\Delta t} X \right)$$

where X is a standard normal, μ the drift of the underlying, σ_S the volatility we use in the model chosen for the simulation of the risk factor and $S_{t_0} = S_0$ given.

Finite Differences for Sensitivities Computation. As previously remarked, the hedging of the market price of potential future default events can be done via a dynamic approach rebalancing the portfolio. Since we have used a delta hedging strategy it is necessary to compute some risk sensitivities in each time instant. This can be done via a simple, brute force, bump and revalue mechanism, getting the sensitivities by ‘‘bumping’’ the risk factors and re-computing the CVA with the bumped risk factors.

Let us consider a sequence of equispaced times $t_0 < \dots < t_N = T$ on the time interval $[0, T]$ as described in Section 4.1. As seen we use a Monte Carlo method to compute the value of the CVA (we will use the notation $u(t, S) = \mathbb{E}[\text{CVA}^*(t, S)]$ to make the text cleaner. S refers to the value of the underlying in t , i.e. S_t). Assuming that $u(t, S)$ is sufficiently differentiable, we can approximate, in each time instant, its first order partial derivative with respect to the risk factor using for instance with a forward Euler approximation

$$\left. \frac{\partial u(t, S)}{\partial S} \right|_{t_i} \approx \frac{u(t_i, S_i + \Delta S) - u(t_i, S_i)}{\Delta S}$$

or a central difference

$$\left. \frac{\partial u(t, S)}{\partial S} \right|_{t_i} \approx \frac{u(t_i, S_i + \Delta S) - u(t_i, S_i - \Delta S)}{2\Delta S}$$

where we have used the notation $S_i = S(t_i)$.

Second order sensitivities can be computed via a central difference

$$\left. \frac{\partial^2 u(t, S)}{\partial S^2} \right|_{t_i} \approx \frac{u(t_i, S_i + \Delta S) - 2u(t_i, S_i) + u(t_i, S_i - \Delta S)}{\Delta S^2}.$$

We can compute the approximation errors for these finite differences formulae using a Taylor expansion of $u(t, S)$ as follows

$$u(t_i, S_i + \Delta S) = u(t_i, S_i) + \Delta S \left. \frac{\partial u(t, S)}{\partial S} \right|_{t_i} + \frac{\Delta S^2}{2} \left. \frac{\partial^2 u(t, S)}{\partial S^2} \right|_{t_i} + \frac{\Delta S^3}{6} \left. \frac{\partial^3 u(t, S)}{\partial S^3} \right|_{t_i} + \dots$$

and

$$u(t_i, S_i - \Delta S) = u(t_i, S_i) - \Delta S \left. \frac{\partial u(t, S)}{\partial S} \right|_{t_i} + \frac{\Delta S^2}{2} \left. \frac{\partial^2 u(t, S)}{\partial S^2} \right|_{t_i} - \frac{\Delta S^3}{6} \left. \frac{\partial^3 u(t, S)}{\partial S^3} \right|_{t_i} + \dots$$

we get that

$$\frac{u(t_i, S_i + \Delta S) - u(t_i, S_i)}{\Delta S} = \left. \frac{\partial u(t, S)}{\partial S} \right|_{t_i} + \mathcal{O}(\Delta S),$$

$$\frac{u(t_i, S_i + \Delta S) - u(t_i, S_i - \Delta S)}{2\Delta S} = \left. \frac{\partial u(t, S)}{\partial S} \right|_{t_i} + \mathcal{O}(\Delta S^2)$$

and

$$\frac{u(t_i, S_i + \Delta S) - 2u(t_i, S_i) + u(t_i, S_i - \Delta S)}{\Delta S^2} = \left. \frac{\partial^2 u(t, S)}{\partial S^2} \right|_{t_i} + \mathcal{O}(\Delta S^2).$$

To sum up then if $X^{(j)}(t, S + \Delta S)$, $X^{(j)}(t, S)$ and $X^{(j)}(t, S - \delta S)$ are the values of the CVA(t, S) obtained for different MC samples, the central difference estimator for $\partial u(t, S)/\partial S$ is

$$\hat{\Delta} = \frac{1}{2\Delta S \cdot N} \left(\sum_{j=1}^N X^{(j)}(t, S + \Delta S) - \sum_{j=1}^N X^{(j)}(t, S - \Delta S) \right)$$

and the central difference estimator for $\partial^2 u(t, S)/\partial S^2$ is

$$\hat{\Gamma} = \frac{1}{\Delta S^2 \cdot N} \left(\sum_{j=1}^N X^{(j)}(t, S + \Delta S) - 2 \sum_{j=1}^N X^{(j)}(t, S) + \sum_{j=1}^N X^{(j)}(t, S - \Delta S) \right).$$

Remark 4.2.1. The advantage of finite differences is that it is easy to implement, however it can be computationally expensive since we have to do some extra sets of calculations (two for central differences). Furthermore we will have a significant bias error if the shift

ΔS is too large, but a too small ΔS could decrease numerical performance. To optimise the numerical performance the size of the perturbation can be exogenously defined. The resulting sensitivity is then scaled back to a predefined market data perturbation. For the forward difference approximation we get

$$\delta u = \frac{u(t, S + \Delta S) - u(t, S)}{\Delta S} \delta S$$

where S is the market factor, ΔS is the shift used in the calculations and δS is the predefined market data perturbation.

Remark 4.2.2. In general we do not have a closed formula to compute the first order sensitivity for the CVA and this is why the brute force bump approach is very used. Actually in the European Put Option case (that we consider in the examples of this thesis) we could directly apply the closed formula for the delta of a Put option in Equation (4.3) getting

$$(1 - \pi) \mathbb{E}_t^{\mathbb{Q}} \left[\sum_{k=0}^{n-1} \frac{1}{2} \left(B(t, t_k) \Delta_P(t_k) + B(t, t_{k+1}) \Delta_P(t_{k+1}) \right) e^{-h(k\Delta t)} (1 - e^{-h\Delta t}) \right],$$

where $\Delta_P(t_k) = \Phi(d_1) - 1$ (using the notation of section 2.3). We have checked that, considering as shift $\Delta S = 10^{-4}$, using a second order approximation for the finite differences formula we obtain the same result as the analytical one.

Nested Monte-Carlo simulations. One of the biggest challenges in the hedging of the CVA is its computational cost. As a matter of fact to hedge the CVA portfolio at each time instant of our discretisation grid, it is necessary to compute the corresponding sensitivity, which requires to start a new Monte Carlo (MC) loop to simulate the risk factors forward in time from the time instant we are in to maturity. This means that if we consider N_{out} simulations for the outer loop and N scenarios for each of the inner MC loops and we discretise the time horizon with n time intervals, we have a total of

$$N_{out}(nN + (n-1)N + \dots + 2N + N) = N_{out}N \left(\sum_{i=1}^n i \right) = N_{out}N \frac{n(n+1)}{2} \quad (4.4)$$

Monte Carlo simulations which can be very time consuming for a high number of scenarios or intervals. Taking for instance $N_{out} = N = 1000$ and $n = 80$ (as we do in Chapter 5), it takes approximately 45 minutes to run the code.

Let us now focus only on the computation of the CVA, not on its hedging (so we do not have nested MonteCarlo simulations anymore but only one Monte Carlo loop) and investigate the noise introduced in the computation of the CVA. In general for a set of Monte Carlo simulations the numerical noise gets reduced with $\frac{1}{\sqrt{N}}$ where N is the number of scenarios. This means for instance that if we increase the number of simulations from 1000 to 10000 we are multiplying the computing effort by a factor of ten but reducing the noise only by a factor of around three. This non-linear function of the marginal decrease of noise per added scenario is clearly a problem ([16]).

Let us fix the rebalancing trades to $n = 20$ and compute the standard deviation. Here we have chosen smaller number of intervals than what we use in the final case studies of Chapter 5 to be able to make our computations faster even if this will introduce a discretisation error but this will be analysed later. Let us compute the standard deviation

of the CVA MC estimate. If we use N simulations we need to compute the CVA M times using N MC simulations and take a mean of the different standard deviations obtained. To be sure that we actually are considering M different scenario simulations, each time we assign a seed that is different from the one used in the previous step. The results taking $N = [10, 100, 1000, 10000]$ are shown in the following figure.

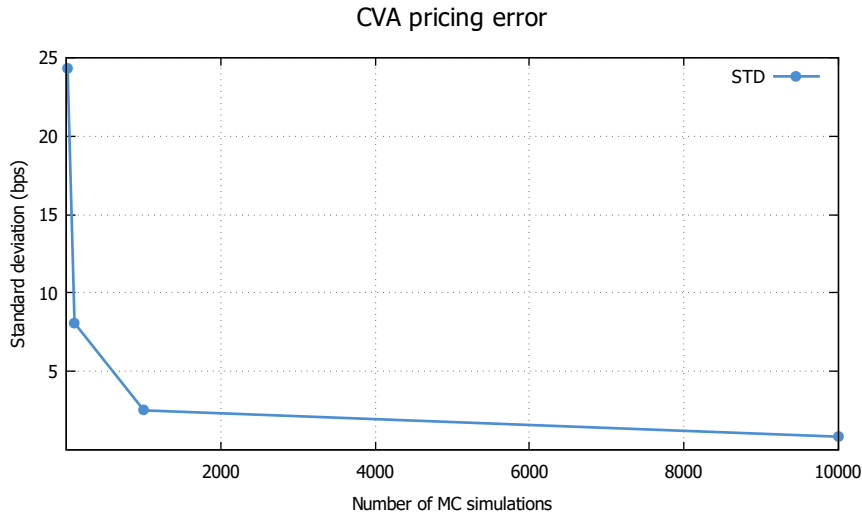


Figure 4.1: CVA pricing error (Standard deviation of the CVA price over several calculations) as a function of the numbers of MC simulations used.

From Figure 4.1 and Table 4.1 we can observe the typical non-linear behaviour of the standard deviation of the Monte-Carlo estimate. As expected, a precise value (low standard deviation) comes at the expense of computational heaviness.

Nr MC simulations	CVA error (bps)
10	24.3
100	8.08
1000	2.50
10000	0.84

Table 4.1: CVA pricing error (Standard deviation of the CVA price over several calculations) with the corresponding number of MC simulations.

4.3 Algorithms

Algorithm 1 Delta Hedging of V_t

```

1: for  $j = 1$  to  $N$  do ▷ MC loop
2:   Compute  $V_0$  and delta =  $\frac{\partial V}{\partial x}(t_0, S_0)$  ▷ Price under  $\mathbb{Q}$  and use  $\sigma_H$ 
3:   bank =  $V_0 - \text{delta} \cdot S_0$  ▷ Put the money in the bank account
4:   Compute gamma =  $\frac{\partial^2 V}{\partial x^2}(t_0, S)$ 
5:   Start computing the analytic P&L  $Z_T$  ▷ First contribution to quadrature rule
6:   for  $i = 1$  to  $n - 1$  do ▷ Time loop
7:     Sample  $Z$  from  $\mathcal{N}(0, 1)$ 
8:      $S_{t_i} = S_{t_{i-1}} \exp((\mu_S - 0.5\sigma_S^2)\Delta t + \sigma_S\sqrt{\Delta t}Z)$  ▷ Simulate under  $\mathbb{P}$ 
9:     Compute newdelta =  $\frac{\partial V}{\partial x}(t_i, S_{t_i})$  ▷ Price under  $\mathbb{Q}$  and use  $\sigma_H$ 
10:    bank = bank  $\cdot e^{r\Delta t} - (\text{newdelta} - \text{delta}) \cdot S_{t_i}$  ▷ Rebalance the portfolio
11:    Compute gamma =  $\frac{\partial^2 V}{\partial x^2}(t_i, S_{t_i})$ 
12:    Compute the  $Z_T$  contribution for  $t_i$  ▷ Trapez. quadrature rule
    return
13:   Sample  $Z$  from  $\mathcal{N}(0, 1)$ 
14:    $S_{t_n} = S_{t_{n-1}} \exp((\mu_S - 0.5\sigma_S^2)\Delta t + \sigma_S\sqrt{\Delta t}Z)$  ▷ Simulate under  $\mathbb{P}$ 
15:   P&L $_j = \text{bank} \cdot e^{r\Delta t} + \text{delta} \cdot S_{t_n} - \text{payoff}$ 
return  $\mathbb{E}[P\&L]$ , STD(P&L) and  $\mathbb{E}[Z_T]$ .

```

Algorithm 2 CVA Computation

```

1: for  $j = 1$  to  $N$  do ▷ MC loop
2:   for  $i = 0$  to  $n - 1$  do ▷ Time loop
3:     Sample  $Z$  from  $\mathcal{N}(0, 1)$ 
4:      $S_{t_i} = S_{t_{i-1}} \exp((r - 0.5\sigma_P^2)\Delta t + \sigma_P\sqrt{\Delta t}Z)$  ▷ Simulate under  $\mathbb{Q}$ 
5:     Compute the shifted  $S_{t_{i-1}}^{s1} = S_{t_{i-1}} + \text{shift}$  and  $S_{t_{i-1}}^{s2} = S_{t_{i-1}} - \text{shift}$ 
6:     Compute the shifted  $S_{t_i}^{s1} = S_{t_i} + \text{shift}$  and  $S_{t_i}^{s2} = S_{t_i} - \text{shift}$ 
7:     Compute  $V_{t_i}, V_{t_{i+1}}$  and the portfolio values with the shifted underlyings
8:     Compute the  $i$ -contribution to  $\text{CVA}_{t_0}$  ▷ Trap. Quad. Rule
9:     Compute the  $i$ -contribution to the two shifted  $\text{CVA}_{t_0}^{s1}$  and  $\text{CVA}_{t_0}^{s2}$ 
   return
10:    $\text{CVA}(t_0) = \text{CVA}(t_0) + \text{CVA}_j(t_0)$ 
11:    $\text{CVA}^{s1}(t_0) = \text{CVA}^{s1}(t_0) + \text{CVA}_j^{s1}(t_0)$ 
12:    $\text{CVA}^{s2}(t_0) = \text{CVA}^{s2}(t_0) + \text{CVA}_j^{s2}(t_0)$ 
return
13: Take the MC mean and compute  $\text{CVA}_{t_0}$ 
14: Compute  $\text{delta}_{\text{CVA}}(t_0) = (\text{CVA}_{t_0}^{s1} - \text{CVA}_{t_0}^{s2}) / (2 \cdot \text{shift})$ 
15: Compute  $\text{gamma}_{\text{CVA}}(t_0) = (\text{CVA}_{t_0}^{s1} - 2\text{CVA}_{t_0} + \text{CVA}_{t_0}^{s2}) / (\text{shift}^2)$ 

```

Algorithm 3 CVA Hedging for V_t

```

1: for  $j = 1$  to  $N$  do ▷ MC loop
2:   Compute  $\text{CVA}_0$  and  $\text{EPE}(t_0, t_1)$ 
3:   Compute  $\text{delta}_{\text{CVA}} = \frac{\partial \text{CVA}}{\partial x}(t_0, S_0)$  and  $\text{gamma}_{\text{CVA}} = \frac{\partial^2 \text{CVA}}{\partial x^2}(t_0, S)$ 
4:    $\text{bank} = \text{CVA}_0 - \text{delta}_{\text{CVA}} \cdot S_0$  ▷ Put the money in the bank account
5:   Start computing the analytic P&L  $Z_T$  ▷ First contribution to quadrature rule
6:   for  $i = 1$  to  $n - 1$  do ▷ Time loop
7:     Sample  $Z$  from  $\mathcal{N}(0, 1)$ 
8:      $S_{t_i} = S_{t_{i-1}} \exp((\mu_S - 0.5\sigma_S^2)\Delta t + \sigma_S\sqrt{\Delta t}Z)$  ▷ Simulate under  $\mathbb{P}$ 
9:     Compute  $\text{EPE}(t_i, t_{i+1})$ 
10:    Compute  $\text{newdelta}_{\text{CVA}} = \frac{\partial \text{CVA}}{\partial x}(t_i, S_{t_i})$  and  $\text{gamma}_{\text{CVA}} = \frac{\partial^2 \text{CVA}}{\partial x^2}(t_i, S_{t_i})$ 
11:     $\text{cflow} = \bar{s}\Delta t(V(t_{i-1}) + \text{EPE}(t_{i-1}, t_i)) / 2$  ▷ Hedge the Credit Risk
12:     $\text{bank} = \text{bank} \cdot e^{r\Delta t} - (\text{newdelta}_{\text{CVA}} - \text{delta}_{\text{CVA}}) \cdot S_{t_i} - \text{cflow}$  ▷ Rebalance
13:    Compute the  $Z_T$  contribution for  $t_i$  ▷ Trapez. quadrature rule
   return
14:   Sample  $Z$  from  $\mathcal{N}(0, 1)$ 
15:    $S_{t_n} = S_{t_{n-1}} \exp((\mu_S - 0.5\sigma_S^2)\Delta t + \sigma_S\sqrt{\Delta t}Z)$  ▷ Simulate under  $\mathbb{P}$ 
16:    $\text{cflow} = \bar{s}\Delta t(V(t_{n-1}) + \text{EPE}(t_{n-1}, t_n)) / 2$ 
17:    $\text{P\&L}_j = \text{bank} \cdot e^{r\Delta t} + \text{delta}_{\text{CVA}} \cdot S_{t_n} - \text{payoff}_{\text{CVA}} - \text{cflow}$ 
return  $\mathbb{E}[P\&L]$ ,  $\text{STD}(P\&L)$  and  $\mathbb{E}[Z_T]$ .

```

Chapter 5

Results

This chapter presents some practical case studies where different types of hedging errors are analysed. First we consider a default free portfolio with one financial derivative then we hedge its CCR. To hedge from the movements of the underlying we follow a Delta hedging strategy and to hedge the credit risk we use different CDSs, as already explained.

5.1 Delta hedging of a European Put Option: Discretisation Error

In this section we investigate the time discretisation error. Let us assume the Black-Scholes setting described in Section 2.3 and that we are short in a European Put Option. As we know, the Black and Scholes hedging strategy works perfectly if we suppose to hedge continuously, assuming we use the same model for simulation and pricing. We have thus followed the algorithm described in Chapter 4 for different numbers of time intervals and analysed the Profit and Loss (P&L) distributions generated by the hedging portfolio in each case.

Table 5.1: P&L statistics summary. Put option value is 5.57353 and $\sigma_S = \sigma_H = 0.2$

Nr Intervals	STD	% STD	Mean	$\mathbb{E}[Z_T]$	$CI_{\text{mean}}^{0.99}$
20	1.5153	27.18	-0.005	0	(-0.0224, 0.0148)
80	0.7631	13.69	-0.001	0	(-0.0098, 0.0078)

In Table 5.1 we show the results for $n = 20$ and $n = 80$ time intervals. We can observe that the mean value lies around zero for both cases. What changes with the number of time steps considered for hedging is the variance. Figure 5.1 shows that, as intuitively expected, the standard deviation of the P&L decreases as the time grid gets thicker and we get closer to the ideal case described by Black-Scholes (i.e. continuous hedging).

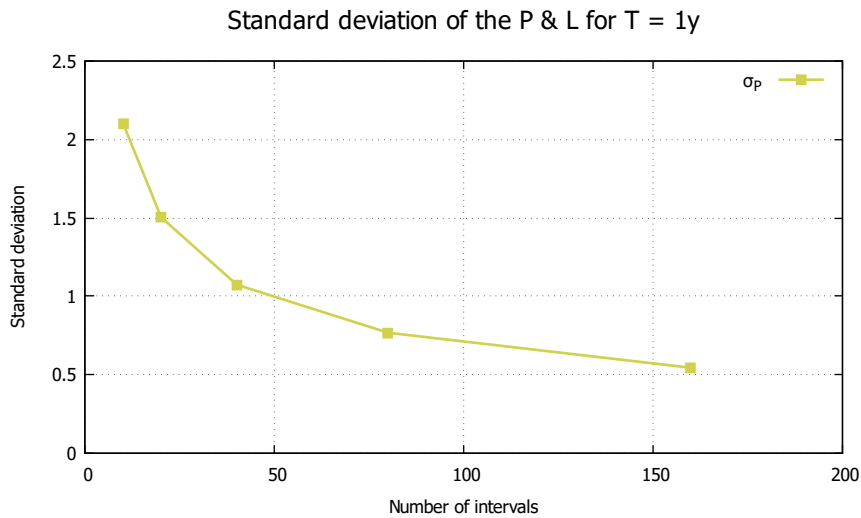


Figure 5.1: Standard deviation of the P&L as a function of the numbers of intervals used in the hedging strategy.

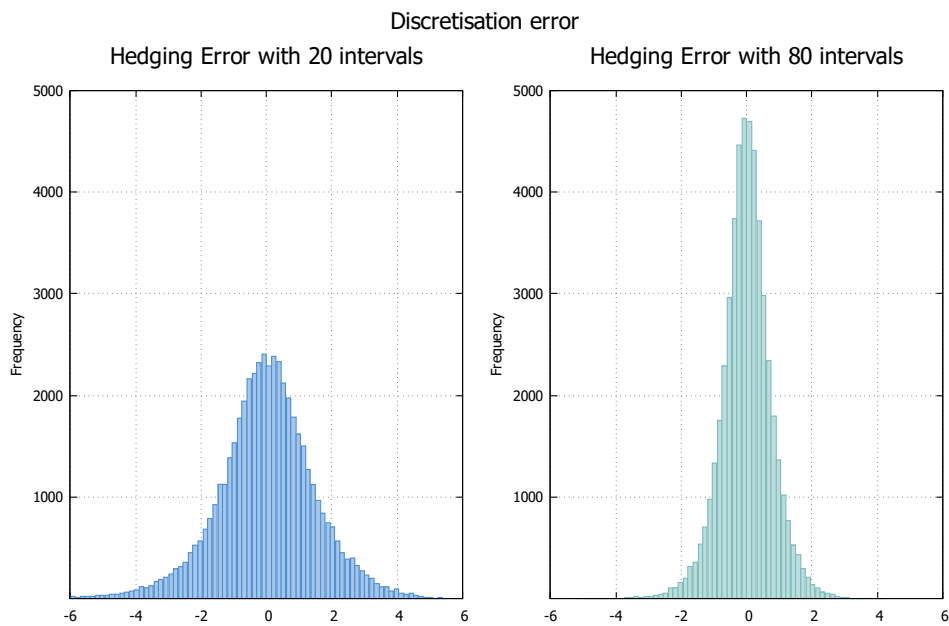


Figure 5.2: P&L of the hedger with $N = 50000$ MC simulations and $\sigma_S = \sigma_H = 0.2$.

Note that looking only at the mean of the P&L could be misleading: as a matter of fact even if the mean value is close to zero, we could still have some very large losses balanced with very large profits, in other words it is important to focus also on the standard deviation of the P&L. As we can see in the following table and from the distributions in Figure 5.2, quadruplicating the number of intervals roughly halves the standard deviation. This is due to the fact that the standard deviation of the P&L distribution behaves as

$constant \cdot \frac{1}{\sqrt{n}}$ (17).

Obviously each of the simulated Monte Carlo paths can look very differently and analysing just one of them can't be considered a complete treatment of the problem as plotting a histogram. However it can still be very interesting to have a look at least at one of them to get a better picture of what our hedging strategy actually does. Figure 5.3 and Figure 5.4 show a generic path for a hedging strategy with respectively 20 and 80 intervals with the corresponding value of the Put (bluish line) we want to hedge: in both cases we can see how the hedging portfolio follows the dynamics of the Put.

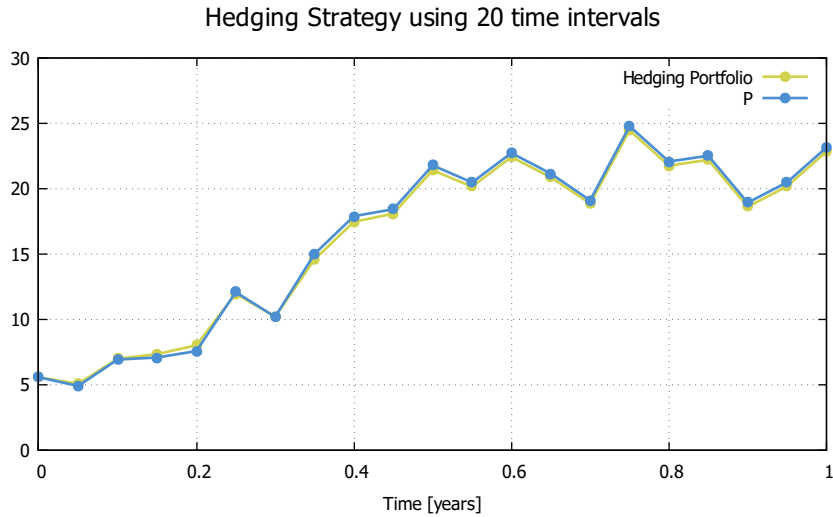


Figure 5.3: Hedging Strategy over one MC path when $\sigma_S = \sigma_H = 0.2$.

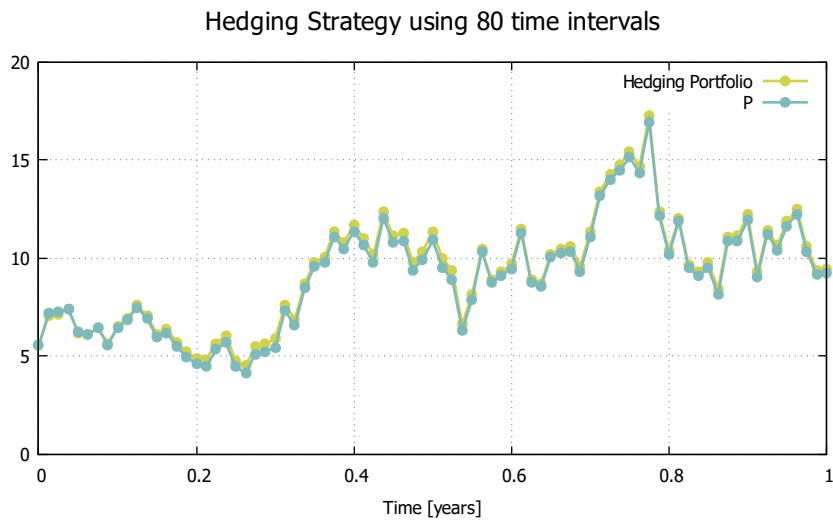


Figure 5.4: Hedging Strategy over one MC path when $\sigma_S = \sigma_H = 0.2$.

5.2 Delta hedging of a European Put Option: Model Error

In this section we check what happens due to a model mismatch for the same portfolio of Example 5.1, comparing the numerical results with the formula derived analytically in Section 3.4.1.

Fixing the number of intervals we can analyze the hedging error generated by simulating the underlying with a different volatility σ_S from the one used for hedging, i.e. σ_H . Taking four different values for the volatility parameter and making the computations both for $n = 20$ and $n = 80$ intervals, we get the following results:

Table 5.2: P&L statistics summary. Put option value is 5.57353 and $\sigma_S = 0.3$

Nr Intervals	STD	% STD	Mean	$\mathbb{E}[Z_T]$	$CI_{\text{mean}}^{0.99}$
20	2.9192	52.38	-3.9858	-3.8951	(-4.0195, -3.9521)
80	2.1385	38.37	-3.9688	-3.9448	(-3.9935, -3.9441)

Table 5.3: P&L statistics summary. Put option value is 5.57353 and $\sigma_S = 0.1$

Nr Intervals	STD	% STD	Mean	$\mathbb{E}[Z_T]$	$CI_{\text{mean}}^{0.99}$
20	1.2789	22.95	3.8333	3.7008	(3.8185, 3.8481)
80	1.1226	20.14	3.8284	3.7949	(3.8154, 3.8414)

Table 5.4: P&L statistics summary. Put option value is 5.57353 and $\sigma_S = 0.21$

Nr Intervals	STD	% STD	Mean	$\mathbb{E}[Z_T]$	$CI_{\text{mean}}^{0.99}$
20	1.6016	28.73	-0.4005	-0.3856	(-0.4190, -0.3820)
80	0.8157	14.64	-0.3959	-0.3919	(-0.4053, -0.3865)

Table 5.5: P&L statistics summary. Put option value is 5.57353 and $\sigma_S = 0.19$

Nr Intervals	STD	% STD	Mean	$\mathbb{E}[Z_T]$	$CI_{\text{mean}}^{0.99}$
20	1.4442	25.91	0.3897	0.3841	(0.3730, 0.4064)
80	0.7403	13.28	0.3917	0.3908	(0.3832, 0.4002)

On the first row of each table we see the case with 20 intervals and on the second the one with 80 rebalancing trades. As expected by the formula (3.17), if the volatility used to simulate the underlying is bigger than the one we use for hedging we get a loss, whereas if we simulate with a lower sigma we get a profit.

The P&L distributions corresponding to the four different cases shown in the previous tables are shown in the following Figure 5.5. To understand what is happening it can be useful to take a look at one example of Monte Carlo path for each case (see Figures 5.6 and 5.7). Since the price of an option increases with volatility, if we simulate the underlying with a σ_S lower (higher) than the σ_P used for pricing, we are hedging assuming that the Put is worth more (less) than what the simulation model says, and will therefore get a profit (loss) from the hedging strategy.

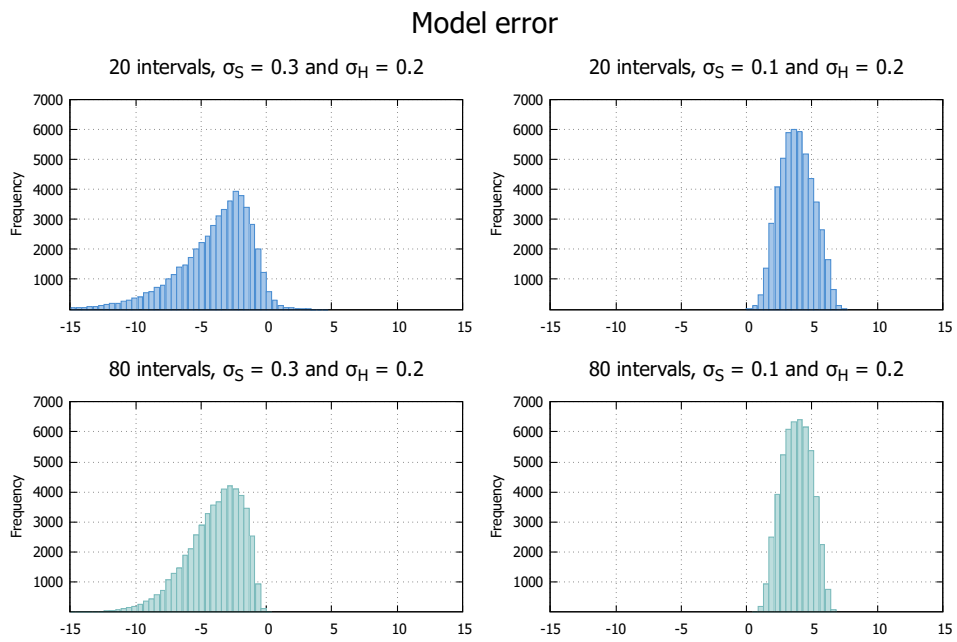


Figure 5.5: P&L of the hedger with $N = 50000$ MC simulations with $\sigma_S = 0.3$ (left plots) and $\sigma_S = 0.1$ (right plots).

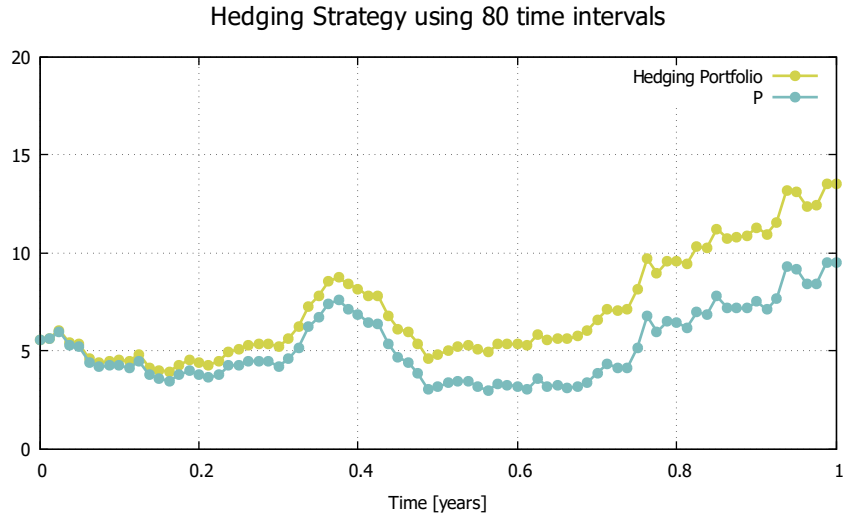


Figure 5.6: Hedging Strategy over one MC path when $\sigma_S = 0.1$ and $\sigma_H = 0.2$.

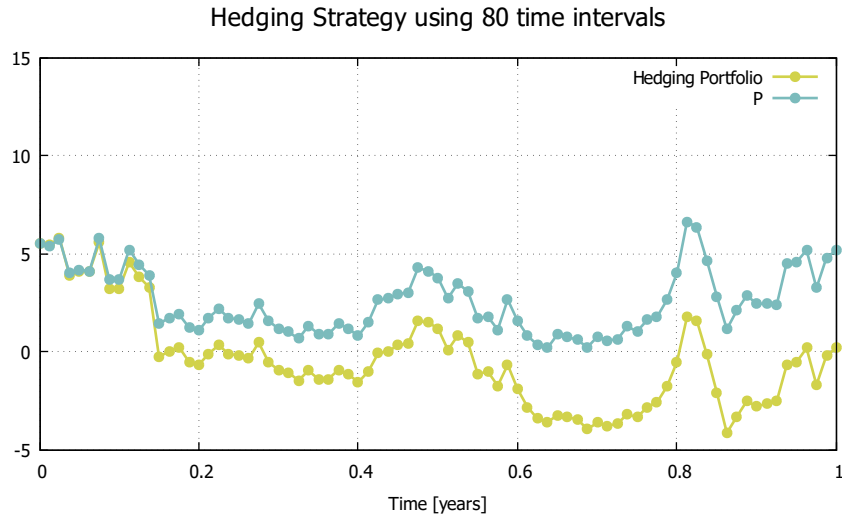


Figure 5.7: Hedging Strategy over one MC path when $\sigma_S = 0.3$ and $\sigma_H = 0.2$.

To summarise, this example shows how the hedging error is influenced by a mismatch in the models for simulation and pricing in a framework where we consider only market risk, i.e. where we have supposed a non-defaultable counterparty. We know that this is actually not a strong assumption anymore and it is important to take into account also the possibility that a counterparty can default. This is in practice done by computing the CVA and considering it into the valuation of the portfolio as described in Section (3.2) and as we will see in the following case study. One may wonder why we included this example without the CVA into this thesis, if the title states clearly that we want to deal with hedging error “in CVA”. The answer is that since the CVA is a price, to hedge it perfectly (or close to it) we must hedge also its market risk, which will be done exactly in the same way as for the European Put case alone.

In the following example we investigate what happens if we “allow” our counterparty to default and focus on the CVA hedging, rather than on the hedging of the Put.

5.3 CVA Hedging for a European Put: Model Error 1

In this example we assume that the counterparty risk is not negligible. Thus the real price $V(t)$ is equal to the risk free price minus the credit value adjustment:

$$V(t) = V_{rf}(t) - \text{CVA}(t).$$

Let us say we are a CVA desk and want to hedge out both the market and the credit risk of the CVA. The first type of hedging is done analogously to the previous example: we consider the portfolio composed only by the CVA and rebalance at each time instant t in order to hold $\Delta_{\text{CVA}}(t)$ units of the underlying, as seen in Section 4.3. However in this case study, to get a portfolio that shrinks to zero with the CVA one must also hedge against the credit risk component: this is done using a CDS as described in Sections 3.5 and 4.3.

As done in the first two case studies we present a table with the statistics summary and a path of a possible scenario to get an idea of what our hedge looks like. From Figure 5.8 we see that, as expected, the CVA shrinks to zero.

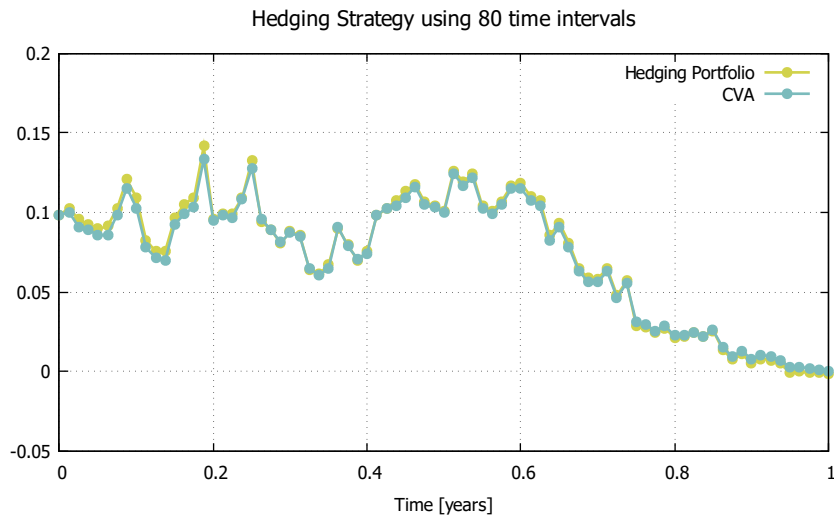


Figure 5.8: Hedging Strategy over one MC path when $\sigma_S = \sigma_H = 0.2$.

Table 5.6: $P\&L$ statistics summary. $\text{CVA} = 0.09939$, $n = 80$ time intervals.

STD	% STD	Mean	$\mathbb{E}[Z_T]$	$\text{CI}_{\text{mean}}^{0.99}$	σ_S	σ_H	σ_P
0.0084	8.42	-0.0011	0	(-0.0018, -0.0004)	0.2	0.2	0.2
0.0166	16.71	-0.0380	-0.0385	(-0.0394, -0.0380)	0.3	0.2	0.2
0.0062	6.23	0.0291	0.0310	(0.0286, 0.0296)	0.1	0.2	0.2

From Table 5.6 we can draw similar conclusions as in the previous example as we can see also from the paths shown in the Figures 5.9 and 5.10: when we use a lower (higher) volatility in the simulation of the underlying we get a profit (loss) due to the fact that we are underestimating (overestimating) the value of the put and consequently also of

the CVA. This corresponds one more time to the analytic formula, i.e. simulating the underlying with a lower (higher) volatility we get, a profit (loss) that is proportional both to the difference between the two volatilities and the Γ_t of the derivative contract.

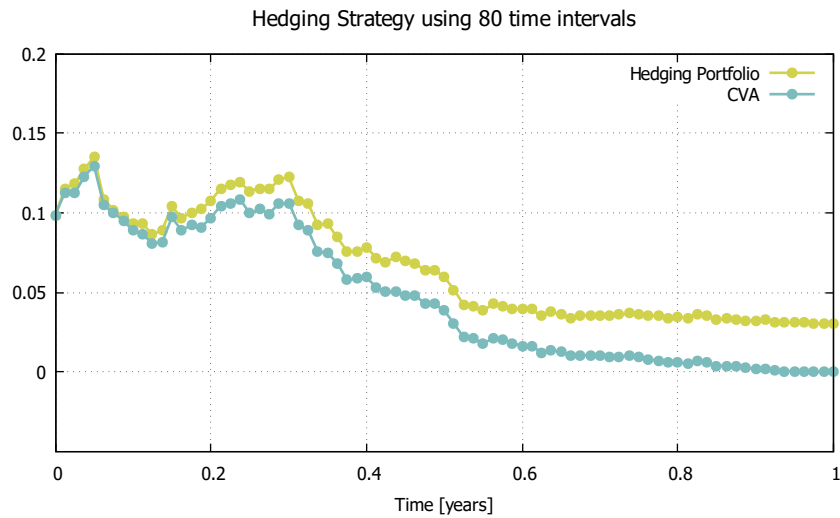


Figure 5.9: Hedging Strategy over one MC path when $\sigma_S = 0.1$ and $\sigma_H = 0.2$.

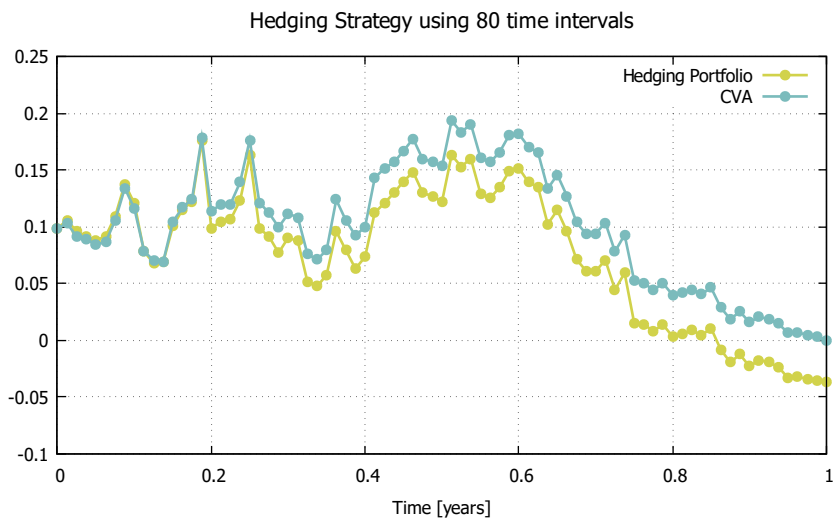


Figure 5.10: Hedging Strategy over one MC path when $\sigma_S = 0.3$ and $\sigma_H = 0.2$.

Figure 5.11 and 5.12 show the P&L distributions for the hedger in the different cases.

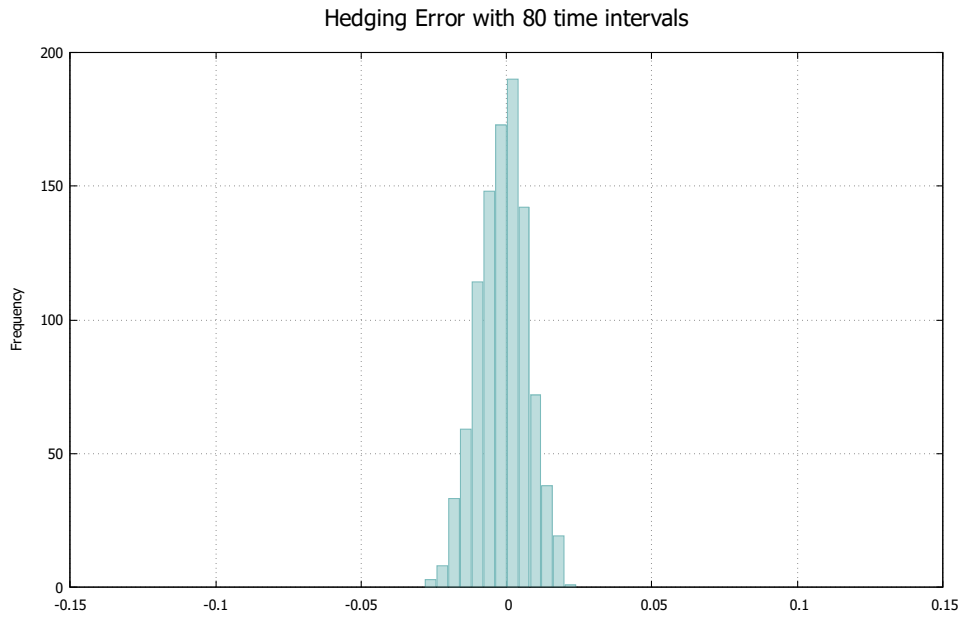


Figure 5.11: $P\&L$ of the hedger with $N = 1000$ MC simulations (both in the inner and in the outer loop) with $\sigma_S = \sigma_H = 0.2$.

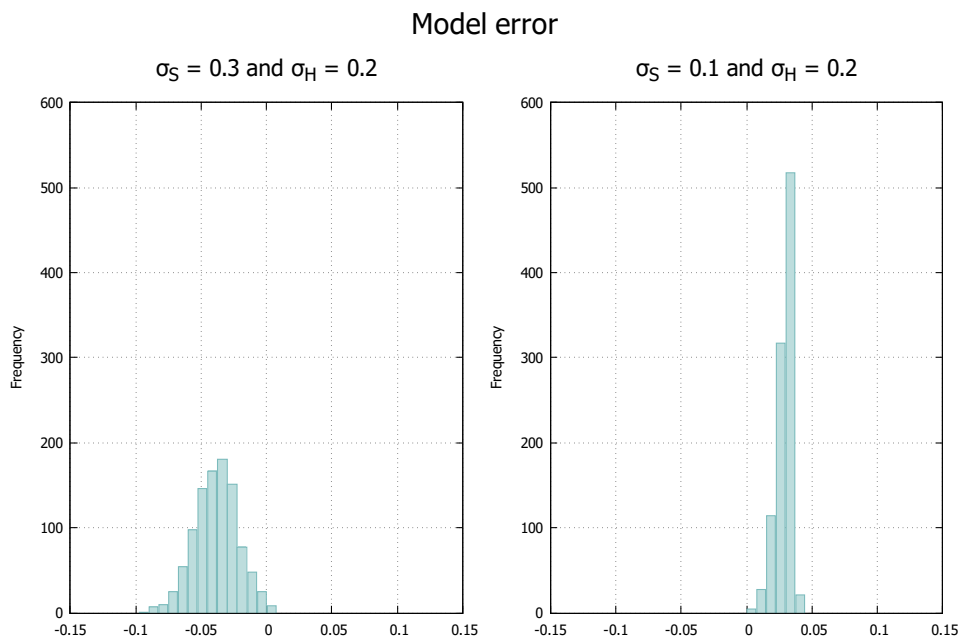


Figure 5.12: Hedging Error in CVA using $N = 1000$ MC simulations (both in the inner and outer loop) as the volatility σ_S used for the simulation changes.

Remark 5.3.1. The fact that we don't get exactly zero as mean value in the P&L when we use the same models for simulating and hedging (Figure 5.11), can be explained mainly by the fact that we are considering less MC simulations: $N = 1000$ for both MC loops while we had 50000 simulations in the first examples. (We were forced to reduce a bit the number of simulations due to the computational heaviness of the code). As a matter of fact, changing the seed for the simulation, we will observe that the mean value of the P&L distribution oscillates between positive and negative values but still close to zero, which suggests there isn't a real bias in the result but it's just a random error made bigger by the fact that we use less simulations compared to the previous case.

Remark 5.3.2. For the case of a European Put Option, as in this example, one could actually have computed the CVA analytically since the Exposure is always positive and there is only one cash flow at maturity. However this example is very useful since it allows us to check that our computations are correct and it is a first application of the mathematical study carried out in this thesis. From here, one could develop many other interesting examples as suggested in Section 6.3.

Finally, this example shows the error we commit in hedging the CVA if the calibrated value of the volatility in the pricing model differs from the real one that actually reflects the underlying movements. Basically it is the error the hedger commits when he believes in a model that is different from the dynamics that the real market factors actually follow. We get coherent results with the analytic formula as we have shown in Table 5.6 and analysing the histograms of Figure 5.12.

5.4 CVA Hedging for a European Put: Model Error 2

Often in a bank, it is common to use a complex model that reflect the real movements of the market factors for pricing and a different model (often less complicated) for the simulation. Here we investigate the performance of the hedging strategy when the CVA Desk, in the pricing loop, simulates with a σ_P that is different from the real σ_H used when pricing the contract. Since we assume that this last model is the correct one, we will simulate the outer MC loop with $\sigma_S = \sigma_H$.

As previously done we summarise our results in a table and plot the histograms corresponding to when we have a higher $\sigma_S = \sigma_H$ and a lower σ_P and viceversa.

Table 5.7: *P&L* statistics summary. $CVA = 0.09939$, $n = 80$ time intervals.

STD	% STD	Mean	$CI_{\text{mean}}^{0.99}$	σ_S	σ_H	σ_P
0.0084	8.42	-0.0011	(-0.0018, -0.0004)	0.2	0.2	0.2
0.0149	14.99	0.0352	(0.0340, 0.0364)	0.2	0.2	0.3
0.0104	10.46	-0.0314	(-0.0322, -0.0306)	0.2	0.2	0.1

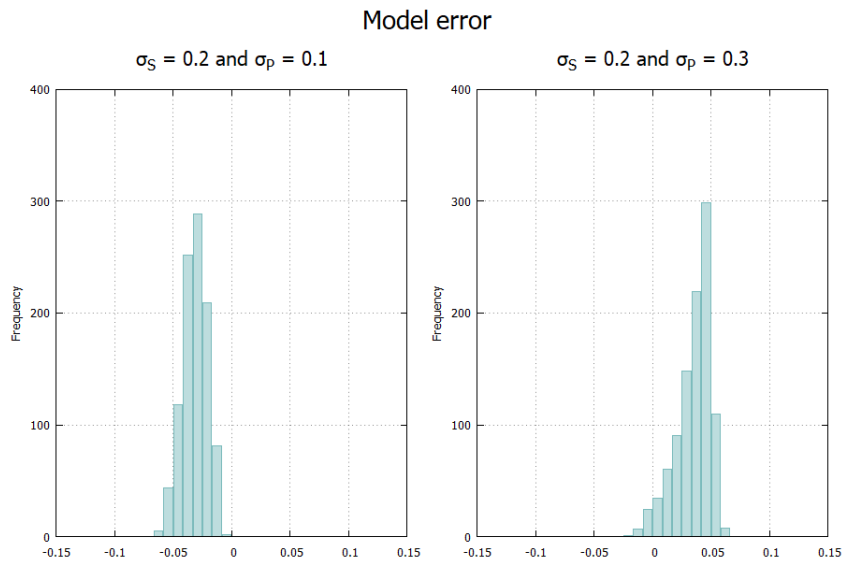


Figure 5.13: Hedging Error in CVA using $N = 1000$ MC simulations (both in the inner and outer loop) as the volatility σ_P used for the inner simulation changes.

From these results we see that using a wrong model for the simulation of the market factors generates an bias in the P&L distribution. This suggests to use the same model for simulation and pricing (if possible).

5.5 Extra Case Study: Model Error 3

This final example is presented to analyse the error produced when you use one model to simulate (both in the inner and outer MC loop), and another one to price the portfolio. So in this case $\sigma_S = \sigma_P$, where the first sigma is the one used in the outer simulation of the market factors (to test our hedging strategy), and the second sigma is the one used when generating the scenarios forward in time to price the CVA. The volatility σ_H , i.e. the "pricing-sigma", differs from the other two.

As previously done we summarise our results in a table and plot the histograms corresponding to when we have a higher $\sigma_S = \sigma_P$ and a lower σ_H and viceversa.

Table 5.8: *P&L* statistics summary. $CVA = 0.09939$, $n = 80$ time intervals.

STD	% STD	Mean	$CI_{\text{mean}}^{0.99}$	σ_S	σ_H	σ_P
0.0084	8.42	-0.0011	(-0.0018, -0.0004)	0.2	0.2	0.2
0.0183	18.41	-0.0014	(-0.0029, -0.00009)	0.3	0.2	0.3
0.0024	2.41	-0.0008	(-0.0009, -0.0006)	0.1	0.2	0.1

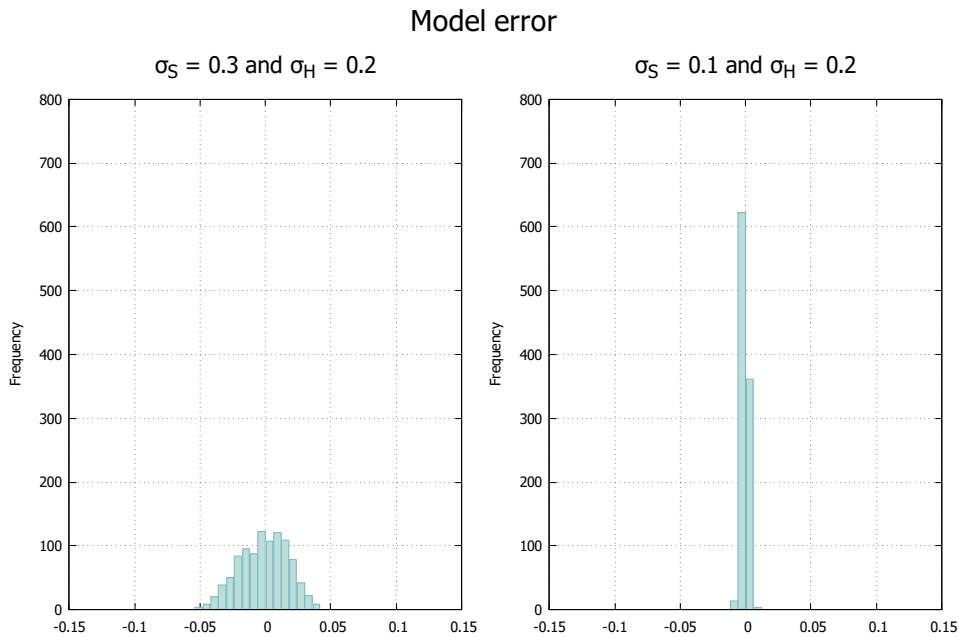


Figure 5.14: Hedging Error in CVA using $N = 1000$ MC simulations (both in the inner and outer loop) as the volatility σ_S used for the simulation changes.

From these results we can see that using the same volatility to simulate the market factors forward in time (both in the outer and inner MC loop) and a different one to hedge and price the portfolio doesn't affect the bias in the profit and loss distribution as much as the other model errors did. The biggest difference produced changing σ_S can be found in the variance of the P&L distribution.

Chapter 6

Discussion

6.1 Summary of findings

When hedging a portfolio we can analyse different types of hedging errors.

First of all we have seen (in Example 5.1) that, as it is impossible to hedge continuously, we are forced to rebalance the portfolio only at discrete time instants. This generates a time discretisation error, that decreases as we tend to the ideal continuous case described by Black-Scholes. The standard deviation of the profit and loss decreases as a constant times $1/\sqrt{n}$, where n refers to the number of intervals. This nonlinear behaviour of the error as a function of the number of intervals shows that even if you put a very high effort in increasing the number of time steps, you don't get the same high benefit in terms of error reduction. For this thesis we have chosen most of the times a number of intervals $n = 80$, which, fixing a time horizon of one year, corresponds to hedging each 4/5 days. We have not considered transaction costs in this thesis, but rebalancing frequently can be very expensive in reality, so the number of intervals can't usually be increased too much.

Analysing the hedging error for a European Put Option (Example 5.2) has been a key in this thesis, since it was a way to check the analytic formula of the hedging error

$$Z_T = \int_0^T e^{r(T-t)} \frac{1}{2} S_t^2 \Gamma_t (\sigma_H^2 - \sigma_S(t)^2) dt,$$

derived in Chapter 3. The numerical study confirmed the theoretical formula and its implications regarding the impacts in hedging error for a derivative instrument due to a model mismatch. This type of mismatch can arise for instance from a wrong calibration in the model parameters and leads to a positive (negative) hedging error if the volatility of the BS model used in the simulation of the underlying is smaller (larger). The volatility that we use to simulate affects also the variance of the P&L distribution, which gets larger if we increase the volatility used in the simulation model and smaller in the other case. Actually, we have observed that if the difference between the two volatilities is not so big (for instance 0.21 or 0.19 instead of 0.2), the hedging error is still present, according to the analytic formula, but is not as large as in the other analysed cases (when we changed the volatility to 0.3 and 0.1). This suggests that as long as you use a model that is very close to the real behaviour of the market factors, you could still hedge your portfolio. However it is recommended to use a correct model when possible since if, on the other hand, you are us-

ing a model which differs more from the real one, the hedging error can be very significant.

It is worth pointing out one more time what we have written in Chapter 3 after the derivation of the analytic formula: the fact that we get an a.s. positive (negative) hedging error doesn't mean we are creating an arbitrage. The point is that, for our purposes of analysing the effect in the model mismatch, we have assumed that we are able to sell the claim at the "hedging price", not the one computed using the dynamics of the underlying with parameters μ_S and σ_S . It is not an arbitrage but it is what the hedger would gain/lose if he managed to sell something at his estimated price on the market and hedging the position with his model and it shows very well the effects of a possible model mismatch.

The third case study (Example 5.3) represents the core of this thesis and showed that what we experienced for a derivative contract (in the Put example 5.2) where we considered only market risk, can be extended to its CVA as well: a model mismatch creates a bias in the profit and loss distribution of the hedging strategy. As in the previous example, we observe that the variance of the P&L distribution is reduced (increased) if we hedge with a model with lower (higher) volatility. However in the CVA case the overall variance is a bit reduced. This could intuitively be explained with the fact that, when we deal with CVA computations, we have an extra inner MC loop when we compute its price, so the MC variance of the outer simulation loop gets reduced as pointed out by Ruiz in [16]. The impact of the hedging error of the credit part is negligible (note that this could change if we used dynamic credit models, but the purpose of this thesis was, as previously remarked, to analyse mainly the error in the market risk hedging).

Example 5.4 is the case study that most stimulates ideas for further research. In practice it usually happens that banks have complex pricing models (corresponding to the real evolution of the market factors), but simpler models to carry out the simulations of the market factors. Here we price the portfolio with a model that we consider to be correct and simulate with a different one. We observe that, changing the volatility σ_P in the inner Monte Carlo loop, we obtain a significant bias in the profit and loss distribution, so this strategy has an impact on the hedging error.

With the final Example 5.5 (that actually represents a case that is not so common in practice) we see what happens when we use a correct simulation model, but a different one for the valuation of the portfolio when computing the CVA in each time step. The biggest difference produced by this model mismatch is in the variance of the P&L distribution and opposite to the results in the previous examples we do not get a significant bias.

6.2 Conclusions

We can conclude that hedging error, not only when hedging the CVA but also in general, can arise from different causes.

The discretisation has an impact on the standard deviation of the profit and loss distribution rather than on its mean. Reducing this error is not always easy. As a matter of fact, taking more intervals is usually too expensive in practice and it increases a lot the overall computational time of the code.

Regarding the first type of model error we have analysed, we have observed similar behaviours both in the case without and with the CVA. Using different models for simulation and hedging causes a bias in the hedging error as expected by the analytic formula. If the difference in the volatility of the two models is small it may still be possible to get a good hedge, but our results suggest that if we have access to the correct model for the evolution of the risk factors, it is recommended to use it. Also the case in which we price with a correct model and simulate with a different one, produces a hedging error that suggests it is better to use the the correct model if we have it.

When instead the two simulations of the market factors are carried out with the same model, different from the pricing model (Example 5.5) we do observe an error but not as significant as the one obtained in the other examples. This could be due to the fact that the two models we have used in this thesis are very similar (we have only changed the volatility parameter) so it seems it is still possible to hedge as long as the dynamics for the simulation is correct. However it is not obvious that we would get the same result if the models were more complex so further studies are needed for this type of model error.

6.3 Future Research

First of all, since we do not have a general closed formula for the hedging error when using one model for the simulation and another one to hedge and price the portfolio (Example 5.4 and 5.5), it could be interesting to develop these cases. It would be particularly interesting to develop the error due to the model mismatch presented in Example 5.4, that represents a common strategy adopted by the banks. For instance, what would happen if we relaxed the Black-Scholes assumption of constant volatility and used a stochastic volatility model for pricing but a BS model in the simulation?

Furthermore one could try to hedge the CVA for a basket of options or for a different financial derivative.

Naturally it would also be interesting to use real market data or to check for instance the performance of a real hedging strategy used by a bank, with their models for simulation and pricing.

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