Quantitative Portfolio Construction Using Stochastic Programming

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Abstract

In this study within quantitative portfolio optimization, stochastic programming is investigated as an investment decision tool. This research takes the direction of scenario based Mean-Absolute Deviation and is compared with the traditional Mean-Variance model and widely used Risk Parity portfolio. Furthermore, this thesis is done in collaboration with the First Swedish National Pension Fund, AP1, and the implemented multi-asset portfolios are thus tailored to match their investment style. The models are evaluated on two different fund management levels, in order to study if the portfolio performance benefits from a more restricted feasible domain. This research concludes that stochastic programming over the investigated time period is inferior to Risk Parity, but outperforms the Mean-Variance Model. The biggest flaw of the model is its poor performance during periods of market stress. However, the model showed superior results during normal market conditions.

Sammanfattning

I denna studie inom kvantitativ portföljoptimering undersöks stokastisk programmering som ett investeringsbeslutsverktyg. Denna studie tar riktningen för scenariobaserad Mean-Absolute Deviation och jämförs med den traditionella Mean-Variance-modellen samt den utbrett använda Risk Parity-portföljen. Avhandlingen görs i samarbete med Första AP-fonden, och de implementerade portföljerna, med flera tillgångsslag, är därför skräddarsydda för att matcha deras investeringsstil. Modellerna utvärderas på två olika fondhanteringsnivåer för att studera om portföljens prestanda drar nytta av en mer restriktiv optimeringsmodell. Den här undersökningen visar att stokastisk programmering under undersökta tidsperioder presterar något sämre än Risk Parity, men överträffar Mean-Variance. Modellens största brist är dess prestanda under perioder av marknadsstress. Modellen visade dock något bättre resultat under normala marknadsförhållanden.

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1 Introduction

No person or institution has the ability to accurately predict future market returns. Constructing the perfectly optimized portfolio with various assets over time is one of the most frequently discussed and written about topics in finance. It is almost impossible to predict the correct returns and most researches considers the case of predicting whether the market will go up or down during the following period.

This is a study within quantitative portfolio optimization investigating if stochastic programming can improve multi-asset portfolio performance. It is compared with other quantitative portfolio allocation models such as the traditional Markowitz's Mean-Variance model. This research takes the direction of scenario based Mean-Absolute Deviation, which with a rigorous mathematical framework efficiently comprehends complex decision making problems. The idea is to simulate various scenarios dynamically with different models in order to transform a stochastic problem into a deterministic one, where the objective is optimized by assigning weights to all included assets. Various methods of generating these scenarios are implemented and analyzed.

The Swedish National Pension Funds manage and allocate capital in the national income pension system. Everyone who works, receives a salary and pays tax in Sweden receives a general pension. This thesis is written in collaboration with the First Swedish National Pension Fund, AP1, one of five buffer funds which as the end of 2017 manages fund capital of $\sim$ SEK 330 billion. Investments are placed worldwide in equites, fixed income and currencies, but also in alternative investments such as real estate, hedge funds, private equity and venture capital funds.

The fund's objective is to maximize its return in relation to the riskiness of its placements. Risk awareness should be neutral between generations, meaning, risks are not to be minimized in the short term if it entails increased risk of adversely affected pensions in the future. Risk is pivotal to the institution's investment strategies as they hold the relatively modest long-term required annual return of 4%.

Personnel at AP1 have carefully selected 17 market indices on which this research is based on. Data has been collected from 2002-2018 and the indicis represent exposure to the fund's holdings as they fully capture its spectrum of assets. Furthermore, the portfolios are tailored with additional constraints and a transaction model to realistically represent the portfolio of AP1. All models are implemented and evaluated on two fund management levels with the aim to determine the optimal asset allocation. To compare the different models, the simulated portfolios are backtested against historical data, as well as tested on metrics such as Sharpe ratio.

1.1 Purpose

In this section the related research questions will be introduced. The objective of this thesis is to develop a reliable framework for stochastic programming in the purpose of portfolio optimization. The following main questions questions are to be answered

- Can stochastic programming improve the performance of a multi-asset portfolio?
  - The model is evaluated and compared with the conventional Markowitz's Mean-Variance model as well as the broadly used Risk Parity model
  - The analysis is performed in a multiperiod setting
- At which level of fund management is the use of optimizers suitable?
  - The aim is to investigate if quantitative allocation function better when the optimizers are free to work by themselves or constrained by investment limits based on rationale
  - The implemented models are evaluated on two levels, namely the minimum regulatory requirements and under the restrictions set up by the board of AP1

1.2 Outline

Chapter 2 introduces the reader to the mathematical background of the thesis. Important concepts on stochastic programming techniques and scenario generation are thoroughly introduced, as well as the dynamic framework that incorporates the portfolios. Chapter 3 describes the data processing and the general methodology of the thesis. Here, the models are implemented with detailed descriptions of all concerning aspects, e.g. transactions costs and specific constraints. Chapter 4 is devoted to showing results corresponding to the considered optimized portfolio models, while the following two chapters, Chapter 5 and Chapter 6, discusses and summarizes the results achieved in this thesis.

2 Theory

The theory implemented in this thesis is presented in the following section.

2.1 Notation and Definitions

In this section the notations used throughout the thesis are stated. Other definitions are introduced in the relevant chapters if necessary.

N - Number of risky assets in the portfolio

$\boldsymbol{w}$ - Vector of size N + 1, containing weights for risk-free asset and each of the N risky assets

w_i - Percentage weight corresponding to asset i

w_0 - Weight corresponding to risk-free asset

R_0 - Risk-free rate

$\boldsymbol{\Sigma}$ - Covariance matrix of asset returns

σ_i - Volatility of asset price i

$\boldsymbol{\mu}$ - Expected returns of risky assets

μ_i - Expected return of asset i

ρ - Correlation coefficient of asset returns

S - Number of scenarios

p_s - Probability of scenario s

$\boldsymbol{R}_s$ - Asset returns for scenario s

$\bar{\boldsymbol{R}}$ - Mean average asset returns across all scenarios

$\phi(\boldsymbol{x})$ - Transaction cost as a function of the amount transacted

T - Rebalancing period

2.2 Portfolio Weighting Strategies

2.2.1 Risk Parity

Two well-documented investment strategies are the Mean-Variance and the equally-weighted portfolios, with main drawbacks being portfolio concentration and limited diversification of risk, respectively. The equally-weighted portfolio is generally considered to be a naive investing approach, as portfolios are constructed by the simple measure of assigning equal weight to all assets [7]. The Mean-Variance approach asses a portfolio of which the expected return of its assembled assets are maximized at a certain level of risk. The method is developed as an extension to the theory focusing on the importance of diversification. An investment strategy that acts as middle ground between the two is called the Equally-weighted Risk Contributions (ERC), and is perhaps the most well-known version of Risk Parity. The main purpose of the strategy is to equalize risk contributions from the different components of the portfolio, i.e $\sigma_i(w) = \sigma_j(w)$, see below for definition. Handling risk properties in this manner has become increasingly popular and goes by the name *risk budgeting*, and the analysis focuses on risk contributions rather than portfolio weights as no asset contributes more than its peers to the total risk of the portfolio.

The marginal risk contribution, $\partial_{w_i}\sigma(w)$, is defined as

$$\partial_{w_i}\sigma(w) = \frac{\partial\sigma(w)}{\partial w_i} = \frac{w_i\sigma_i^2 + \sum_{j\neq i} w_j\sigma_{ij}}{\sigma(w)} \tag{2.2.1}$$

Then by definition, the total risk contribution of the i^{th} asset is given by (2.2.2). and the sum of the total risk contributions by (2.2.3)

$$\sigma_i(w) = w_i \cdot \partial_{w_i}\sigma(w) \tag{2.2.2}$$

Continuing, the sum of the total risk contributions is then given as

$$\sigma(w) = \sum_{i=1}^{n} \sigma_i(w) \tag{2.2.3}$$

From (2.2.3), it can be concluded that the risk of the portfolio is equal to the sum of the total risk contributions.

Assuming the scenario with equal correlation for every couple of variables, meaning $\rho_{i,j} = \rho$ for all i, j, the total risk contribution of component i becomes

$$\sigma_i(w) = \frac{w_i^2\sigma_i^2 + \rho\sum_{j\neq i} w_i w_j\sigma_i\sigma_j}{\sigma(w)} \tag{2.2.4}$$

The assumption of **equal correlation** coupled with the budget constraint of the portfolio weights equaling one, yields the following weights to the ERC portfolio

$$w_i = \frac{\sigma_i^{-1}}{\sigma_1^{-1} + ... + \sigma_n^{-1}} \tag{2.2.5}$$

In other words, each components' weight is deduced from the ratio of the inverse of its volatility and the harmonic average of the volatilities. The weight of each component i decreases with increased volatility.

While the previous solution measures each component's risk in relation to its corresponding portfolio, it does not provide a closed-form solution. In order to do so, the construction of a **numerical algorithm** is required. One approach to constructing the ERC portfolio is to consider the following optimization problem

$$\begin{aligned} \underset{\boldsymbol{y}}{\text{Minimize}} \quad & \sqrt{\boldsymbol{y}^T \Sigma \boldsymbol{y}} \\ \text{Subject to} \quad & \sum_{i=1}^{N} ln\, y_i \geq c, \quad \forall\, i \\ & y_i \geq 0, \quad \forall\, i \end{aligned} \tag{2.2.6}$$

where c is an arbitrary constant that corresponds to a particular value such that $-\infty \leq c \leq -N\ln(N)$. The first constraint implies sufficient diversification of weights, whilst the second implies the exclusion of short-selling. The portfolio weights are then obtained by normalizing the help variable $\boldsymbol{y}$, $\boldsymbol{w} = \frac{\boldsymbol{y}}{\sum_{i=1}^{N} y_i}$. Note that the weights corresponding to the risk free asset is not included here. If $\sum_{i=1}^{N} w_i < 1$, the rest of the capital is put to the risk free-asset such that $w_0 = 1 - \sum_{i=1}^{N} w_i$. Furthermore, the optimization problem in (2.2.6) yields a unique solution as long as the covariance matrix Σ is positive-definite [18, 14].

2.2.2 Mean-Variance

In 1952, Harry Markowitz published his seminal work on portfolio selection, in which he established a framework for investment decisions. The study has its limitations, but it was nonetheless groundbreaking for its day and awarded with the Nobel Prize 1990. This model for portfolio choice is known as the Mean-Variance model due to the fact that it optimizes the expected return and variance of the portfolio. The model can be formulated in various ways where one can weigh the model to maximize the expected return or minimize the risk. This trade-off between return and risk is regulated by the tuning parameter c in (2.2.7). This problem formulation includes a risk-free asset where the risk-free return is given by R_0. The corresponding weight is given by $w_0 = 1 - \boldsymbol{w}^T\mathbf{1}$ [11, 12].

$$\underset{\boldsymbol{w}}{\text{Minimize}} \quad \frac{c}{2}\boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w} - \boldsymbol{w}^T \boldsymbol{\mu} - w_0 R_0 \tag{2.2.7}$$

By varying the trade-off parameter c $\geq$ 0, one can obtain pairs $(\sigma_p(c), \mu_p(c))$ of the expected optimal portfolio values. The set of these optimal portfolios is called the efficient frontier. The frontier is a modern portfolio theory tool that represents the optimal parts of the risk-return spectrum. It is used to analyze portfolios to determine the best combination of underlying assets that has the best expected return for its level of risk. The pairs $(\sigma_p(c), \mu_p(c))$ that construct the efficient frontier are given by equation 2.2.8 [11, 12].

$$\begin{aligned} \sigma_p(c) &= (\boldsymbol{w}^{\boldsymbol{T}} \boldsymbol{\Sigma} \boldsymbol{w})^{1/2} \\ \mu_p(c) &= w_0 R_0 + \boldsymbol{w}^{\boldsymbol{T}} \boldsymbol{\mu} \end{aligned} \tag{2.2.8}$$

For practical implementation, the weights are functions of sample estimates of expected returns and covariances. The optimizer tend to pick the assets with very attractive features, i.e. high returns and low variances, and deselect those with the worst features. For this reason, the weights of the assets with high returns and low variances are likely overestimated. As a result the effect of the estimation error is maximized since these cases usually correspond to the highest estimation errors. This is a known flaw of Markowitz's model that is known as *error maximization* [12].

2.3 Stochastic Programming

There is often some degree of uncertainty or randomness to certain variables in mathematical modeling. An approach to overcome this predicament, or at least, to reduce the risk of misinterpreting the data is to perform stochastic programming. The method investigates how different solutions behave in various scenarios in order to find the optimal solution. By creating and simulating different scenarios for the given assets, the technique transforms the stochastic nature of the variables into a deterministic problem. The model is then left with the task of determining weights to the given assets in order to optimize the objective. The generated problem can now in most cases be solved by applying linear or quadratic programming techniques. Scenarios are created as to reflect likely future outcomes by change in input variables. A few number of scenarios may overadjust the data, while too many projected scenarios can result in a computationally intractable problem [8].

2.3.1 Scenario Properties

Scenario generation is the process of creating a finite set of scenarios which describe the distribution of the random parameters in the optimization formulation. This collection represents the stochastic model parameters where each scenario is a possible realization of them, which is weighted by its probability of occurrence. This process is related, but should not be confused with forecasting. Forecasting is the prediction of the most likely value for a random variable while a scenario set can be viewed as a density forecast [19].

Scenario optimization techniques are only useful if the simulated scenarios are of high quality. Some key aspects to keep in mind while conducting them are

- Parsimonious – the importance of creating a relatively small set of scenarios
- Representative – scenarios must represent a realistic form of the problem at hand
- Arbitrage-free – scenarios that create great arbitrage opportunities are regarded as unrealistic and should therefore be avoided
- Numerical stability - the optimal solutions for different scenario sets should not vary significantly

There are numerous approaches to generate scenarios. The most widely used generators are based on either historical observations or model-based methods that describe the price-dynamics of the assets. This report will focus on two techniques for scenario optimization, namely moment-matching and multivariate GARCH. [8, 10, 25, 20]

2.3.2 Moment-matching

The method of utilizing an algorithm for moment-matching scenario generation produces scenarios as well as corresponding probability weights that match the given mean, the covariance matrix, the average of the marginal skewness and the marginal kurtosis of each component of a random vector. This algorithm has two major advantages over other scenario generating methods:

1. Computationally modest - the moment-matching model excludes optimisation in contrast to other similar methods
2. The generated scenarios come with corresponding probability weights, making it unnecessary to attach user-defined probabilities and saving time for solving the stochastic program

The method generates $2Ns+3$ scenarios and their corresponding probabilities. Here, n is the dimension of the random vector and s is an arbitrary positive integer. The three extra scenarios are generated in order to match the average skewness and the average marginal kurtosis for each component of the random vector r with dimension N. The mean vector $\boldsymbol{\mu}$, the covariance matrix $\boldsymbol{\Sigma}$ and the marginal third and forth central moments, κ_j ζ_j are used as inputs for the algorithm.

The average marginal moments are defined as

$$\frac{1}{N}\sum_{j=1}^{N}\kappa_j = \boldsymbol{\kappa}, \qquad \frac{1}{N}\sum_{j=1}^{N}\zeta_j = \boldsymbol{\zeta}$$

The algorithm begins with the choice of arbitrary positive integer s, an arbitrary non-zero deterministic such that $\boldsymbol{\Sigma}-\boldsymbol{Z}\boldsymbol{Z}^T > 0$ and a scalar $p \in (0,1)$. The remaining parameters to be calculated then are

$$\alpha = \frac{1}{2}\phi_1 + \frac{1}{2}\sqrt{4\phi_2 - 3\phi_1^2}, \tag{2.3.1}$$

$$\beta = -\frac{1}{2}\phi_1 + \frac{1}{2}\sqrt{4\phi_2 - 3\phi_1^2}, \tag{2.3.2}$$

$$w_0 = 1 - \frac{1}{\alpha\beta}, \tag{2.3.3}$$

$$w_1 = \frac{1}{\alpha(\alpha+\beta}, \tag{2.3.4}$$

$$w_2 = \frac{1}{\beta(\alpha+\beta}, \tag{2.3.5}$$

$$\gamma = 2s^2\frac{N\boldsymbol{\zeta} - \frac{3}{4}\sum_{j=1}^{N} Z_j^4\left(\frac{N\boldsymbol{\kappa}}{\sum_{i=1}^{N} Z_j^3}\right)^2}{\sum_{l,k} L_{lk}^4}, \tag{2.3.6}$$

where α and β are the symmetric parameter and w_0, w_1 and w_2 are coefficients. The upper constraint γ is set to make certain of real numbers for α and β for the expression under square root. $\boldsymbol{L}$ is a positive definite matrix such that $\boldsymbol{\Sigma} = \boldsymbol{L}\boldsymbol{L}^T + \boldsymbol{Z}\boldsymbol{Z}^T$ holds. Furthermore, the auxiliary parameters, ϕ_1 and ϕ_2, are defined as

$$\phi_1 = \frac{N\boldsymbol{\kappa}\sqrt{p_{s+1}}}{\sum_{j=1}^{N} Z_j^3}, \tag{2.3.7}$$

$$\phi_2 = p_{s+1}\frac{N\boldsymbol{\zeta} - \frac{1}{2s^2}\sum_{l,k} L_{lk}^4\left(\sum_{i=1}^{s}\frac{1}{p_i}\right)}{\sum_{j=1}^{N} Z_j^4}, \tag{2.3.8}$$

The probability weights can now be generated. First off, real scalars $p_i \in (0,1)$ for $i = 1, 2, ..., s$ are created under the conditions

$$\sum_{i=1}^{s} p_i < \frac{1}{2n} \tag{2.3.9}$$

$$\sum_{i=1}^{s} \frac{1}{p_i} < \gamma \tag{2.3.10}$$

$$p_{s+1} = 1 - 2N \sum_{i=1}^{s} p_i, \tag{2.3.11}$$

where the probability weights of the first $2Ns$ scenarios are represented by p_i and the additional three scenarios' weights are represented by p_{s+1}, corresponding to those of higher order moments. Equation 2.3.9 restricts the probabilities to make certain of real numbers of α and β as previously mentioned.

With the use of p_1, $p_2 ... p_{1+s}$ and equations 2.3.3-2.3.5, the following weights are generated

$$P = \{p_1, p_2, ..., p_s, p_1, p_2, ..., p_s, ..., p_{s+1}w_o, p_{s+1}w_1, p_{s+1}w_2\} \tag{2.3.12}$$

In the final step, support points r_k of the multivariate distribution of returns are generated. $S = 2Ns+3$ support points are created using their respective probability weights P_k from equation (2.3.12). The procedure is conducted as follows

$$P\Big(r_{(i-1)N+c} = \mu + \frac{1}{\sqrt{2sp_i}}\boldsymbol{L_c}\Big) = p_i = P_{i+s(c-1)}, \tag{2.3.13}$$

$$P\Big(r_{(s+i-1)N+c} = \mu - \frac{1}{\sqrt{2sp_i}}\boldsymbol{L_c}\Big) = p_i = P_{i+s(N+c-1)}, \tag{2.3.14}$$

$$P(r_{2Ns+1} = \mu) = p_{s+1}w_0 = P_{2Ns+1}, \tag{2.3.15}$$

$$P\Big(r_{2Ns+2} = \mu + \frac{\alpha}{\sqrt{p_{s+1}}}\boldsymbol{Z}\Big) = p_{s+1}w_1 = P_{2Ns+2}, \tag{2.3.16}$$

$$P\Big(r_{2Ns+3} = \mu + \frac{\beta}{\sqrt{p_{s+1}}}\boldsymbol{Z}\Big) = p_{s+1}w_2 = P_{2Ns+3}, \tag{2.3.17}$$

where $i = 1,2, ..., s$, $c = 1,2, ..., N$ and $\boldsymbol{L_c}$ is the c:th column of $\boldsymbol{L}$. Vector r_k yields the outcome of the scenario generation algorithm from the $2Ns + 3$ support points, shown below

$$r_k = (r_1, r_2, ..., r_{2Ns+1}, r_{2Ns+2}, r_{2Ns+3}) \tag{2.3.18}$$

This is the final step of the algorithm [22].

2.3.3 Multivariate GARCH

Generalized autoregressive conditional heteroskedasticity, or GARCH, is a method of estimating the stylized features of a return process $\{Z_t\}$. It has been proven to be successful in determining volatility as well as estimating future volatility. The main use of the model is to predict future volatility $k-$step ahead, i.e. h_{t+k}. Volatilities in the financial market are generally known to move closely together over time. The covariance matrix among assets can be estimated by the univariate GARCH model, but by extending it to a multivariate GARCH model better decision tools can be obtained, especially in financial applications in areas such as hedging and portfolio selection [5].

In the multivariate case, a GARCH model is defined as

$$\boldsymbol{r}_t = \boldsymbol{\mu}_t + \boldsymbol{a}_t \tag{2.3.19}$$

$$\boldsymbol{a}_t = \boldsymbol{H}_t^{1/2}\boldsymbol{z}_t, \tag{2.3.20}$$

where $\boldsymbol{r}_t$ is a $n \times 1$ vector of returns at time t, $\boldsymbol{a}_t$ is a $n \times 1$ vector of mean-corrected returns of n assets at time t and $\boldsymbol{\mu}_t$ is a $n \times 1$ vector of the expected value of the conditional r_t. Then the conditional variances of $\boldsymbol{a}_t$ at time t is given by the $n \times n$ matrix $\boldsymbol{H}_t$, where $\boldsymbol{H}_t^{1/2}$ is obtained by Cholesky factorization of $\boldsymbol{H}_t$. Lastly, $\boldsymbol{z}_t$ is given as a $n \times 1$ vector of IID errors such that $E[\boldsymbol{z}_t] = 0$ and $E[\boldsymbol{z}_t\boldsymbol{z}_t^T] = I$ [9].

2.3.4 DCC-GARCH

Models of conditional variances and correlations are built on the thesis of modelling the conditional variances and correlations rather than the corresponding conditional covariance matrix. The basic idea behind the Dynamic Conditional Correlation (DCC) GARCH model is hence to decompose the covariance matrix, $\boldsymbol{H}_t$ into conditional standard deviations, $\boldsymbol{D}_t$ and a correlation matrix, $\boldsymbol{R}_t$. They are both constructed to be time-varying variables.

Given returns $\boldsymbol{a}_t$ from n assets where $E[\boldsymbol{a}_t] = 0$ and with covariance matrix $\boldsymbol{H}_t$, the DCC-GARCH model is given by

$$\boldsymbol{r}_t = \boldsymbol{\mu}_t + \boldsymbol{a}_t \tag{2.3.21}$$

$$\boldsymbol{a}_t = \boldsymbol{H}_t^{1/2}\boldsymbol{z}_t \tag{2.3.22}$$

$$\boldsymbol{H}_t = \boldsymbol{D}_t\boldsymbol{R}_t\boldsymbol{D}_t, \tag{2.3.23}$$

where $\boldsymbol{r}_t$, $\boldsymbol{a}_t$, $\boldsymbol{\mu}_t$, $\boldsymbol{H}_t$ and $\boldsymbol{z}_t$ are given as before. Here, $\boldsymbol{D}_t$ is a $n \times n$ diagonal matrix of conditional standard deviations of $\boldsymbol{a}_t$ at time t, whereas $\boldsymbol{R}_t$ is a $n \times n$ conditional correlation matrix of $\boldsymbol{a}_t$ at time t.

The diagonal matrix $\boldsymbol{D}_t$ is constructed of standard deviations from univariate GARCH models, such that

$$\boldsymbol{D}_t = \begin{bmatrix} \sqrt{h_{1t}} & 0 & \dots & 0 \\ 0 & \sqrt{h_{2t}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \sqrt{h_{nt}} \end{bmatrix}$$

where,

$$h_{it} = \alpha_{i0} + \sum_{q=1}^{Q_i} \alpha_{iq} a_{i,t-q}^2 + \sum_{p=1}^{P_i} \beta_{ip} h_{i,t-p} \tag{2.3.24}$$

A simplification of the model is when the conditional correlation matrix is time invariant i.e. [9],

$$\boldsymbol{H}_t = \boldsymbol{D}_t \boldsymbol{R} \boldsymbol{D}_t \tag{2.3.25}$$

2.3.5 Student's t Copula

In the case of constructing a multivariate model, suppose a random vector $\boldsymbol{X} = (X_1, ..., X_d)$ with non-trivial dependence among components and with marginal distribution functions defined as $F_1, ..., F_d$. Then $\boldsymbol{X}$ can with quantile transformation be defined as

$$\boldsymbol{X} = (F_1^{-1}(U_1), ..., F_d^{-1}(U_d)) \tag{2.3.26}$$

where the vector $\boldsymbol{U} = (U_1, ..., U_d)$ has uniformly distributed components on (0,1). The copula $C^t_{\nu,\boldsymbol{R}}$ of a d-dimensional standard Student's t distribution with $\nu > 0$ degrees of freedom and linear correlation matrix $\boldsymbol{R}$ is the distribution of the random vector $(t_\nu(X_1), \dots, t_\nu(X_d))$, where $\boldsymbol{X}$ has a $t_{d(\mathbf{0},\boldsymbol{R},\nu)}$ distribution and t_ν is the univariate standard Student's t_ν distribution function. Thus we get the Student's t copula as

$$C^t_{\nu,\boldsymbol{R}}(\boldsymbol{u}) = P(t_\nu(X_1) \leq u_1, \dots, t_\nu(X_d) \leq u_d) = t^d_{\nu,\boldsymbol{R}}(t_\nu^{-1}(u_1), \dots, t_\nu^{-1}(u_d)), \tag{2.3.27}$$

where $t^d_{\nu,\boldsymbol{R}}$ is the distribution function of $\mathbf{X}$ [12].

2.3.6 Mean-Absolute Deviation

As opposed to the Mean-Variance model where risk is a squared deviation, the Mean-Absolute Deviation model measures the risk as an absolute deviation from the mean. It is defined as

$$\begin{aligned} \underset{\boldsymbol{w}}{\text{Minimize}} \quad & MAD \\ \text{subject to} \quad & w_0 R_0 + \boldsymbol{w}^T \bar{\boldsymbol{R}} \geq R_{min} \end{aligned} \tag{2.3.28}$$

where

$$MAD = \sum_{s=1}^{S} p_s |\boldsymbol{w}^T (\boldsymbol{R}_s - \bar{\boldsymbol{R}})|. \tag{2.3.29}$$

Here, p_s denotes the probability of scenario s, $\mathbf{w}$ denotes the vector of portfolio weights, $\mathbf{R_s}$ denotes the vector of returns in scenario s and $\bar{\mathbf{R}}$ represents the vector of return means for all given scenarios [25].

The underlying distribution of asset returns often exhibit heavy tails, which indicates non-normality. Unlike Mean-Variance, mean-absolute deviation as a sample statistic measure, does not assume normally distributed sample population [26].

2.4 Dynamic Portfolio Construction

To maintain the investment discipline over time rebalancing must be considered. This means that the assets of the portfolio are periodically bought and sold. Henceforth, the portfolio selection strategies described in section 2.2 shall be reformulated to match a multi-period setting.

At each time the portfolio is rebalanced, transaction costs are particularly important since the possible return is penalized twice. First when assets are sold to finance the new investments and then when the new securities are bought. To capture this in the weighting strategies, a transaction-cost function is introduced to the allocation models [4].

2.4.1 Incorporating Transaction Costs

A trade should only be transacted if the investor believes that it will generate greater return than the cost of it. By introducing transaction costs to the model, a more realistic decomposition of assets is obtained by penalizing those associated with high costs. This is especially important for the multi-period model, where each rebalancing of the portfolio is associated with buying and selling [4].

Typically transaction fees can firstly be divided into two parts, fixed and variable costs. The fixed costs are those which do not depend on transacted amount, e.g. broker fees [21, 15].

Assuming the transaction costs are separable, the total cost is obtained as the sum of the transaction costs associated with each trade, i.e.

$$\phi(\boldsymbol{x}) = \sum_{i=1}^{N} \phi_i(x_i) = \begin{cases} \sum_{i=1}^{N} \phi_{v,i}(x_i) + \beta_i^+ \\ \sum_{i=1}^{N} \phi_{v,i}(x_i) + \beta_i^- \end{cases} \tag{2.4.1}$$

where $\boldsymbol{x} = (x_1, ..., x_N)^T$ is the amount transacted, $\beta_i^{+/-}$ and $\phi_{v,i}(x_i)$ are the fixed and variable costs respectively. Note that β_i^+ is the fixed cost for buying an asset and β_i^- the fixed cost of selling. Accordingly to the asset type, the costs are modeled as a proportion of the transaction in this thesis [24]. Further information about the assets can be found in section 3.1 and A.1.

$$\phi_{v,i}(x_i) = \begin{cases} \alpha_i^+ x_i, & x_i \geq 0 \\ -\alpha_i^- x_i, & x_i \leq 0 \end{cases} \tag{2.4.2}$$

Furthermore, the choice of a transaction model can affect the feasibility of the optimization problems. Both the objective function to be optimized and the set over which the optimization is performed, must all be convex. Therefore, it is required that the transaction cost function, 2.4.1, is convex to be directly solvable by linear or quadratic programming [17]. .

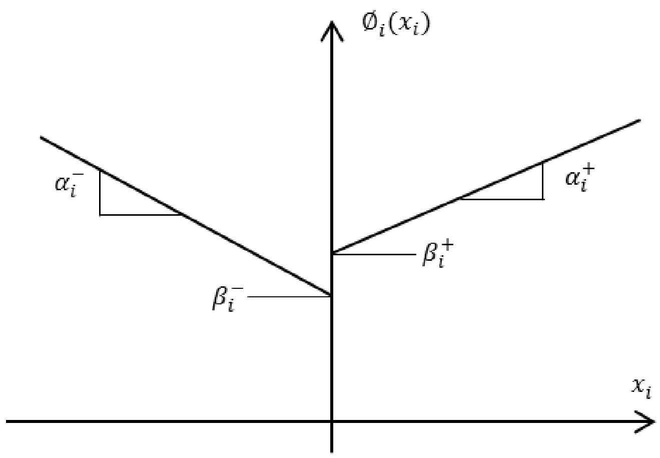

Figure 2.1: Transaction cost as a piece-wise linear function with non-zero fixed costs.

Consider a linear variable cost function, then the components of 2.4.1 are given by

$$\gamma_{v,i}(x_i) = \begin{cases} \beta_i^- - \alpha_i^- x_i, & x_i < 0 \\ 0, & x_i = 0 \\ \beta_i^+ + \alpha_i^+ x_i, & x_i > 0. \end{cases} \tag{2.4.3}$$

Clearly, this function is not convex, unless the fixed cost is set to zero. Therefore, the transaction model used in this thesis will assume $\beta^+ = \beta^- = 0$. How the cost coefficients α_i are set can be read in section 3.4.

2.4.2 Rebalancing of Portfolio Weights

The dynamic portfolio selection model consists of a set of the optimization problems in section 2.2 and 2.3, each at a different point in time with period.

Consider a portfolio which is rebalanced in the beginning of each period. If the initial investments are performed at $t = t_0$, the consecutive weights are computed at the discrete points of time $t_n = t_0 + nT$ for n = 0,1,...,M, where M is the total number of periods. The corresponding portfolio weights are denoted by $\boldsymbol{w}(t_n) = \boldsymbol{w}_n$. At each rebalancing stage, a budget constraint must be satisfied such that the rebalancing is self-financing. This constraint applies to all models and is given by the equation below [3, 17].

$$\mathbf{1}^T(\boldsymbol{w}_n - \boldsymbol{w}_{n-1}) + \phi(\boldsymbol{w}_n - \boldsymbol{w}_{n-1}) \leq 0. \tag{2.4.4}$$

Risk Parity

Since the aim of the Risk Parity portfolio is to assign equal risk to all assets the transaction costs do not affect the objective function of the optimization problem. However, the self-financing condition 2.4.4 is added as a constraint to the original problem. This means that 2.2.6 is translated to

$$\begin{aligned} \underset{\boldsymbol{y}_n}{\text{Minimize}} \quad & \sqrt{\boldsymbol{y}_n^T \boldsymbol{\Sigma}_n \boldsymbol{y}_n} \\ \text{Subject to} \quad & \sum_{i=1}^{N} ln\, y_{n,i} \geq c, \quad \forall\, i \\ & y_{n,i} \geq 0, \quad \forall\, i, \\ & \mathbf{1}^T(\boldsymbol{w}_n - \boldsymbol{w}_{n-1}) + \phi(\boldsymbol{w}_n - \boldsymbol{w}_{n-1}) \leq 0 \end{aligned} \tag{2.4.5}$$

in a multi-period setting. The corresponding weights are obtained in the same way as the single-period model, i.e [3],

$$\boldsymbol{w_n} = \frac{\boldsymbol{y_n}}{\mathbf{1}^T \cdot \boldsymbol{y_n}}. \tag{2.4.6}$$

Mean-Variance

In contrast to Risk Parity, the Mean-Variance model has an objective function dependent on the expected return. This means that the transaction cost must be subtracted from the expected return in equation 2.2.7 as well as adding 2.4.4 as a constraint.

$$\begin{aligned} \underset{\boldsymbol{w}_n}{\text{Minimize}} \quad & \frac{c}{2}\boldsymbol{w}_n^T\boldsymbol{\Sigma}_n\boldsymbol{w}_n - \boldsymbol{w}_n^T\boldsymbol{\mu}_n - w_{n,0}R_{n,0} + \phi(\boldsymbol{w}_n - \boldsymbol{w}_{n-1}) \\ \text{Subject to} \quad & \mathbf{1}^T(\boldsymbol{w}_n - \boldsymbol{w}_{n-1}) + \phi(\boldsymbol{w}_n - \boldsymbol{w}_{n-1}) \leq 0. \end{aligned} \tag{2.4.7}$$

The dynamic portfolio selection formulation according to Markowitz's model is then given by equation 2.4.7 above.

Mean-Absolute Deviation

In a similar fashion as for the Mean-Variance model, the transaction cost is subtracted from the expected return in the constraint of the optimization problem. More specifically, the formulation of this particular investment decision problem transforms into [16]

$$\begin{aligned} \underset{\boldsymbol{w}_n}{\text{Minimize}} \quad & \sum_{s=1}^{S} p_{n,s}|\boldsymbol{w}_n^T(\boldsymbol{R}_{n,s} - \bar{\boldsymbol{R}}_n)| \\ \text{subject to} \quad & w_0R_0 + \boldsymbol{w}_n^T\bar{\boldsymbol{R}}_n - \phi(\boldsymbol{w}_n - \boldsymbol{w}_{n-1}) \geq R_{min} \\ & \mathbf{1}^T(\boldsymbol{w}_n - \boldsymbol{w}_{n-1}) + \phi(\boldsymbol{w}_n - \boldsymbol{w}_{n-1}) \leq 0. \end{aligned} \tag{2.4.8}$$

2.5 Portfolio Evaluation

2.5.1 Sharpe Ratio

The Sharpe Ratio is a way of evaluating an investment's performance by taking its risk in consideration. Pension fund investors are generally risk averse. When faced with two investments with similar expected returns, the risk averse investor almost certainly chooses the portfolio with less risk. The Sharpe Ratio helps investors evaluate portfolios with different expected returns and levels of risk. It is defined as

$$SR = \frac{E[R_P^A - r_F^A]}{\sigma_P^A} \tag{2.5.1}$$

where R_P^A is the annual asset return, r_F^A is the risk-free rate and σ_P^A the standard deviation of the portfolio's returns. $E[R_P^A - r_F^A]$ is then a measure of the expected excess return of the portfolio, given a risk-free benchmark level. In general, the attractiveness of the risk-adjusted return increases with the value of the Sharpe Ratio [12].

2.5.2 Value-at-Risk

The Value-at-Risk (VaR) is in general terms defined as a measurement of potential loss for investments. In other words, VaR indicates a quantile of the probability distribution for a portfolio's potential loss given a specific time period.

The Value-at-Risk of a portfolio with value X at time 1 at level $p \in (0,1)$ is given as

$$\mathrm{VaR}_p(X) = \min\{m : \mathrm{P}(mR_0 + X < 0) \leq p\}, \tag{2.5.2}$$

where R_0 is the percentage return of a risk-free asset. It can also be expressed as

$$\mathrm{VaR}_p(X) = F_L^{-1}(1-p), \tag{2.5.3}$$

where $L = -X/R_0$ [12].

3 Data and Methodology

3.1 Data

The data used in the analysis is provided by AP1 which all is obtained from Bloomberg. The basket of assets of the fund's portfolio is represented by 17 indices which are presented in Table 3.1. Additional to the indices below the 3 month rate for SEK, USD, EUR and JPY is used to correct the portfolio for FX hedging costs which is further described in section 3.3. See Appendix A.1 for further information regarding the assets of the portfolio.

Weigth	**Asset**	**Currency**
w_0	Bloomberg Barclays 1-3 Month T-Bill ETF	USD
w_1	Bloomberg Barclays US Govt	USD
w_2	Swedish Government Bond Total Index	SEK
w_3	Bloomberg Barclays US Corporate High Yield	USD
w_4	S&P 500	USD
w_5	OMX Stockholm 30	SEK
w_6	Topix Index	JPY
w_7	Russell 2000	USD
w_8	MSCI World	USD
w_9	MSCI Emerging Markets	USD
w_{10}	MSCI World Small Cap	USD
w_{11}	MSCI Sweden Small Cap	SEK
w_{12}	US REIT	USD
w_{13}	LPX Composite	EUR
w_{14}	USD/SEK	-
w_{15}	EUR/SEK	-
w_{16}	JPY/SEK	-

Table 3.1: Portfolio assets. Datasource: Bloomberg.

The analysis is performed using *R 3.5.0*, *Matlab R2017a* and *Python 3.6.4* with packages *numpy*, *scipy*, *pandas* and *pickle*.

3.2 Additional Constraints

The models described in section 2 are all optimization-problems with constraints. To better match the requirements that asset managers face, additional constraints must be considered. The regarded constraints in this thesis are added to tailor the portfolio such that it matches AP1's investment style and complies with regulatory requirements.

Long only Portfolio

Real money investors hold the securities they buy and are generally long only on their investments. Therefore, the long-only constraint,

$$w_i \geq 0, \quad \forall i, \tag{3.2.1}$$

will apply for all portfolios.

Exposure Limits

AP1 are obligated to follow APL (AP-fund-law), a law regulating the allocations of the fund. APL specifies requirements on the allocations between asset classes. This is the minimum regulatory requirements and will be the first level of fund management that will be considered. Applied on the assets in Table 3.1, APL translates to the following constraints

$$\begin{aligned} w_1 + w_2 + w_3 &\geq 0.3 \\ w_{13} &\leq 0.05 \\ w_{14} + w_{15} + w_{16} &\leq 0.4. \end{aligned} \tag{3.2.2}$$

Moreover, the asset allocation of AP1 is decided by different decision levels. Based on the overall asset allocation, exposure limits are specified for asset classes as well as for illiquid investments. These exposure limits are defined by the Board's risk preference, henceforth denoted BRP. The BRP also determines AP1's strategic long term benchmark based on its investment beliefs and general guidelines [2]. This is the second level of fund management that will be analyzed in this thesis.

In terms of the optimization problems specified in section 2, these requirements translates to the constraints below.

$$\begin{aligned} 0.25 \leq w_1 + w_2 + w_3 + w_4 &\leq 0.35 \\ 0.30 \leq w_5 + w_6 + w_7 + w_8 + w_9 + w_{10} + w_{11} &\leq 0.55 \\ w_{12} &\leq 0.15 \\ w13 &\leq 0.05 \\ 0.15 \leq w_{14} + w_{15} + w_{16} &\leq 0.38 \end{aligned} \tag{3.2.3}$$

In this research, the portfolio optimization will be constructed using constraints represented by both 3.2.2 and 3.2.3, in order to review the effect of the additional constraints.

Optimal Currency Hedge Ratio

In reality, the investments are made in the currencies of each local asset holding. These positions can thereafter be hedged to limit the risk of undesired moves in the currency rates. Due to the fact that the weights corresponding to the given FX-rates represent the unhedged positions in the portfolio one can easily integrate the currency exposure to the optimization problem. This thesis is written in collaboration with AP1 and the analyzed portfolios must therefore replicate their actual portfolio as good as possible. This is the **only** reason why this constraint is added, such that the analysis is of value for the fund. The portfolio results will be evaluated as a whole and the optimal currency hedge ratio will not be considered alone.

By simply modifying the constraints such that the sum of the weights is less or equal to a number greater than one, the optimal currency exposure is jointly determined with the portfolio's local asset holdings. Thus, the optimal hedge ratio is determined by adding the constraints

$$\begin{aligned} \sum_{i=1}^{N} w_i &\leq 1.4 \\ \sum_{i=1}^{N-n} w_i &\leq 1.0 \\ \sum_{i=n}^{N} w_i &\leq 0.4, \end{aligned} \tag{3.2.4}$$

where n is the number of currencies [23]. The currency pairs used in this thesis can be found in Table 3.1.

Note that the weights sum up to 1.4 to comply with APL specified in equation 3.2.2. This means that the maximum currency exposure is 40 %, i.e. 60 % of the portfolio is FX hedged as a minimum. Since the sum of the weights are greater than one, one must not forget to normalize the return.

3.3 Data Processing

Before the analysis can be performed the data needs to be modified. Firstly, a risk-free interest rate must be determined such that cash can be present in the portfolio. The second constraint in equation 3.2.4 states that the sum of the assets in the portfolio must be less or equal to one. In other words, if those weights do not sum up to one, the containing capital is kept in cash, which yields the risk-free return.

In theory this rate is the rate of return of a hypothetical investment with no risk of financial loss, over a given period of time. To infer the risk-free interest rate in the optimization problems in section 2, a money-market ETF is used, corresponding to weight w_0 in Table 3.1. The ETF consists of US Treasury Bills with maturities between 1-3 months. Since the default risk of the US government is so low it can be negligible, it seems like a plausible choice to approximate the risk-free asset with this index.

Equally important is the fact that the variance of a risk-free asset and correlations with risky assets are all equal to zero. As can bee seen in Appendix A.2, the correlations with the risky assets are all low and the variance is obtained to $1.0941 \cdot 10^{-7}$. Hence, it is seems that the choice of asset corresponding to w_0 is a credible approximation of a risk-free asset.

Secondly, the time series have to be corrected for currency hedging costs. Accordingly to the constraints 3.2.4, one can assume that the portfolio can be divided into two parts; the fully FX-hedged portfolio and the currency exposure. Here, weight w_1-w_{13} corresponds to the fully hedged portfolio while w_{14}-w_{16} resembles the proportion of the holdings that are left unhedged. Because of this assumption, the time series have to be corrected for the cost of hedging. To obtain the final time series the corrections in the table below are performed.

Table 3.2: To obtain the time series for the currency hedged portfolio, the original series are corrected through short rates in the affected currencies.

Weigth	**Correction**
w_0	-
w_1	+ 3m STIBOR - 3m USD LIBOR
w_2	-
w_3	+ 3m STIBOR - 3m USD LIBOR
w_4	+ 3m STIBOR - 3m USD LIBOR
w_5	-
w_6	+ 3m STIBOR - 3m JPY LIBOR
w_7	+ 3m STIBOR - 3m USD LIBOR
w_8	+ 3m STIBOR - 3m USD LIBOR
w_9	x(SEK/USD)
w_{10}	+ 3m STIBOR - 3m USD LIBOR
w_{11}	-
w_{12}	+ 3m STIBOR - 3m USD LIBOR
w_{13}	+ 3m STIBOR - 3m EURBIOR
w_{14}	+ 3m USD LIBOR - 3m STIBOR
w_{15}	+ 3m EURIBOR - 3m STIBOR
w_{16}	+ 3m JPY LIBOR - 3m STIBOR

STIBOR (Stockholm Interbank Offered Rate), LIBOR (London Interbank Offered Rate) and EURIBOR (Euro Interbank Offered Rate) are reference rates based on averaged interest rates at which banks in each respective market are willing to lend funds to another, without collateral, at different maturities.

The following and final step of correcting the time series is to handle missing values. Considering that the availability of data varies between the indices, a start date is chosen such that the data is aligned. In this case the start date is set to 2002-01-07 since from that date forward, weekly data is available for all assets. However, some values are still missing for various reasons and can cause issues in the computations to be performed. Therefore, missing values are handled by linear interpolation using adjacent data points.

3.4 Transaction Cost Coefficients

In order to evaluate the performance of each investment strategy with all different assets, one needs to not only consider the returns of the individual assets, but also the transaction costs that accompanies each rebalancing occasion. For the purpose of this thesis, interviews have been conducted with asset managers at AP1 to get a better understanding of how to most correctly assign transaction coefficients to each of the different asset indices.

Table 3.3: Datasource: The Journal of Portfolio Management [13].

Asset Class	**Time Period**	**Transaction costs**
Equities	1880-1992	0.34%
	1993-2002	0.11%
	2003-2016	0.06%
Bonds	1880-1992	0.06%
	1993-2002	0.02%
	2003-2016	0.01%
Commodities	1880-1992	0.58%
	1993-2002	0.19%
	2003-2016	0.10%
Currencies	1880-1992	0.18%
	1993-2002	0.06%
	2003-2016	0.03%

In addition, Hurst B. et.al (2017) recently made a study on trend-following investment strategies, where Table 3.3 was used to describe different asset classes' simulated transaction costs. These costs are based on proprietary estimates made in 2012, of average transaction costs for each asset class specified including both market impact and commissions.

Equities

The S&P 500 index is constructed to measure performance of the US economy as it weighs 500 large US equities representing all major industries. With the free float-adjusted market capitalization representing over 90% of the index, S&P is considered to have the low transaction coefficient of 0.05%.

The MSCI World equity index consists of large and mid-cap equity performances across 23 developed markets. With approximately 85% of the free float market capitalization of each country and the exclusion of emerging markets, MSCI World is assigned the transaction cost coefficient of 0.06%. MSCI Emerging Markets index consists of 24 countries representing 10% of the world market capitalization with the free float-adjusted market capitalization exceeding 80%. Due to the increased risk of securities in emerging markets, the index is assigned the higher transactions cost coefficient of 0.10%. Furthermore, the MSCI World Small Cap index captures small cap representation in 23 developed markets. With only 14% of the free float-adjusted market capitalization of each country, and lower overall trading volumes that smaller companies endure, the index is assigned transaction cost coefficient of 0.08%. By the same token, the MSCI Sweden Small Cap index with its largest contributor only carrying a market capitalization of USD 6.7 billion, the index is assigned the relatively high transaction cost coefficient of 0.14%.

The Japanese stock market is represented by Topix Index, a metric that lists all of the largest companies (firms in the *first section* of the Tokyo Stock Exchange) in the country. The transaction coefficient is set to 0.06%. Finally, the Russell 2000 index represents the bottom 2000 stocks in the Russell 3000 index (representing the 3000 largest companies in the US). The index serves as benchmark for small-cap stocks in the US, and is assigned the transaction cost coefficient 0.07%.

Bonds

The Bloomberg Barclays US Govt is assigned transaction cost value 0.01%, as determined by Hurst B. et.al. [13]. Considering global presence and higher volumes transacted in this than in Swedish Government Bond Total Index, the latter is assigned the higher coefficient of 0.05%. Finally, Bloomberg Barclays US Corporate High Yield is assigned the highest coefficient in this asset class, 0.06%.

Alternative Investments

The LPX Composite represents all listed private equity companies that fulfill some explicit liquidity constraints. The second index in this asset class, US REIT, is an index constructed to capture the investable universe of publicly traded property companies. Although these indices represent sections of the stock market, they serve as proxies for investments in illiquid assets, e.g. apartments or land. Consequently, the transaction cost model intents to penalize these assets as they are assigned higher transaction cost coefficients. US REIT receives coefficient 0.20%, while LPX Composite that represents more complex and illiquid investments in private equity is assigned coefficient 0.30%.

Currencies

Since the weights w_{14}, w_{15} and w_{16} are representing the open currency exposure of the portfolio rather than positions in the currency pairs specified in Table 3.1, the corresponding cost coefficients are set to zero.

All finalized transaction cost values are presented in Table 3.4.

Table 3.4: Transaction cost coefficients used in this thesis.

	Simulated Transaction Costs	
Asset Class	**Asset**	**Transaction costs**
Bonds	Bloomberg 1-3 Month T-Bill ETF	0.00%
	Bloomberg US Govt	0.01%
	Swedish Government Bond Index	0.05%
	Bloomberg US Corporate High Yield	0.06%
Equities	S&P 500	0.05%
	OMX Stockholm 30	0.08%
	Topix Index	0.06%
	Russell 2000	0.07%
	MSCI World	0.06%
	MSCI Emerging Markets	0.10%
	MSCI World Small Cap	0.08%
	MSCI Sweden Small Cap	0.14%
Alternative Investments	US REIT	0.20%
	LPX Composite	0.30%
Currencies	USD/SEK	0.00%
	EUR/SEK	0.00%
	JPY/SEK	0.00%

3.5 Scenario Generation

Three scenario generation models are implemented in this thesis where one is based on estimation of population parameters and the other two on time series theory. The moment-matching model, described in section 2.3.2, failed to produce scenarios in the real domain for this particular choice of assets. This is further described in section 3.5.1.

Furthermore, DCC-GARCH and copula-GARCH are used to produce inputs to the stochastic programming based decision model. The latter is multivariate GARCH with constant conditional correlations, equation 2.3.25, using a copula function as it's distribution model which is described in section 2.3.5. In section 3.5.2 the implementation of these time series models and the procedure of generating scenarios are outlined in detail.

3.5.1 Moment Matching

Certain scenario generation models are based on matching a specific class of statistical parameters, e.g moments as in the case with the moment-matching models.

The aim of the moment-matching procedure is to generate scenarios as well as their corresponding probability weights that match exactly the given mean, the covariance matrix, the average of the marginal skewness and the marginal kurtosis of each component of a random vector. The model is implemented in Python.

The mean vector $\boldsymbol{\mu}$, the covariance matrix $\boldsymbol{\Sigma}$ and the marginal third and forth central moments, κ_j and ζ_j are used as inputs for the algorithm. Next, the user chooses an arbitrary non zero deterministic vector $\boldsymbol{Z}$, such that

$$\boldsymbol{\Sigma} - \boldsymbol{Z}\boldsymbol{Z}^T > 0 \tag{3.5.1}$$

The vector $\boldsymbol{Z}$ is later used to determine ϕ_1, ϕ_2 from equations 2.3.7 in order to eventually determine coefficients α and β form equations 2.3.1-2.3.2. $\boldsymbol{Z}$ has been determined with two different methods.

In the first method, proposed in '*An algorithm for moment-mathing scenario generation with application to financial portfolio optimisation*' by Ponomareva et.al. (2014), it is calculated as

$$Z_j = \rho\sqrt{R_{jj}}, \tag{3.5.2}$$

where R_{jj} is the diagonal of the covariance matrix and $\rho \in (0, 1)$ is chosen for some sufficiently small values [22].

The second approach to generate the vector $\boldsymbol{Z}$ was proposed in '*Comment on "An algorithm for moment-mathing scenario generation with application to financial portfolio optimisation"* by Contreras et.al. (2018), where eigenvalues and eigenvectors are used to construct the covariance matrix. One lets $\lambda_1 \leq ... \leq \lambda_n$ be the eigenvalues and $\boldsymbol{v}^1, ..., \boldsymbol{v}^n$ the corresponding eigenvectors of the matrix $\boldsymbol{\Sigma}$. Then $\boldsymbol{Z}$ is generating using

$$Z_j = \sqrt{\lambda_l}\boldsymbol{l}, \tag{3.5.3}$$

Then

$$(\Sigma - ZZ^T)\boldsymbol{v}^j = \begin{cases} \lambda_j \boldsymbol{v}^j, & \text{if } j \neq l \\ \lambda_l(1-\rho^2)\boldsymbol{v}^l, & \text{if } j = 0. \end{cases} \tag{3.5.4}$$

The eigenvalues are positive and as a consequence, $\boldsymbol{\Sigma} - \boldsymbol{Z}\boldsymbol{Z}^T > 0$ should hold [6].

Unfortunately, none of the methods were able to generate scenarios in the real domain. This is further discussed in section 5.1.1.

3.5.2 Multivariate GARCH

Two multivariate GARCH models are used to generate scenarios, DCC-GARCH and copula-GARCH. Copula-GARCH is a mixture model between a multivariate GARCH model with constant conditional correlation and a copula. Both models are implemented in *R* using the package *rmgarch* written by Alexios Ghalanos. For specifics about the implementation, see the sections below.

The procedure for generating scenarios is similar for both models. A parameter estimation is performed in the beginning of each year using a window length of l weeks. Furthermore, the fitted models are used to simulate S scenarios a year ahead, where every fourth time step is considered which corresponds to each rebalancing occasion. In this way the path-dependency is preserved for the scenarios. The starting values are given by the last observations to fit the models. After a year has past, a significant amount of new historical data is available and new time series models are fitted to simulate new scenarios for the consecutive year.

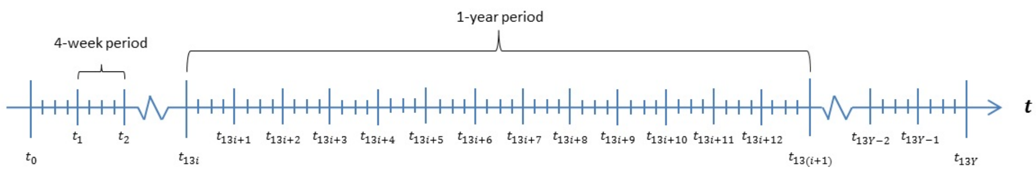

Figure 3.1: An illustration of the discretization used in the scenario generation.

Given m historical weekly returns, this procedure is repeated for Y years, where $Y = \left\lfloor \frac{M-l}{52} \right\rfloor$.

DCC-GARCH

The DCC-GARCH model is used to simulate scenarios with the procedure described above. The implemented method is based on a two-step Maximum-Likelihood fit followed by a simulation of S scenarios.

Table 3.5: Implementation specifications of DCC-GARCH.

Input	Historical returns of assets in Table 3.1 using a rolling window length of $l = 520$. This means that at each point of time $t = t_{13i}$ a DCC-GARCH model is fitted using the 520 antecedent historical asset returns.
Start Date: End Date:	2012-01-23 2017-01-16
Output	S scenarios of assets returns for the sequent year. To use computer resources efficiently, only every fourth week is considered. The scenarios are assigned equal probabilities.
Technical Specifications	Implemented in *R* using package *rmgarch*
Order: Fit Model: Distribution Model: Simulation Model:	(1,1) *dccfit*[1] Multivariate Normal *dccsim*[1]

Moreover, to ensure that the scenarios are of high quality, the generated scenario sets are tested for the properties described in section 2.3.1.

Parsimonious — Scenario sets of size S = 100, 500 and 1000 are generated. By producing results with different data sets, one can test which size is large enough such that it renders sufficiently small errors and small enough from a computational perspective.

Representative — To establish a representative set of scenarios, the model of choice must capture important aspects of the assets' price dynamics. Since the return volatility varies over time, DCC-GARCH seems like a plausible choice.

Arbitrage free — The conditional variance of any asset with respect to all other assets is positive. In other words, no asset can be perfectly explained by a combination of the others, i.e. replicated, and thus rules out arbitrage.

[1]Object in package *rmgarch*. URL http://www.unstarched.net, https://bitbucket.org/alexiosg

Numerically Stable — The numerical stability is considered in respect of a particular decision model, in this case equation 2.4.8. The combined use of a scenario generator and decision model should lead to numerically stable results. To test this, the optimal solutions for two different scenario sets are compared. Generating two sets of the same size using DCC-GARCH, the optimal solutions are computed according to equation 2.3.28. By comparing the outcomes one can conclude that the results are numerically stable. For more information, see Appendix B.2.

Copula-GARCH

Copula functions were introduced as a tool to connect disparate marginal distribution together, to form a joint multivariate distribution. Since they where introduced, they have become popular in analyzing financial time series.

Due to the fact that *Student's t* copulas captures tail events for n-variate superior to Normal, this particular copula has been chosen as the distribution model in the multivarie GARCH model with constant correlations. Together, they form Copula-GARCH which is the second time series model to be used for scenario generation in this thesis [1].

Table 3.6: Implementation specifications of Copula-GARCH.

Input	Historical returns using a rolling window length of $l = 156$ corresponding to 3 years of data
Start Date: End Date:	2005-01-31 2017-01-16
Output	S scenarios of assets returns for the sequent year. The scenarios are assigned equal probabilities.
Technical Specifications	Implemented in *R* using package *rmgarch*.
Order: Fit Model: Distribution Model: Simulation Model:	(1,1) *cgarchfit*[1] *Student's t* copula *cgarchsim*[1]

[1]Object in package *rmgarch*. URL http://www.unstarched.net, https://bitbucket.org/alexiosg

Parsimonious	As for DCC-GARCH, scenario sets of size 100, 500 and 1000 are generated to ensure both accuracy and computational tractability.
Representative	Copula-GARCH also captures the varying volatility and thus represents a realistic form of the problem at hand.
Arbitrage free	Also here, arbitrage is ruled out by positive conditional variances.
Numerically Stable	The stability for this model is considered as for DCC-GARCH. The solutions for the two sets do not vary and the scenario generator is therefore considered to be numerically stable. More information can be found in Appendix B.3.

3.6 Allocation Models

The asset allocation models in section 2 are all formulated as optimization problems with some arbitrary parameters. These parameters are chosen such that the models fulfill the requirements of the investor, e.g. risk appetite, diversification. This section demonstrates how the models are calibrated and specifies how they are implemented.

3.6.1 Risk Parity

The objective of the Risk Parity portfolio is to allocate risk equally to all assets given certain constraints, presented in section 2.4.2. In other words, the portfolio may be viewed as variance-minimizing subject to a constraint of sufficient diversification in terms of component weights. This results in allocations with the aim to maintain equal risk, measured as volatility. The optimization of the objective function given in 2.4.2 results in the initial constraint of $-\infty \leq c \leq -N\ln(N)$. In this thesis, the arbitrary constant, c, will be chosen such that $c = -N\ln(N)$.

Furthermore, the allocation is performed dynamically during 2005-2018 according to the scheme described in section 2.4.2. Because of the diversification constraint, the APL and BRP constraints do not give unique outcomes and thus Risk Parity will only be evaluated at one fund management level. Specifications can be found in Table 3.7.

Table 3.7: Implementation specifications of Risk Parity.

Input	Rolling covariance estimate using a window length of 60 data points. The estimation is performed using *numpy*
Start Date: End Date:	2005-01-31 2018-01-08
Output	Vector of percentage weights $\boldsymbol{w}$.
Technical Specifications	Implemented in *Python* using packages *numpy* and *scipy.optimize*
Optimization method: Rebalancing:	Sequential Least Squares Programming[2] 676 times during specified time period

3.6.2 Mean-Variance

Due to the fact that the asset allocation is performed dynamically, the formulation according to equation 2.4.7 is considered. As described in section 2.2.2, the parameter c determines the trade-off between risk and expected return. In the final analysis, the objective is to maximize the return with a boundary on the volatility such that it renders a suitable risk appetite. This boundary is chosen to be the annualized volatility of 10 %, i.e. $\sigma_P^A \leq 0.10$. To calibrate the model, c is be chosen such that it yields an estimated annualized portfolio volatility of 10%.

Table 3.8: Implementation specifications of Mean-Variance.

Input	Rolling covariance and mean estimates using a window length of 60 data points. The estimation is performed using *numpy*
Start Date: End Date:	2005-01-31 2018-01-08
Output	Vector of percentage weights $\boldsymbol{w}$.
Technical Specifications	Implemented in *Python* using packages *numpy* and *scipy.optimize*
Optimization method: Rebalancing:	Sequential Least Squares Programming[2] 676 times during specified time period

The Mean-Variance model is evaluated with both APL, equation 3.2.2, and BRP constraints, equation 3.2.3, described in section 3.2. These portfolios will be referred to as MV APL and MV BRP respectively.

[2] Optimizer SLSQP in package *scipy.optimize*. URL http://www.pyopt.org/reference/optimizers.slsqp.html

3.6.3 Mean-Absolute Deviation

As for the Mean-Variance model, Mean-Absolute Deviation, as it is formulated in 2.4.8, optimizes two metrics simultaneously. Similarly, the optimization is performed such that it yields an estimated mean-absolute deviation of 10%. Specifications about the implementation can be found in Table 3.9.

Table 3.9: Implementation specifications of Mean-Absolute Deviation.

Input	Scenarios generated with methods described in (3.5.2).
Start Date: End Date:	2005-01-31 (2012-01-23)[3] 2018-01-08
Output	Vector of percentage weights $\boldsymbol{w}$.
Technical Specifications	Implemented in *Python* using packages *numpy* and *scipy.optimize*
Optimization method: Rebalancing:	Sequential Least Squares Programming[2] 676 (312) times during specified time period[3]

Moreover, the Mean-Absolute model is evaluated with APL and BRP constraint, referred to as MAD APL and MAD BRP.

3.7 Rebalancing and Portfolio Drift

Over time, the value of each individual investment move up and down which result in a *drift* of each weight w_i. This results in another composition of securities than intended when the investments were done and is the reason why rebalancing is needed. Between rebalancing occasions the changing weights consequently drift the value of the portfolio. How the portfolio drift is modeled in this report is demonstrated below.

The portfolio is chosen to be rebalanced every fourth week. According to the notation specified in section 2.4.2 this correspond to T = 4 since the analysis is performed on weekly data. With the discretization of time outlined in 2.4.2, each rebalancing is performed at $t_n = t_0 + nT$ with $k = 1, 2, 3, 4$ stages between each occasion. A weight at time t_n and stage k is denoted by $\boldsymbol{w}_n(k)$. The drift is considered for all 4 stages between t_n and t_{n-1} and is calculated according to the following procedure for n = 1, 2, ..., M.

[2] Optimizer SLSQP in package *scipy.optimize*. URL http://www.pyopt.org/reference/optimizers.slsqp.html
[3] 2005-01-31 using scenarios generated with Copula-GARCH and 2012-01-23 with DCC-GARCH.

k = 1

At this instant a rebalancing is considered and the current positions should be valuated. To do this, the drift of the weights is calculated according to

$$\boldsymbol{w}_n^*(1) = [w_0^{(n-1)}(4)R_0^{(n)}(1) \quad w_1^{(n-1)}(4)R_1^{(n)}(1) \quad \cdots \quad w_{16}^{(n-1)}(4)R_{16}^{(n)}(1)]^T$$

where the star denotes that the weights are not normalized.

The weights can be divided into two subsets, the actual assets of the portfolio, $\boldsymbol{w}_{n,P}$, and the weights representing the open currency exposure, $\boldsymbol{w}_{n,UH}$. The corresponding unnormalized weights at this stage are given by

$$\boldsymbol{w}_{n,P}^*(1) = [w_0^{(n-1)}(4)R_0^{(n)}(1) \quad \cdots \quad w_{13}^{(n-1)}(4)R_{13}^{(n)}(1)]^T$$
$$\boldsymbol{w}_{n,UH}^*(1) = [w_{14}^{(n-1)}(4)R_{14}^{(n)}(1) \quad \cdots \quad w_{16}^{(n-1)}(4)R_{16}^{(n)}(1)]^T.$$

Due to the fact that $\boldsymbol{w}_{n,UH}^*$ are not actual holdings of the assets, the value change has to be divided by 1 added with the un-hedged porportion of the portfolio, i.e.

$$\Delta V_n(1) \; = \; \frac{\mathbf{1}^T \cdot \boldsymbol{w}_n^*(1)}{\mathbf{1}^T \cdot \boldsymbol{w}(4)_{n-1}} \Rightarrow \quad V_n \; = \; V_n(1) \; = \; V_{n-1}(4)\Delta V_n(1)$$

Note that this is not necessary for n = 1. Then V_1 is given as 100 % of the initial investment, i.e. $V_1 = 1$.

The normalized weights at this stage, $\boldsymbol{w}_n(1)$, are obtained with the optimizer of choice.

k = 2, 3, 4

Start by calculating the drifting weights.

$$\boldsymbol{w}_n^*(k) = [w_0^{(n-1)}(k-1)R_0^{(n)}(k) \quad \cdots \quad w_{16}^{(n-1)}(k-1)R_{16}^{(n)}(k)]^T$$

Compute the portfolio value as for k = 1.

$$\Delta V_n(k) \; = \; \frac{\mathbf{1}^T \cdot \boldsymbol{w}_n^*(k)}{\mathbf{1}^T \cdot \boldsymbol{w}(k-1)_n} \Rightarrow V_n(k) \; = \; V_n(k-1)\Delta V_n(k)$$

The normalized weights at this stage are necessary to compute the value change for k+1 and are simply obtained by

$$\boldsymbol{w}_n(k) = \frac{\boldsymbol{w}_n^*(k)}{V_n(k)}$$

3.8 Portfolio Evaluation

The portfolio evaluation is implemented in MATLAB for the time periods specified in section 3.6. The performance of the weighting strategies are compared in terms of the metrics described in section 3.8.

Table 3.10: Implementation specifications of the portfolio evaluation.

Metric	**Window length**	**Technical Specification**
Annualized Return	52 data points	Return defined as 1 + value change. *mean()*[4] is used to compute the average.
Annualized Volatility	52 data points	*std()*[4] is used to compute the volatility
Sharpe Ratio	52 data points	*mean()* and *std()* are used to calculate the quotient
Value-at-Risk	52 data points	Computed at level p = 0.05 using the normal distribution method[5]

[4]MATLAB built-in function.

[5]Value-at-Risk estimation under the assumption that portfolio profits and losses are normally distributed. URL: https://se.mathworks.com/help/risk/value-at-risk-estimation-and-backtesting-1.html

4 Results

This section will present the quantitative results obtained that will be used to answer the research questions. The section begins by outlying the results acquired from the different optimization strategies. It continues by presenting results from the different scenario generators implemented in this research, how the number of scenarios effect the decision model's outcomes and how the transaction cost model impact the results of the optimizers.

4.1 Comparison of allocation strategies

The results of the proposed optimization strategies are presented in Table 4.1 together with some key performance indicators.

Table 4.1: Results of the different optimization strategies.

Strategy	Return	Volatility	Sharpe Ratio	$\mathbf{VaR_{0.05}}$
MAD APL	1.0956	0.1057	1.2919	0.1738
MAD BRP	1.0975	0.1031	1.3072	0.1695
MV APL	1.0790	0.0706	1.2087	0.1161
MV BRP	1.0899	0.0754	1.3063	0.1240
RP	1.1063	0.1026	1.3770	0.1687

Here, return describes the average annual return of each strategy. All metrics are extracted as annual averages over the time period 2006-2017.

Table 4.1 indicates that Risk Parity with an average annual return of 1.1063 has outperformed all other strategies, closely followed by Mean-Absolute Deviation BRP with an average annual return of 1.0975. Mean-Absolute Deviation APL has proven to be the most volatile strategy, followed by Mean-Absolute Deviation BRP and Risk Parity. All strategies show fairly high Sharpe ratio values, indicating acceptable levels of risk for investors.

Figure 4.1 exhibits the different portfolio strategy's performance in terms of annualized return between years 2006-2017. The portfolios lose significant value during the exceptionally distressed time period of 2007-2008, in the midst of the prevalent financial crisis. However, Mean-Variance APL and Mean-Variance BRP perform considerably better than the other strategies. All portfolios seem to behave relatively similarly from 2012 and onwards.

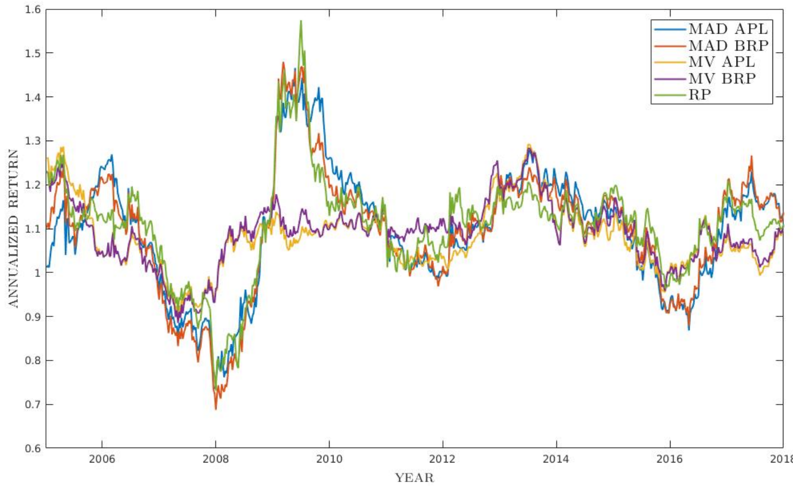

Figure 4.1: Comparison between different optimization strategies in terms of annualized return years 2005-2017.

As shown in Figure 4.2, all portfolios experience high volatility during the distressed period of 2007-2008. The turmoil caused by the recession inevitably made the overall market more volatile. Contrary to expectations however, the Risk Parity portfolio shows the highest volatility during this period, almost reaching the annualized volatility of 0.25. The 5% Value-at-Risk performance displayed in Figure 4.3 show identical behavior.

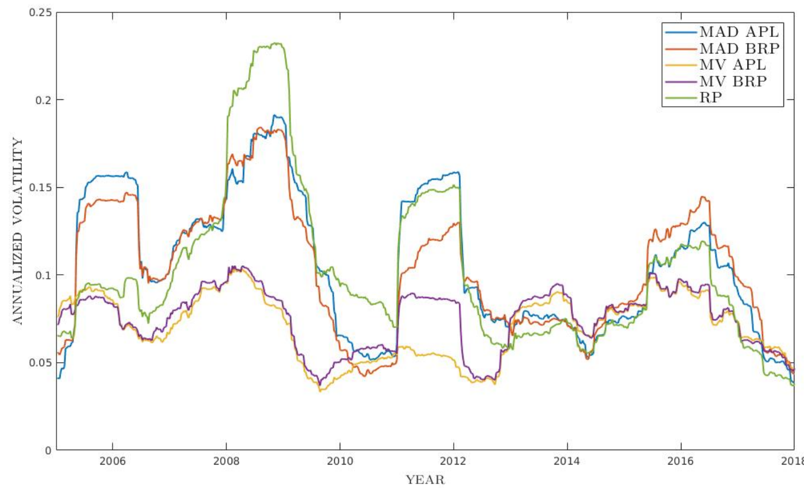

Figure 4.2: Comparison between different optimization strategies in terms of annualized volatility years 2005-2017.

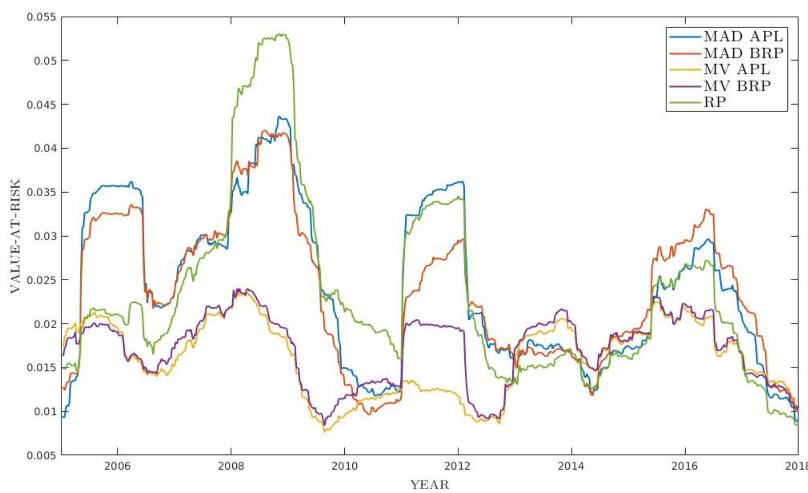

Figure 4.3: Comparison between different optimization strategies in terms of 5% value-at-risk in years 2005-2017.

Figure 4.4 displays the different strategy's Sharp ratio performance. The largest discrepancies seem to occur during years 2009-2011. As expected, after investigating the annualized returns in Figure 4.1, it is clear that Mean-Variance APL and Mean-Variance BRP outperform the other strategies during distressed market conditions.

It is also noteworthy that all strategies at some point in the examined time period experience a Sharpe ratio as high as 4.0.

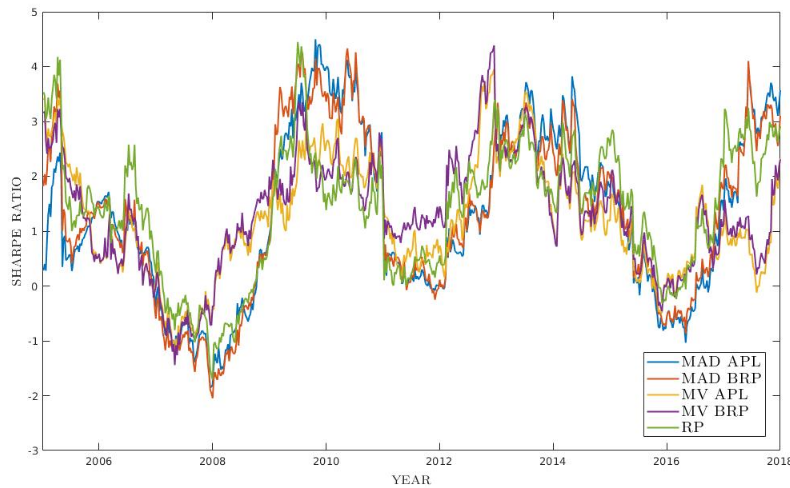

Figure 4.4: Comparison between different optimization strategies in terms of Sharpe ratio in years 2005-2017.

4.2 Scenario Generators

In this section results for the different scenario generators are presented. To begin with, a comparison of the two methods outlined in section 3.5.2 is performed in section 4.2.1. Further, scenario sizes of $S = 100$, 500 and 100 are evaluated for both models in section 4.2.2.

4.2.1 Methods

As described in section 3.6.3, the computations are performed both on the scenarios generated with Copula-GARCH and DCC-GARCH. When evaluating the scenario generation methods; return, volatility, Sharpe ratio and Value-at-Risk are considered for multi-period Mean-Absolute Deviation with APL constraints and scenario size $S = 500$. For the portfolios with BRP constraints and analysis of individual years, more information can be found in Appendix C.1.

As can be seen in Figure 4.5, the Mean-Absolute Deviation portfolio based on Copula-GARCH produces a higher rate of return for almost all years, with an exception of mid 2015 to mid 2016. This is also clear by the 4.8% higher average compared to DCC-GARCH.

Table 4.2: Performance averages for scenario generators for years 2013-2017.

Model	**Return**	**Volatility**	**Sharpe Ratio**	**Value-at-Risk$_{0.05}$**
Copula-GARCH	1.1114	0.0819	1.6968	0.1123
DCC-GARCH	1.0629	0.0785	0.8558	0.1077

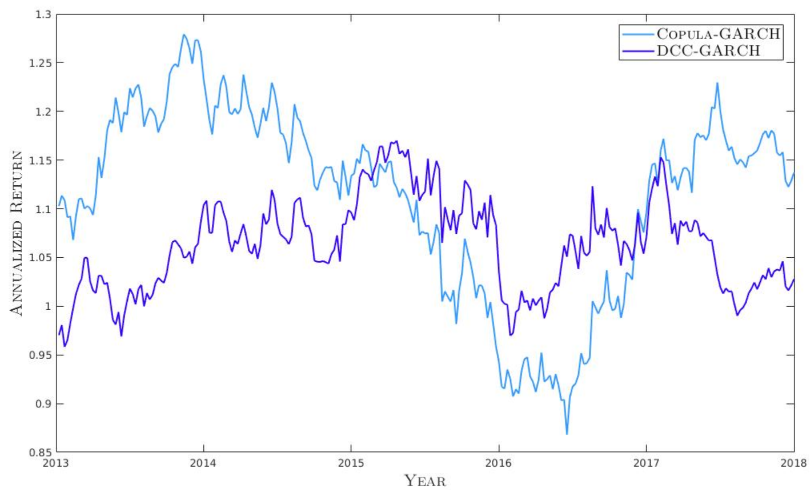

Figure 4.5: Comparison of scenario generators with APL constraints in terms of annualized return for 2013-2017.

In terms of risk, the methods perform similarly during years 2013-2017. As Figure 4.6 indicates, the volatility is similar with a slightly smaller overall risk for DCC-GARCH. The second risk measure, Value-at-Risk, also shows a similar performance which can be seen in Figure 4.7.

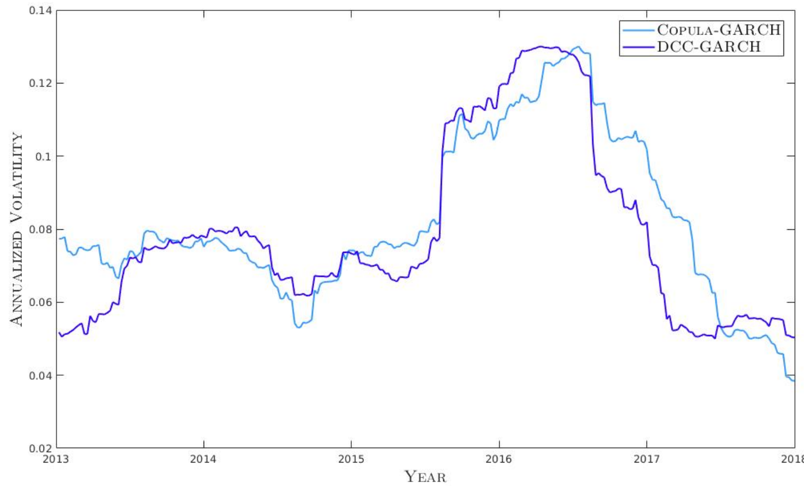

Figure 4.6: Comparison of scenario generators with APL constraints in terms of annualized volatility for 2013-2017.

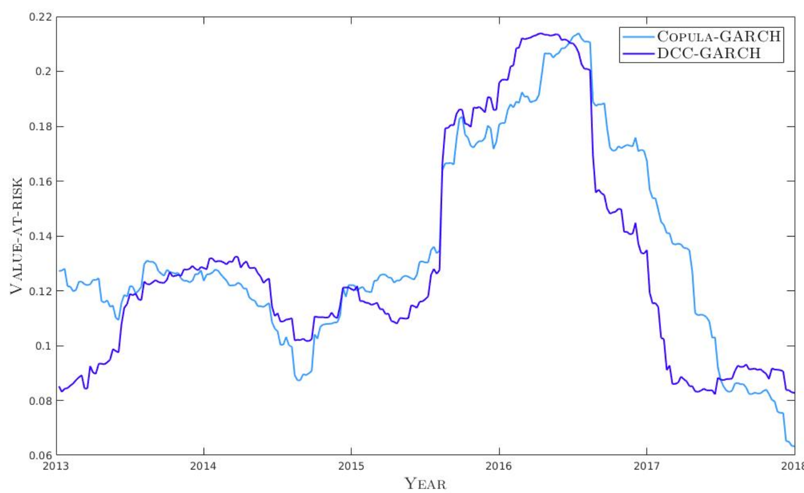

Figure 4.7: Comparison of scenario generators in terms of 5% Value-at-Risk for 2013-2017.

Consequently, Copula-GARCH outperforms DCC-GARCH in terms of Sharpe ratio, with an exception for the time period between 2015-2016. Since the models generate similar levels of volatility but Copula-GARCH performs better in terms of return, it is clear that the Sharpe ratio is higher. This result can be seen in Figure 4.8.

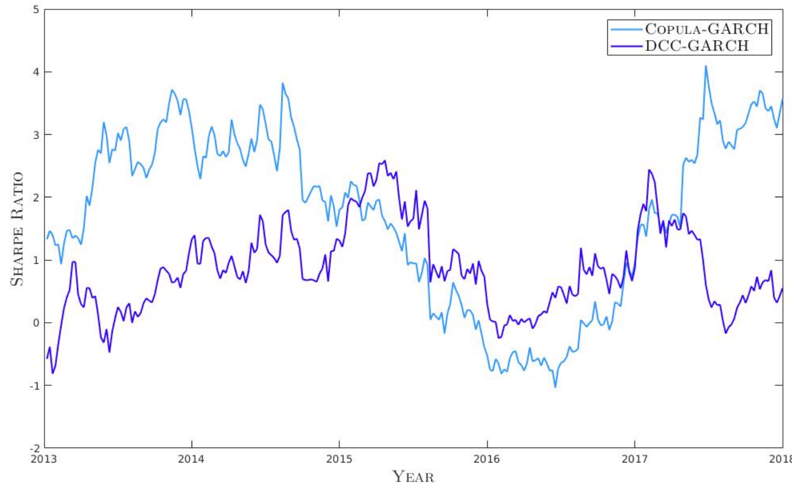

Figure 4.8: Comparison of scenario generators with APL constraints in terms of Sharpe ratio for 2013-2017.

4.2.2 Scenario Size

The scenario generations performed with both Copula-GARCH and DCC-GARCH are implemented with three distinct scenario sizes, S = 100, 500 and 1000, as described in section 3.5.2. The purpose is to learn how the increase in quantity of scenarios affect each model's performance.

The results are summarized in Table 4.3 below. Values in Table 4.3 corresponding to the Copula-GARCH model are extracted as averages between years 2006-2017, whereas values corresponding to the DCC-GARCH model are calculated as averages between years 2013-2017. The reason for the latter model's shorter time period is that it requires a longer window length due to the large number of estimation parameters. Hence, the availability of scenarios starts later than for copula-GARCH.

Table 4.3: Summary of Copula-GARCH and DCC-GARCH performance with different scenario sizes.

	Copula-GARCH				**DCC-GARCH**			
Size	**Return**	**Vol**	**SR**	$\mathbf{VaR_{0.05}}$	**Return**	**Vol**	**SR**	$\mathbf{VaR_{0.05}}$
100	1.1135	0.1032	1.4359	0.1868	1.1041	0.0764	1.2667	0.1257
500	1.1220	0.0991	1.6394	0.1761	1.0629	0.0785	0.8558	0.1292
1000	1.1220	0.0991	1.6394	0.1761	1.0629	0.0785	0.8558	0.1292

The results show a slight increase in return for the Copula-GARCH model as the number of scenarios increases from 100 to 500. Contrarily, the DCC-GARCH model experiences a decreased return as the scenario size increases by the same amount. Volatility decreases from 0.1032 to 0.0991 for the Copula-GARCH model, while it increases from 0.0764 to 0.0785 for the DCC-GARCH model, in both cases the scenario size is increased from 100 to 500 scenarios. The Sharpe ratio is subsequently increased for the Copula-GARCH, while the DCC-GARCH model experiences a decrease. Finally, the Value-at-Risk is, using a confidence level of 95%, decreased for the Copula-GARCH model, while it is increased for the DCC-GARCH model.

All results remain constant as the scenario size is increased from 500 till 1000. Outcomes from Table 4.3 will be further discussed in Chapter 5.

DCC-GARCH

Figures 4.9-4.11 below graphically examine the behavior of the DCC-GARCH model between years 2012-2017 in terms of annualized return, annualized volatility, Sharpe ratio and Value-at-Risk. As demonstrated in the previous section by Table 4.3, an increase in scenario sets from 500 to 1000 scenarios has no effect on the outcome, regardless of investigated variable. Thus, the data points representing 500 scenarios coincide with the ones representing 1000 scenarios. As shown in Figure 4.9, the DCC-GARCH model implementing 100 scenarios outperforms the others over the total time period, except between years 2012-2014.

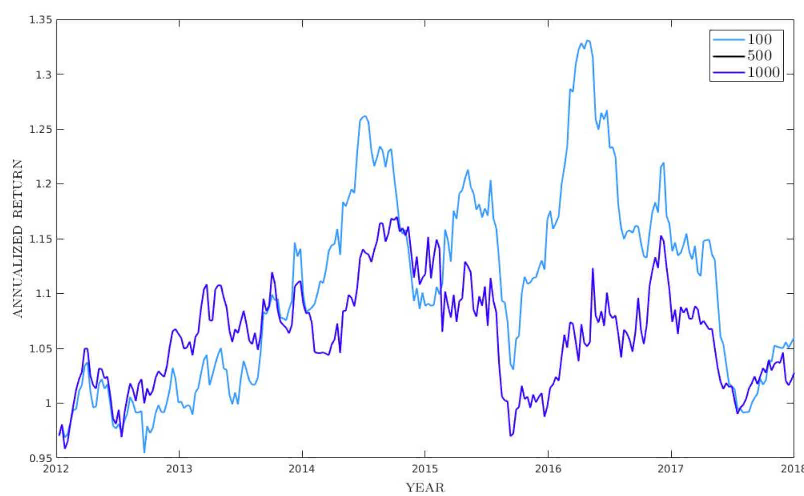

Figure 4.9: Comparison of scenario sizes for DCC-GARCH in terms of annualized return over total time period.

Figures 4.10 and 4.11 show seemingly identical behavior, a natural occurrence as risk-measures correlates well with volatility. As shown in both figures, there is a significant increase between years 2015-2017 for the model with 500 scenarios, as opposed to the model using 100 scenarios.

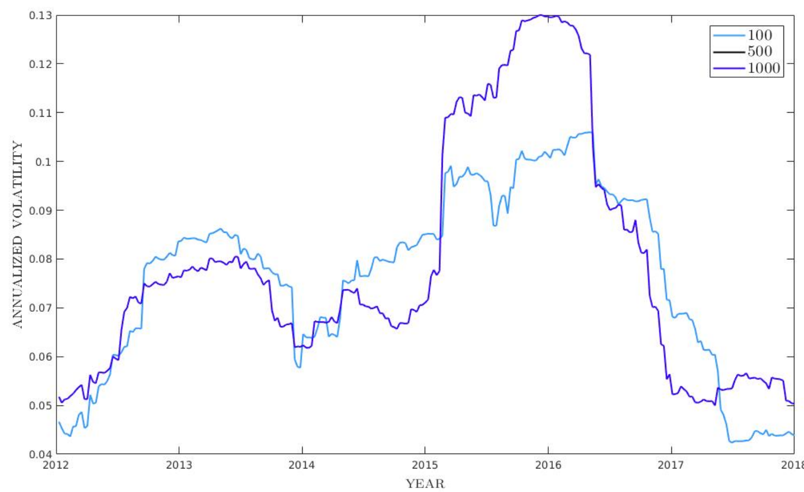

Figure 4.10: Comparison of scenario sizes for DCC-GARCH in terms of annualized volatility over total time period.

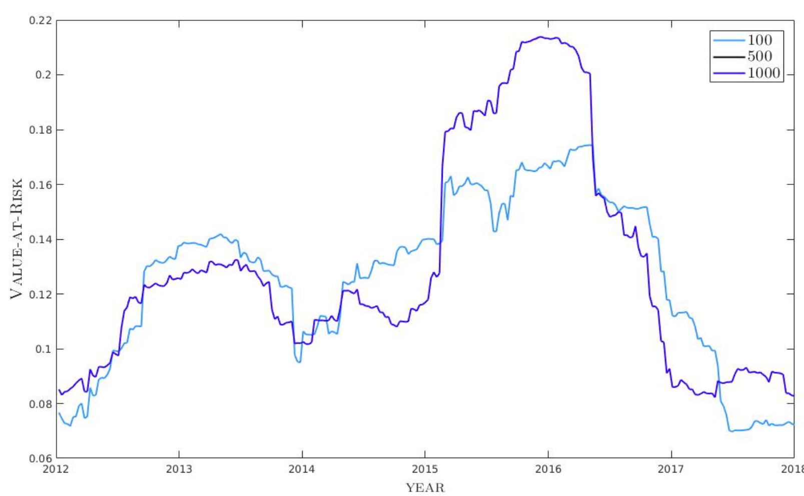

Figure 4.11: Comparison of scenario sizes for DCC-GARCH in terms of 5 % Value-at-Risk over total time period.

As depicted in Figure 4.12, the DCC-GARCH model using 100 scenarios outperforms the others in terms of Sharpe ratio over the majority of the time period, except around years 2012-2014 where it experiences lower returns, shown earlier in Figure 4.9.

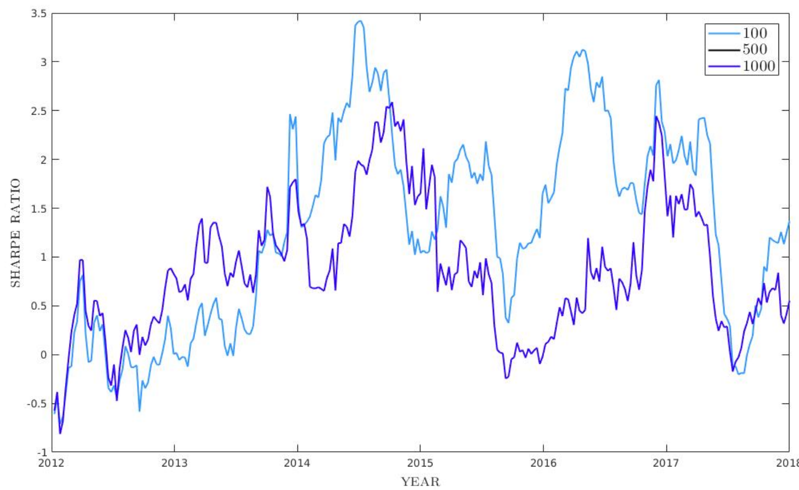

Figure 4.12: Comparison of scenario sizes for DCC-GARCH in terms of sharpe ratio over total time period.

Copula-GARCH

Figures 4.13-4.16 below graphically examine the behavior of the DCC-GARCH model between years 2012-2017 in terms of annualized return, annualized volatility, Sharpe ratio and Value-at-Risk. Again, an increase in quantity of scenarios from 500 to 1000 unveil no effect on the results.

The most noticeable and interesting difference between these results and the ones presented for DCC-GARCH, is how correlated the annualized returns for the Copula-GARCH generated by 100 and 500 scenarios are.

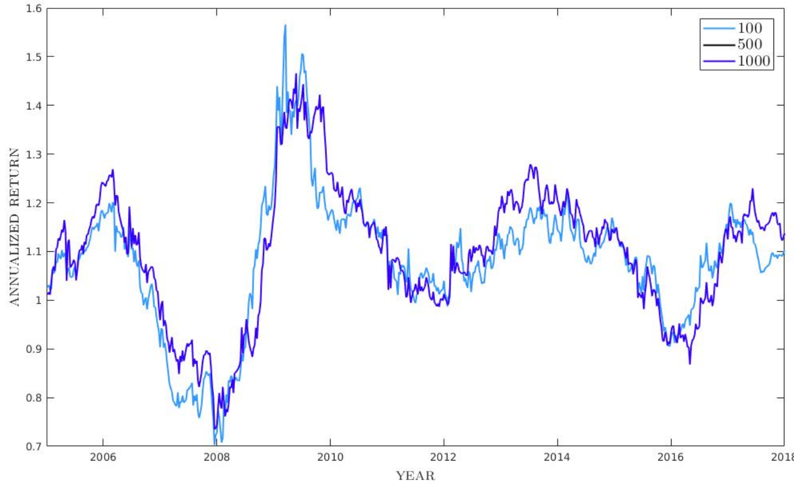

Figure 4.13: Comparison of scenario sizes for Copula-GARCH in terms of annualized return over total time period.

Figures 4.14 and 4.15 show that the Copula-GARCH model using larger number of scenarios, i.e. 500, outperforms the model with only 100 scenarios during the period of distressed market conditions between years 2008-2009.

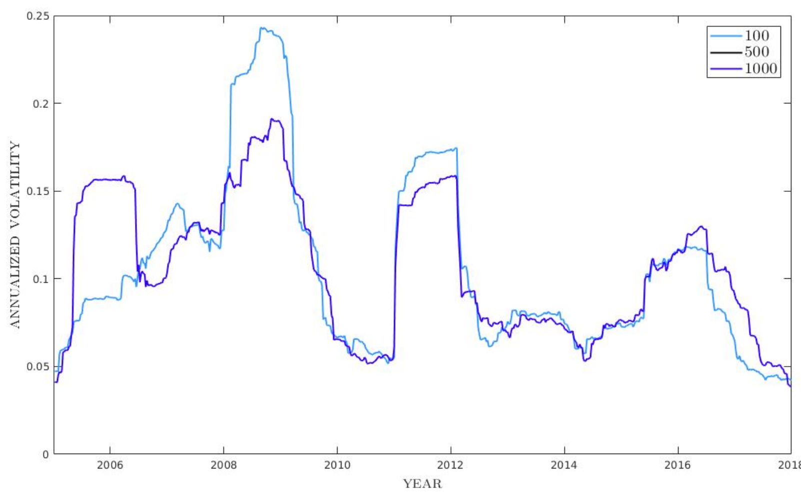

Figure 4.14: Comparison of scenario sizes for Copula-GARCH in terms of annualized volatility over total time period.

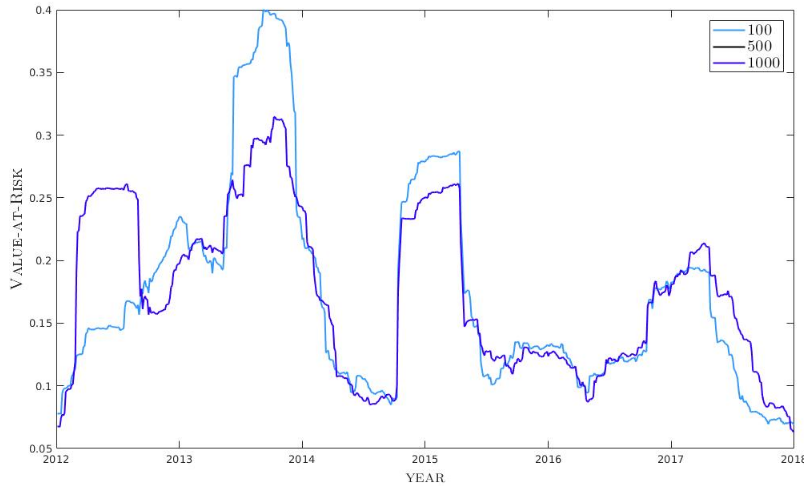

Figure 4.15: Comparison of scenario sizes for Copula-GARCH in terms of 5 % Value-at-Risk over total time period.

Results of annualized returns observed in Figure 4.13 naturally create the same pattern in Sharpe ratio, presented in Figure 4.16, as the metric is highly dependent on return.

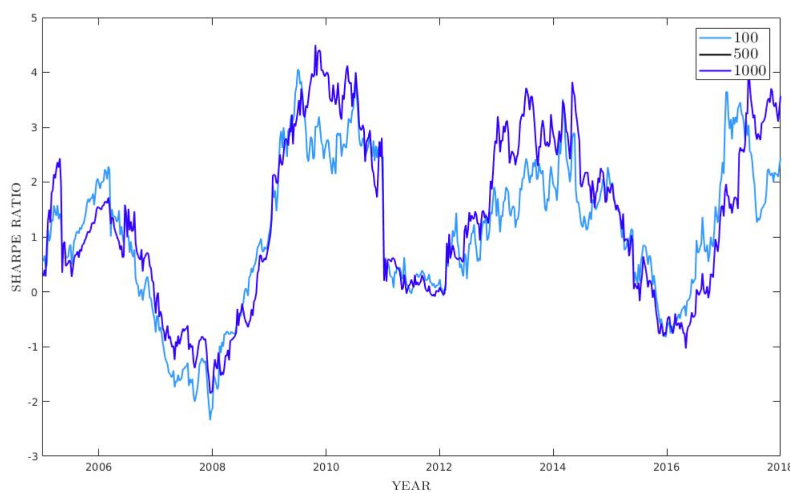

Figure 4.16: Comparison of scenario sizes for Copula-GARCH in terms of sharpe ratio over total time period.

4.2.3 Transaction Costs

In this thesis, portfolio performance has also been evaluated based on the associated transaction costs that accompanies each rebalancing occasion. Table 4.18 examines the impact transaction costs have on the average portfolio turnover for each individual allocation strategy.

Portfolio turnover is defined as either the total amount of new securities purchased or the amount of securities sold at each rebalancing occasion, whichever is less, expressed as proportion of the total portfolio value. Mean-Absolute Deviation APL and Mean-Absolute Deviation BRP both measure extremely high quantities of portfolio turnover, as the investigated models suggest >40% turnover including transaction costs, and nearly 50% turnover excluding transaction costs. Mean-Variance APL and Mean-Variance BRP however, experience significantly lower volumes of portfolio turnover. Mean-Variance APL measures 12.57% including transaction costs and 14.75% excluding transaction costs, whereas Mean-Variance BRP register slightly lower measurements of 10.04% including transaction costs and 12.02% portfolio turnover excluding transaction costs. The study further suggests that portfolio turnover is unaffected by transaction costs for the Risk Parity portfolio, as it is held consistent at 2.10% both including and excluding transaction costs.

Table 4.4: Average portfolio turnover at each rebalancing occasion for years 2009-2017.

	Average Portfolio Turnover	
Strategy	**Excl. transaction costs**	**Incl. transaction costs**
MAD APL	0.4548	0.4154
MAD BRP	0.4802	0.4647
MV APL	0.1475	0.1257
MV BRP	0.1202	0.1004
RP	0.0210	0.0210

The impact of a transaction cost model is also examined in terms of sharpe ratio for the Mean-Variance and Mean-Absolute Deviation model with APL constraints. Since transaction costs do not impact the Risk Parity model, it is excluded from this analysis.

As seen in Figure 4.17, the transactions costs do not affect the Mean-Variance model significantly. The Sharpe ratios overlap each other the four first years and are very similar for the whole period. Only during 2011-2013 one can see a gain in Sharpe ratio for the model including transaction costs.

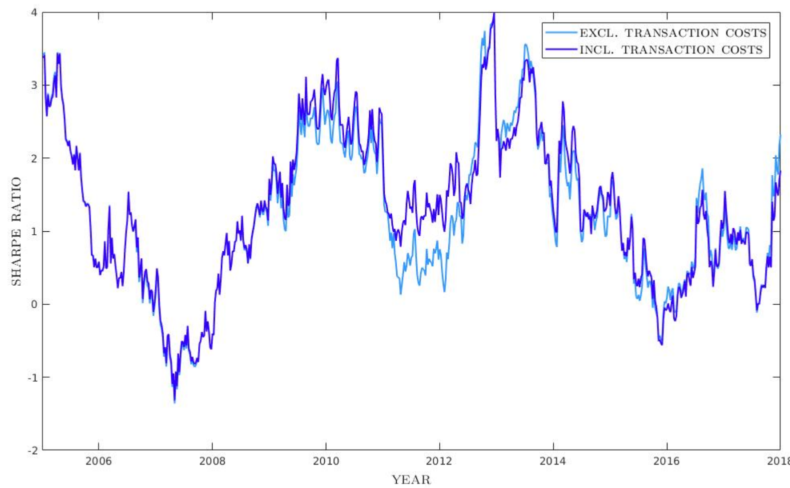

Figure 4.17: Comparison of MV excluding and including transaction costs in terms of Sharpe ratio for 2017.

As for Mean-Variance, transaction costs have a limited impact on Mean-Absolute Deviation. This can be graphically examined in Figure 4.18 where the curves closely follow each other with an exception of 2009-2011.

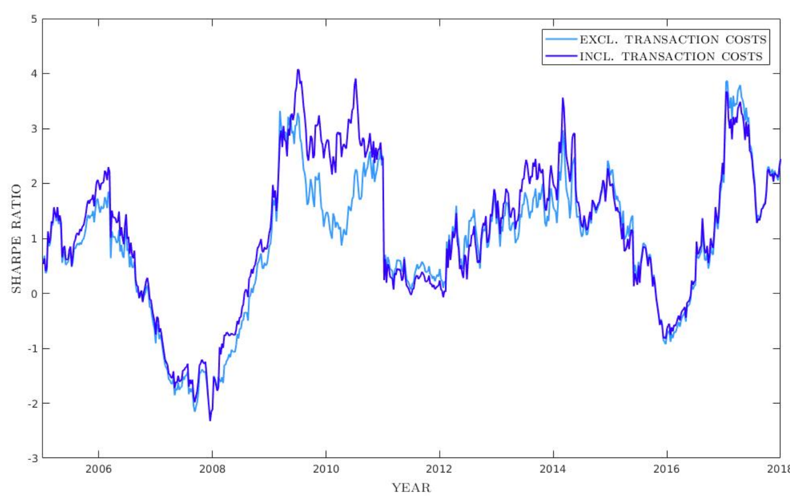

Figure 4.18: Comparison of MAD excluding and including transaction costs in terms of sharpe ratio for 2017.

5 Discussion

5.1 Scenario Generation

5.1.1 Moment Matching

Two different approaches in implementing the theory of moment-matching scenario generation has been performed in this thesis. The first one was introduced by Ponomareva et.al. (2014) in [22], and the second one by Contreras et.al. (2018) in [6]. The research performed by Contreras et.al. (2018) was made with intention to improve the work of Ponomareva et.al. (2014), suggesting their method of generating scenarios was conducted with inconsistent assumptions, causing the algorithm to produce negative probabilities, ultimately altering the results as consequence.

Unfortunately, both methods of determining the deterministic vector $\boldsymbol{Z}$ lead to violation of equation 3.5.1, with corresponding matrix shown in Appendix B.1. Consequently, γ from equation 2.3.6 becomes negative, violating condition $\sum_{i=1}^{s} \frac{1}{p_i} < \gamma$ and the subsequent coefficients α and β from equations 2.3.1-2.3.2 end up in the complex domain. The model has been applied for many different periods, all with the same outcome of incompleteness. Ponomareva et.al. (2014) failed to mention their implemented asset types, however their covariance matrix $\boldsymbol{\Sigma}$ consists of all positive entries. This report is diversely based on a study of 17 assets from a variety of underlying markets, natural yielding negative covariances between certain asset classes. The occurrence of negative covariances from the assets used in this research might explain the outcome of non-real solutions from the proposed algorithm.

5.1.2 Multivariate GARCH

Unlike the moment matching model, both GARCH models successfully generated scenarios representing the asset returns. It could also be said that the GARCH models fulfill the properties of a satisfactory scenario generator. Especially in terms of generating representative scenarios as they preserve the path-dependency of the asset returns and models volatilities and covariances that varies over time.

Notwithstanding these advantages, it is worth to highlight some key aspects observed in the results. Scenario generation is a tool to model uncertainty and randomness in decision policy and thus susceptible to estimation issues as any other estimator. As the GARCH models are fitted, the estimation is vulnerable to two sources of error, namely sampling error and non-stationarity of the time series. The cause of the sampling error is insufficient data so a solution would be to simply use more data. On the other hand, the non-stationarity is due to changes in the variance and mean over time, i.e. lag-dependency. By shortening the time windows the lag-dependency is likely to be reduced whereas the contribution of the sampling error is anticipated to increase. Hence, the choice of window length is an important point of consideration as can be seen in from the above analysis.

A further point to be considered is the trade-off between solution tractability and model accuracy. The scenarios are the foundation of the decision making and are evaluated as the combined use of a scenario generator and decision model. As the number of scenarios increase the model accuracy is expected to increase, resulting in superior investment decisions. Nonetheless, given finite computational resources, the scenario size has to be small enough such that the problem does not become intractable. Since the decision making model is based on a iterative optimization algorithm, the computational complexity increases exponentially with scenario size.

DCC-GARCH

As has been seen from the results in section 4.2.1, the decision model based on DCC-GARCH resulted, on average, in inferior results compared to the alternative scenario generator. Although the model accomplished to generate a lower risk of 0.34% on average, it only achieved an average return slightly higher than half of Copula-GARCH.

A significant aspect of the underlying causes of the results is the long window length when fitting the DCC-GARCH model. As noted in the definition of the model in section 2.3.4, the model requires a large number of parameters to be estimated. This makes it more easily affected by the *curse of dimensionality*, i.e. the fact that the amount of data needed to support the result grows exponentially with the dimensionality. For this reason, a window length of 10 years was required to estimate the DCC-GARCH parameters without encountering singularities. This means that the model is likely to be built on non-stationary time series. Having this in mind, the investment decision for the whole time period were based on data that included the 2008 financial crisis. This might explain the risk-avert allocations of the decision model and the reason that it managed to maintain a lower risk.

The claim that DCC-GARCH poorly models the asset returns is also supported by the results in section 4.2.2. The decision model is expected to perform better with increased scenario size but the results shows the opposite. This indicates that the method fails to model the uncertainty in the decision making and that the higher performance of $S = 100$ is caused by mere chance.

This critique, unfortunately, implies that DCC-GARCH is not a credible choice for scenario generation of n-variate data. The method should be discarded unless clustering is considered such that a shorter window length can be used for the estimation.

Copula-GARCH

The fitting of Copula-GARCH required a shorter window length compared to DCC-GARCH. Instead of 10 years, only 3 years were needed for the estimation of model parameters. The optimizer based on Copula-GARCH was able to produce an average Sharpe ratio twice as large compared to the other scenario generator. Hence this model did not seem to encounter the same issues of non-stationary time series described above.

As expected, the model accuracy increased with scenario size which can be seen in section 4.2.2. The gains of larger scenario sets are most observable during time periods of distressed market conditions and high volatility. The risk is significantly lower for larger scenario sets, both in terms of volatility and Value-at-Risk, for years 2008 and 2011, which are years were spikes in volatility could be observed in the markets.

Moreover, one can see a saturation of performance at $S = 500$ as the decision model based on the scenario set of $S = 1000$ yields identical allocations. In other words, once a certain tolerance is reached, a higher accuracy do not influence the decisions.

5.2 Allocation models

5.2.1 Risk Parity

The Risk Parity model by definition aims at making the total risk contributions of all assets equal among them, with volatility used as risk measure. In other words, the investment strategy is formalized such that the quantity of capital invested in each asset should be determined such that the total risk contribution for all assets within the portfolio are equal.

According to Table 4.1, the Risk Parity constructed portfolio in this thesis scored the highest average annual return during the investigated time period, although only slightly higher than Mean-Absolute Deviation APL. Before performing the simulations, the superior returns realized for the Risk Parity portfolio was not expected. An explanation for the unconventional results might be found in the definition of the portfolio's construction. The optimization problem does not take into account expected return, but solely focuses on the assets' volatility. Since the mean vector $\boldsymbol{\mu}$ is more complicated to estimate than the covariance matrix $\boldsymbol{\Sigma}$, the risk parity by nature has less exposure to significant estimation errors. In other words, the difference between expected and realized return is generally larger than the difference between expected and realized volatility.

Another interesting discovery from the results of the model (Table 4.1) is its relatively high annualized volatility, only scoring lower than Mean-Absolute Deviation

APL. This somewhat unexpected result might be explained by the model's performance during the distressed market conditions experienced in the critical years of 2008-2009 (Figure 4.2). Here, it measured significantly higher volatility than all other models due to its definition concerning equal risk contribution across all assets. Therefore, the model has the possibility to reduce its exposure in certain (poor) investments, but is limited from excluding them completely, an option all other models are programed to have.

Furthermore, Table 4.4 suggests to the reader that the Risk Parity model outperforms all other models in terms of portfolio turnover, with an average portfolio turnover of only $\sim 2\%$, which is a clear advantage.

5.2.2 Mean-Variance

As one could have expect, the Mean-Variance portfolios had on average, the poorest performance in terms of Sharpe ratio. However, the portfolio strategy managed to keep the lowest levels of risk, which is an important property especially during periods of market stress. As observed, Mean-Variance was by far the superior choice under the financial crisis of 2008.

It is also worth to mention that Mean-Variance was the model that gained the most of a more restrictive feasible region. In absolute measures, the Sharpe ratio rose $\sim$ 10 % with BRP constrains compared to the same portfolio under APL. This can be explained by the fact that the model suffers from error maximization and thus benefits the most when sources of errors are restricted.

5.2.3 Mean-Absolute Deviation

It has been shown that this particular decision model has on average the highest risk, in terms of value-at-risk, and suffers by far the biggest losses during periods of market stress. However, the model seems to outperform the other two under normal market conditions, in terms of Sharpe ratio, and only Mean-Variance managed to keep a lower risk. Furthermore, the portfolio with BRP constraints outperformed the less constrained one. Despite the fact that Mean-Absolute Deviation BRP suffered bigger losses during periods of market stress, it still managed to maintain a lower level of risk. Having considered this, it is important to acknowledge that the Mean-Absolute Deviation model is based on stochastic programming with mean-absolute deviation as it's measure of risk. What differs mean-absolute deviation from variance, which is the risk measure for the other two implemented models, is that the statistic is not squared.

In statistics, the sample variance is most commonly preferred since the aim is only to estimate. If the estimates are off by a small margin, it is acceptable, but if they deviate a lot, they have a bigger impact on the estimate and should therefore be treated as such. Hence, the variance penalizes the larger deviations as smaller ones do not cause as much damage as the larger ones do. This property might be preferable in many cases, but if the underlying distribution has heavy tails it renders in an unjust assessment. If this is the case, the sample mean-absolute deviation may be more robust and may exist when the second central moment is not defined.

Since asset returns often exhibit non-normal behavior, mean-absolute deviation might seem like a plausible choice. However, this choice also results in a substantial disadvantage during periods of market stress, which is clearly observable in the results. Considering that the assets with the highest risk are not as penalized in Mean-Absolute Deviation as in the other two models, this is not a remarkable discovery. Through the objective function, optimality is not as sensitive to substantial market moves and thus less reactive under distressed market conditions.

Moreover, the model performs better with BRP constraints compared to the minimum regulatory requirements, APL. This indicates that this quantitative allocation model benefits from investment constraints based on rationale. Quantitative models often lack a long-term return and risk level and thus more prone to errors in the models. They are mainly based on statistical methods of which historical data is the input. The past is in many cases a poor indicator of the future but historical data remains the best way to forecast it with the available tools today and the only objective way to measure risk. However, as the results show, the less constrained optimizers are more susceptible to sources of error and it is thus beneficial to limit the feasible region of the optimization.

Lastly, it is also important to highlight that Mean-Absolute Deviation had by far the highest portfolio turnover. Reaching nearly 63 % as an maximum annual average, compared to Mean-Variance and Risk Parity with 25 % and 5 % respectively. To keep a low portfolio turnover is important for two reasons. A fund with a high turnover rate will incur more transaction costs than a fund with a lower rate. Despite the higher costs, a fund can still generate a higher return through superior asset selection the high turnover rate might entail. However, unless fully automated, these transactions have to be executed and monitored by staff, which is another source of cost. This is the second drawback of a high turnover rate, the fund has to be more actively managed which results in higher costs of human resources. The intention with a transaction cost model is to limit this effect but also to penalize the assets with high costs and thus a more realistic problem. However, it has been shown that it has a limited effect both on portfolio turnover and performance.

Computational Issues

Due to the high computational complexity of stochastic programming, processing power can be an issue. The implemented models were executed on a Intel Core i7 6700K 4 GHz processor. By reason of the high complexity, the computations were spread out such that there was only one optimizer per processor. The model with the largest scenario set resulted in a running time of 123 hours and 36 minutes for the whole time period (2005-01-31 - 2018-01-08).

To boost computational efficiency one can consider the data partition proposed in Figure 5.1. By reason of the difficulty of implementation, this partition was not considered in this thesis. However, if computational efficiency is of importance, this particular partition reduces the computational complexity of the problem at hand and maintains the important property of path dependency of the scenarios.

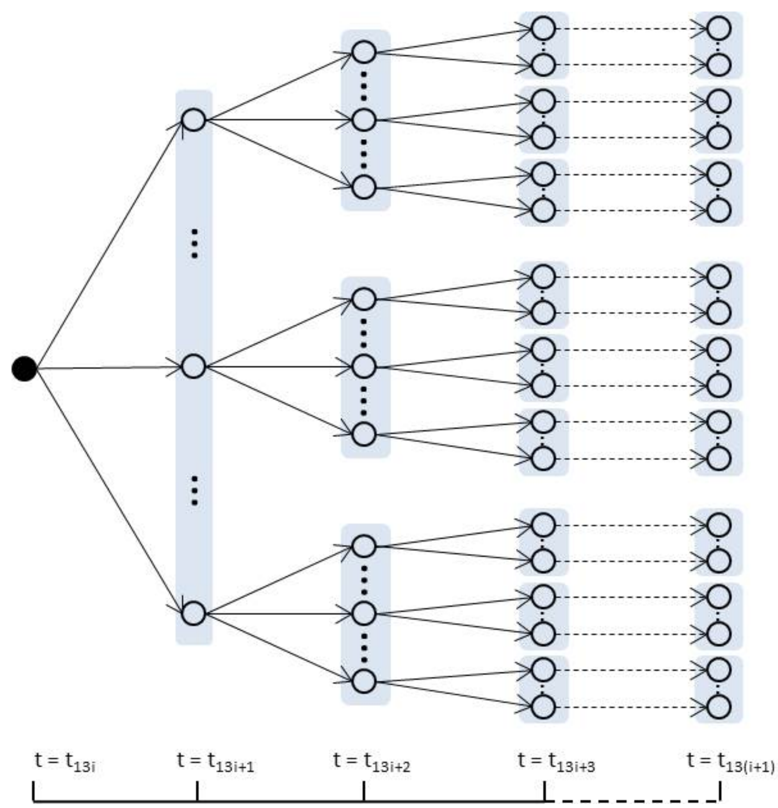

Figure 5.1: Partition of the scenario set to boost computational efficiency.

6 Conclusion and Future Research

6.1 Answering the Research Questions

The purpose of this study has been to develop a reliable framework for stochastic programming in regards of portfolio optimization and to investigate several different allocation models. The following research questions are answered with the use of quantitative analysis.

"Can stochastic programming improve the performance of a multi-asset portfolio?"

This research concludes that stochastic programming at the investigated time period of 2005-01-31 - 2018-01-08 is inferior to Risk Parity, but outperforms the Mean-Variance Model. The biggest flaw of the model is its poor performance during periods of market stress. Compared with the other two implemented models, the Mean-Absolute Deviation suffered the greatest losses during these periods.

"At which level of fund management is the use of optimizers suitable?"

This research also concludes that the optimizer with the more restrictive constraints, BRP, performs better than the minimum regulatory requirements, APL. This implies that these quantitative models benefit from restricted feasible regions based on a long-term investment rationale.

6.2 Future Research

As discussed, the poor performance of the Mean-Absolute Deviation model may be caused by the fact that the metric of its objective function does not penalize large deviations as much as the other two models and thus less reactive under distressed market conditions. Although this might explain the problem, it has not been isolated in this thesis. Therefore, it would be of interest to study if stochastic programming could benefit from a more sensible metric of risk applied on a multi-asset portfolio.

Another interesting aspect would be to investigate at which quantity of scenarios saturation of model accuracy is achieved. In this study scenario sets of $S = 100$, 500 and 1000 were implemented and it has been shown that no more than 500 were needed to achieve equivalent results. However, the limit of saturation lies between 100-500 and because of the high computational complexity it would be of value to determine it.

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A Appendix: Portfolio Assets

A.1 Asset Descriptions

A.2 Asset Correlations

The correlations between the assets described in section 3 Table 3.1 are presented in the figure below.

	w_0	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_10	w_11	w_12	w_13	w_14	w_15	w_16
w_0	1.0000	-0.4493	0.0428	-0.4503	-0.4513	-0.0771	0.4394	-0.4513	-0.4514	-0.0638	-0.4518	-0.0641	-0.4507	-0.4380	0.4496	0.4335	0.4344
w_1	-0.4493	1.0000	0.0865	1.0000	0.9999	-0.0531	0.4976	0.9998	0.9999	-0.0850	0.9999	-0.0455	0.9997	0.8037	-0.9999	-0.8070	-0.8056
w_2	0.0428	0.0865	1.0000	0.0835	0.0794	-0.3652	0.0699	0.0781	0.0787	-0.2148	0.0787	-0.3601	0.0799	0.0266	-0.0817	-0.0400	-0.0299
w_3	-0.4503	1.0000	0.0835	1.0000	0.9999	-0.0496	0.4967	0.9998	0.9999	-0.0814	0.9999	-0.0418	0.9997	0.8040	-0.9999	-0.8071	-0.8057
w_4	-0.4513	0.9999	0.0794	0.9999	1.0000	-0.0406	0.4964	1.0000	1.0000	-0.0753	0.9999	-0.0342	0.9999	0.8048	-0.9999	-0.8075	-0.8063
w_5	-0.0771	-0.0531	-0.3652	-0.0496	-0.0406	1.0000	-0.0726	-0.0383	-0.0394	0.6457	-0.0404	0.7929	-0.0384	0.0031	0.0496	0.0297	0.0221
w_6	0.4394	0.4976	0.0699	0.4967	0.4964	-0.0726	1.0000	0.4964	0.4966	-0.0618	0.4964	-0.0569	0.4964	0.3202	-0.4975	-0.3255	-0.3250
w_7	-0.4513	0.9998	0.0781	0.9998	1.0000	-0.0383	0.4964	1.0000	0.9999	-0.0728	0.9999	-0.0313	0.9999	0.8047	-0.9998	-0.8073	-0.8061
w_8	-0.4514	0.9999	0.0787	0.9999	1.0000	-0.0394	0.4966	0.9999	1.0000	-0.0738	1.0000	-0.0334	0.9998	0.8048	-0.9999	-0.8075	-0.8062
w_9	-0.0638	-0.0850	-0.2148	-0.0814	-0.0753	0.6457	-0.0618	-0.0728	-0.0738	1.0000	-0.0731	0.5317	-0.0739	-0.0386	0.0848	0.0726	0.0674
w_10	-0.4518	0.9999	0.0787	0.9999	0.9999	-0.0404	0.4964	0.9999	1.0000	-0.0731	1.0000	-0.0334	0.9998	0.8048	-0.9999	-0.8074	-0.8061
w_11	-0.0641	-0.0455	-0.3601	-0.0418	-0.0342	0.7929	-0.0569	-0.0313	-0.0334	0.5317	-0.0334	1.0000	-0.0302	0.0087	0.0385	0.0194	0.0073
w_12	-0.4507	0.9997	0.0799	0.9997	0.9999	-0.0384	0.4964	0.9999	0.9998	-0.0739	0.9998	-0.0302	1.0000	0.8042	-0.9997	-0.8068	-0.8056
w_13	-0.4380	0.8037	0.0266	0.8040	0.8048	0.0031	0.3202	0.8047	0.8048	-0.0386	0.8048	0.0087	0.8042	1.0000	-0.8041	-0.9989	-0.9988
w_14	0.4496	-0.9999	-0.0817	-0.9999	-0.9999	0.0496	-0.4975	-0.9998	-0.9999	0.0848	-0.9999	0.0385	-0.9997	-0.8041	1.0000	0.8073	0.8061
w_15	0.4335	-0.8070	-0.0400	-0.8071	-0.8075	0.0297	-0.3255	-0.8073	-0.8075	0.0726	-0.8074	0.0194	-0.8068	-0.9989	0.8073	1.0000	0.9996
w_16	0.4344	-0.8056	-0.0299	-0.8057	-0.8063	0.0221	-0.3250	-0.8061	-0.8062	0.0674	-0.8061	0.0073	-0.8056	-0.9988	0.8061	0.9996	1.0000

Figure A.1: Asset correlations.

B Appendix: Scenario Generation

B.1 Moment Matching

4,531E-05	3,050E-05	1,216E-05	-4,497E-05	-8,867E-05	-7,391E-05	-4,222E-05	-5,863E-05	-1,092E-04	-5,251E-05	-1,083E-05	-1,639E-05	-6,758E-05	-5,268E-05	-2,234E-06	-2,622E-05
3,050E-05	3,404E-05	1,331E-05	-3,062E-05	-9,437E-05	-6,314E-05	-2,326E-05	-4,811E-05	-8,661E-05	-4,631E-05	-1,351E-05	-1,797E-05	-5,100E-05	-5,014E-05	-2,864E-06	-1,867E-05
1,216E-05	1,331E-05	4,692E-05	2,255E-05	1,164E-05	2,538E-06	3,424E-05	1,419E-05	-1,228E-05	9,001E-06	2,459E-05	-1,496E-05	8,550E-07	-2,722E-05	-4,687E-06	-1,031E-05
-4,497E-05	-3,062E-05	2,255E-05	4,770E-04	4,759E-04	1,885E-04	4,984E-04	4,655E-04	3,939E-04	2,164E-04	2,232E-04	2,056E-04	3,239E-04	4,863E-05	-4,094E-05	-5,523E-06
-8,867E-05	-9,437E-05	1,164E-05	4,759E-04	1,085E-03	3,881E-04	4,978E-04	5,959E-04	6,722E-04	3,985E-04	3,901E-04	1,969E-04	5,108E-04	1,907E-04	-4,121E-05	5,594E-05
-7,391E-05	-6,314E-05	2,538E-06	1,885E-04	3,881E-04	8,386E-04	2,977E-04	2,862E-04	6,064E-04	4,542E-04	1,732E-04	1,469E-04	4,475E-04	1,043E-04	3,675E-06	4,099E-05
-4,222E-05	-2,326E-05	3,424E-05	4,984E-04	4,978E-04	2,977E-04	6,824E-04	4,917E-04	4,741E-04	3,082E-04	2,778E-04	2,304E-04	4,150E-04	3,455E-05	-4,557E-05	-2,866E-05
-5,863E-05	-4,811E-05	1,419E-05	4,655E-04	5,959E-04	2,862E-04	4,917E-04	5,027E-04	4,906E-04	2,871E-04	2,565E-04	2,112E-04	3,920E-04	8,492E-05	-4,399E-05	1,043E-05
-1,092E-04	-8,661E-05	-1,228E-05	3,939E-04	6,722E-04	6,064E-04	4,741E-04	4,906E-04	9,475E-04	4,556E-04	1,866E-04	1,995E-04	5,630E-04	2,647E-04	7,241E-06	1,348E-04
-5,251E-05	-4,631E-05	9,001E-06	2,164E-04	3,985E-04	4,542E-04	3,082E-04	2,871E-04	4,556E-04	3,714E-04	2,047E-04	1,385E-04	3,757E-04	8,049E-05	-1,661E-05	8,555E-06
-1,083E-05	-1,351E-05	2,459E-05	2,232E-04	3,901E-04	1,732E-04	2,778E-04	2,565E-04	1,866E-04	2,047E-04	4,336E-04	1,245E-04	1,804E-04	-1,229E-04	-5,143E-05	-1,181E-04
-1,639E-05	-1,797E-05	-1,496E-05	2,056E-04	1,969E-04	1,469E-04	2,304E-04	2,112E-04	1,995E-04	1,385E-04	1,245E-04	2,414E-04	1,823E-04	3,995E-06	-2,611E-05	-2,268E-05
-6,758E-05	-5,100E-05	8,550E-07	3,239E-04	5,108E-04	4,475E-04	4,150E-04	3,920E-04	5,630E-04	3,757E-04	1,804E-04	1,823E-04	5,866E-04	1,591E-04	-3,782E-05	8,579E-05
-5,268E-05	-5,014E-05	-2,722E-05	4,863E-05	1,907E-04	1,043E-04	3,455E-05	8,492E-05	2,647E-04	8,049E-05	-1,229E-04	3,995E-06	1,591E-04	2,379E-04	3,499E-05	1,492E-04
-2,234E-06	-2,864E-06	-4,687E-06	-4,094E-05	-4,121E-05	3,675E-06	-4,557E-05	-4,399E-05	7,241E-06	-1,661E-05	-5,143E-05	-2,611E-05	-3,782E-05	3,499E-05	4,018E-05	2,200E-05
-2,622E-05	-1,867E-05	-1,031E-05	-5,523E-06	5,594E-05	4,099E-05	-2,866E-05	1,043E-05	1,348E-04	8,555E-06	-1,181E-04	-2,268E-05	8,579E-05	1,492E-04	2,200E-05	2,119E-04

Figure B.1: Numerical values of the violated condition $\boldsymbol{\Sigma} - \boldsymbol{Z}\boldsymbol{Z}^T > 0$ from section 2.3.2.

B.2 DCC-GARCH

To test the stability of DCC-GARCH as a scenario generator in respect of 2.4.8 as a decision model, two independent sets of size S = 100 are generated. The optimal solutions are computed for both data sets and are presented in Table B.1.

Table B.1: Stability analysis of GARCH models.

	DCC		**Copula**	
Data set	**1**	**2**	**1**	**2**
$\mathbf{w_0}$	0.005	0.005	0.075	0.075
$\mathbf{w_1}$	0.000	0.000	0.000	0.000
$\mathbf{w_2}$	0.295	0.295	0.225	0.225
$\mathbf{w_3}$	0.000	0.000	0.000	0.000
$\mathbf{w_4}$	0.000	0.000	0.000	0.000
$\mathbf{w_5}$	0.000	0.000	0.000	0.000
$\mathbf{w_6}$	0.000	0.000	0.000	0.000
$\mathbf{w_7}$	0.000	0.000	0.000	0.000
$\mathbf{w_8}$	0.000	0.000	0.000	0.000
$\mathbf{w_9}$	0.000	0.000	0.000	0.000
$\mathbf{w_{10}}$	0.000	0.000	0.000	0.000
$\mathbf{w_{11}}$	0.000	0.000	0.000	0.000
$\mathbf{w_{12}}$	0.700	0.700	0.700	0.700
$\mathbf{w_{13}}$	0.000	0.000	0.000	0.000
$\mathbf{w_{14}}$	0.400	0.400	0.400	0.400
$\mathbf{w_{15}}$	0.000	0.000	0.000	0.000
$\mathbf{w_{16}}$	0.000	0.000	0.000	0.000

The optimal solutions are obtained with additional constraints 3.2.1, 3.2.2 and 3.2.4 for the last 4-week period of 2017.

B.3 Copula-GARCH

The numerical stability is tested as for DCC-GARCH. The optimal solutions for the two sets are given in Table B.1.
The same constraints and time period as in B.2 are considered.

C Appendix: Results

Figures C.2-C.4 below give a closer look at each strategy's performance during the critical year of 2008.

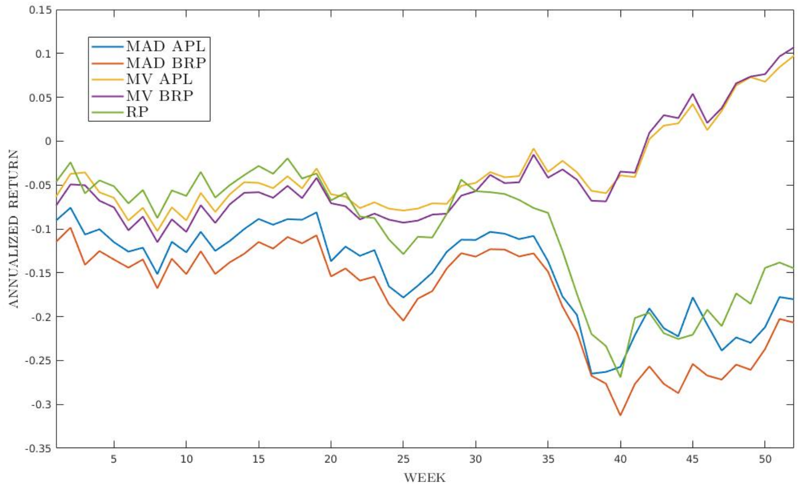

Figure C.1: Comparison between different optimization strategies in terms of annualized return in distressed year of 2008.

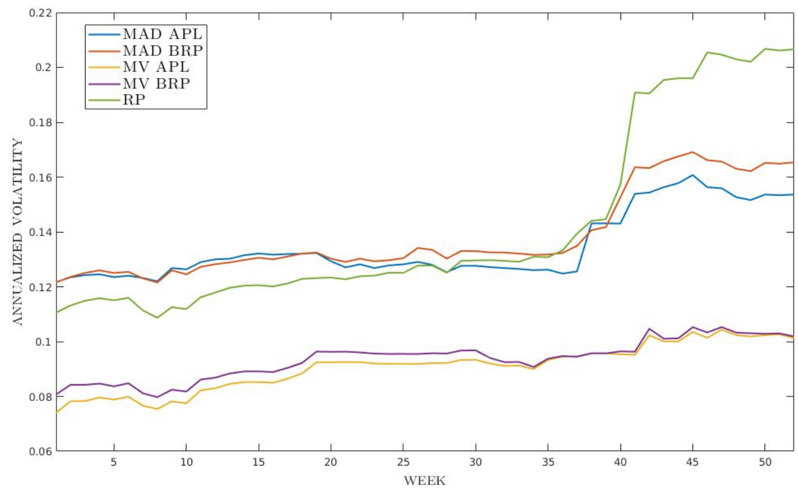

Figure C.2: Comparison between different optimization strategies in terms of annualized volatility in distressed year of 2008.

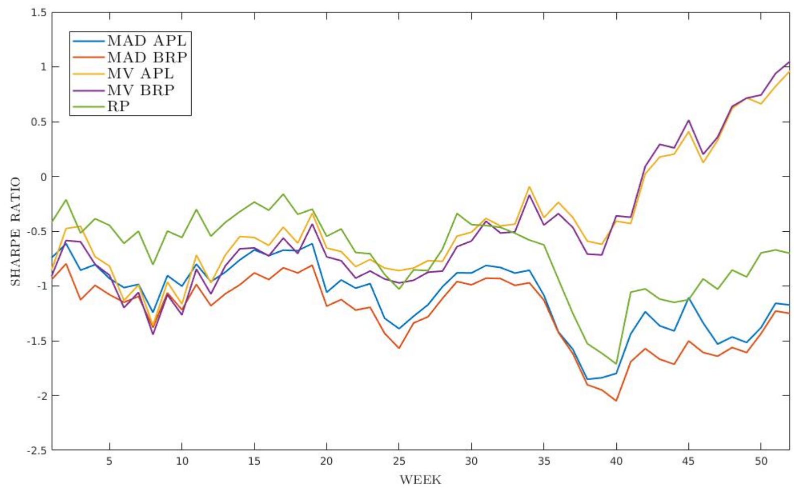

Figure C.3: Comparison between different optimization strategies in terms of sharpe ratio in distressed year of 2008.

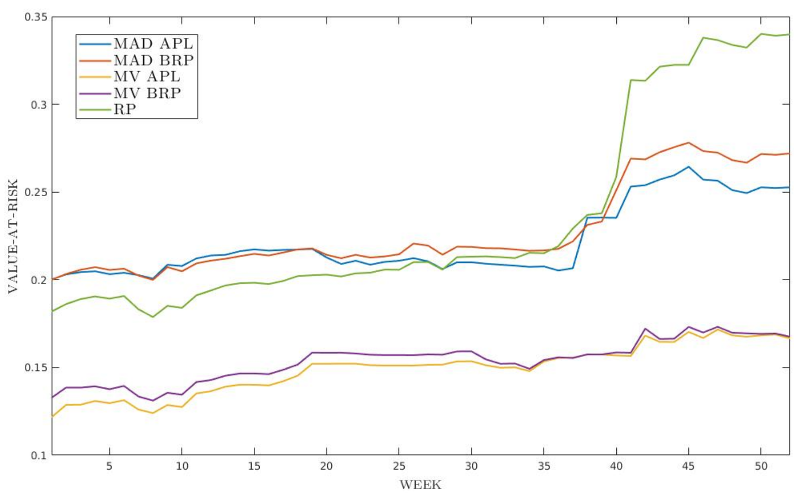

Figure C.4: Comparison between different optimization strategies in terms of annualized volatility in distressed year of 2008.

Tables C.1-C.3 summarizes numerical results from each strategy's returns, shape ratio and volatility between years 2006-2017.

Table C.1: Annualized returns for each strategy between 2006-2017.

Year	MAD APL	MAD BRP	MV APL	MV BRP	RP
2006	1.1002	1.1322	1.1707	1.1460	1.1643
2007	1.0448	1.0423	1.0301	1.0308	1.1018
2008	0.7938	0.7599	0.9652	0.9588	0.8948
2009	1.1454	0.9765	1.0889	1.1121	1.0924
2010	1.2574	1.1559	1.0985	1.1086	1.2580
2011	1.1083	1.0745	1.0795	1.1030	1.0865
2012	1.0492	1.0330	1.0391	1.1007	1.0998
2013	1.1172	1.1257	1.1988	1.2071	1.1541
2014	1.1377	1.0982	1.1253	1.1292	1.1269
2015	1.0836	1.0950	1.0631	1.0870	1.1265
2016	1.0026	0.9731	1.0379	1.0333	1.0395
2017	1.1198	1.1394	1.0504	1.0614	1.1312

Table C.2: Sharpe ratio for each strategy between 2006-2017.

Year	MAD APL	MAD BRP	MV APL	MV BRP	RP
2006	1.2525	1.7036	1.9598	1.7785	2.0867
2007	0.5165	0.4832	0.4613	0.4706	1.2046
2008	-1.5059	-1.5687	-0.4214	-0.4742	-0.6996
2009	0.7641	0.0256	1.1385	1.3705	0.5428
2010	2.9040	2.1996	2.3533	2.2613	2.4018
2011	1.6592	1.2124	1.4926	1.5455	1.0082
2012	0.4402	0.3655	0.8714	1.6182	1.0123
2013	1.5367	1.6529	2.9133	2.9734	2.4262
2014	1.9770	1.3843	1.6144	1.6327	1.9428
2015	1.0600	1.0808	0.7691	1.0405	1.5796
2016	0.1292	-0.1462	0.4927	0.4238	0.4398
2017	2.4528	2.6696	0.8600	1.0702	2.5787

Table C.3: Volatility for each strategy between 2006-2017.

Year	MAD APL	MAD BRP	MV APL	MV BRP	RP
2006	0.0776	0.0821	0.0876	0.0832	0.0844
2007	0.1127	0.1110	0.0670	0.0705	0.0890
2008	0.1415	0.1580	0.0907	0.0935	0.1417
2009	0.2182	0.2159	0.0811	0.0852	0.2113
2010	0.0859	0.0712	0.0421	0.0487	0.1040
2011	0.1031	0.0934	0.0543	0.0714	0.1069
2012	0.1374	0.1211	0.0473	0.0680	0.1201
2013	0.0748	0.0762	0.0688	0.0717	0.0636
2014	0.0701	0.0715	0.0772	0.0794	0.0661
2015	0.0873	0.0976	0.0860	0.0871	0.0874
2016	0.1039	0.1166	0.0850	0.0874	0.1054
2017	0.0483	0.0524	0.0602	0.0585	0.0509

C.1 Scenario Generators

Figures C.5-C.7 below give a closer look at each scenario generator's performance with APL constraints during the most recent year of 2017.

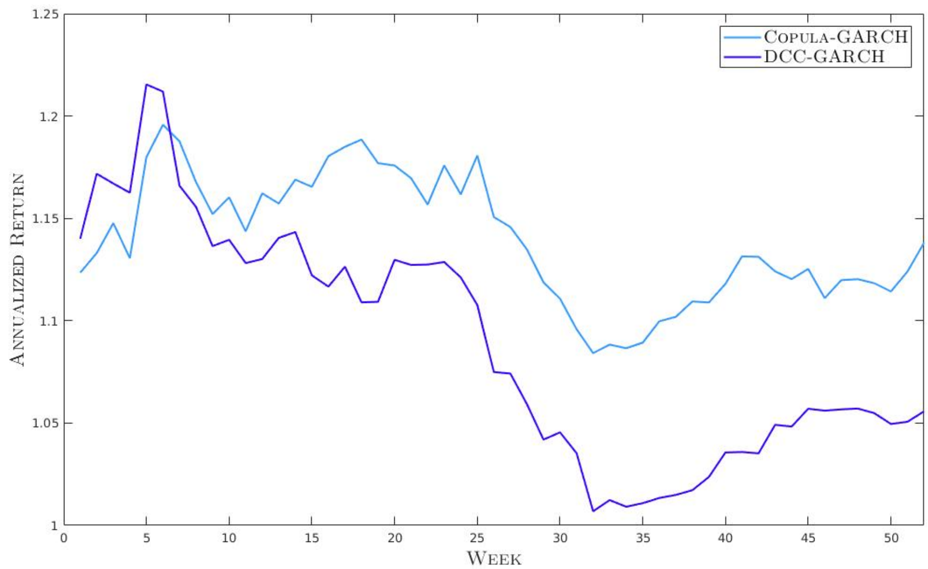

Figure C.5: Comparison of scenario generators with APL constraints in terms of annualized return for 2017.

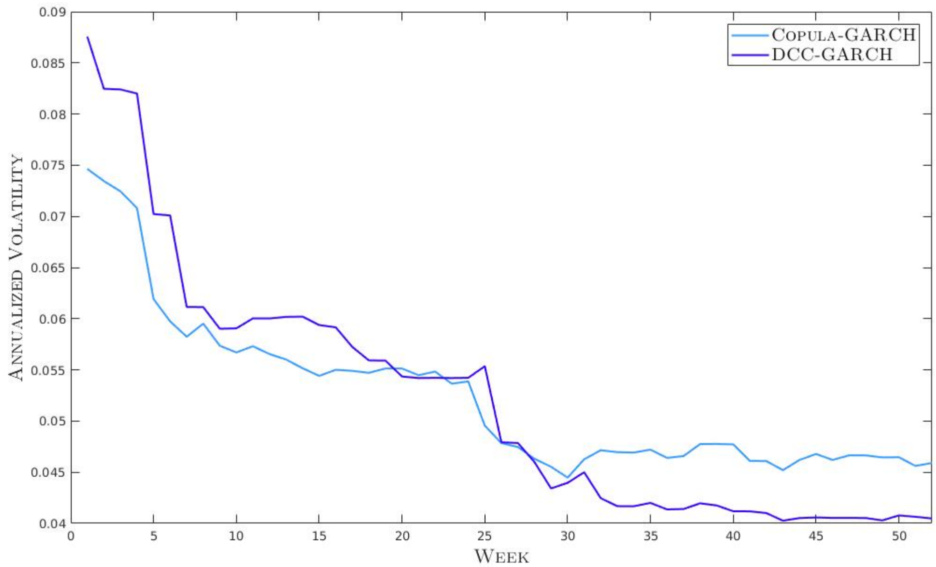

Figure C.6: Comparison of scenario generators with APL constraints in terms of annualized volatility for 2017.

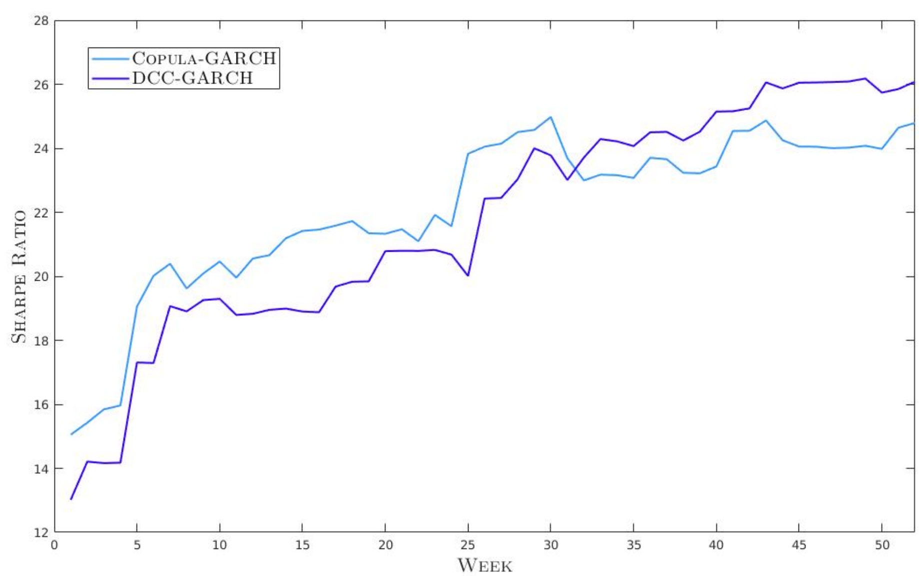

Figure C.7: Comparison of scenario generators with APL constraints in terms of Sharpe ratio for 2017.

Table C.4: Performance averages for scenario generators with BRP constraints for years 2013-2018.

Model	**Return**	**Volatility**	**Sharpe Ratio**	**Value-at-Risk**
Copula-GARCH	1.1156	0.0881	1.6469	0.1207
DCC-GARCH	1.0783	0.0781	1.0971	0.1071

Figures C.8-C.11 exhibits the different scenario generator's performance with BRP constraints in terms of annualized return, annualized volatility, Value-at-Risk and Sharpe ratio between years 2013-2017.

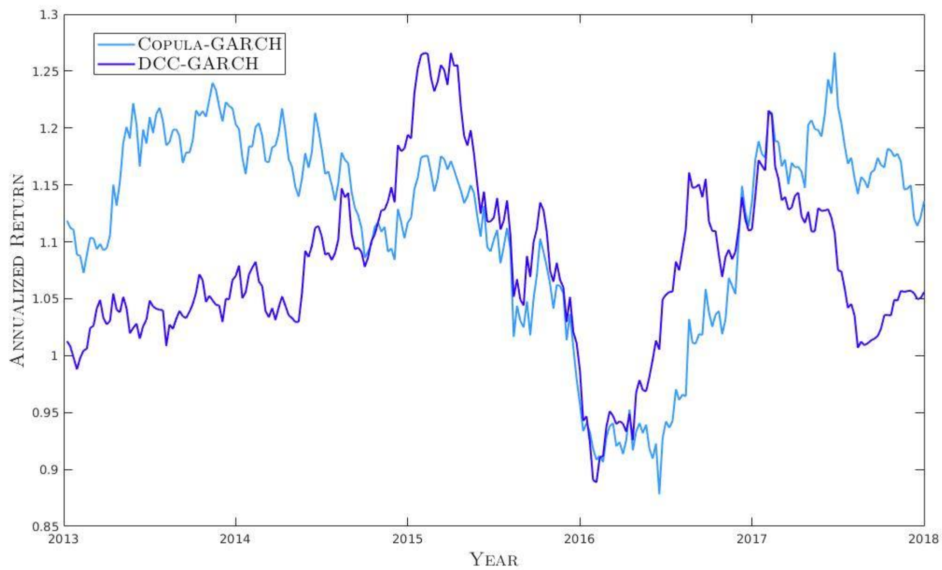

Figure C.8: Comparison of scenario generators with BRP constraints in terms of annualized return for 2013-2017.

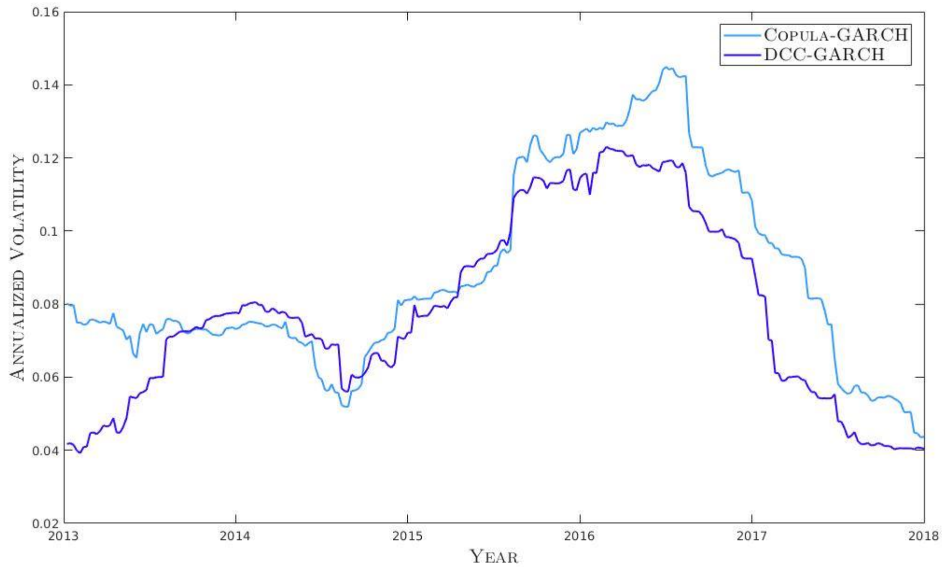

Figure C.9: Comparison of scenario generators with BRP constraints in terms of annualized volatility for 2013-2017.

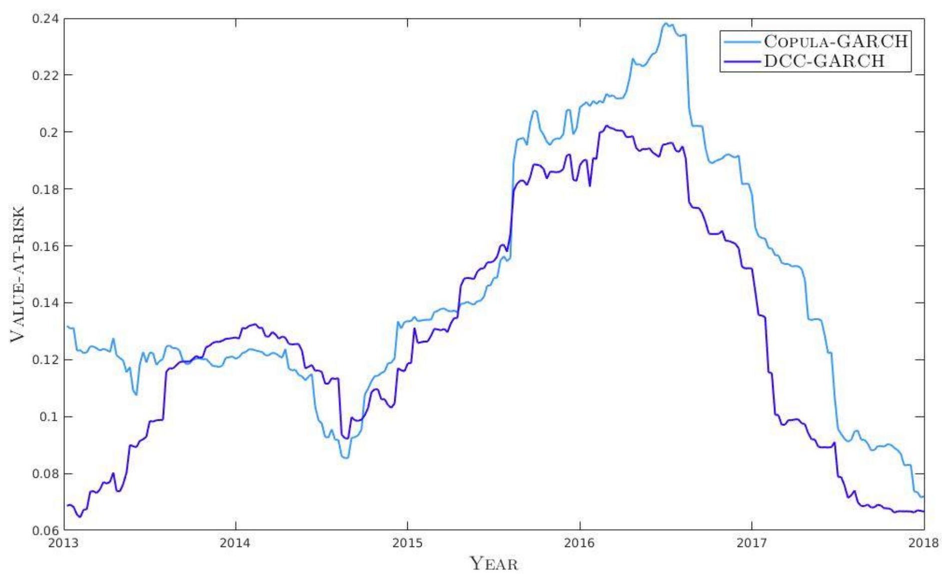

Figure C.10: Comparison of scenario generators with BRP constraints in terms of 5% Value-at-Risk for 2013-2017.

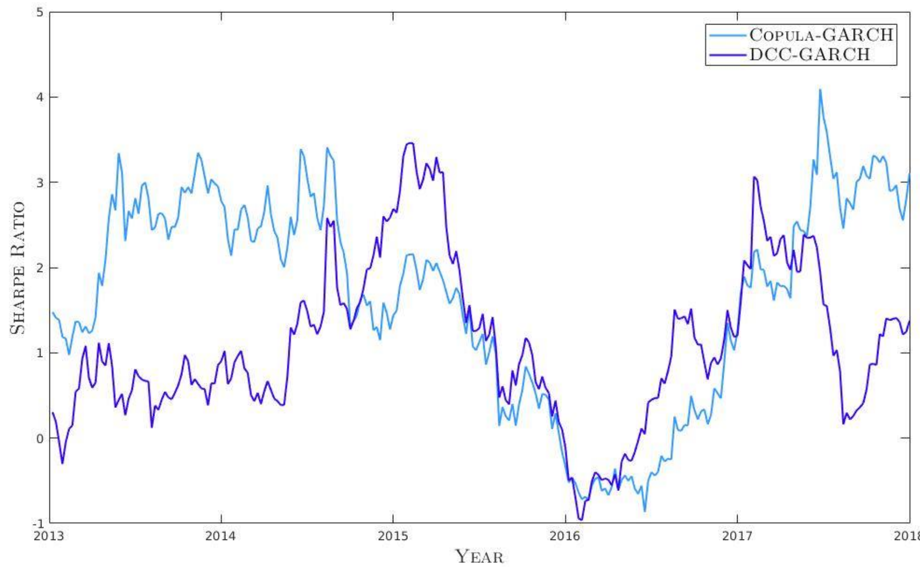

Figure C.11: Comparison of scenario generators with BRP constraints in terms of Sharpe ratio for 2013-2017.

C.2 Scenario Sizes

Tables C.5-C.7 below summarizes the numerical results between different scenario sizes annualized return, Sharpe ratio and annualized volatility produced by DCC-GARCH between years 2013-2017.

Table C.5: Comparison of scenario sizes for DCC-GARCH in terms of annualized return between 2013-2017.

Year	**100**	**500**	**1000**
2013	1.0287	1.0362	1.0362
2014	1.1546	1.1257	1.1257
2015	1.1611	1.1281	1.1281
2016	1.0061	0.9536	0.9536
2017	1.0334	1.0356	1.0356

Table C.6: Comparison of scenario sizes for DCC-GARCH in terms of Sharpe ratio between 2013-2017.

Year	**100**	**500**	**1000**
2013	0.4107	0.3852	0.3852
2014	2.3427	1.6325	1.6325
2015	1.9490	1.5258	1.5258
2016	0.0737	-0.4301	-0.4301
2017	0.7484	0.6620	0.6620

Table C.7: Comparison of scenario sizes for DCC-GARCH in terms of annualized volatility between 2013-2017.

Year	**100**	**500**	**1000**
2013	0.0652	0.0662	0.0662
2014	0.0750	0.0720	0.0720
2015	0.0868	0.0871	0.0871
2016	0.0982	0.1123	0.1123
2017	0.0564	0.0551	0.0551

Tables C.8-C.12 below summarizes the numerical results between different scenario sizes of annualized return, Shape ratio, annualized volatility, 5% Value-at-Risk and 1% Value-at-Risk produced by Copula-GARCH between years 2006-2017.

Table C.8: Annualized returns from Copula-GARCH for different scenario sizes between 2006-2017.

Year	**100**	**500**	**1000**
2006	1.1002	1.1284	1.1284
2007	1.0448	1.0916	1.0916
2008	0.7938	0.8508	0.8508
2009	1.1454	1.0863	1.0863
2010	1.2574	1.3110	1.3110
2011	1.1083	1.1035	1.1035
2012	1.0492	1.0405	1.0405
2013	1.1172	1.1843	1.1843
2014	1.1377	1.1779	1.1779
2015	1.0836	1.0736	1.0736
2016	1.0026	0.9638	0.9638
2017	1.1198	1.1575	1.1575

Table C.9: Sharpe Ratio from Copula-GARCH for different scenario sizes between 2006-2017.

Year	100	500	1000
2006	1.2525	1.1349	1.1349
2007	0.5165	0.7072	0.7072
2008	-1.5059	-1.0930	-1.0930
2009	0.7641	0.5449	0.5449
2010	2.9040	3.6325	3.6325
2011	1.6592	1.6667	1.6667
2012	0.4402	0.4262	0.4262
2013	1.5367	2.4676	2.4676
2014	1.9770	2.6470	2.6470
2015	1.0600	0.9562	0.9562
2016	0.1292	-0.2849	-0.2849
2017	2.4528	2.6981	2.6981

Table C.10: Annualized volatility from Copula-GARCH for different scenario sizes between 2006-2017.

Year	100	500	1000
2006	0.0776	0.1215	0.1215
2007	0.1127	0.1206	0.1206
2008	0.1415	0.1346	0.1346
2009	0.2182	0.1731	0.1731
2010	0.0859	0.0872	0.0872
2011	0.1031	0.0948	0.0948
2012	0.1374	0.1273	0.1273
2013	0.0748	0.0747	0.0747
2014	0.0701	0.0677	0.0677
2015	0.0873	0.0833	0.0833
2016	0.1039	0.1160	0.1160
2017	0.0483	0.0630	0.0630

Table C.11: Value-at-Risk calculations at 95% confidence level from Copula-GARCH and DCC-GARCH simulations.

$\mathbf{VaR_{0.05}}$						
	Copula-GARCH			**DCC-GARCH**		
Year	**100**	**500**	**1000**	**100**	**500**	**1000**
2007	0.1276	0.1999	0.1999	-	-	-
2008	0.1853	0.1983	0.1983	-	-	-
2009	0.2328	0.2214	0.2214	-	-	-
2010	0.3589	0.2848	0.2848	-	-	-
2011	0.1412	0.1434	0.1434	-	-	-
2012	0.1696	0.1559	0.1559	-	-	-
2013	0.2260	0.2094	0.2094	-	-	-
2014	0.1231	0.1229	0.1229	0.1083	0.1089	0.1089
2015	0.1153	0.1113	0.1113	0.1233	0.1184	0.1184
2016	0.1436	0.1452	0.1452	0.1427	0.1432	0.1432
2017	0.1708	0.1907	0.1907	0.1615	0.1847	0.1847

Table C.12: Value-at-Risk calculations at 99% confidence level from Copula-GARCH and DCC-GARCH simulations.

$\mathbf{VaR_{0.01}}$						
	Copula-GARCH			**DCC-GARCH**		
Year	**100**	**500**	**1000**	**100**	**500**	**1000**
2007	0.1804	0.2828	0.2828	-	-	-
2008	0.2621	0.2805	0.2805	-	-	-
2009	0.3292	0.3131	0.3131	-	-	-
2010	0.5076	0.4028	0.4028	-	-	-
2011	0.1997	0.2028	0.2028	-	-	-
2012	0.2399	0.2205	0.2205	-	-	-
2013	0.3196	0.2962	0.2962	-	-	-
2014	0.1741	0.1738	0.1738	0.1531	0.1541	0.1541
2015	0.1630	0.1574	0.1574	0.1744	0.1675	0.1675
2016	0.2030	0.2054	0.2054	0.2284	0.2026	0.2026
2017	0.2416	0.2698	0.2698	0.1313	0.2612	0.2612

Table C.13 summarizes the averages of the different scenario generator's performances.

Table C.13: Performance averages for scenario generators with APL constraints for years 2013-2018.

	COPULA-GARCH				DCC-GARCH			
Size	**Return**	**Vol**	**SR**	$\mathbf{VaR_{0.95}}$	**Return**	**Vol**	**SR**	$\mathbf{VaR_{0.95}}$
100	1.1135	0.1032	1.4359	0.1868	1.1041	0.1778	1.2667	0.1257
500	1.1220	0.0991	1.6394	0.1761	1.1063	0.1827	0.8558	0.1292
1000	1.1220	0.0991	1.6394	0.1761	1.0663	0.1827	0.8558	0.1292