Flat families by strongly stable ideals and a generalization of Gröbner bases

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Setting

$$S := K[x_0, \ldots, x_n], \quad x_0 < x_1 < \ldots < x_n,$$
$$x^\alpha = x_0^{\alpha_0} x_1^{\alpha_1} \ldots x_n^{\alpha_n}, \quad \min(x^\alpha) := \min\{x_i : \alpha_i > 0\}$$

- J any monomial ideal and B_J its monomial (minimal) basis
- $\mathcal{N}(J)$ the set of terms outside J (sous-escalier)
- $Supp(h)$ the support of a polynomial h, i.e., the set of terms that occur in h with non-zero coefficients
- Given an ideal I, a *J-reduced form modulo I* of a polynomial h is a polynomial h_0 such that $Supp(h_0) \subseteq \mathcal{N}(J)$ and $h - h_0 \in I$

Framework

AIM: to determine the set $\mathcal{M}\mathrm{f}(J)$ of all the *homogeneous* ideals I of S such that $S = I \oplus \langle \mathcal{N}(J) \rangle$ as a K-vector space (i.e., S/I has $\mathcal{N}(J)$ as K-vector basis)

TOOL: J-marked basis; when J is strongly stable, we have a version of the division algorithm *without* term orders and a Buchberger-like Criterion

IMPLICATIONS: $\mathcal{M}\mathrm{f}(J)$ is a *family* (J-marked family) that can be endowed with a structure of an affine scheme which is homogeneous with respect to a non-standard grading and flat at the origin

APPLICATION: computation in Hilbert schemes

J-marked set

- (Reeves and Sturmfels, 1993) A *marked polynomial* is a polynomial $f \in S$ together with a specified term $Ht(f)$ of $Supp(f)$ that will be called *head term of f*. A *reduction relation* $\xrightarrow{\mathcal{F}}$ modulo a given set $\mathcal{F}$ of marked polynomials can be defined in the usual sense of Gröbner bases theory.
- Let G be a finite set of *homogeneous* marked polynomials $f_\alpha = x^\alpha - \sum c_{\alpha\gamma} x^\gamma$, with $Ht(f_\alpha) = x^\alpha$,
  J monomial ideal

  G is a *J-marked set* if: $Ht(f_\alpha) \neq Ht(f_{\alpha'})$,
  $\{Ht(f_\alpha), \ldots\} = B_J$ and every x^γ belongs to $\mathcal{N}(J)$

J-marked set and J-marked basis: properties

A J-marked set G is a *J-marked basis* if $\mathcal{N}(J)$ is a basis of $S/(G)$ as a K-vector space, i.e. $S = (G) \oplus \langle \mathcal{N}(J) \rangle$ as a K-vector space
$\Leftrightarrow$ $(G) \in \mathcal{M}\mathrm{f}(J)$
$\Leftrightarrow$ any polynomial h of S has a unique J-reduced form modulo (G)

- The ideal (G) generated by a J-marked basis has the same Hilbert function as J;
- $S = I \oplus \langle \mathcal{N}(J) \rangle$ (i.e. $I \in \mathcal{M}\mathrm{f}(J)$) $\Rightarrow$ I contains a J-marked set G (but $\not\Rightarrow$ $I = (G)$)
- a J-marked family $\mathcal{M}\mathrm{f}(J)$ contains all the homogeneous ideals with J as initial ideal (w.r.t. some term order), but also more . . .

Example

$K[x,y,z]$, $J=(xy,z^2)$
$G_1=\{f_1=xy+yz, f_2=z^2+xz\}$, $f_3=xyz$;
$G_2=\{f_1'=xy+x^2-yz, f_2'=z^2+y^2-xz\}$

(1) $I_1=(G_1)$ and J have not the same Hilbert function: G_1 *is NOT a J-marked basis*

(2) $I_2=(G_2)$ complete intersection,
I_2 and J have the same Hilbert function, but $\mathcal{N}(J)$ is not free in $K[x,y,z]/I_2$ [because $zf_1'+yf_2'=x^2z+y^3\in I_2$ is a sum of terms in $\mathcal{N}(J)$]: G_2 *is NOT a J-marked basis*

(3) $I_3=(f_1,f_2,f_3)$ belongs to $\mathcal{M}\mathrm{f}(J)$, but $(G_2)\neq I_3$ (and G_2 is not a J-marked basis)

J-marked sets: reduction relation

$K[x, y, z]$, $J = (xy, z^2)$

$G_1 = \{f_1 = xy + yz, f_2 = z^2 + xz\}$ J-marked set

$$xyz \xrightarrow{G_1} xyz - zf_1 = -yz^2 \xrightarrow{G_1} yz^2 + yf_2 = xyz$$

The reduction relation $\xrightarrow{G_1}$ is not Noetherian.

In this case $J = (xy, z^2)$ is not strongly stable, but $\xrightarrow{G}$ can be non-Noetherian also if J is strongly stable

J strongly stable

$x_0 < x_1 < \ldots < x_n$

J strongly stable:

$$x_0^{\alpha_0} \ldots x_n^{\alpha_n} \in J \quad \Rightarrow \quad x_0^{\alpha_0} \ldots x_i^{\alpha_i - 1} \ldots x_j^{\alpha_j + 1} \ldots x_n^{\alpha_n} \in J$$
$$\forall\ 0 \leq i < j \leq n : \alpha_i > 0$$

or, equivalently,

$$x_0^{\beta_0} \ldots x_n^{\beta_n} \in \mathcal{N}(J) \quad \Rightarrow \quad x_0^{\beta_0} \ldots x_h^{\beta_h + 1} \ldots x_k^{\beta_k - 1} \ldots x_n^{\beta_n} \in \mathcal{N}(J)$$
$$\forall\ 0 \leq h < k \leq n : \beta_k > 0$$

Example

$S := K[x,y,z]$ $(x < y < z)$
$J = (z^3, z^2y, zy^2, y^5)_{\geq 4}$ strongly stable ideal
$f = zy^2x - y^4 - z^2x^2$ with $Ht(f) = zy^2x$,
$G = B_J \cup \{f\} \setminus \{zy^2x\}$ a J-marked set (that is a J-marked basis)

We have the loop:

$$z^2y^2x^3 \xrightarrow{f} zy^4x^2 + z^3x^4 \xrightarrow{z^3x^2} zy^4x^2 \xrightarrow{f} y^6x + z^2y^2x^3 \xrightarrow{y^5} z^2y^2x^3$$

The reduction relation $\xrightarrow{G}$ is not Noetherian.

G is not a Gröbner basis with respect to any term order $\prec$. Indeed, $zy^2x^2 \succ y^4x$ and $zy^2x^2 \succ z^2x^3$ would be in contradiction with the equality $(zy^2x^2)^2 = z^2x^3 \cdot y^4x$!!!

Existence of J-reduced forms

J strongly stable
$G = \{f_\alpha = x^\alpha - \sum c_{\alpha\gamma} x^\gamma : Ht(f_\alpha) = x^\alpha \in B_J\}$ a J-marked set.

Theorem

Every polynomial of S has a J-reduced form modulo (G)

Any J:

- G J-marked basis $\Rightarrow$ (G) and J have the same Hilbert function
- I has a J-marked basis $\Rightarrow$ $I \in \mathcal{M}\mathrm{f}(J)$

J strongly stable:

- G J-marked basis $\Leftrightarrow$ $dim_K(G)_m = dim_K J_m$, for every $m \geq 0$ ($\mathcal{N}(J)$ is free in $S/(G)$)
- I has a J-marked basis $\Leftrightarrow$ $I \in \mathcal{M}\mathrm{f}(J)$

Existence of J-reduced forms: sketch of the proof

Hp: J strongly stable, G J-marked set
Th: $E := \{x^\beta : x^\beta$ has not a J-reduced form modulo $(G)\} = \emptyset$

$E \subset J \setminus \{B_J\}$: $x^\beta \in E \Rightarrow x^\beta = x_i x^\delta$ with $x^\delta \in J$
such that: $m = deg(x^\beta)$ minimum possible in E, x_i minimum possible in E_m
hence, $x^\delta \notin E$: $x^\delta - \sum c_{\delta\gamma} x^\gamma \in (G)$, with $x^\gamma \in \mathcal{N}(J)$

$$x^\beta - \left(x_i x^\delta - x_i \sum c_{\delta\gamma} x^\gamma\right) = \sum c_{\delta\gamma} x_i x^\gamma$$

Claim: $x_i x^\gamma \notin E$

Otherwise, $x_i x^\gamma \in E$, so that $x_i x^\gamma = x_j x^\epsilon$ with $x^\epsilon \in J$;
we have $x_i > x_j$, otherwise $x^\epsilon = \dfrac{x^\gamma}{x_j} x_i \in \mathcal{N}(J)$ (contradiction!)

Construction of J-reduced forms

J strongly stable, G J-marked set, $I = (G)$

Lemma

$x^\beta \in J_m \setminus B_J$, $x_i = \min(x^\beta) \Rightarrow x^\beta / x_i \in J_{m-1}$

$m = \alpha_J$ initial degree of J; $V_m := G_m$

$m > \alpha_J + 1$; for every $x^\beta \in J_m \setminus B_J$, we take

- $x_i = \min(x^\beta)$, and
- g_ϵ the *unique* polynomial of V_{m-1} with head term $x^\epsilon = x^\beta / x_i$

thus $g_\beta := x_i g_\epsilon$ with $Ht(g_\beta) = x^\beta = x_i x^\epsilon$, and
$V_m := G_m \cup \{g_\beta \ : \ x^\beta \in J_m \setminus B_J\}$

Proposition

The reduction relation $\xrightarrow{V_m}$ *is Noetherian in* S_m

The reduction relation $\xrightarrow{V_m}$ is Noetherian because of the following *order* $\preceq_m$ *on* V_m:

let $\geq$ be any order on G_m

$$f_\alpha, f_{\alpha'} \in G_m, \quad f_\alpha \succeq_m f_{\alpha'} \Leftrightarrow f_\alpha \geq f_{\alpha'}$$

$$g_\beta \in V_m \setminus G_m, \quad f_\alpha \in G_m \Rightarrow g_\beta \succ_m f_\alpha$$

$$x_i g_\epsilon \succeq_m x_j g_\eta \Leftrightarrow x_i > x_j \text{ or } g_\epsilon \succeq_{m-1} g_\eta$$

Property:

$$x_i g_\epsilon \in V_m \setminus G_m \text{ and } x^\beta \in (Supp(x_i g_\epsilon) \cap J_m) \setminus \{x_i x^\epsilon\} \Rightarrow x_i g_\epsilon \succ_m g_\beta$$

ReducedFormConstructor$(h, V_m) \rightarrow \bar{h}$
INPUT: a homogeneous polynomial h of degree m
the list V_m ordered by $\succeq_m$
OUTPUT: V_m-reduction $\bar{h}$ of h

```
begin
    L := |V_m|;
    for K = 1 to L do
        x^η := Ht(V_m[K]);
        a:=coefficient of x^η in h;
        if a ≠ 0 then
            h := h − a · V_m[K];
        endif;
    endfor;
    return h;
end;
```

Buchberger-like criterion

The *S-polynomial* of two elements f_α, $f_{\alpha'}$ of a J-marked set G is $S(f_\alpha, f_{\alpha'}) := x^\beta f_\alpha - x^{\beta'} f_{\alpha'}$, where $x^{\beta+\alpha} = x^{\beta'+\alpha'} = lcm(x^\alpha, x^{\alpha'})$.

Theorem

Let J be a strongly stable ideal and I the homogeneous ideal generated by a J-marked set G. With the above notation:

$$I \in \mathcal{M}\mathrm{f}(J) \Leftrightarrow \overline{S(f_\alpha, f_{\alpha'})} = 0, \forall f_\alpha, f_{\alpha'} \in G$$

To prove the Theorem we consider the set of marked polynomials

$$W_m = \{x^\delta f_\alpha : x^{\delta+\alpha} \text{ has degree } m, f_\alpha \in G\},$$

with $Ht(x^\delta f_\alpha) = x^{\delta+\alpha}$, for which the following property holds.

Property

If $x^\delta f_\alpha \in W_m$, $x^\epsilon \in Supp(f_\alpha)$ and $x^{\delta+\epsilon} = x^{\alpha'+\delta'}$, with $x^\delta f_\alpha \neq x^{\delta'} f_{\alpha'} \in V_m$, then $\max\left(\frac{x^{\delta'}}{gcd(x^\delta, x^{\delta'})}\right) < \max\left(\frac{x^\delta}{gcd(x^\delta, x^{\delta'})}\right)$.

Then we can order W_m, for instance, in the following way:

$$x^\delta f_\alpha \geq_m x^{\delta'} f_{\alpha'} \Leftrightarrow x^\delta >_{lex} x^{\delta'} \text{ or } x^\delta = x^{\delta'} \text{ and } f_\alpha \geq f_{\alpha'}$$

Buchberger-like criterion: sketch of the proof

Hp: $\overline{S(f_\alpha, f_{\alpha'})} = 0, \forall f_\alpha, f_{\alpha'} \in G$
Th: $x^\delta f_\alpha \in W_m \Rightarrow x^\delta f_\alpha \in \langle V_m \rangle$

The thesis holds for every $x^{\bar\delta} f_{\bar\alpha} \in W_m$ such that $x^{\bar\delta} f_{\bar\alpha} <_m x^\delta f_\alpha$
(the minimum polynomial of W_m belongs to V_m by construction)

Let $x^{\delta'} f_{\alpha'} \in V_m$ such that $x^\delta x^\alpha = x^{\delta'} x^{\alpha'}$

- $x^\delta f_\alpha - x^{\delta'} f_{\alpha'} = S(f_\alpha, f_{\alpha'})$: it is enough to apply the hypothesis;
- $x^\delta f_\alpha - x^{\delta'} f_{\alpha'} = x^\beta S(f_\alpha, f_{\alpha'}) = x^\beta (x^\eta f_\alpha - x^{\eta'} f_{\alpha'})$, $x^\beta \neq 1$:
  $x^\eta f_\alpha = x^{\beta'} f_{\alpha'} + \sum c_i g_{\eta_i}$, with $g_{\eta_i} <_m x^\eta f_\alpha$ (by the hypothesis)

  hence:
  $x^\delta f_\alpha = x^{\delta'} f_{\alpha'} + \sum c_i x^\beta g_{\eta_i}$, with $x^\beta g_{\eta_i} <_m x^\delta f_\alpha$

Syzygies

J strongly stable
G J-marked basis

Corollary

Every homogeneous syzygy of J lifts to a syzygy of G.

Corollary

Let $\{M_1, \dots, M_t\}$ be a set of homogeneous generators of the module of syzygies of J. Then, a set $\{K_1, \dots, K_t\}$ of liftings of the M_i generates the module of syzygies of G.

Gröbner-like bases

Staggered Bases
(Gebauer, Möller 1986) (Möller, Mora, Traverso 1992)

Border Bases
(Auzinger, Stetter 1988) (Marinari, Möller, Mora 1993) (Mourrain 1999, Mourrain, Trèbuchet 2005, 2008)

Involutive Bases (Janet Bases, Pommaret Bases)
(Janet 1920, 1929) (Pommaret 1978) (Gerdt 2000, Gerdt, Blinkov 1998)

In group rings: (Madlener, Reinert 1991, 1993)

J-marked family as an affine scheme

J strongly stable, $x^\alpha \in B_J$

$F_\alpha := x^\alpha - \sum C_{\alpha\gamma} x^\gamma \in K[C][X]$, where $x^\gamma \in \mathcal{N}(J)_{|\alpha|}$, $N := |C|$.

$\mathcal{G} := \{F_\alpha : x^\alpha \in B_J\}$ is a J-marked set letting $Ht(F_\alpha) = x^\alpha$.

Remark

$\mathcal{G}$ becomes the J-marked basis of an ideal $I \in \mathcal{M}\mathrm{f}(J)$ for a (unique) specialization of the variables C in K^N (every ideal $I \in \mathcal{M}\mathrm{f}(J)$ has a unique J-marked basis)

But not every specialization gives rise to an ideal of $\mathcal{M}\mathrm{f}(J)$

We use the Buchberger-like criterion

$\mathfrak{R} :=$ (coefficients in K[C] of $\overline{S(F_\alpha, F_{\alpha'})} : F_\alpha, F_{\alpha'} \in \mathcal{G}$)

Theorem

There is a one to one correspondence between the ideals of $\mathcal{M}\mathrm{f}(J)$ and the points of the affine scheme in K^N defined by the ideal $\mathfrak{R}$

If J is considered as an initial ideal:
(Carrà Ferro 1988; Notari, Spreafico 2000; Ferrarese, Roggero 2009; Lella, Roggero 2009; Robbiano 2009; Roggero, Terracini 2010)

J-marked schemes

Let A_m be the matrix whose *columns* correspond to the terms of degree m in S and whose *rows* contain the coefficients of the terms in all $x^\delta F_\alpha$ with $m = |\delta + \alpha|$

(every entry of the matrix A_m is 1, 0 or one of the variables C)

$\mathfrak{A} \subset K[C]$ ideal generated by the minors of order $dim_K J_m + 1$ of A_m, for every m.

Lemma

The ideal $\mathfrak{A}$ is equal to the ideal $\mathfrak{R}$: the construction of the ideal $\mathfrak{R}$ does not depend on the procedure of reduction

The ideal $\mathfrak{R} = \mathfrak{A}$ defines an affine scheme called *J-marked scheme*

J-marked schemes: some properties

Theorem

The J-marked scheme is homogeneous with respect to a non-standard grading λ of $K[C]$ over the group $\mathbb{Z}^{n+1}$ given by $\lambda(C_{\alpha\gamma}) = \alpha - \gamma$.

$reg(I)$ Castelnuovo-Mumford regularity of I

Proposition

The J-marked family $\mathcal{M}\mathrm{f}(J)$ is flat at the origin. In particular, for every ideal I in $\mathcal{M}\mathrm{f}(J)$, we get $reg(J) \geq reg(I)$.

An explicit computation in $\mathcal{H}ilb^2_8$

$J = (z^3, z^2y, zy^2, y^5)_{\geq 4} \subset K[x, y, z]$, K algebraically closed (where $x < y < z$ and $ch(K) = 0$)

J is strongly stable, not a segment (in the usual meaning) w.r.t. any term order

$$F_1 = z^4 + c_1x^2z^2 + c_2y^4 + c_3x^2yz + c_4xy^3 + c_5x^3z + c_6x^2y^2 + c_7x^3y + c_8x^4,$$

$$F_2 = z^3y + c_9x^2z^2 + c_{10}y^4 + c_{11}x^2yz + c_{12}xy^3 + c_{13}x^3z + c_{14}x^2y^2 + c_{15}x^3y + c_{16}x^4,$$

$$F_3 = z^2y^2 + c_{17}x^2z^2 + c_{18}y^4 + c_{19}x^2yz + c_{20}xy^3 + c_{21}x^3z + c_{22}x^2y^2 + c_{23}x^3y + c_{24}x^4,$$

$$F_4 = zy^3 + c_{25}x^2z^2 + c_{26}y^4 + c_{27}x^2yz + c_{28}xy^3 + c_{29}x^3z + c_{30}x^2y^2 + c_{31}x^3y + c_{32}x^4,$$

$$F_5 = z^3x + c_{33}x^2z^2 + c_{34}y^4 + c_{35}x^2yz + c_{36}xy^3 + c_{37}x^3z + c_{38}x^2y^2 + c_{39}x^3y + c_{40}x^4,$$

$$F_6 = z^2yx + c_{41}x^2z^2 + c_{42}y^4 + c_{43}x^2yz + c_{44}xy^3 + c_{45}x^3z + c_{46}x^2y^2 + c_{47}x^3y + c_{48}x^4,$$

$$F_7 = zy^2x + c_{49}x^2z^2 + c_{50}y_4 + c_{51}x^2yz + c_{52}xy^3 + c_{53}x^3z + c_{54}x^2y^2 + c_{55}x^3y + c_{56}x^4,$$

$$F_8 = y^5 + c_{57}x^3z^2 + c_{58}xy^4 + c_{59}x^3yz + c_{60}x^2y^3 + c_{61}x^4z + c_{62}x^3y^2 + c_{63}x^4y + c_{64}x^5.$$

The affine scheme $\mathcal{M}f(J)$ can be embedded as an open set in $\mathcal{H}ilb_8^2$

By our computations we check that:

- $\mathcal{M}f(J)$ is smooth and rational;
- $\mathcal{M}f(J)$ has dimension 16.

Moreover, $\mathcal{M}f(J)$ can be embedded in a $\mathbb{A}^{19}$ and is homogeneous with respect to a non-positive grading

$\#$ global Plücker coordinates $= \binom{45}{37} - 1 > 2 * 10^8$

$\#$ local Plücker coordinates $= 37 * 8 = 296$

$N = \#$ variables of type $C = 8 * 8 = 64$

that becomes 19 after repeated eliminations

Example

$J = (x_3^2, x_3x_2, x_2^3) \subset K[x_0, \ldots, x_3]$

Hilbert polynomial $= 4t$

Gotzmann number $= 6$

$\#$ global Plücker coordinates $= \binom{84}{60} - 1 > 641 * 10^{18}$

$\#$ local Plücker coordinates $= 60 * 24 = 1440$

$N = \#$ variables of type $C = 2 * 8 + 12 = 28$

Future work:

- Relations among J-marked bases and the other Gröbner-like bases
- Improvements of the constructions
- Borel open covering of Hilbert schemes (Bertone, Lella, Roggero)

THANK YOU