# On irregular binomial D-modules

María Cruz Fernández Fernández
(joint with Francisco Jesús Castro Jiménez)

University of Extremadura (Spain)

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# A $D$-module

The Weyl Algebra:
$D = \mathbb{C}[x_1, \ldots, x_n]\langle\partial_1, \ldots, \partial_n\rangle$ with $\partial_i x_i = x_i \partial_i + 1$.

If $J \subseteq D$ is a left ideal, then $M = D/J$ is a $D$-module.

Example: If $n = 1$, $D = \mathbb{C}[x_1]\langle\partial_1\rangle$, $P = x_1\partial_1 - 1 \in D$, $J = DP$ then

$$M = D/J = \frac{D}{D(x_1\partial_1 - 1)}$$

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# Some properties for $D$-modules.

Holonomicity: a property for $D$-modules that implies finite dimension of their solution space.

Regularity: a property for holonomic $D$-modules that implies that any formal series solution is convergent.

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# $A$-gradings on $D$

A matrix $A = (a_1 \; a_2 \cdots a_n) \in \mathbb{Z}^{d \times n}$ defines a grading on the Weyl Algebra $D$ by setting:

$$\deg(x_i) = a_i \in \mathbb{Z}^d$$

$$\deg(\partial_i) = -a_i \in \mathbb{Z}^d$$

# Binomial $D$-modules (Dickenstein-Matusevich-Miller, 2010)

Binomial $D$-modules are determined by $(A, I, \beta)$:

- $A = (a_1 \cdots a_n) \in \mathbb{Z}^{d \times n}$ pointed matrix, $\operatorname{rank}(A) = d \leq n$.
- $I \subseteq \mathbb{C}[\partial]$ binomial $A$-graded ideal (i.e. generated by some binomials $\partial^u - \lambda \partial^v$ with $Au = Av$ and $\lambda \in \mathbb{C}$).
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# Binomial $D$-modules (Dickenstein-Matusevich-Miller, 2010)

Euler operators: $E_i = \sum_{j=1}^{n} a_{ij} x_j \partial_j$, $i = 1, \ldots, d$.

Binomial $D$-module:

$$\mathcal{M}_A(I, \beta) := \frac{D}{DI + D\langle E_1 - \beta_1, \ldots, E_d - \beta_d \rangle}$$

Example:

$A = (1, 2)$, $I = \langle \partial_1^4 - \partial_2^2 \rangle$, $E - \beta = x_1\partial_1 + 2x_2\partial_2 - \beta$

$\mathcal{M}_A(\beta) = \frac{D}{(D(\partial_1^4 - \partial_2^2) + D(x_1\partial_1 + 2x_2\partial_2 - \beta))}$

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Binomial $D$-modules generalize:

- If $I = I_A$ is a toric ideal (binomial prime ideal) then $M_A(I, \beta)$ is a hypergeometric system $M_A(\beta)$ (Gel'fand, Graev, Kapranov, Zelevinsky).
- If $I$ is a lattice basis ideal, $M_A(I, \beta)$ is (roughly speaking) a Horn system.

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# Quasidegrees

Consider the restriction of the $A$-grading to $R = \mathbb{C}[\partial_1, \ldots, \partial_n]$: $\deg(\partial_i) = -a_i$.

**Definition**

Let $V$ be an $A$-graded $R$-module. The set of *true degrees* of $V$ is

$$\operatorname{tdeg}(V) = \{\alpha \in \mathbb{Z}^d : \ V_\alpha \neq 0\}$$

The set of *quasi-degrees* of $V$ is

$$\operatorname{qdeg}(V) = \overline{\operatorname{tdeg}(V)}^{\text{Zariski}}$$

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# Example:

Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 & 0 \\ 1 & 2 & 0 & 2 \end{pmatrix}$$

and the ideal $I = \langle \partial_2, \partial_3, \partial_4 \rangle$, then $R/I = \mathbb{C}[\partial_1]$.
Thus

$$-\operatorname{tdeg}(R/I) = \{-\deg(\partial_1^m) = m \begin{pmatrix} 1 \\ 1 \end{pmatrix} : \; m \in \mathbb{N}\} = \mathbb{N} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

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and the ideal $I = \langle \partial_2, \partial_1^2 - \partial_3\partial_4 \rangle$, then all the monomials $\partial_1^{m_1}\partial_3^{m_3}\partial_4^{m_4}$ are nonzero in $R/I$ and $-\deg(\partial_1^{m_1}\partial_3^{m_3}\partial_4^{m_4}) = \sum_{i=1,3,4} m_i a_i \in \mathbb{N}a_1 + \mathbb{N}a_3 + \mathbb{N}a_4$.
In fact

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# Holonomic binomial $D$-modules

The Andean arrangement $\mathcal{Z}_{Andean}$ of $I$ is certain affine subspace arrangement defined using quasi-degrees of quotients $R/C$ for some primary components $C$ of $I$.

**Theorem (Dickenstein-Matusevich-Miller, 2010)**

$M_A(I, \beta)$ is holonomic if and only if $-\beta \notin \mathcal{Z}_{Andean}$.
If $I$ is standard $\mathbb{Z}$-graded and $M_A(I, \beta)$ is holonomic then it is regular holonomic.

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**Theorem (Hotta, 1998; Saito-Sturmfels-Takayama, 2000; Schulze-Walther, 2008)**

$M_A(\beta)$ is holonomic for all $\beta$.
$M_A(\beta)$ is regular holonomic iff $I$ is standard $\mathbb{Z}$-graded.

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# Example 1

If $A = (1\ 1\ 2\ 3)$ and $I = \langle \partial_1 - \partial_2, \partial_3^4, \partial_4^3, \partial_3^3 - \partial_4^2 \rangle$ then

$$M_A(I,\beta) = \frac{D}{DI + D(x_1\partial_1 + x_2\partial_2 + 2x_3\partial_3 + 3x_4\partial_4 - \beta)}$$

is regular holonomic.

Note: $I$ is not standard $\mathbb{Z}$-graded but its associated prime $\sqrt{I} = \langle \partial_1 - \partial_2, \partial_3, \partial_4 \rangle$ is.

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# Characterizing regular holonomicity

**Theorem**

Let $I \subseteq R$ be an $A$-graded binomial ideal such that $M_A(I, \beta)$ is holonomic. Then $M_A(I, \beta)$ is regular holonomic if and only if for all the binomial primary componets $C$ of $I$ such that $-\beta \in \mathrm{qdeg}(R/C)$ the associated prime $\sqrt{C}$ is standard $\mathbb{Z}$-graded.

# Example 2

$I = \langle \partial_1^2\partial_2 - \partial_2^2, \partial_2\partial_3, \partial_2\partial_4, \partial_1^2\partial_3 - \partial_3^2\partial_4, \partial_1^2\partial_4 - \partial_3\partial_4^2 \rangle$.
$I$ is $A$-graded for the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 & 0 \\ 1 & 2 & 0 & 2 \end{pmatrix}$$

but $I$ is not standard $\mathbb{Z}$-graded.

We have the prime decomposition $I = I_1 \cap I_2 \cap I_3$ where
$I_1 = \langle \partial_2, \partial_3, \partial_4 \rangle$, $I_2 = \langle \partial_1^2 - \partial_2, \partial_3, \partial_4 \rangle$ and $I_3 = \langle \partial_2, \partial_1^2 - \partial_3\partial_4 \rangle$.

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# Example 2 (continuation)

In this example $M_A(I,\beta)$ is holonomic for all $\beta \in \mathbb{C}$.

$\text{qdeg}(R/I_j) = \mathbb{C}\binom{1}{1}$ for $j = 1, 2$ and $\text{qdeg}(R/I_3) = \mathbb{C}^2$.

Since $I_3 = \langle \partial_2, \partial_1^2 - \partial_3\partial_4 \rangle$ is $\mathbb{Z}$-graded we have that $M_A(I,\beta)$ is regular holonomic for $\beta \in \mathbb{C}^2 \setminus \mathbb{C}\binom{1}{1}$.

On the other hand, if $\beta \in \text{qdeg}(R/I_2) = \mathbb{C}\binom{1}{1}$ we have that $M_A(I,\beta)$ is irregular because $I_2 = \langle \partial_1^2 - \partial_2, \partial_3, \partial_4 \rangle$ is not $\mathbb{Z}$-graded.

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# $L$-Characteristic varieties and slopes of binomial $D$-modules.

**Theorem**

*If $M_A(I, \beta)$ is holonomic then the L-characteristic variety of $M_A(I, \beta)$ is the union of the L-characteristic varieties of $M_A(J, 0)$ for all associated primes $J$ of $I$ such that $-\beta \in \text{qdeg}(R/C)$ with $C$ the $J$–primary component of $I$.*

*In particular, we have the analogous result for the slopes of $M_A(I, \beta)$ along coordinate subspaces.*

Note that each $M_A(J, 0)$ is basicly a hypergeometric system because $J$ is a binomial prime ideal.

$L$-characteristic varieties and slopes of hypergeometric systems do not depend on $\beta$ and they were described by Schulze and Walther (2008).

# $L$-Characteristic varieties and slopes of binomial $D$-modules.

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*If $M_A(I,\beta)$ is holonomic then the L-characteristic variety of $M_A(I,\beta)$ is the union of the L-characteristic varieties of $M_A(J,0)$ for all associated primes $J$ of $I$ such that $-\beta \in \operatorname{qdeg}(R/C)$ with $C$ the $J$–primary component of $I$.*
*In particular, we have the analogous result for the slopes of $M_A(I,\beta)$ along coordinate subspaces.*

Note that each $M_A(J,0)$ is basicly a hypergeometric system because $J$ is a binomial prime ideal.
$L$-characteristic varieties and slopes of hypergeometric systems do not depend on $\beta$ and they were described by Schulze and Walther (2008).

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# Gevrey solutions of binomial $D$-modules.

Fix a binomial primary decomposition of $I$ and consider $J$ varying between the associated primes of $I$ such that $-\beta \in \mathrm{qdeg}(R/C)$ with $C$ the $J$–primary component.

**Theorem**

*If $M_A(I, \beta)$ is holonomic and $\beta$ is generic, the dimension of the space of Gevrey solutions of $M_A(I, \beta)$ along a coordinate hiperplane is explicitely determined in terms of the corresponding dimensions for $M_A(J, \beta)$ and the multiplicities of $J$ in $I$.*

In fact, for very generic paramenter $\beta$ any Gevrey solution of $M_A(I, \beta)$ is a linear combination of series $x^{\gamma}\phi$ with $\phi$ a Gevrey solution of $M_A(J, \beta)$.
Gevrey solutions of hypergeometric systems can be explicitly described for very generic parameters (F.-F., 2010).

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