Binary cumulant varieties

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General idea

- Binary moments: $M = [m_I]_{I\subseteq[n]} \in \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 = \mathbb{C}^{2^n}$ such that $m_\emptyset = 1$
  - e.g. $n = 2$ then $M = (m_\emptyset, m_1, m_2, m_{12})$
- let $V \subseteq \mathbb{C}^{2^n}$ invariant under $SL(2)^n$
  - Change to cumulants $K = [k_I]_{I\subseteq[n]}$ to obtain a possibly "nicer" description of V

Motivating example

- Let $M \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. The $2 \times 2 \times 2$ hyperdeterminant given by

$$\begin{aligned}\mathrm{Det}(M) &= m_1^2 m_{23}^2 + m_2^2 m_{13}^2 + m_3^2 m_{12}^2 + m_{123}^2 m_\emptyset^2 - 2(m_1 m_2 m_{13} m_{23} + m_1 m_3 m_{12} m_{23} \\ &+ m_2 m_3 m_{12} m_{13} + m_\emptyset m_1 m_{23} m_{123} + m_\emptyset m_2 m_{13} m_{123} + m_\emptyset m_3 m_{12} m_{123}) + \\ &+ 4(m_1 m_2 m_3 m_{123} + m_\emptyset m_{12} m_{13} m_{23}).\end{aligned}$$

- If $m_\emptyset = 1$ then in cumulants

$$\mathrm{Det}(M) = k_{123}^2 + 4 k_{12} k_{13} k_{23}.$$

Basic definitions

- Moments m_I, for $I \subseteq [n]$, $m_\emptyset = 1$.
  - MGF: $M(x) = \sum_{I \subseteq [n]} m_I \prod_{i \in I} x_i = 1 + \sum_{I \neq \emptyset} m_I \prod_{i \in I} x_i$
    - $n = 2$: $M(x) = 1 + m_1 x_1 + m_2 x_2 + m_{12} x_1 x_2$

- Cumulants $K = [k_I]$, $k_\emptyset = 0$.
  - CGF $K(x) = \log M(x) = \sum_{I \subseteq [n]} k_I \prod_{i \in I} x_i$ modulo $\langle x_1^2, \ldots, x_n^2 \rangle$
    - note: $M(x) = \exp K(x)$ modulo $\langle x_1^2, \ldots, x_n^2 \rangle$

Moment-cumulant formula

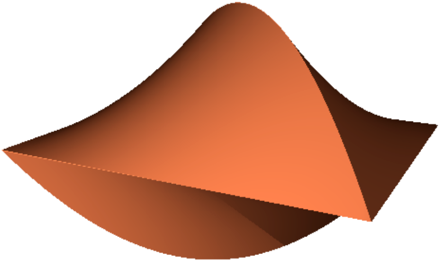
- Let $\Pi(I)$ be the set of partitions of $I \subseteq [n]$
- $k_I = \sum_{\pi \in \Pi(I)} (-1)^{|\pi|-1}(|\pi|-1)! \prod_{B \in \pi} m_B$

- inverse: $m_I = \sum_{\pi \in \Pi(I)} \prod_{B \in \pi} k_B$

- $k_i = m_i$, $k_{ij} = m_{ij} - m_i m_j$
- $\Pi([3]) = \{1|2|3, 1|23, 2|13, 3|12, 123\}$
- $k_{123} = m_{123} - m_1 m_{23} - m_2 m_{13} - m_3 m_{12} + 2 m_1 m_2 m_3$
- $m_{123} = k_{123} + k_1 k_{23} + k_2 k_{13} + k_3 k_{12} + k_1 k_2 k_3$

The image of the probability simplex

- $n = 2$, $\mathrm{Seg}(\mathbb{P}^1 \times \mathbb{P}^1)_{\geq} \subseteq \mathbb{RP}^3_{\geq} \simeq \Delta_3$



Group actions

- the ith component of $\mathbb{C}^{2^n} = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$ is $\mathbb{C}^2 = \text{span}_{\mathbb{C}}\{e_0^{(i)}, e_1^{(i)}\}$
- Let $M = (M_1, \ldots, M_n) \in GL(2)^n$, where $M_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix}$.
- M acts on $v \in \mathbb{C}^{2^n}$ by $e_0^{(i)} \mapsto a_i e_0^{(i)} + c_i e_1^{(i)}$, $e_1^{(i)} \mapsto b_i e_0^{(i)} + d_i e_1^{(i)}$.

- Diagonal (D): $M_i = \begin{bmatrix} 1 & 0 \\ 0 & \lambda_i \end{bmatrix}$ for $i = 1, \ldots, n$.
- Unipotent (U): $M_i = \begin{bmatrix} 1 & c_i \\ 0 & 1 \end{bmatrix}$ for $i = 1, \ldots, n$.

Change of values of X

- random vector (multivariate umbra):

  $X = (X_1, \ldots, X_n)$, s.t. $\quad m_I \;=\; \mathbb{E}(\prod_{i \in I} X_i), \quad \mathbb{E}(\cdot)$ is linear.

- D: $D \cdot [m_I] \;=\; [\prod_{i \in I} \lambda_i m_I]$ for $\lambda \in (\mathbb{C}^*)^n$
- $D \cdot X \;=\; (\lambda_1 X_1, \ldots, \lambda_n X_n)$

- U: $e_0^{(i)} \mapsto e_0^{(i)}$ and $e_1^{(i)} \mapsto c_i e_0^{(i)} + e_1^{(i)}$
- e.g. $(1, m_1, m_2, m_{12}) \mapsto (1, m_1 + c_1, m_2 + c_2, m_{12} + c_1 m_2 + c_2 m_1 + c_1 c_2)$
- $U \cdot X \;=\; (X_1 + c_1, \ldots, X_n + c_n)$

Invariance of cumulants

- $D \cdot k_I \quad = \quad \prod_{i \in I} \lambda_i k_I$
- $U \cdot k_i = k_i + c_i$ and $U \cdot k_I = k_I$ if $|I| \geq 2$

Theorem (Sturmfels,Z.)

A subvariety of $\{m_\emptyset = 1\}$ is invariant under these two groups if and only if it is defined by $\mathbb{Z}^n$-homogeneous polynomials in k_I for $|I| \geq 2$ (higher order cumulants).

- e.g. $\quad \mathrm{Det}(M) \quad = \quad k_{123}^2 + 4k_{12}k_{13}k_{23}$

Projective version

Corollary

Let V be a subvariety of the open affine subset $\{m_\emptyset = 1\}$ in $\mathbb{P}(\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2)$ and let $\overline{V}$ be its Zariski closure. If $\overline{V}$ is invariant under $SL(2)^n$ then $\mathcal{I}_V$ is generated by $\mathbb{Z}^n$-homogeneous polynomials in k_I for $|I| \geq 2$.

Hyperdeterminant

- $\mathrm{Det}(M) =$ irreducible polynomial in $[m_I]$ that vanishes when $M(x) = \sum_I m_I \prod_{i \in I} x_i = 0$ has a singular point in $\mathbb{C}^n$ [GKZ]
- For $n = 4$ has $2{,}894{,}276$ terms, degree 24 (see [Huggins et al])

Theorem (Sturmfels,Z.)

- The $2 \times \cdots \times 2$ hyperdeterminant $\mathrm{Det}(M)$ is a polynomial function in the $2^n - n - 1$ higher order cumulants $\{k_I : |I| \geq 2\}$.
- It is $\mathbb{Z}^n$-homogeneous of degree $\frac{1}{2}(C_n, \ldots, C_n)$ (for $n = 2, 3, 4, 5$ the corresponding $C_n = 2, 4, 24, 128$).
- For $n = 4$ the hyperdeterminant has precisely $13{,}819$ monomials in the 11 unknowns, degree $(12, 12, 12, 12)$.

Segre embedding of $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$

- $(\mathbb{P}^1)^n \quad = \quad (\mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \hookrightarrow \mathbb{P}(\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2))$
- in moments: $m_I = \prod_{i \in I} t_i$ (in $m_\emptyset = 1$)
- in cumulants: the equations are $k_I = 0$ for all $|I| \geq 2$

- $\mathrm{Tan}((\mathbb{P}^1)^n) = \overline{\{x \in \mathbb{P}^{2^n - 1} : x \text{ lies on a line tangent to } (\mathbb{P}^1)^n\}}$

Tangent is toric

- parametrization: let $(a_0^{(i)}, a_1^{(i)}), (b_0^{(i)}, b_1^{(i)}) \in \mathbb{C}^2$ for $i = 1, \ldots, n$

$$M(x) = \sum_{j=1}^{n} (1 + b_1^{(j)} x_j) \prod_{i \neq j} (1 + a_1^{(i)} x_i).$$

Theorem (Sturmfels, Z.)

The image of $\mathrm{Tan}((\mathbb{P}^1)^n)$ in the space of higher order cumulants $\mathbb{C}^{2^n-n-1}$ is an n-dimensional toric variety. It is isomorphic to the affine toric variety parametrized by $k_I = \prod_{i \in J} s_i, \;\; s_i = \frac{b_1^{(i)} - a_1^{(i)}}{n}$.

Further work

- Cremona transformation $\mathbb{P}^{2^n-1} \dashrightarrow \mathbb{P}^{2^n-1}$
  - the exceptional locus $m_\emptyset = m_1 \cdots m_n = 0$ as a set
- $\mathbb{C}^k \otimes \cdots \otimes \mathbb{C}^k = \mathbb{C}^{k^n}$ for $k \geq 3$
- understand the semialgebraic structure of probabilisticically meaningful cumulants

Thank you!

B. Sturmfels and P. Zwiernik, ***Binary cumulant varieties.***
`arXiv:1103.0153`, March 2011.