Software for computing multiplier ideals

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Outline

What is a multiplier ideal?

Ideal sheaf associated to a singularity.

- Captures subtle information about singularity
- Useful in geometry, especially birational geometry
- Defined in terms of resolution of singularities
- Usually hard to compute

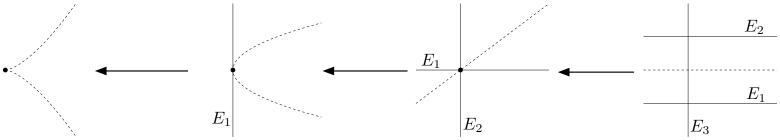
Definition

Given ideal I on $X = \mathbb{C}^n$, real number $t \geq 0$

- $f : Y \to X$ resolution of singularities
- $I\mathcal{O}_Y = \mathcal{O}_Y(-F)$ total transform
- $K = K_{Y/X} = V(\det df)$ relative canonical divisor
- The t'th multiplier ideal is $\mathcal{J}(I^t) = f_*\mathcal{O}_Y(K - \lfloor tF \rfloor)$

Example

$I = (y^2 - x^3)$. Resolution:



$$K = E_1 + 2E_2 + 4E_3, \qquad F = 2E_1 + 3E_2 + 6E_3 + C'$$

$$\begin{aligned} \mathcal{J}(I^{5/6}) &= f_*\mathcal{O}_Y((1 - \lfloor \tfrac{5}{6}2 \rfloor)E_1 + (2 - \lfloor \tfrac{5}{6}3 \rfloor)E_2 + (4 - \lfloor \tfrac{5}{6}6 \rfloor)E_3) \\ &= f_*\mathcal{O}_Y((1-1)E_1 + (2-2)E_2 + (4-5)E_3) \\ &= f_*\mathcal{O}_Y(-E_3) \\ &= (x, y) \end{aligned}$$

Why study multiplier ideals?

- Deformation invariance of plurigenera
- Existence of flips
- Asymptotic base loci of linear series
- Chow stability criterion
- Uniform results in commutative algebra

Why compute multiplier ideals?

- Study jumping numbers: values of t at which $\mathcal{J}(I^t)$ changes.
  E.g., diagonal arrangement in $(\mathbb{C}^n)^{\otimes k}$
- Characterize essential exceptional divisors:
  which components of $K - \lfloor tF \rfloor$ are needed?
- Conjectural strong bounds on symbolic powers:
  strengthen results of the form $I^{(m)} \subseteq \mathcal{J} \subseteq I^r$

Computing multiplier ideals

An existing implementation:

- `Dmodules` (Berkesch, Leykin)

Alternative approach: via resolution of singularities.

- `desing.lib` by G. Bodnár and J. Schicho
- `resolve.lib` by A. Frühbis–Krüger and G. Pfister

Outline of computation

1. Find resolution of singularities.
2. In each chart,
   1. find equations for exceptional divisors E_i,
   2. write total transform $F = \sum a_i E_i$,
   3. write relative canonical divisor $K = \sum b_i E_i$,
   4. form divisor $K - \lfloor tF \rfloor$,
   5. push this ideal down to the root chart.
3. Intersect these pushdowns.

First package: multiplierideals.lib

Work in progress, but partially working:

```
> LIB "multiplierideals.lib";
> multiplieridealinit();
> ring R = 0,(x(1..2)),dp;
> ideal I = x(1)^3,x(2)^2;
> desing(I); numericaldataallcharts(I);
> lct(I);
5/6
> multiplierideal(I,5/6);
_[1]=x(2)
_[2]=x(1)
```

$$\mathcal{J}(I^{5/6}) = (x_1, x_2)$$

```
> multiplierideal(I,11/6);
_[1]=x(2)^3
_[2]=x(1)*x(2)^2
_[3]=x(1)^2*x(2)
_[4]=x(1)^4
```

$$\mathcal{J}(I^{11/6}) = (x_1^4, x_1^2 x_2, x_1 x_2^2, x_2^3)$$

Second package: MonomialMultiplierIdeals.m2

- Howald's theorem: multiplier ideal of monomial ideal
- Combinatorial description via Newton polyhedron
- $\mathcal{J}(I^t)$ is the monomial ideal

$$x^v \in \mathcal{J}(I^t) \iff v + (1, \dots, 1) \in \mathrm{Int}(t \cdot \mathrm{Newt}(I))$$

- Use `Normaliz` (Bruns et al) to manipulate Newton polyhedron
- `MonomialMultiplierIdeals` package for `Macaulay2` now available.

A question of Lazarsfeld

What do multiplier ideals of $\mathrm{gin}(I)$ say about $V(I)$?

Example: I = ideal of curve parametrized by (t^a, t^b, t^c)

- Ideal of curve (t^4, t^5, t^{11}) is $I = (y^3 - xz, x^4 - yz, x^3y^2 - z^2)$
- $J = \mathrm{gin}(I) = (x^3, x^2y^2, xy^4, y^5)$
- Can find log canonical threshold $\mathrm{lct}(J) = 8/15$

Example

```
i1 : loadPackage "MonomialMultiplierIdeals";
  loadPackage "GenericInitialIdeal";
i2 : S = QQ[t]; R = QQ[x,y,z];
i3 : curveIdeal = (a,b,c) -> ker map(S,R,{t^a,t^b,t^c});
  -- ideal of curve parametrized by (t^a,t^b,t^c)
i4 : curveGinLCT = (a,b,c) ->
  monomialLCT monomialIdeal gin curveIdeal(a,b,c);
i5 : (5..10) / (i -> curveGinLCT(4,i,11)) / toString
o5 = (8/15, 8/15, 9/14, 2/3, 17/30, 6/11)
```

Outline of computation

To compute

$$x^v \in \mathcal{J}(I^t) \iff v + (1, \ldots, 1) \in \mathrm{Int}(t \cdot \mathrm{Newt}(I))$$

1. Find defining inequalities for Newt(I), $A\mathbf{v} \geq \mathbf{b}$.
2. Scale by $t = p/q$, yielding $qA\mathbf{v} \geq p\mathbf{b}$.
3. Add 1 to entries of $p\mathbf{b}$ corresponding to non-coordinate faces $\rightsquigarrow \widetilde{p\mathbf{b}}$
4. Find monomial ideal generated by lattice points in $qA\mathbf{v} \geq \widetilde{p\mathbf{b}}$.
5. Compute ideal quotient by $x^{(1,\ldots,1)} = x_1 \cdots x_n$.

Thank you!