Stable Complete Intersections

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  - ... devise a strategy to make the situation reasonably stable?
  - ... change the generating polynomials s.t. the stability of the zeros increases?
- **Aim** of the talk is to address these problems using a mixture of methods from classical algebraic geometry and numerical algebra.

Motivating example - The Linear Case

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Solution $= (1000, -1001)$

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- It is well-known that for a **linear system** with n equations and n unknowns, the **most stable** situation occurs when the coefficient matrix is **orthonormal** (the 2-norm **condition number** of the coefficient matrix is 1).
- So another question arises: is there an **analogue to orthonormality** when we deal with polynomial systems?

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  - ◇ **condition number Shub-Smale....**
- **Philosopy:** introduce a measure of stability which is a generalization of the classical notion of condition number for linear systems.

Notations

- K field ($K = \mathbb{C}$ or $K = \mathbb{R}$)
- $\mathbf{x} = (x_1, \ldots, x_n)$ unknowns and $\mathbf{a} = (a_1, \ldots, a_m)$ parameters
- $\mathbf{f}(\mathbf{x}) = \{f_1(\mathbf{x}), \ldots, f_n(\mathbf{x})\} \subset K[\mathbf{x}]$ zero-dimensional complete intersection
- $I \subset K[\mathbf{x}]$ ideal generated by $\mathbf{f}(\mathbf{x})$
- $F(\mathbf{a}, \mathbf{x}) = \{F_1(\mathbf{a}, \mathbf{x}), \ldots, F_n(\mathbf{a}, \mathbf{x})\} \subset K[\mathbf{a}, \mathbf{x}]$
- $I(\mathbf{a}, \mathbf{x}) \subset K[\mathbf{a}, \mathbf{x}]$ ideal generated by $F(\mathbf{a}, \mathbf{x})$
- $\mathcal{S} = \mathbb{A}^m_K$ scheme of the **a**-parameters
- $\Phi : \mathrm{Spec}(K[\mathbf{a}, \mathbf{x}]/I(\mathbf{a}, \mathbf{x})) \longrightarrow \mathcal{S}$ morphism of schemes

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Proposition

Let σ be a term ordering on $\mathbb{T}^n$
let $G(\mathbf{a},\mathbf{x})$ be the reduced σ-Gröbner basis of $I(\mathbf{a},\mathbf{x})K(\mathbf{a})[\mathbf{x}]$
let $d(\mathbf{a})$ be the l.c.m. of all denominators of the polys coefficients of $G(\mathbf{a},\mathbf{x})$
let $T = \mathbb{T}^n \setminus \mathrm{LT}_\sigma(I(\mathbf{a},\mathbf{x})K(\mathbf{a})[\mathbf{x}])$. Then:

(a) The open subscheme $\mathcal{U}$ of $\mathcal{S}$ defined by $d(\mathbf{a}) \neq 0$ is I-free.

(b) The multiplicity of each fiber over $\mathcal{U}$ coincides with the cardinality of T.

Smoothness

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A dense Zariski-open subscheme $\mathcal{U}$ of $\mathcal{S}$ such that $\Phi^{-1}(\mathcal{U}) \longrightarrow \mathcal{U}$ is smooth, i.e. all the fibers of $\Phi^{-1}(\mathcal{U}) \longrightarrow \mathcal{U}$ are zero-dimensional smooth complete intersections, is said to be an I-smooth subscheme of $\mathcal{S}$.

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Theorem

Let $D(\mathbf{a}, \mathbf{x}) = \det(\mathrm{Jac}_F(\mathbf{a}, \mathbf{x}))$ be the det of the Jacobian of $F(\mathbf{a}, \mathbf{x}) \in K[\mathbf{a}][\mathbf{x}]$
let $J(\mathbf{a}, \mathbf{x}) = I(\mathbf{a}, \mathbf{x}) + (D(\mathbf{a}, \mathbf{x}))$ in $K[\mathbf{a}, \mathbf{x}]$
let $H = J(\mathbf{a}, \mathbf{x}) \cap K[\mathbf{a}]$. Then:

(a) There exists an I-smooth subscheme of $\mathcal{S}$ if and only if $H \neq (0)$.

(b) Let $0 \neq h(\mathbf{a}) \in H$; the open subscheme $\mathcal{U}$ of $\mathcal{S}$ defined by $h(\mathbf{a}) \neq 0$ is I-smooth.

Freeness + Smoothness = Optimality

Definition

A dense Zariski-open subscheme $\mathcal{U}$ of $\mathcal{S}$ such $\Phi^{-1}(\mathcal{U}) \longrightarrow \mathcal{U}$ is free and smooth is said to be ***I*-optimal**.

Theorem

There is an **algorithm** which computes an ***I*-optimal** subscheme of $\mathcal{S}$.

An example

We consider the ideal $I = (xy + 1,\ x^2 + y^2 - 5) \in \mathbb{R}[x, y]$ embedded into the family $I(\mathbf{a}, x, y) = (xy + a_1x + 1,\ x^2 + y^2 + a_2)$.

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Freeness

We compute the reduced Lex-Gröbner basis of $I(\mathbf{a}, \mathbf{x})K(\mathbf{a})[x, y]$

$$\begin{aligned} g_1 &= x - y^3 - a_1 y^2 - a_2 y - a_1 a_2, \\ g_2 &= y^4 + 2a_1 y^3 + (a_1^2 + a_2)y^2 + 2a_1 a_2 y + (a_1^2 a_2 + 1) \end{aligned}$$

There is no condition for the free locus.

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Smoothness

We compute $D(\mathbf{a}, x, y) = \det(\mathrm{Jac}_F(\mathbf{a}, x, y)) = -2x^2 + 2y^2 + 2a_1 y$, $J(\mathbf{a}, x, y) = I(\mathbf{a}, x, y) + (D(\mathbf{a}, x, y))$ and $H = J(\mathbf{a}, x, y) \cap K[\mathbf{a}] = (h(\mathbf{a}))$

$$h(\mathbf{a}) = a_1^6 a_2 + 3a_1^4 a_2^2 + a_1^4 + 3a_1^2 a_2^3 + 20a_1^2 a_2 + a_2^4 - 8a_2^2 + 16$$

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Optimality

An I-**optimal** subscheme is $\mathcal{U} = \{\alpha \in \mathbb{A}^2_{\mathbb{R}}\ :\ h(\alpha_1, \alpha_2) \neq 0\}$.

What about real zeros?

Let $\mathbf{f}(\mathbf{x})$ be a zero-dimensional complete intersection in $\mathbb{R}[\mathbf{x}]$
let $F(\mathbf{a}, \mathbf{x}) \in \mathbb{R}[\mathbf{a}, \mathbf{x}]$ be generically zero-dimensional family containing $\mathbf{f}(\mathbf{x})$.

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Assume that there exists an I-optimal subscheme $\mathcal{U}$ of $\mathbb{A}^m_{\mathbb{R}}$;
let $\alpha_I \in \mathcal{U}$ be the point in the parameter space corresponding to $I = (\mathbf{f}(\mathbf{x}))$
let $\mu_{\mathbb{R},I}$ be the number of real points in the fiber over α_I (= zeroes of I)

Then:
there exists an open semi-algebraic subscheme $\mathcal{V}$ of $\mathcal{U}$ s.t.
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- proof **Shape Lemma + theorem Basu-Pollack-Roy**

An example: continuation 1

Going back to the example ...

- An I-optimal subscheme is $\mathcal{U} = \{\alpha \in \mathbb{A}^2_{\mathbb{R}} : h(\alpha) \neq 0\}$
- Therefore, for $h(\mathbf{a}) \neq 0$ each fiber is **smooth** and has **multiplicity** 4 hence it consists of 4 distinct complex points.

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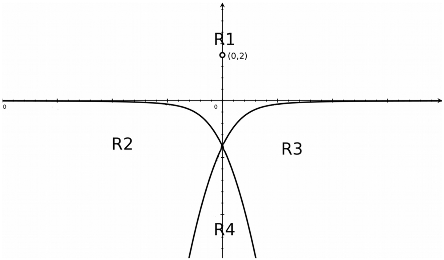

What about real points?

- $\alpha_I = (0, -5) \in \mathcal{U}$ and $\mu_{\mathbb{R},I} = 4$
- The real curve $h(\mathbf{a}) = 0$ is the union of **two branches** and **point** $(0, 2)$.

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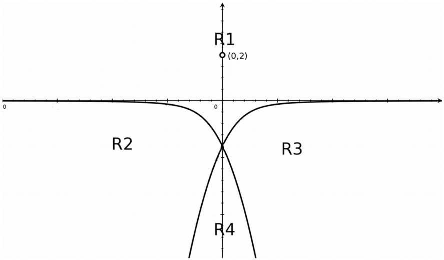

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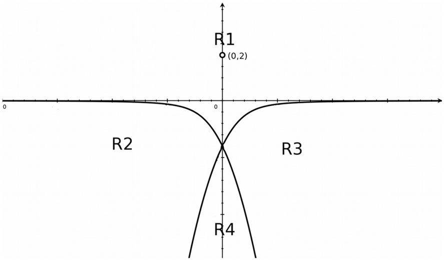

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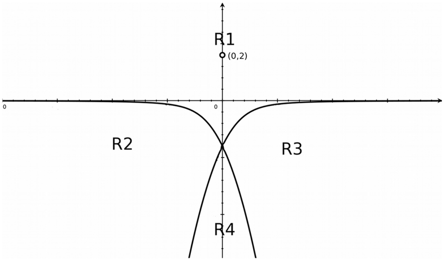

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- Region R4: **four real points**.

An example: continuation 2

- To describe the four regions algebraically, we use the **Sturm-Habicht sequence** of $g_2 = y^4 + 2a_1y^3 + (a_1^2 + a_2)y^2 + 2a_1a_2y + (a_1^2a_2 + 1) \in \mathbb{R}(\mathbf{a})[y]$.

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- The leading monomials of the sequence are

$$y^4, \;\; 4y^3, \;\; 4r(\mathbf{a})y^2, \;\; -8\ell(\mathbf{a})y, \;\; 16h(\mathbf{a})$$

where $r(\mathbf{a}) = a_1^2 - 2a_2$ and $\ell(\mathbf{a}) = a_1^4a_2 + 2a_1^2a_2^2 + 2a_1^2 + a_2^3 - 4a_2$.

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- We get

$$\mathrm{R4} = \{\alpha \in \mathbb{R}^2 \mid r(\alpha) > 0, \; \ell(\alpha) < 0, \; h(\alpha) > 0\}$$

  which **is semi-algebraic open, not Zariski-open**.

Admissible Perturbations

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Definition

Let $\varepsilon(\mathbf{x}) = \{\varepsilon_1(\mathbf{x}), \ldots, \varepsilon_n(\mathbf{x})\}$ be a set of polynomials in $\mathbb{R}[\mathbf{x}]$.
Let $\mathcal{V} \subset \mathbb{A}^m_{\mathbb{R}}$ be an open semi-algebraic subset of $\mathcal{U}$ such that

- $\alpha_I \in \mathcal{V}$
- $\forall\, \alpha \in \mathcal{V}$ the number of real roots of $F(\alpha, \mathbf{x}) = 0$ is $\mu_{\mathbb{R},I}$.

If there exists $\alpha \in \mathcal{V}$ such that $(\mathbf{f} + \varepsilon)(\mathbf{x}) = F(\alpha, \mathbf{x})$,
then $\varepsilon(\mathbf{x})$ is called an admissible perturbation of $\mathbf{f}(\mathbf{x})$.

Example of admissible perturbation

Let $\mathbf{f} = \{f_1, f_2\} \in \mathbb{R}[x, y]$ and $I = (f_1, f_2)$ where

$$\begin{aligned} f_1 &= xy - 6 \\ f_2 &= x^2 + y^2 - 13 \end{aligned} \qquad \text{and} \qquad \mathcal{Z}_{\mathbb{R}}(\mathbf{f}) = \{(-3, -2), (3, 2), (-2, -3), (2, 3)\}$$

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$$\begin{array}{l} f_1 = xy - 6 \\ f_2 = x^2 + y^2 - 13 \end{array} \qquad \text{and} \qquad \mathcal{Z}_{\mathbb{R}}(\mathbf{f}) = \{(-3, -2), (3, 2), (-2, -3), (2, 3)\}$$

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- The **semi-algebraic open set**

$$\mathcal{V} = \{\alpha \in \mathbb{R}^3 \mid \alpha_3^2 - 4\alpha_1^2\alpha_2 > 0, \alpha_2 > 0, \alpha_3 < 0\}$$

is a subset of the I-**optimal scheme** $\mathcal{U} = \{\alpha \in A^3_{\mathbb{R}} \mid \alpha_2(\alpha_3^2 - 4\alpha_1^2\alpha_2) \neq 0\}$

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The polynomial set $\varepsilon(x, y) = \{2,\ \frac{5}{4}y^2\}$ is an admissible perturbation of $\mathbf{f}(x, y)$. The real roots of $(\mathbf{f} + \varepsilon)(\mathbf{x}) = 0$ are

$$\mathcal{Z}_{\mathbb{R}}(\mathbf{f} + \varepsilon) = \left\{\left(-3, -\frac{4}{3}\right), \left(3, \frac{4}{3}\right), (-2, -2), (2, 2)\right\}$$

Local Condition Number

Let p be a nonsingular real solution of $\mathbf{f}(\mathbf{x}) = 0$
let $\varepsilon(\mathbf{x})$ be an admissible perturbation of $\mathbf{f}(\mathbf{x})$

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Corollary

Let $p + \Delta p$ be the real solution of $(\mathbf{f} + \varepsilon)(\mathbf{x}) = \mathbf{0}$ corresponding to p. Then

$$0 = (\mathbf{f} + \varepsilon)(p + \Delta p) \approx \varepsilon(p) + \mathrm{Jac}_{\mathbf{f}+\varepsilon}(p)\, \Delta p \qquad (1)$$

and the approximate solution of (1) is denoted by $\Delta p^1 = -\mathrm{Jac}_{\mathbf{f}+\varepsilon}(p)^{-1} \varepsilon(p)$.

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Theorem (Local Condition Number)

Under the condition $\| \operatorname{Jac}_{\mathbf{f}}(p)^{-1} \operatorname{Jac}_{\varepsilon}(p) \| < 1$, we have

$$\frac{\|\Delta p^1\|}{\|p\|} \leq \Lambda(\mathbf{f}, \varepsilon, p)\, \kappa(\mathbf{f}, p) \left(\frac{\| \operatorname{Jac}_{\varepsilon}(p) \|}{\| \operatorname{Jac}_{\mathbf{f}}(p) \|} + \frac{\|\varepsilon(0) - \varepsilon^{\geq 2}(0, p)\|}{\|\mathbf{f}(0) - \mathbf{f}^{\geq 2}(0, p)\|} \right) \tag{2}$$

where $\Lambda(\mathbf{f}, \varepsilon, p) = 1/(1 - \| \operatorname{Jac}_{\mathbf{f}}(p)^{-1} \operatorname{Jac}_{\varepsilon}(p) \|)$.

Generalization of the Linear Case

The notion of local condition number is a generalization of the classical notion of condition number of linear systems.

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  and so $\quad \kappa(\mathbf{f}, p) = \|\,\mathrm{Jac}_{\mathbf{f}}(p)^{-1}\|\|\,\mathrm{Jac}_{\mathbf{f}}(p)\| = \|A^{-1}\|\|A\|$

- Further, using the perturbation $\varepsilon(\mathbf{x}) = \Delta A\mathbf{x} - \Delta b$, relation (2) becomes

$$\frac{\|\Delta p\|}{\|p\|} \le \frac{1}{1 - \|A^{-1}\|\ \|\Delta A\|}\|A^{-1}\|\ \|A\| \left(\frac{\|\Delta A\|}{\|A\|} + \frac{\|\Delta b\|}{\|b\|} \right)$$

  which is the relation that quantifies the sensitivity of the $Ax = b$ problem.

Further properties and Optimization

Further properties

- $\|\cdot\|$ is and induced matrix norm $\Rightarrow \kappa(\mathbf{f}, p) \geq 1$
- $\kappa_2(\mathbf{f}, p) = \frac{\sigma_{\max}(\mathrm{Jac}_{\mathbf{f}}(p))}{\sigma_{\min}(\mathrm{Jac}_{\mathbf{f}}(p))}$
- $\kappa_2(\mathbf{f}, p) = 1 \iff \mathrm{Jac}_{\mathbf{f}}(p)$ orthonormal
- $\kappa(\mathbf{f}, p)$ invariant under multiplication by unique nonzero scalar $\gamma \in \mathbb{R}$

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Proposition

Suppose that $\deg(f_1) = \ldots = \deg(f_n)$
let $C = (c_{ij}) \in \mathrm{Mat}_n(\mathbb{R})$ invertible, and $\mathbf{g}$ be defined by $\mathbf{g}^{tr} = C \cdot \mathbf{f}^{tr}$.
Then the following conditions are equivalent:

(a) $\kappa_2(\mathbf{g}, p) = 1$, the minimum possible

(b) $C^t C = (\mathrm{Jac}_{\mathbf{f}}(p)\,\mathrm{Jac}_{\mathbf{f}}(p)^t)^{-1}$

Experiments I

We consider $\mathbf{f} = \{f_1, f_2\} \in \mathbb{R}[x,y]$ and $I = (f_1, f_2)$ where

$$\begin{aligned} f_1 &= \tfrac{1}{4}x^2y + xy^2 + \tfrac{1}{4}y^3 + \tfrac{1}{5}x^2 - \tfrac{5}{8}xy + \tfrac{13}{40}y^2 + \tfrac{9}{40}x - \tfrac{3}{5}y + \tfrac{1}{40} \\ f_2 &= x^3 + \tfrac{14}{13}xy^2 + \tfrac{57}{52}x^2 - \tfrac{25}{52}xy + \tfrac{8}{13}y^2 - \tfrac{11}{52}x - \tfrac{4}{13}y - \tfrac{4}{13} \end{aligned}$$

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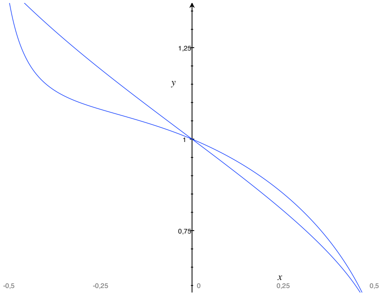

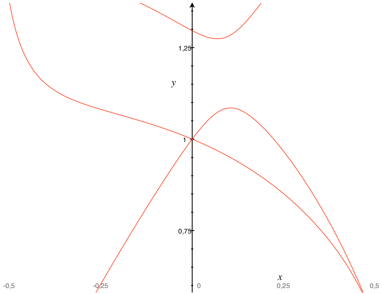

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- $q \in \mathcal{Z}_{\mathbb{R}}(\mathbf{f} + \varepsilon)$ and $r \in \mathcal{Z}_{\mathbb{R}}(\mathbf{g} + \varepsilon)$ are perturbations of p for different admissible perturbations ε

$\kappa_2(\mathbf{f}, p)$	$\frac{\|q-p\|_2}{\|p\|_2}$
8	0.000097

$\kappa_2(\mathbf{g}, p)$	$\frac{\|r-p\|_2}{\|p\|_2}$
1	0.000023

Experiments II

We consider $\mathbf{f} = \{f_1, f_2, f_3\} \in \mathbb{R}[x, y, z]$ and $I = (f_1, f_2, f_3)$ where

$$\begin{aligned} f_1 &= \tfrac{6}{17}x^2 + xy - \tfrac{24}{85}x - \tfrac{8}{85}y - \tfrac{6}{85} \\ f_2 &= \tfrac{39}{89}x^2 + \tfrac{70}{89}xy + yz - \tfrac{39}{89}x + \tfrac{10}{89}y \\ f_3 &= y^2 + 2xz + z^2 - z \end{aligned}$$

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$\kappa_2(\mathbf{f}, p)$	$\frac{\|q-p\|_2}{\|p\|_2}$
123	0.0436

$\kappa_2(\mathbf{g}, p)$	$\frac{\|r-p\|_2}{\|p\|_2}$
1	0.0221

Future work

- Optimization in the case of arbitrary degrees
- Definition of global condition number
- ...

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Thank you!