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SIXTY STUDY QUESTIONS TO THE COURSE NUMERISK BEHANDLING AV DIFFERENTIALEKVATIONER I

Parameter values and functions occurring in the questions below will be exchanged at the exam.

(4) 1. Given a system of ODE's $\dot{y} = Ay$, where A is an $n \times n$-matrix. Give a sufficient condition that the analytical solution is bounded for all $t > 0$. Will there be bounded solutions when the matrix $A = ...$?

(4) 2. For a nonlinear system of ODE's $\dot{y} = f(y)$, what is meant by the critical points of the system? Give a sufficient condition that a critical point is stable. Are the critical point of the following system stable? ....

(4) 3. Given a nonlinear system of ODE's $\dot{y} = f(y)$. Assume that the right hand side consists of differentiable functions. Describe a Newton's method for computing the critical points of the system. For the following system, choose an initial vector and make one iteration with Newton's method. ....

(3) 4. What does it mean that an $n \times n$ matrix A is diagonalizable? Is the following matrix A diagonalizable? ....

(4) 5. For a system of ODE's $\dot{y} = Ay, y(0) = y_0$, the solution can be written as $y(t) = e^{At}y_0$. What is meant by the matrix e^{At}? What is e^{At} when $A = ...$?

(1) 6. For a system of ODE's $\dot{y} = f(y)$ what is meant by an initial value problem?

(4) 7. What is meant by the eigenvalues and the eigenvectors of an $n \times n$ matrix A? The eigenvectors are roots of the characteristic equation. What is the characteristic equation of A? Which are the eigenvalues and eigenvectors for the matrix $A = ...$?

(4) 8. Given a difference equation $y_{n+1} = a_1 y_n + a_2 y_{n-1} + ... + a_{k+1} y_{n-k}$ where $a_1, a_2, ...a_{k+1}$ are real. Formulate a sufficient condition that the sequence y_n is bounded for all n. Is the solution bounded when $n > 0$ for the following difference equation? ....

(4) 9. Given a difference equation on the form $\nabla y_n + \frac{1}{2}\nabla^2 y_n = 0$. Is the solution y_n bounded for all $n > 0$?

(2) 10. Formulate the following 3rd order ODE as a system of three first order ODE's ......

(3) 11. Give the recursion formula when the Euler backward method with time step h is used on e.g. the ODE-problem $y' = -y^3 + t, y(0) = 1$. In each time step t_{n+1} a nonlinear

equation must be solved. Formulate that equation on the form $F(y_{n+1}) = 0$ and the iteration formula when Newton's method is applied to this equation.

(2) 12. What is meant by an explicit method for solving an initial value problem

$$\dot{y} = f(y), y(0) = y_0$$

Give at least two examples (with their formulas) of explicit methods often used to solve such a problem.

(2) 13. What is meant by an implicit method for solving an initial value problem

$$\dot{y} = f(y), y(0) = y_0$$

Give at least two examples of implicit methods (with their formulas) often used to solve such a problem.

(2) 14. What is meant by the order of accuracy for a method used to solve an initial value problem $\dot{y} = f(y), y(0) = y_0$? What is the order of the Euler forward method and the classical Runge-Kutta method?

(2) 15. What is meant by automatic stepsize control for a method used to solve an initial value problem $\dot{y} = f(y), y(0) = y_0$?

(4) 16. Why is the midpoint method not suited for ODE-systems where the eigenvalues of the jacobian are real and negative? Are there any ODE-systems for which the method could be suitable?

(2) 17. Explain the difference between local error and global error for the explicit Euler method.

(2) 18. An ODE-problem, initial value and boundary value problem, can be solved by a discretization method or an ansatz method. Give a brief description of what is meant by these two method classes.

(4) 19. When an initial value problem $\dot{y} = f(y), y(0) = y_0$ is solved with a discretization method, what is meant by the stability area in the complex hq-plane for the method? Give a sketch of the stability area for the method ....

(2) 20. What is meant by a stiff system of ODE's $\dot{y} = f(y)$?

(2) 21. For a linear system of ODE's $\dot{y} = Ay$, where the eigenvalues of A are $\lambda_1, \lambda_2, \ldots, \lambda_n$, when is the system stable? Assume that all eigenvalues are real and negative, when is the system stiff?

(2) 22. Describe some discretization methods that are suitable for stiff initial value ODE problems.

(2) 23. Which is the computational problem when a stiff initial value ODE-system is solved with an explicit method?

(3) 24. Given e.g. the following ODE-system $\dot{y} = -100y + z, y(0) = 1, \dot{z} = -0.1z, z(0) = 1$. For which values of the stepsize h is the Euler forward method stable? Same question for the Euler backward method.

(4) 25. Given the vibration equation

$$m\ddot{x} + c\dot{x} + kx = 0, \quad x(0) = 0, \quad \dot{x}(0) = v_0$$

Use scaling of x and t to formulate this ODE on dimensionless form. Determine scaling factors so that the scaled equation contains as few parameters as possible.

(2) 26. Describe the finite difference method used to solve a boundary value problem $y'' + a(x)y' + b(x)y = c(x), y'(0) = a, y'(1) + y(1) = b$.

(4) 27. Verify with Taylor expansion that the following two approximations are of second order, i.e. $O(h^2)$. 1) $y'(x) \approx (y(x+h) - y(x-h))/2h$, 2) $y''(x) \approx (y(x+h) - 2y(x) + y(x-h))/h^2$

(4) 28. Derive a second order difference approximation to $y^{(4)}(x)$ using the values $y(x + 2h), y(x+h), y(x), y(x-h)$ and $y(x-2h)$.

(4) 29. Derive a second order difference approximation to $y'(x)$ using the values $y(x), y(x-h)$ and $y(x-2h)$.

(2) 30. Given a second order ODE $y'' = f(x, y, y')$. Assume a Dirichlet boundary value is given in the left interval point $y(0) = 1$. Present two other ways in which a boundary contition can be given in the right boundary point.

(4) 31. When a boundary value problem $y'' = p(x)y' + q(x)y + r(x), y(0) = 1, y(1) = 0$ is solved with discretization based on the approximations $y''(x_n) \approx (y_{n+1} - 2y_n + y_{n-1})/h^2$ and $y'(x_n) \approx (y_{n+1} - y_{n-1})/2h$ we obtain a linear system $Ay = b$ of equations to be solved. Set up this system. Which special structure does the matrix A have?

(2) 32. What is meant by a tridiagonal $n \times n$ matrix A? The number of flops needed to solve a corresponding linear system of equations $Ax = b$ can be expressed as $O(n^p)$. What is the value of p? The number of bytes needed in the memory of a computer to store such a system can be expressed as $O(n^q)$. What is the value of q?

(2) 33. What is meant by a banded $n \times n$ matrix A? Describe some way of storing such a matrix in a sparse way. Same question for a profile matrix.

(2) 34. What input and output data are suitable for a Matlab function solving a tridiagonal system of linear equations $Ax = b$ if the goal is to save number of flops and computer memory?

(4) 35. The boundary value problem $-y'' = f(x), y(0) = y(1) = 0$ can be solved with an ansatz method based on Galerkin's method. Formulate the Galerkin method for this problem.

(4) 36. Classify the following PDEs with respect to linearity (linear or nonlinear), order (first, second), type (elliptic, parabolic, hyperbolic):

a) $u_t = u_{xx} + u$, b) $u_{xx} + 2u_{yy} = 0$, c) $u_t = (a(u)u_x)_x$, d) $u_t + uu_x = 0$, e) $u_y = u_x x + x$

(4) 37. Formulate the ODE-system $\dot{u} = Au + b$ when the Method of Lines is applied to the PDE-problem

$$u_t = u_{xx} + u, t > 0, 0 \leq x \leq 1, u(0, t) = 0, u(1, t) = 1, u(x, 0) = x$$

u_{xx} is approximated by the central difference formula.

(4) 38. Formulate the ODE-system $\dot{u} = Au + b$ when the Method of Lines is applied to the PDE-problem

$$u_t + u_x = 0, t > 0, 0 \leq x \leq 1, u(0, t) = 0, u(x, 0) = x$$

u_x is to be approximated by the backward Euler difference formula.

(2) 39. What is meant by an upwind scheme for solving $u_t + au_x = 0, a > 0$?

(2) 40. Derive a difference approximation and the corresponding stencil to e.g. the Laplace operator

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$$

(2) 41. What is the order of accuracy in t and x when the Method of Lines with central differences in the x-variable and the implicit Euler method is applied to e.g. the heat equation $u_t = u_{xx}$?

(2) 42. Give a second order accurate approximation difference formula of $(d(x)u_x)_x$.

(2) 43. Describe the Crank-Nicolson method for solving the heat equation.

(2) 44. Formulate the Galerkin method for the elliptic problem

$$u_{xx} + u_{yy} = f(x, y), (x, y) \in \Omega$$

$$u(x, y) = 0, (x, y) \in \partial\Omega$$

(2) 45. Formulate the Galerkin method for the parabolic problem

$$u_t = u_{xx}, u(x, 0) = f(x), u(0, t) = u(1, t) = 0$$

(2) 46. In an ansatz method for a 1D problem the solution $u(x)$ is approximated by the linear expression $u_h(x) = \sum \alpha_i \varphi_i(x)$, where the basis functions $\varphi_i(x)$ can be chosen differently. Give a description of the "roof-functions", i.e. the piecewise linear basis functions in x.

(2) 47. In an ansatz method for a 2D problem the solution $u(x, y)$ is approximated by the linear expression $u_h(x, y) = \sum \alpha_i \varphi_i(x, y)$, where the basis functions $\varphi_i(x, y)$ can be chosen differently. Give a description of the "pyramid-functions", i.e. the piecewise linear basis functions in x, y.

(2) 48. When solving PDE-problems in 2D and 3D with difference or ansatz methods we are lead to solving large linear systems of equations $Ax = b$. What is meant by a direct method and an iterative method for solving $Ax = b$? Give examples by name of some direct methods and some iterative methods.

(1) 49. What is meant by dissipation and dispersion when a conservation law is solved numerically?

(2) 50. What is meant by "fill-in" when solving a sparse linear system of equations $Ax = b$ with a direct method?

(3) 51. What is meant by the Cholesky-factorization of a symmetric positive definite matrix A? Can the following matrix A be Cholesky factorized?

(2) 52. If $Ax = b$ is rewritten on the form $x = Hx + c$ and the iteration $x_{k+1} = Hx_k + c$ is defined, what is the condition on H for convergence of the iterations to the solution of $Ax = b$? Also formulate a condition on H for fast convergence.

(2) 53. What is meant by the steepest descent method for solving $Ax = b$, where A is symmetric and positive definite?

(4) 54. Describe preconditioning when used with the steepest descent method. Assume that the matrix A is symmetric and positive definite.

(3) 55. For a hyperbolic PDE in x and t the solution $u(x,t)$ is constant along certain curves in the x,t-plane. What are these curves called? Give the analytic expression for these curves belonging to $u_t + 2u_x = 0$.

(4) 56. Given the system of PDEs

$$u_t + \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} u_x = 0$$

Is the system hyperbolic? Which are the characteristics?

(3) 57. Given the systems of PDEs

$$u_t + \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} u_x = 0$$

A solution is wanted on the interval $0 \leq x \leq 1$, $t \geq 0$. Suggest initial and boundary conditions that give a mathematically well posed problem.

(2) 58. What is meant by a numerical boundary condition?

(2) 59. Use Neumann analysis to verify that the upwind scheme is unstable when applied to $u_t = au_x, a > 0$.

(2) 60. What is meant by artificial diffusion? Why is that sometimes used when solving hyperbolic PDEs?