NADA, KTH, Lennart Edsberg

Tentamen i Kursen DN2221
Tillämpade Numeriska Metoder I
Thursday 2012-12-13 kl 14–19

No means of help allowed. To pass 13 credits of max 29 is needed. Present your answers in English or Swedish.

1. Given an $n \times n$ matrix A with eigenvalues λ_i and eigenvectors $\mathbf{c}_i$ (column vectors).

(1) a) A diagonalization transformation of A is the so called similarity transformation $S^{-1}AS = D$, where D is a diagonal matrix. How are S and D related to the eigenvalues and the eigenvectors of A?

(2) b) Assume

$$A = \begin{pmatrix} -1 & \alpha \\ \alpha & -1 \end{pmatrix},$$

where α is a real parameter. Which are the eigenvalues of A? Can A be transformed to diagonal form for all values of α? Motivate your answer.

2. Given the ODE $\ddot{u} + au = 0 \quad (*)$, where the parameter a is a real number different from zero.

(1) a) For which values of a are the solutions of $(*)$ stable, and for which unstable?

(2) b) When Euler's explicit method is applied to $(*)$ for which values of the constant stepsize h are the numerical solutions stable?

(2) c) Same question as b) when Euler's implicit method is applied.

3. Compute the coefficients a,b and c in the approximation formula $y''(0) = ay(0) + by(h) + cy(2h) + O(h^p)$ so that the order p is as high as possible.

(2) a) Which are the values of a,b and c?

(1) b) What is the value of p?

4. The following boundary value problem (BVP) occurs in chemical engineering:

$$\frac{1}{P}\frac{d^2u}{dx^2} - \frac{du}{dx} - Du = 0, \quad u(0) = 1 + \frac{1}{P}\frac{du}{dx}(0), \quad \frac{du}{dx}(1) = 0$$

where P is called the Peclet number and D the Dahmköhler number. This problem can be solved with e.g. the shooting method.

(2) a) Formulate the shooting method for this problem, i.e. formulate the initial value problem (IVP) when introducing $\frac{du}{dx}(0) = k$

(2) b) When the IVP is solved numerically for $P = 100$ and $D = 0.21$, the u-value at $x = 1$ turns out to be of order 10^{43} almost independently of k. This result came up both for a nonstiff method and a stiff method. Explain why! (this part b) can be solved without having solved a))

5. Consider the first-order PDE-system

$$\frac{\partial \mathbf{u}}{\partial t} + A\frac{\partial \mathbf{u}}{\partial x} = 0$$

where A is the matrix in 1b) above (α is real).

(1) a) Verify that the system is hyperbolic for all values of α.

(2) b) Use the diagonalization transformation in 1a) to decouple the system into two scalar hyperbolic PDE's.

(1) c) Let $\alpha = 3$. Which are the characteristics of the two decoupled PDEs in 5b).

6. Consider the space between two long concentric cylinders, the inner with radius a the outer with radius R. In that space there is a gas with variable concentration $c(r,z)$. The gas is transported with diffusion in the r-direction and convection in the z-direction. The walls of the cylinders are isolated, which means that $(\partial c/\partial r)_{r=a} = 0$ and $(\partial c/\partial r)_{r=R} = 0$. The concentration $c(r,z)$ is modeled by the following PDE:

$$v\frac{\partial c}{\partial z} = D(\frac{\partial^2 c}{\partial r^2} + \frac{1}{r}\frac{\partial c}{\partial r})$$

At the entrance of the space the concentration is $c(r,0) = c_0$. The parameters v and D are constants.

(1) a) Is this PDE elliptic, parabolic or hyperbolic?

(5) b) Use the Method of Lines to discretize the r-interval $[a,R]$ into discrete points $r_1 = a, r_2 = a+h_r, r_3 = a+2h_r, \ldots, r_N = R$. Add yourself ghost points if necessary. Formulate the system of ODEs

$$\frac{d\mathbf{c}}{dz} = B\mathbf{c} + \mathbf{q}, \mathbf{c}(0) = \mathbf{c}_0,$$

i.e. what is B, $\mathbf{q}$ and $\mathbf{c}_0$? What is the relation between a, R, N and h_r?

(4) c) Assume that the implicit Euler method with constant stepsize h_t is used to solve the ODE-system in 6b). A linear system of equations $T\mathbf{c}_{k+1} = M\mathbf{c}_k + \mathbf{g}, \quad \mathbf{c}_0 = \mathbf{c}(0)$ is obtained. Express how T, M and $\mathbf{g}$ are related to B, $\mathbf{q}$ and h_t.