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Tentamen i Kursen DN2221, SF2520 och SF1536
Tillämpade Numeriska Metoder I
Wednesday 2014-01-15 kl 14–19

No means of help allowed. To pass 13 credits of max 29 is needed. Present your answers in English or Swedish.

1. Given the differential equation $y'' = 4y' + 5y$.

(1) a) Calculate and give the general solution formula of this ODE.

(2) b) Assume $y(0) = 1$. Which value of $y'(0)$ gives a solution $y(x)$ for which $y(\infty) = 0$? Is this solution analytically stable?

(2) c) Assume we use implicit Euler to solve numerically the given ODE-problem for arbitrary initial values. What condition on the stepsize h is necessary in order to obtain a numerically *unstable* solution?

2. Given a differential equation of the form

$$\frac{d^2r}{dt^2} = a(t), \quad r(0) = 0, v(0) = 0 \qquad (*)$$

(The problem can be interpreted as Newton's second law for a particle with mass $m = 1$ where $a(t)$ is the time-dependent acceleration, $v(t)$ the velocity and $r(t)$ the position). Examples of two numerical methods for solving this problem is 1) Verlet's method and 2) Euler's explicit method.
Verlet's method can be derived from the two central difference approximations

$$r_{k+1} - 2r_k + r_{k-1} = h^2 a_k, \qquad v_k = (r_{k+1} - r_{k-1})/(2h)$$

where r is the position and v is the velocity.

(1) a) Rewrite the original ODE $(*)$ as a system of two first order ODEs and formulate Euler's explicit method using the variables r and v.

(1) b) Verify that Verlet's method can also be written

$$r_{k+1} = r_k + hv_k + \frac{h^2}{2}a_k, \qquad v_{k+1} = v_k + \frac{h}{2}(a_{k+1} + a_k)$$

(3) c) The local error is defined as the residual obtained when the exact solution is inserted into the method. Verify that Euler's and Verlet's method for the problem $(*)$ has local error $O(h^2)$ and $O(h^3)$ respectively.

3. (3) Assume we want to solve Poisson's equation on a quarter of a circle using the fivepoint method.

$$\triangle u = f(x, y), (x, y) \in \Omega$$

$$u = g(x, y), (x, y) \in \partial\Omega$$

If we use a quadratic grid, the boundary points will usually not be situated on the grid points. Therefore the molecule must be modified to an unsymmetric molecule where α and β are chosen so that the molecule fits to boundary points correctly.

Derive the coefficients a, b, c, d, e (they will depend on α, β) in the difference approximation

$$\triangle u(x_i, y_j) = \frac{au_{i,j-1} + bu_{i-1,j} + cu_{i,j} + du_{i+1,j} + eu_{i,j+1}}{h^2} + O(h^p)$$

so that the order p of approximation is as high as possible. What is p for the formula you derive? Hint: derive difference approximations for the x and y directions separately then add.

4. (2) What is meant by a stiff system of ODEs? Why is the classical Runge-Kutta method (and the ode23 and ode45 functions in MATLAB) not suitable for stiff problems?

5. Consider the first-order PDE-system

$$\frac{\partial \mathbf{u}}{\partial t} + \begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix} \frac{\partial \mathbf{u}}{\partial x} = 0$$

(1) a) Verify that the system is hyperbolic.

(2) b) Make a similarity transformation of the matrix to diagonal form and decouple the system into two scalar hyperbolic PDE's.

6. (3) The following PDE models the cooling of a hot metal sphere of radius R

$$\frac{\partial T}{\partial t} = \kappa \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 \frac{\partial T}{\partial r}), \quad 0 \leq r \leq R, \quad t \geq 0 \tag{1}$$

The initial condition states that the sphere has the same temperature T_0 at time $t = 0$ i.e.

$$T(r, 0) = T_0$$

The boundary conditions of the problem are

$$\frac{\partial T}{\partial r}(0, t) = 0 \qquad k \frac{\partial T}{\partial r}(R, t) = -\beta(T(R, t) - T_1)$$

The units of the variables and parameters of this problem are:
T, T_0, T_1 $[{}^0C]$, where T_1 is the temperaure of the environment, r, R $[m]$, t $[s]$, κ $[m^2/s]$, thermal diffusivity, k $[J/(m\,{}^0C)]$, conductivity, β $[J/(m^2\ {}^0C)]$, heat transfer coefficient. By a proper scaling of the variables the number of parameters can be reduced to only one parameter a:

$$\frac{\partial u}{\partial \tau} = \frac{1}{x^2} \frac{\partial}{\partial x}(x^2 \frac{\partial u}{\partial x}), \quad u(x, 0) = 1, \quad \frac{\partial u}{\partial x}(0, \tau) = 0, \frac{\partial u}{\partial x}(1, \tau) = au(1, \tau) \tag{2}$$

Find this scaling and show that it gives as result the dimensionless PDE problem given above. What is the algebraic relation between a and the original parameters? Show also that a is dimensionless.

Hint: A proper scaling for T is $T = T_1 + (T_0 - T_1)u$

7. The scaled version of the problem given above, i.e. eqn (2) is to be solved numerically with the Method of Lines.

(1) a) Introduce a suitable discretization of the x-axis. Number the the grid-points suitably and give the relation between the stepsize h and the number N of grid-points.

(2) b) At $x = 0$ the PDE in (2) is singular. By using l'Hõpital's rule the singularity at $x = 0$ can be removed. What form does the PDE take at $x = 0$ after this calculation?

(2) b) With the grid given in a) use the Method of Lines to discretize the x-variable in the PDE to a system of ODE's. Discretize also the boundary conditions of the problem.

(3) d) Formulate the resulting ODE-system with initial values.