NADA, KTH, Lennart Edsberg

Tentamen i Kursen DN2221
Tillämpade Numeriska Metoder I
Thursday 2009-12-14 kl 14–19

SOLUTIONS

1. a) The definition of e^{At} involving eigenvalues and eigenvectors is

$$e^{At} = Se^{Dt}S^{-1}$$

where $e^{Dt} = diag(e^{\lambda_1 t}, e^{\lambda_2 t}, \dots, e^{\lambda_n t})$

b) The eigenvalues of A are i and $-i$ The homogeneous linear systems of equations giving the eigenvectors are

$$\lambda_1 = i \quad \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = 0, \quad \lambda_2 = -i \quad \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = 0$$

giving $\mathbf{s}_1 = (i, -1)^T$ and $\mathbf{s}_2 = (i, 1)^T$. We obtain

$$S = \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}, \quad S^{-1} = \frac{1}{2i} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

Some calculations give

$$e^{At} = Se^{Dt}S^{-1} = \begin{pmatrix} cos(t) & sin(t) \\ -sin(t) & cos(t) \end{pmatrix}$$

2. a) The general solution is (for simple eigenvalues) $\mathbf{x}(t) = \sum \alpha_k \mathbf{s}_k e^{\lambda_k t}$. Since $e^{\lambda_k t} = \cos(t) + i\sin(t)$ there is no term that grows unboundedly as $t \to \infty$, hence the solution is stable (bounded).

b) Since all $h\lambda_k$ are situated on the imaginary axis and the stability region S_{EE} for Euler's explicit method does not contain the imaginary axis, the numerical solution will be unstable.

c) The stability region for the explicit midpoint method is the interval $[-i, i]$ on the imaginary axis, see the book pg 187. Hence if $h \leq 1/max\mu_k$ the numerical solution will be stable.

3. Symmetry properties give that $a = e$ and $b = d$. Taylor expansion of $ay(-2h) + by(-h) + cy(0) + by(h) + ay(2h)$ gives $(a + b + c + b + a)y(0) + h^2(2a + b/2 + b/2 + 2a)y''(0) + h^4(16a/12 + b/12) + O(h^6)$.The linear system of equations is $2a + 2b + c = 0, \quad 4a + b = 1/h^2, \quad 16a + b = 0$ and the solution $a = -1/12h^2$, $b = 16/12h^2$, $c = -30/12h^2$. Hence

$$y''(0) = \frac{-y(-2h) + 16y(h) - 30y(0) + 16y(h) - y(2h)}{12h^2} + O(h^4)$$

and the approximation is of fourth order.

4. See the book, pg 152. The characteristics are the parallell straight lines $x = -t + C$. See also formula (8.31) on page 158.

5. See the book chapter 6.5.

6.