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Tentamen i Kursen DN2221
Tillämpade Numeriska Metoder I
Thursday 2014-01-15 kl 14–19

SOLUTIONS

1. a) The characteristic equation is $\lambda^2 - 4\lambda - 5 = 0$. The characteristic roots are $\lambda_1 = 5$ and $\lambda_2 = -1$. Hence the general solution is $y(x) = Ae^{5x} + Be^{-x}$

b) $y(0) = 1 \rightarrow A + B = 1$. $y(\infty) = 0 \rightarrow A = 0$. Hence the particular solution of this boundary value problem is $y_p(x) = e^{-x}$. This solution is NOT stable since arbitrarily close to the trajectory $y_p(x) = e^{-x}$ there are solution curves containing some multiple of e^{5x} which makes $y_p(x)$ UNSTABLE.

c) The INSTABILITY region of implicit Euler is the disc $|h\lambda - 1| < 1$ in the $h\lambda$-plane, hence, if λ is real, the interval $0 < h\lambda < 2$. In our case $h\lambda_2$ is always in the STABILITY region while $h\lambda_1$ is in the INSTABILITY region if $0 < h < 2/5$.

2. a) The first order system is $\dot{r} = v \qquad \dot{v} = a(t)$, hence in matrix form

$$\begin{pmatrix} \dot{r} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ a(t) \end{pmatrix}$$

Eulers explicit method applied to this system gives

$$r_{k+1} = r_k + hv_k, \quad r_0 = 0 \qquad v_{k+1} = v_k + ha_k, \quad v_0 = 0$$

b) From the given central difference formulas $r_{k+1} - r_{k-1} = 2hv_k \rightarrow r_{k-1} = r_{k+1} - 2hv_k$. Inserted into the first difference formula this gives $r_{k+1} - 2r_k + r_{k+1} - 2hv_k = h^2 a_k \rightarrow r_{k+1} = r_k + hv_k + (h^2/2)a_k$, the formula for the position. The velocity formula can be obtained from the two difference equations: $r_{k+1} - 2r_k + r_{k-1} = h^2 a_k$ and $r_{k+2} - 2r_{k+1} + r_k = h^2 a_{k+1}$ Adding them together gives $(r_{k+2} - r_{k+1}) - (r_{k+1} - r_{k-1}) = h^2(a_k + a_{k+1})$ which can also be written $v_{k+1} = v_k + (h/2)(a_k + a_{k+1})$

c) Explicit Euler: the residual for the position variable $r(t)$ is

$$r(t+h) - r(t) - h\dot{r}(t) = r(t) + h\dot{r}(t) + (h^2/2)\ddot{r}(t) - r(t) - h\dot{r}(t) = O(h^2)$$

. The residual for the velocity variable $v(t)$ is

$$v(t+h) - v(t) - h\dot{v}(t) = v(t) + h\dot{v}(t) + (h^2/2)\ddot{v}(t) - v(t) - h\dot{v}(t) = O(h^2)$$

In the book (pg 45) the local error is derived to be of order one, but there the definition of the local error is a little different!
Verlet's method: the residual for the position variable $r(t)$ is

$$r(t+h) - r(t) - h\dot{r}(t) - (h^2/2)\ddot{r}(t) = r(t) + h\dot{r}(t) + (h^2/2)\ddot{r}(t) + O(h^3) - r(t) - h\dot{r}(t) - (h^2/2)\ddot{r}(t) = O(h^3)$$

. The residual of the velocity variable $v(t)$ is

$$v(t+h) - v(t) - (h/2)(\dot{v}(t+h) + \dot{v}(t)) =$$

$$v(t) + h\dot{v}(t) + (h^2/2)\ddot{v}(t) + O(h^3) - v(t) - (h/2)(\dot{v}(t) + h\ddot{v}(t) + O(h^2) + \dot{v}(t)) = O(h^3)$$

3. Start with the difference formula in the x-direction:

$$\frac{\partial^2 u}{\partial x^2}(x,y) \approx \frac{bu(x-h,y)+cu(x,y)+du(x+\alpha h,y)}{h^2}$$

Taylorexpansion around (x,y) in the x-direction gives the linear system of three algebraic equations: $b+c+d=0$, $-bh+d\alpha h=0$ and $((h^2/2)b+(h^2/2)\alpha^2 d)/h^2=1$ with the solution $b=2\alpha/(\alpha+\alpha^2)$, $d=2/(\alpha+\alpha^2)$ och $c=-2/\alpha$. When $\alpha=1$ this is in accordance with the usual central difference formula!
Now replace x by y and α by β and we get a similar formula $a=2\beta/(\beta+\beta^2)$, $e=2/(\beta+\beta^2)$ and $c=-2/\beta$. The sum of these two gives a modified 5-point formula for the laplace operator:

$$\triangle u(x_i,y_j) \approx \frac{1}{h^2}(\frac{2}{1+\alpha}u_{i-1,j}+\frac{2}{1+\beta}u_{i,j-1}+\frac{2}{\alpha(1+\alpha)}u_{i+1,j}+\frac{2}{\beta(1+\beta)}u_{i,j+1}-2(\frac{1}{\alpha}+\frac{1}{\beta})u_{i,j})$$

4. See the book, pg 52

5. a) The matrix A is diagonalizable with real eigenvalues $\lambda_1=1$ and $\lambda_2=5$, hence the system is hyperbolic.

b) The eigenvalue problem $AS=AD$ gives $S^{-1}AS=D$, where

$$D=\begin{pmatrix}1&0\\0&5\end{pmatrix},\qquad S=\begin{pmatrix}-1&1\\2&2\end{pmatrix},\qquad S^{-1}=-\frac{1}{4}\begin{pmatrix}2&-1\\-2&-1\end{pmatrix}$$

With the transformation $\mathbf{u}=S\mathbf{v}$ we get the two uncoupled hyperbolic PDEs:

$$\frac{\partial v_1}{\partial t}+\frac{\partial v_1}{\partial x}=0,\qquad \frac{\partial v_2}{\partial t}+5\frac{\partial v_2}{\partial x}=0$$

6. With the transformations $T=T_1+(T_0-T_1)u$, $r=Rx$ and $t=\alpha\tau$, where α is to be determined, we get for the PDE:

$$\frac{T_0-T_1}{\alpha}\frac{\partial u}{\partial \tau}=\frac{\kappa}{R^2}\frac{1}{R}\frac{\partial}{\partial x}(R^2x^2(T_0-T_1)\frac{1}{R}\frac{\partial u}{\partial x})$$

. With $\alpha=R^2/\kappa$ the PDE becomes dimensionsless.
The initial condition: $T_1+(T_0-T_1)u(x,0)=T_1$, i.e. $u(x,0)=1$.
The boundary conditions: $\frac{T_0-T_1}{R}\frac{\partial u}{\partial x}(0,\tau)=0$ and $k\frac{T_0-T_1}{R}\frac{\partial u}{\partial x}(1,\tau)=-\beta(T_1+(T_0-T_1)u(1,\tau)-T_1)$ which gives $\frac{\partial u}{\partial x}(1,\tau)=-\frac{\beta R}{k}u(1,\tau)$ and $a=-\beta R/k$.

7) a) Introduce the grid points $x_i=(i-1)h, i=0,1,\ldots,N,N+1$, where x_0 and x_{N+1} are ghost points, $x_1=0$ and $x_N=1$. Hence $(N-1)h=1$. This is one possible way of introducing an equidistant grid for the problem.

b) Write the right hand side as

$$\frac{1}{r^2}\frac{\partial}{\partial x}(x^2\frac{\partial u}{\partial x})=\frac{2}{x}\frac{\partial u}{\partial x}+\frac{\partial^2 u}{\partial x^2}$$

At $x=0$ the first term takes the form $2\frac{\partial^2 u}{\partial x^2}$ using l'Hôpital's rule. Hence the PDE at $x=0$ takes the form

$$\frac{\partial u}{\partial \tau}=3\frac{\partial^2 u}{\partial x^2}$$

c) With the MoL the PDE turns into a system of ODEs:

$$\frac{du_1}{d\tau} = 3\frac{u_2 - 2u_1 + u_0}{h^2}, \quad u_1(0) = 1$$

$$\frac{du_i}{d\tau} = \frac{2}{x_i}\frac{T_{i+1} - T_{i-1}}{2h} + \frac{T_{i+1} - 2T_i + T_{i-1}}{h^2}, \quad u_i(0) = 1, \quad i = 2, 3, \ldots, N$$

The boundary conditions are discretized:

$$\frac{u_2 - u_0}{2h} = 0, \qquad \frac{u_{N+1} - u_{N-1}}{2h} = au_N$$

d) The systems of ODEs written on matrix-vector form:

$$\frac{d\mathbf{u}}{d\tau} = A\mathbf{u} + \mathbf{b}, \quad \mathbf{u}(0) = \mathbf{u}_0$$

where A is tridiagonal, $\mathbf{b} = 0$ and $\mathbf{u}_0 = (1, 1, ..., 1)^T$. The nonzero elements $a_{i,j}$ of A are: the first row $a_{1,1} = -6/h^2$ and $a_{1,2} = 6/h^2$, for row number $i = 2, 3, ..., N-1$ $a_{i,i-1} = 1/h^2 - 1/(x_i h)$, $a_{i,i} = -2/h^2$,$a_{i,i+1} = 1/h^2 + 1/(x_i h)$. The nonzero elements of the last row are $a_{N,N-1} = 2/h^2$ and $a_{N,N} = 2a + (2ah - 2)/h^2$.