Introduction to Arnoldi method

SF2524 - Matrix Computations for Large-scale Systems

KTH Royal Institute of Technology (Elias Jarlebring)

2014-11-07

Agenda lecture 2

- Introduction to Arnoldi method
- Gram-Schmidt - efficiency and roundoff errors
- Derivation of Arnoldi method
- (Convergence characterization)

Main eigenvalue algorithms in this course

- Fundamental eigenvalue techniques (Lecture 1)
- Arnoldi method (Lecture 2-3).

- QR-method (Lecture 8-9).

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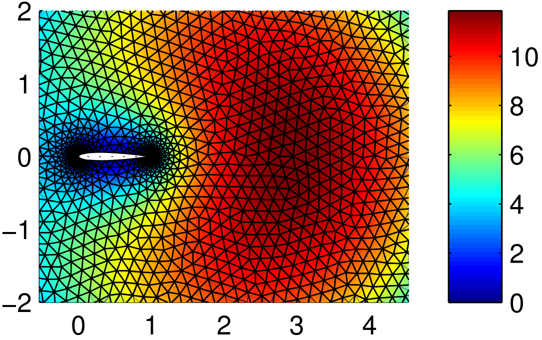
- Fundamental eigenvalue techniques (Lecture 1)
- Arnoldi method (Lecture 2-3).
  Typically suitable when
  - we are interested in a small number of eigenvalues,
  - the matrix is large and sparse
  - Currently solvable size on desktop $m \sim 10^6$ (depending on structure)
- QR-method (Lecture 8-9).
  Typically suitable when
  - we want to compute all eigenvalues,
  - the matrix does not have any particular easy structure.
  - Solvable size on desktop $m \sim 1000$.

Finite-element method example (slide 1/2)

Laplace eigenvalue problem

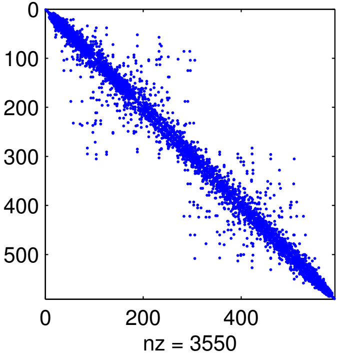
$$\Delta u = \lambda u$$

with Dirichlet boundary conditions outside wing profile.



Finite-element method example (slide 2/2)

$\Rightarrow$ Problem: Compute eigenvalues of matrix $A \in \mathbb{R}^{m\times m}$ where m large but few non-zero elements:



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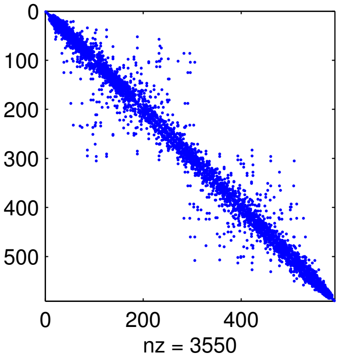

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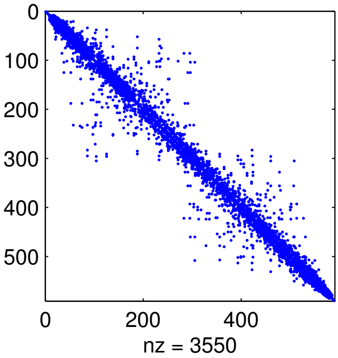

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More than 10 hours $\Rightarrow$ we need a different method.

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are called *Ritz pairs*.

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- Use Rayleigh-Ritz on $Q = (q_1, \ldots, q_n)$ where

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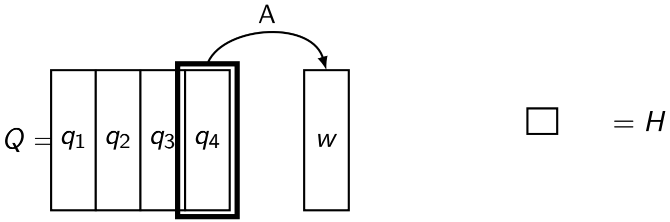
Arnoldi method graphically

Graphical illustration of algorithm:

$$Q = \begin{array}{|c|c|c|c|} \hline q_1 & q_2 & q_3 & q_4 \\ \hline \end{array} \qquad\qquad \square \quad = H$$

Arnoldi method graphically

Graphical illustration of algorithm:



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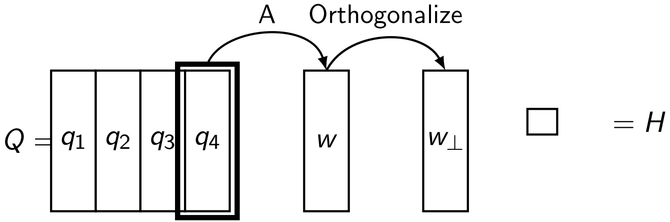

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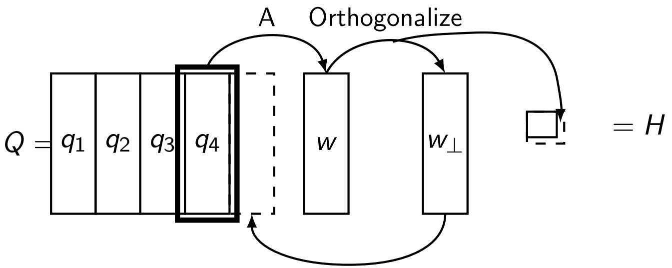

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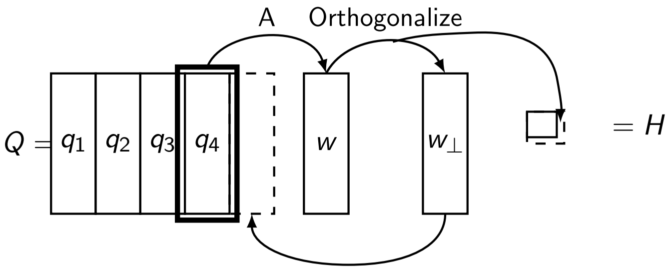

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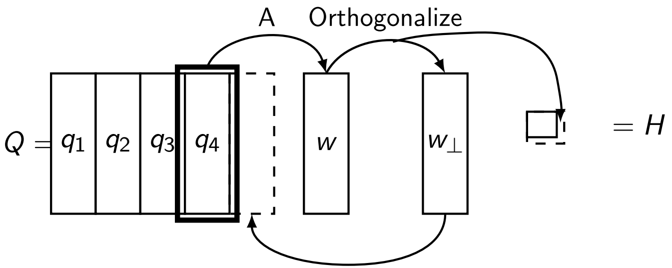

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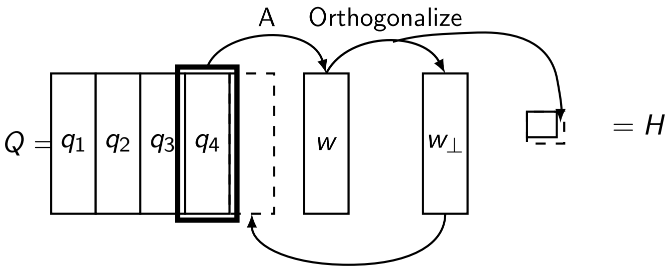

After iteration: Take eigenvalues of H as approximate eigenvalues.

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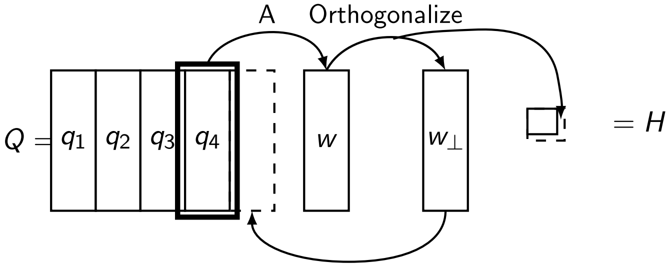

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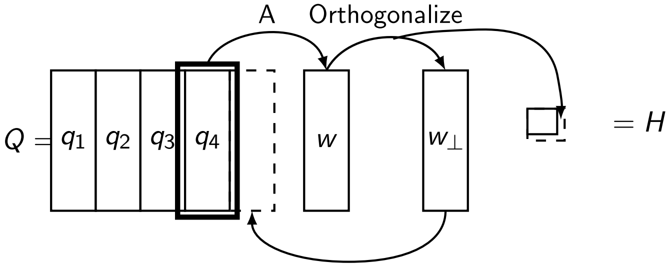

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We will now...

(1) derive a good orthogonalization procedure: variants of Gram-Schmidt,

(2) show that Arnoldi generates a Rayleigh-Ritz approximation,

(3) characterize the convergence.