# Introduction to functions of matrices

SF2524 - Matrix Computations for Large-scale Systems

## Functions of matrices

Matrix functions (or functions of matrices) will in this block refer to a certain class of functions

$$f : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$$

that are consistent extensions of scalar functions.

## Simplest examples

- If $f(t) = b_0 + b_1 t + \cdots + b_m t^m$ it is natural to define

$$f(A) = b_0 I + b_1 A + \cdots + b_m A^m.$$

- If $f(t) = \frac{\alpha+t}{\beta+t}$ it is natural to define

$$f(A) = (\alpha I + A)^{-1}(\beta I + A) = (\beta I + A)(\alpha I + A)^{-1}.$$

Not matrix functions:

$$f(A) = \det(A), \quad f(A) = \|A\|$$

# Definitions

Essentially equivalent definitions:

- Jordan based: §9.1.1
- Taylor serites: §9.1.2
- (Cauchy integral: §9.2.7)

**Definition (Taylor series §9.1.2)**

Consider an analytic function $f : \mathbb{C} \to \mathbb{C}$, with a Taylor expansion

$$f(t) = f(0) + \frac{f'(0)}{1!} t + \cdots .$$

# Definitions

Essentially equivalent definitions:

- Jordan based: §9.1.1
- Taylor serites: §9.1.2
- (Cauchy integral: §9.2.7)

## Definition (Taylor series §9.1.2)

Consider an analytic function $f : \mathbb{C} \to \mathbb{C}$, with a Taylor expansion

$$f(t) = f(0) + \frac{f'(0)}{1!}t + \cdots .$$

The matrix function $f(A)$ is defined as

$$f(A) := \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!}A^i = f(0)I + \frac{f'(0)}{1!}A + \cdots .$$

# Applications

## The most well-known non-trivial matrix function

Consider the linear autonomous ODE

$$y'(t) = Ay(t), \quad y(0) = y_0$$

The **matrix exponential** is the function that

$$y(t) = \exp(tA)y_0$$

More generally, the solution to

$$y'(t) = Ay(t) + f(t)$$

satisfies

$$y(t) = \exp(tA)y_0 + \int_0^t \exp(A(t-s))f(s)\, ds$$

For some problems much better than traditional time-stepping methods.

## Trigonometric matrix functions and square roots

Suppose $y(t) \in \mathbb{R}^n$ satisfies

$$y''(x) + Au(x) = 0 \;\; y(0) = y_0, \;\; y'(0) = y_0'.$$

The solution is explicitly given by

$$y(t) = \cos(\sqrt{A}t)y_0 + (\sqrt{A})^{-1}\sin(\sqrt{(}A)t)y_0'.$$

## Matrix logarithm in Markov chains

The transition probability matrix $P(t)$ is related to the transition intensity matrix $Q$ with

$$P(t) = \exp(Qt)$$

were $Q$ satisfies certain properties. Inverse problem: Given $P(1)$ is there $Q$ such that the properties are satisfied. Method: Compute

$$Q = \log(P(0))$$

and check properties.

## Further applications in

- Solving the Riccati equation (in control theory)
- Study of stability of time-delay systems
- Orthogonal procrustes problems
- Geometric mean
- $\cdots$