Addition to GVL 9.4.1

Analysis of Newton method for matrix sign function

(A) S_k invertible

and $\mathrm{sign}(A) = \mathrm{sign}(A)$

(B) local quadratic convergence

(C) global convergence

(A)

Lemma: Suppose $A \in \mathbb{C}^{n \times n}$ and $\lambda(A) \subset \mathbb{C} \setminus i\mathbb{R}$. Then,

a) $\lambda(S_k) \subset \mathbb{C} \setminus i\mathbb{R}$ for $k = 1, \dots$

b) $\text{sign}(A) = \text{sign}(S_0) = \text{sign}(S_k)$ for $k = 1, \dots$

Proof: Let $S_0 = A = X \Lambda_0 X^{-1}$ be Jordan form of A. Hence,

$$S_1 = \frac{1}{2}(S_0 + S_0^{-1})$$
$$= X\left(\frac{1}{2}(\Lambda_0 + \Lambda_0^{-1})\right) X^{-1}$$
$$= X \Lambda_1 X^{-1}.$$

where $\Lambda_1 = \frac{1}{2}(\Lambda_0 + \Lambda_0^{-1})$ is upper triangular. Analogously

$$S_k = X \Lambda_k X^{-1}$$

where $\Lambda_k = \frac{1}{2}(\Lambda_{k-1} + \Lambda_{k-1}^{-1})$ is upper triangular.

Hence, for any $\lambda \in \lambda(S_k)$

$$\lambda = \frac{1}{2}\left(a+bi + \frac{1}{a+bi}\right) = (*)$$

where $a+bi \in \lambda(S_{k-1})$.

Moreover,

$(*)$ $\lambda = \frac{1}{2} a \left(1 + \frac{1}{a^2+b^2}\right) +$

$\frac{1}{2} b \left(1 - \frac{1}{a^2+b^2}\right) i.$

In order to show (a) suppose

$\lambda(S_{k-1}) \subset \mathbb{C} \setminus i\mathbb{R}$. Any $\lambda \in \lambda(S_k)$

satisfies

$$\operatorname{Re} \lambda = \frac{1}{2} a \cdot \left(1 + \frac{1}{a^2+b^2}\right) \neq 0$$

since $a \neq 0$ (because $a+bi \in \lambda(S_{k-1}) \subset \mathbb{C} \setminus i\mathbb{R}$).

Hence, $\lambda \notin i\mathbb{R}$.

In order to show (b); note

that $\operatorname{sign}(S_{k+1}) = \operatorname{sign}\left(\frac{1}{2}(S_k + S_k^{-1})\right)$

$$= \operatorname{sign}\left(\frac{1}{2}\left(X \Lambda_k X^{-1} + X \Lambda_k^{-1} X^{-1}\right)\right)$$

$$= X \operatorname{Sign}\left(\frac{1}{2}\left(\Lambda_k + \Lambda_k^{-1}\right)\right) X^{-1}$$

$$= X \operatorname{Sign}(\Lambda_k) X^{-1}$$

Since from $(*)$

$$\operatorname{sign}(\lambda) = \operatorname{sign}\left(\frac{1}{2} a \left(1 + \frac{1}{a^2+b^2}\right)\right) =$$

$$= \operatorname{sign}(a).$$

(13)

Local quadratic convergence

Let $S = \text{sign}(A)$. Then,

$S^2 = I$ and $S_k S = S S_k$.

Moreover,

$$(S_k - S)^2 = S_k^2 - 2 S S_k + \underbrace{S^2}_{=I}$$

$$= 2 S_k (S_{k+1} - S)$$

Hence,

$$\| S_{k+1} - S \| \leq \frac{1}{2} \| S_k^{-1} \| \, \| S_k - S \|^2$$

(c) Global convergence

See details on GVL pg 538

Define: $G_k := (S_k - S)(S_k + S)^{-1}$

Show $G_k = G_0^{2^k}$

Show $G_k \to 0$ since

$|\lambda(G_0)| < 1$

Hence, $G_k \cdot (S_k + S) = (S_k - S)$

$$S_k = S(I + \underbrace{G_k}_{\to 0})(I - \underbrace{G_k}_{\to})^{-1} \to S.$$