SF2524 Matrix Computations for Large-scale Systems
Exam

Aids: None   Time: Four hours

Grades: E: 16 points, D: 19 points, C: 22 points, B: 25 points, A: 28 points (out of the possible 35 points, including bonus points from homeworks).
Notation in exam / course: $P_m := \{$ polynomials of degree $m\}$, $P_m^0 := \{p \in P_m : p(0) = 1\}$.

Problem 1 (4p) We consider the special case of Rayleigh quotient for real matrices and real eigenpairs. Let $r(v) := v^T Av/v^T v$ be the Rayleigh quotient where $A \in \mathbb{R}^{n\times n}$. Suppose $v_* \in \mathbb{R}^n$ is a normalized real eigenvector corresponding to a real eigenvalue $\lambda_* \in \mathbb{R}$.

(a) State the Rayleigh quotient iteration.
(b) Derive formula for $w \in \mathbb{R}^n$ such that $r(v) = \lambda_* + w^T(v - v_*) + \mathcal{O}(\|v - v_*\|^2)$.
Hint: For sufficiently small x, we have $\frac{1}{1+x} = 1 - x + \mathcal{O}(x^2)$
(c) In what sense is the Rayleigh quotient better for symmetric matrices?

Problem 2 (3p) Erik the engineer discretizes a partial differential equation and finds out that he needs to solve $Ax = b$ where $A \in \mathbb{R}^{n\times n}$ is a large sparse symmetric positive definite matrix with condition number $\kappa(A) := \|A\|\|A^{-1}\| \gg 1$. He needs to decide if he should use an implementation of CG (Conjugate Gradients) or CGN (Conjugate Gradients Normal equations) for his problem.

(a) What is the relationship between CG and CGN?
(b) According to a theorem in the course, the error of CG is bounded by

$$\frac{\|e_n\|_A}{\|e_0\|_A} \le 2\left(\frac{\sqrt{\kappa(A)}-1}{\sqrt{\kappa(A)}+1}\right)^m. \tag{1}$$

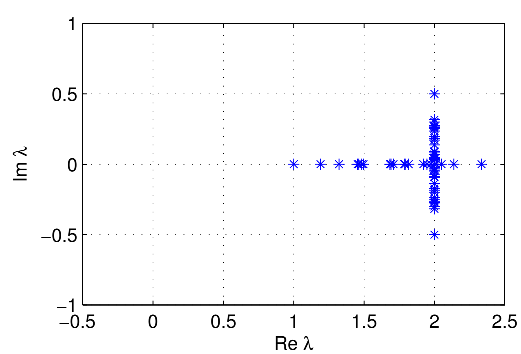
Derive a bound for the error in CGN in terms of $\kappa(A)$ based on (1). Which method has in general faster convergence in Erik's case?

Problem 3 (4p) Suppose the eigenvalues of a matrix A are given as in the figure to the right. Suppose eigenvalues are distinct and $\kappa(V) = 1$.

(a) State a definition of the approximation generated by GMRES.
(b) We apply m steps of GMRES to $Ax = b$ and get approximation x_m. Derive a α and β such that

$$\frac{\|Ax_m - b\|}{\|b\|} \le \alpha\beta^m.$$

Clearly specify which theorems/results you use and what quantities you observe in the figure. You may use any theorem/result derived in the course.



Problem 4 (4p)
Let

$$A = \begin{bmatrix} 1 & -3 \\ -1 & 1 \end{bmatrix}.$$

(a) Describe the basic QR-method and use the output to the right to compute one step for A. Describe clearly how you use the output.
(b) Describe the shifted QR-method and compute one step for A.

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>> F=[1 1; -1 ,1];
>> T=[1,-2;0,-1];
>> F*T
ans =
     1    -3
    -1     1
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Problem 5 (4p) Let $T \in \mathbb{R}^{3\times 3}$ be an **upper triangular matrix** with distinct eigenvalues and let

$$f(T) = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{13} \\ f_{31} & f_{32} & f_{33} \end{bmatrix},$$

where the matrix function is defined via the Jordan definition or Taylor definition.

(a) Derive explicit formulas for f_{11}, f_{21}, f_{31}, f_{22}, f_{32}, f_{33} involving only elements of T.
(b) Derive an explicit formula for f_{12} involving only elements of T and the values in (a).

Problem 6 (5p) Let Q_m and $\underline{H}_m$ be an Arnoldi factorization of A.

(a) How are the eigenvalue approximations computed from the Arnoldi factorization in *Arnoldi's method for eigenvalue problems*?
(b) State the Krylov approximation f_m of $f(A)b$.
(c) Under certain conditions on A and f, the error of the Krylov approximation is bounded by

$$\|f(A)b - f_m\| \leq 2\|b\| \min_{p\in P_{m-1}} \max_{z\in\Omega} |f(z) - p(z)|$$

where Ω is a compact set containing all eigenvalues. Suppose the eigenvalues are real and in the interval $I = (0.5, 1.5)$. Determine α and β such that

$$\|f(A)b - f_m\| \leq \alpha \frac{\beta^m}{m!},$$

for all m and for any function satisfying $|f^{(k)}(x)| \leq C$ for all $x \in I$, $k \in \mathbb{Z}$.
Hint: The remainder of the truncated Taylor series satisfies $f(x) - \sum_{k=0}^{m-1} (x-\mu)^k \frac{f^{(k)}(\mu)}{k!} = (x-\mu)^m \frac{f^{(m)}(\xi)}{m!}$*, for some value* $\xi \in [x, \mu]$.

Problem 7 (5p) Suppose $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix, such that $A = V\Lambda V^T$ where $V^TV = I$ and the columns of V are eigenvectors. We start the Arnoldi method with a vector b such that it is orthogonal to the first eigenvector: $x_1^T b = 0$. In this case, the error indicator for Arnoldi's method for eigenvalue problems (for the second eigenvalue λ_2) is bounded by

$$\|(I - Q_m Q_m^T)x_2\| \leq \xi_2 \tilde{\varepsilon}_2^{(m)} \tag{2}$$

for some constant ξ_2, where

$$\tilde{\varepsilon}_2^{(m)} := \min_{\substack{p\in P_{m-1} \\ p(\lambda_2)=1}} \max(|p(\lambda_3)|, \ldots, |p(\lambda_n)|). \tag{3}$$

(a) Suppose $\lambda_k = 1 + \sin\left(\frac{(k-1)\pi}{2(n-1)}\right)$, $k = 1, \ldots n$. Use (2) to derive β such that $\|(I - QQ_m^T)x_2\| \leq \alpha\beta^{m-1}$ for some α.
(b) Prove (2) and (3) and derive a formula for the constant ξ_2.